Symbolic dynamics and simulation theorems

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Séminaire AUTOMATA, IRIF
May, 2017

Motivation

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$T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ given by $T(x, y) = (2x + y, x + y) \mod 1$.

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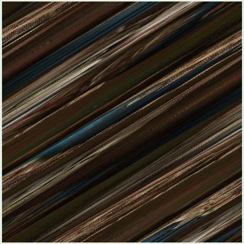

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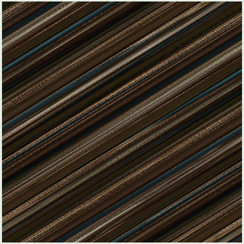

Coding of an orbit

A dynamical system might be complicated. A good idea is to code its trajectories using a partition.

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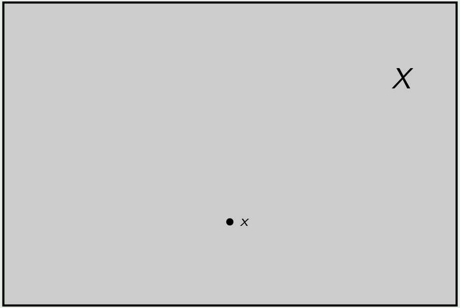

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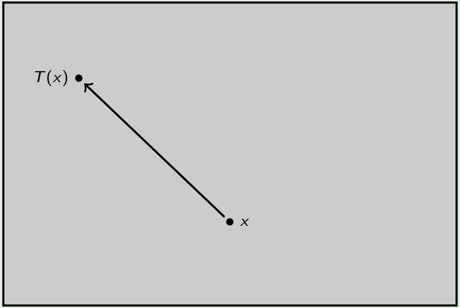

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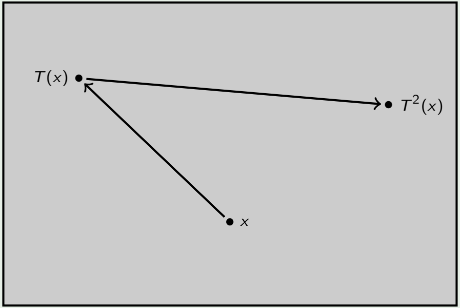

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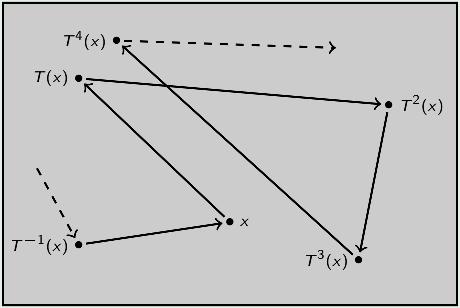

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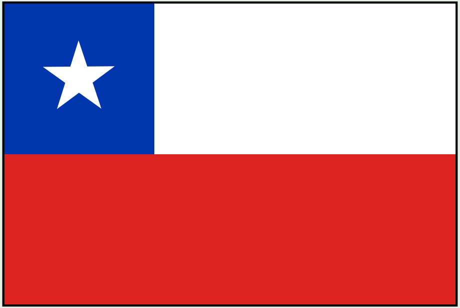

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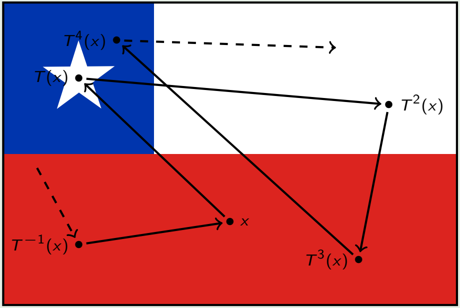

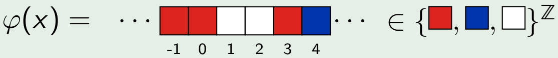

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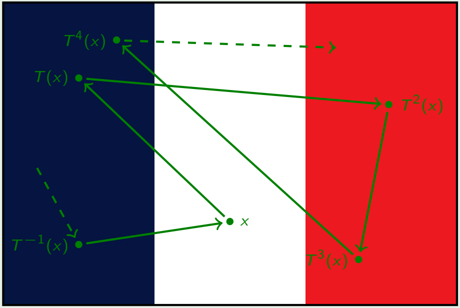

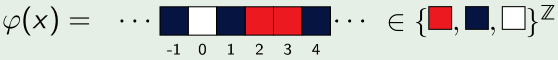

Motivation

Why would coding be a good idea?

