The torsion problem for the automorphism group of a full $\mathbb{Z}^d$-shift and its topological fullgroup.

Sebastián Barbieri

From a joint work with Jarkko Kari and Ville Salo
LIP, ENS de Lyon – CNRS – INRIA – UCBL – Université de Lyon

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Motivation

Given a fullshift $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ recall that its automorphism group is given by

$$\mathrm{Aut}(\mathcal{A}^{\mathbb{Z}}) = \{\phi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}} \text{ homeomorpism}, [\sigma, \phi] = \mathrm{id}\}$$

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It is still unknown whether $\mathrm{Aut}(\{0,1\}^{\mathbb{Z}}) \cong \mathrm{Aut}(\{0,1,2\}^{\mathbb{Z}})$, but we know that for any pair of alphabets $\mathcal{A}, \mathcal{B}$ with at least two elements

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Nevertheless, we know that $\mathrm{Aut}(\{0,1\}^{\mathbb{Z}}) \ncong \mathrm{Aut}(\{0,1,2,3\}^{\mathbb{Z}})$. The proof comes from studying the roots of elements in the center.

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It might be a good idea to understand torsion in these groups.

Definition (Torsion problem)

Let $G = \langle S \mid R \rangle$ be a finitely generated group. The torsion problem of G is the language $\mathtt{TP}(G)$ where

$$\mathtt{TP}(G) = \{w \in (S \cup S^{-1})^* \mid \exists n \in \mathbb{N} \text{ such that } w^n =_G 1\}$$

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Example

Let $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \langle a, b \mid [a, b], b^3 \rangle$. Then

$$\mathtt{TP}(\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) = \{w \in \{a, b, a^{-1}, b^{-1}\}^* \mid |w|_a = |w|_{a^{-1}}\}$$

Talk highlights

Theorem (B, Kari, Salo)

For any finite alphabet $|A| \geq 2$, $\mathrm{Aut}(A^{\mathbb{Z}})$ *contains a finitely generated subgroup with undecidable torsion problem. The same result also holds for any sofic subshift of positive entropy.*

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The topological fullgroup of a dynamical system (X