- Instead of a complicated homeomorphism we get a shift action.
- If the coding is “good”, dynamical properties are preserved.
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Theorem

If X is a Cantor space and T is an expansive action then (X, T) is conjugate to a symbolic system (a subshift).

Definitions

- G is a countable group.
- $\mathcal{A}$ is a finite alphabet. Ex: $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of configurations, $x : G \to \mathcal{A}$
- $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is the left shift action given by:

$$\sigma(h, x)_g := \sigma^h(x)_g = x_{h^{-1}g}.$$

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Definition: full G-shift

The pair $(\mathcal{A}^G, \sigma)$ is called the *full G-shift*.

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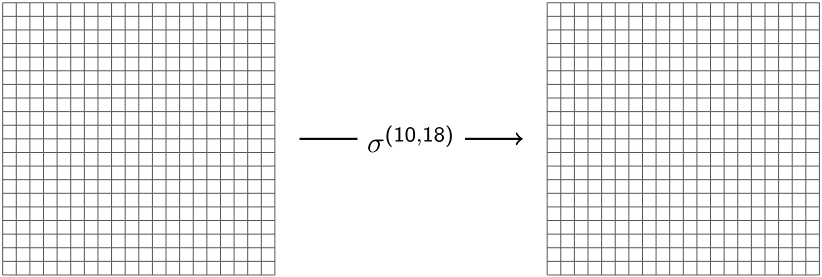

Figure: A random configuration $x \in \{\blacksquare, \square\}^{\mathbb{Z}^2/20\mathbb{Z}^2}$ and its image by $\sigma^{(10,18)}$.

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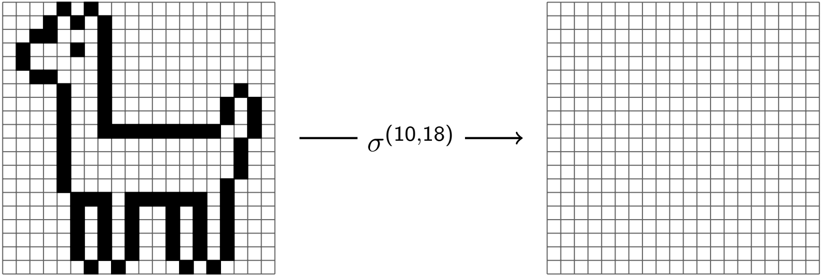

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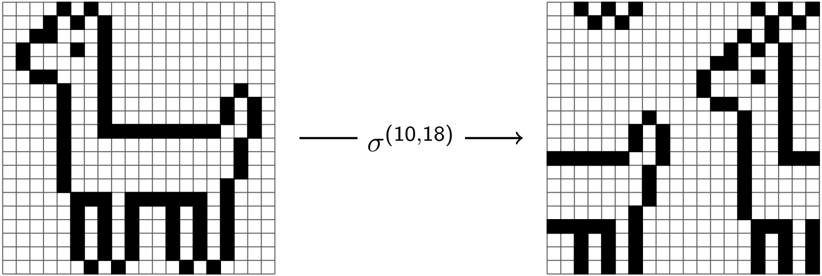

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Examples:

- $X = \left\{ x \in \{0,1\}^{\mathbb{Z}} \mid \text{no two consecutive 1's in } x \right\}$
- $X = \left\{ x \in \{0,1\}^{G} \mid \text{finite CC of 1's are of even length} \right\}$

Definitions

Luckily, subshifts can also be described in a combinatorial way.

- A *pattern* is a finite configuration, i.e. $p \in \mathcal{A}^F$ where $F \subset G$ and $|F| < \infty$. We denote $\mathrm{supp}(p) = F$.
- A *cylinder* is the set $[a]_g := \{x \in \mathcal{A}^G \mid x_g = a\}$.
- $$[p] := \bigcap_{g \in \mathrm{supp}(p)} [p_g]_g.$$

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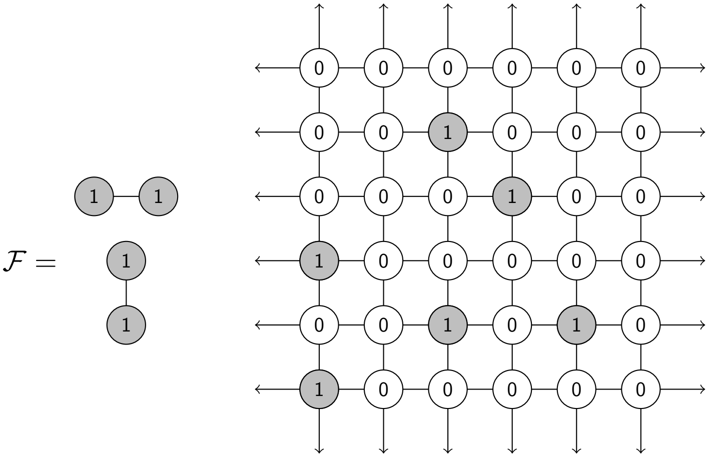
Proposition

A subshift is a set of configurations avoiding patterns from a set $\mathcal{F}$.

$$X = X_{\mathcal{F}} := \mathcal{A}^G \setminus \bigcup_{g \in G, p \in \mathcal{F}} \sigma^g([p])$$

Example in $\mathbb{Z}^2$: Hard-square shift

Example: Hard-square shift. X is the set of assignments of $\mathbb{Z}^2$ to $\{0, 1\}$ such that there are no two adjacent ones.



Example: one-or-less subshift

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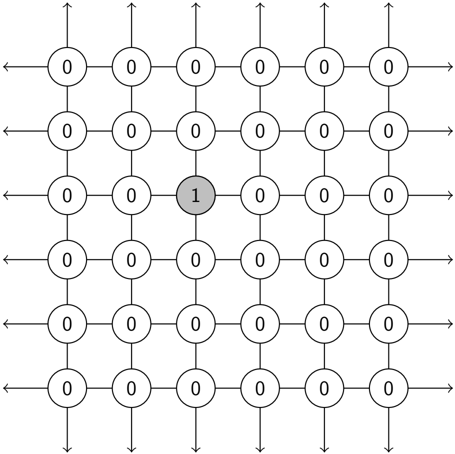
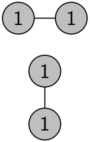
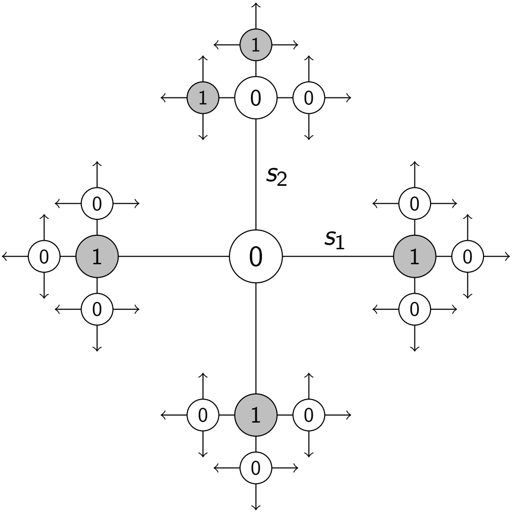
$$X_{\leq 1} := \{x \in \{0,1\}^G \mid 0 \notin \{x_u, x_v\} \implies u = v\}.$$



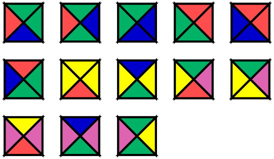
Example: Fibonacci in F_2.

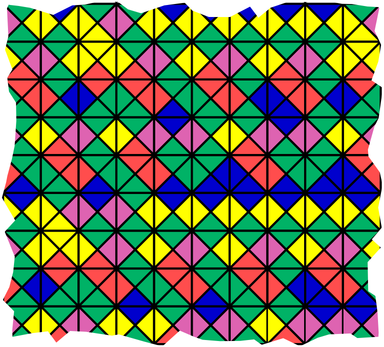
$\mathcal{F} =$ 



Example: Wang tiling

A subshift defined by Wang tiles: two tiles can be put next to each other only their adjacent colors match.





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- A simple class with respect to the combinatorial definition
- 2D-SFT $\equiv$ Wang tilings.

Strongly aperiodic subshifts

Definition (Strongly aperiodic subshift)

A subshift $X \subset A^G$ is *strongly aperiodic* if all its configurations have trivial stabilizer

$$\forall x \in X, \forall g \in G,\ \sigma^g(x) = x \Rightarrow g = 1_G.$$

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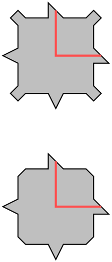
Theorem (Berger 1966, Robinson 1971, Kari 1996, Jeandel & Rao 2015)

There exist strongly aperiodic SFTs on $\mathbb{Z}^2$.

Example of strongly aperiodic $\mathbb{Z}^2$-SFT: Robinson tileset

The Robinson tileset, where tiles can be rotated and reflected.



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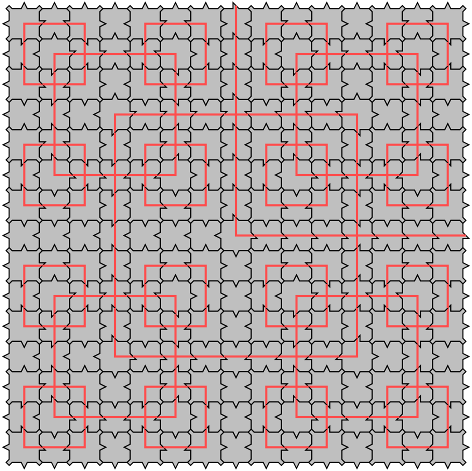

Some recent results

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- Surface groups (Cohen & Goodman-Strauss, 2015).
- groups $\mathbb{Z}^2 \rtimes H$ where H has decidable **WP** (B & Sablik, 2016).

Simulation Theorems

What is a simulation theorem?

Finitely presented group

A group G is finitely presented if it can be described as $G = \langle S | R \rangle$ where both S and $R \subset (S \cup S^{-1})^*$ are finite.

$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$$

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Recursively presented group

A group G is recursively presented if it can be described as $G = \langle S | R \rangle$ where $S \subset \mathbb{N}$ and $R \subset (S \cup S^{-1})^*$ are recursive sets.

$$L = \langle a, t \mid (at^n at^{-n})^2, n \in \mathbb{N} \rangle$$

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Corollary [Theorem: Novikov 1955, Boone 1958]

There are finitely presented groups with undecidable word problem

Apply Highman's theorem to
$G = \langle a, b, c, d \mid b^{-n}ab^{n} = c^{-n}dc^{n}, n \in \mathtt{HALT} \rangle$.

A dynamical simulation theorem

Effectively closed dynamical system

An action $T : \mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{N}}$ is *effectively closed* if there exists a Turing machine which on entry $w \in \{0,1\}^*$ enumerates a language L such that:

$$T([w]) = \{0,1\}^{\mathbb{N}} \setminus \bigcup_{u \in L} [u].$$

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Theorem (Hochman, 2009)

Let $T : \mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{N}}$ be an effectively closed action. There exists a $\mathbb{Z}^3$-SFT such that its $\mathbb{Z}$-subaction is an almost 1-1 extension (very close) of T.

The case of subshifts

sofic subshift

A subshift is called *sofic* if it is the image of an SFT by a local recoding.

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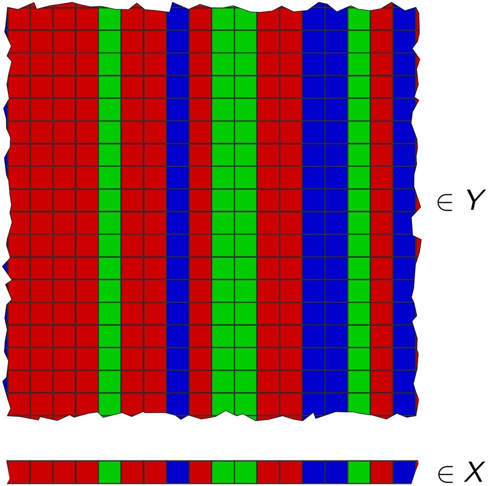
Theorem (Aubrun-Sablik, Durand-Romaschenko-Shen 2010)

For effectively closed $\mathbb{Z}$-subshift X there exists a $\mathbb{Z}^2$-sofic subshift Y such that every $y \in Y$ is a periodic vertical extension of a configuration $x \in X$.

The case of subshifts



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It is complicated to come up with $\mathbb{Z}^2$-SFTs which are strongly aperiodic, however, finding a $\mathbb{Z}$-effectively closed subshift which is aperiodic is easy.

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Example

Let x be a fixed point of the Thue-Morse substitution.

$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \dots$$

Then $X = \overline{\mathrm{Orb}_\sigma(x)}$ is strongly aperiodic and effectively closed.

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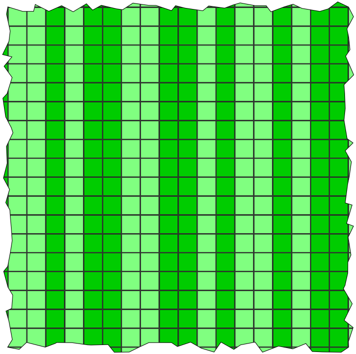

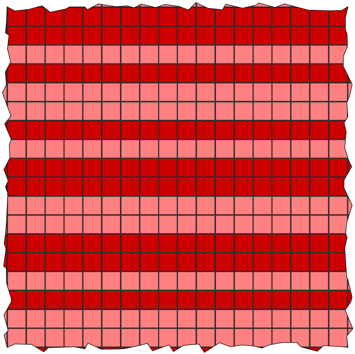

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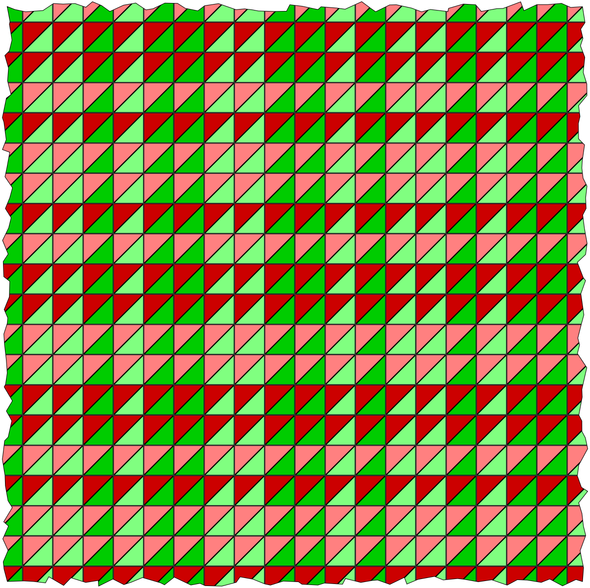

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- Easy construction of strongly aperiodic $\mathbb{Z}^2$-SFTs
- $\mathbb{Z}^2$-SFTs with no computable configurations (Original result by Hanf-Myers 1974)
- Classifying the entropies of $\mathbb{Z}^2$-SFTs (Original result by Hochman-Meyerovitch 2010)

Two new results in general groups

Let $T : G \curvearrowright \{0,1\}^{\mathbb{N}}$ be an effectively closed action of a finitely generated group.

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For any semidirect product $\mathbb{Z}^2 \rtimes G$ there exists a $\mathbb{Z}^2 \rtimes G$-SFT such that its G-subaction is an extension of T.

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Theorem (B, 2017)

For any pair of infinite and finitely generated groups H_1, H_2 there exists a $(G \times H_1 \times H_2)$-SFT such that its G-subaction is an extension of T.

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$\Psi : \{0,1\}^{\mathbb{N}} \to \{0,1,\$\}^{\mathbb{Z}}$ given by:

$$\Psi(x)_j = \begin{cases} x_n & \text{if } j = 3^n \mod 3^{n+1} \\ \$ & \text{in the contrary case.} \end{cases}$$

How does one prove such a thing?

Let's keep it simple, let's do $G \times \mathbb{Z}^2$. Consider

$\Psi : \{0,1\}^{\mathbb{N}} \to \{0,1,\$\}^{\mathbb{Z}}$ given by:

$$\Psi(x)_j = \begin{cases} x_n & \text{if } j = 3^n \mod 3^{n+1} \\ \$ & \text{in the contrary case.} \end{cases}$$

Example

If we write $x = x_0x_1x_2x_3\dots$ we obtain,

$$\Psi(x) = \dots \$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\dots$$

How does one prove such a thing?

$$\ldots \$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\$\ldots$$

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$$\downarrow$$

$$\ldots \${\color{red}x_0\$}x_1{\color{red}x_0\$}\${\color{red}x_0\$}x_2{\color{red}x_0\$}x_1{\color{red}x_0\$}\${\color{red}x_0\$}\${\color{red}x_0\$}x_1{\color{red}x_0\$}\${\color{red}x_0\$}x_3{\color{red}x_0\$} \ldots$$

How does one prove such a thing?

$$\ldots\$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\$\ldots$$

$$\downarrow$$

$$\ldots\$\color{red}{x_0\$}\color{black}{x_1}\color{red}{x_0\$}\color{black}{\$}\color{red}{x_0\$}\color{black}{x_2}\color{red}{x_0\$}\color{black}{x_1}\color{red}{x_0\$}\color{black}{\$}\color{red}{x_0\$}\color{black}{\$}\color{red}{x_0\$}\color{black}{x_1}\color{red}{x_0\$}\color{black}{\$}\color{red}{x_0\$}\color{black}{x_3}\color{red}{x_0\$}\color{black}{\ldots}$$

$$\downarrow$$

$$\ldots\$ \quad x_1 \quad \$ \quad x_2 \quad x_1 \quad \$ \quad \$ \quad x_1 \quad \$ \quad x_3 \quad \ldots$$

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$$\downarrow$$

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$$\downarrow$$

$$\ldots\$\quad x_1\quad \$\quad x_2\quad x_1\quad \$\quad \$\quad x_1\quad \$\quad x_3\quad \ldots$$

$$\downarrow$$

$$\ldots\$x_1\$x_2x_1\$\$x_1\$x_3x_1\$x_2x_1\$\$x_1\$\$x_1\$x_2x_1\$\$x_1\$x_4x_1\$\ldots$$

How does one prove such a thing?

▷ pick afinite set of generators S of G.
▷ construct a subshift Π where every configuration is (up to shifts and a set of measure 0) an S-tuple of configurations of the previous form.

$$S = \{1_G, s_1, \ldots s_n\}$$

$$(\Psi(x), \Psi(T^{s_1}(x), \ldots, \Psi(T^{s_n}(x)) \in \Pi$$

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Claim

If T is an effectively closed action, Π is effectively closed.

How does one prove such a thing?

- Take Π and construct a sofic $\mathbb{Z}^2$ subshift $\widetilde{\Pi}$ having Π in every horizontal row.

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- $\triangleright$ Take Π and construct a sofic $\mathbb{Z}^2$ subshift $\widetilde{\Pi}$ having Π in every horizontal row.
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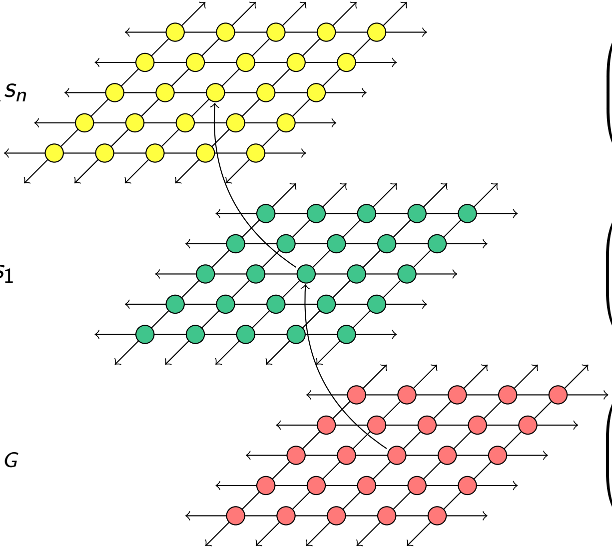
- ▷ Take Π and construct a sofic $\mathbb{Z}^2$ subshift $\widetilde{\Pi}$ having Π in every horizontal row.
- ▷ Using the decoding argument, construct a map from Π to X.
- ▷ Put in every G-coset of $G \times \mathbb{Z}^2$ a configuration of $\widetilde{\Pi}$.

How does one prove such a thing?



$s_1 s_n$

$$\begin{pmatrix} \Psi(z) \\ \Psi(T^{s_1}(z)) \\ \vdots \\ \Psi(T^{s_n}(z)) \end{pmatrix} \in \Pi$$

s_1

$$\begin{pmatrix} \Psi(y) \\ \Psi(T^{s_1}(y)) \\ \vdots \\ \Psi(T^{s_n}(y)) \end{pmatrix} \in \Pi$$

1_G

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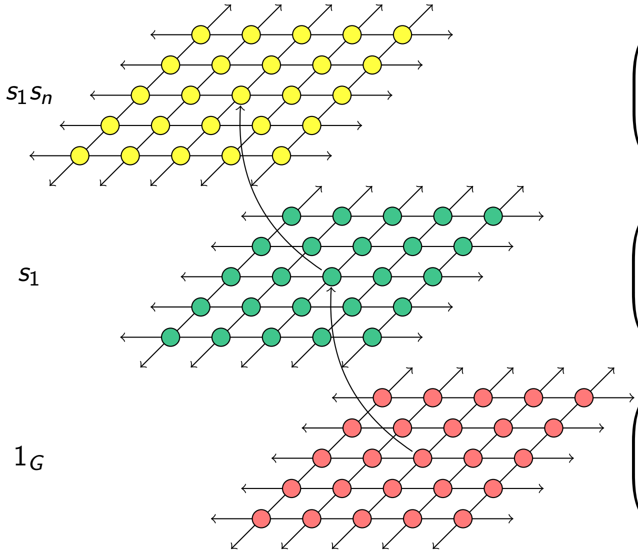

$$\begin{pmatrix} \Psi(z) \\ \Psi(T^{s_1}(z)) \\ \vdots \\ \Psi(T^{s_n}(z)) \end{pmatrix} \in \Pi$$

$$\begin{pmatrix} \color{red}{\Psi(y)} \\ \Psi(T^{s_1}(y)) \\ \vdots \\ \Psi(T^{s_n}(y)) \end{pmatrix} \in \Pi$$

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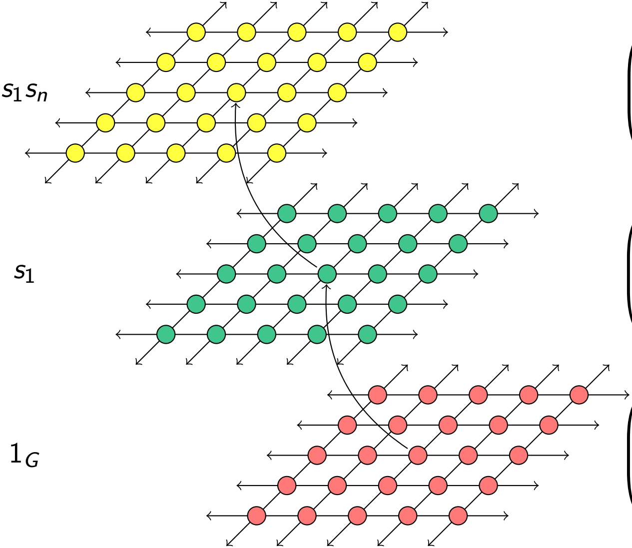

$$\begin{pmatrix} \color{blue}{\Psi(z)} \\ \Psi(T^{s_1}(z)) \\ \vdots \\ \Psi(T^{s_n}(z)) \end{pmatrix} \in \Pi$$

$$\begin{pmatrix} \color{red}{\Psi(y)} \\ \Psi(T^{s_1}(y)) \\ \vdots \\ \color{blue}{\Psi(T^{s_n}(y))} \end{pmatrix} \in \Pi$$

$$\begin{pmatrix} \Psi(x) \\ \color{red}{\Psi(T^{s_1}(x))} \\ \vdots \\ \Psi(T^{s_n}(x)) \end{pmatrix} \in \Pi$$

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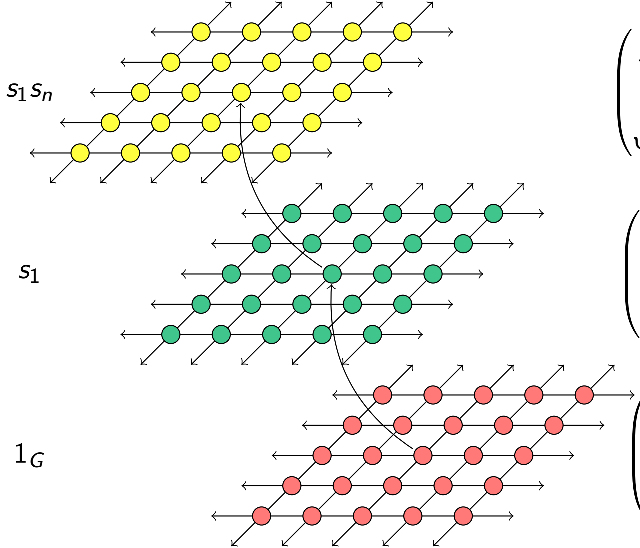

$$\begin{pmatrix} \Psi(T^{s_1 s_n}(x)) \\ \Psi(T^{s_1 s_1 s_n}(x)) \\ \vdots \\ \Psi(T^{s_n s_1 s_n}(x)) \end{pmatrix} \in \Pi$$

$$\begin{pmatrix} \Psi(T^{s_1}(x)) \\ \Psi(T^{s_1 s_1}(x)) \\ \vdots \\ \Psi(T^{s_1 s_n}(x)) \end{pmatrix} \in \Pi$$

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Two corollaries

Theorem (B, Sablik 2016)

If G is finitely generated, $\mathtt{WP}(G)$ is decidable and $d > 1$. Then $G \rtimes \mathbb{Z}^d$ admits a SA SFT.

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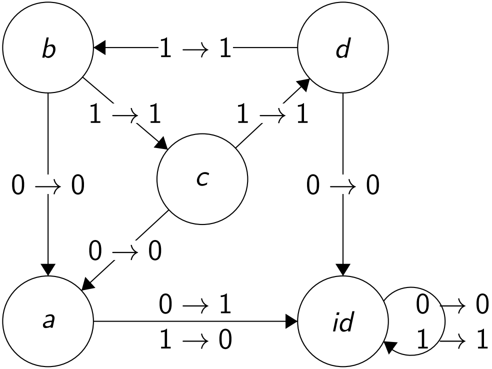
Theorem (B 2017)

If G_i are at least three infinite and finitely generated groups with decidable word problem. Then $G_1 \times \cdots \times G_n$ admits a SA SFT.

What about the Grigorchuk group?



The Grigorchuk group is generated by the actions a, b, c, d over $\{0,1\}^{\mathbb{N}}$.

What about the Grigorchuk group?

- The Grigorchuk group is infinite and finitely generated.
- It contains no copy of $\mathbb{Z}$ as a subgroup. For every $g \in G$, there is $n \in \mathbb{N}$ such that $g^n = 1_G$.
- Decidable word problem (and conjugacy problem).
- It has intermediate growth.
- It is commensurable to its square. ie: G and $G \times G$ have an isomorphic finite index subgroup.

What about the Grigorchuk group?

- ▷ If G is commensurable to $G \times G$, then G is also commensurable to $G \times G \times G$.

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Admitting a strongly aperiodic SFT is a commensurability invariant.

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Theorem (Carroll-Penland, 2015)

Admitting a strongly aperiodic SFT is a commensurability invariant.

Theorem (B, 2017)

The Grigorchuk group admits a strongly aperiodic SFT.

Thank you for your attention!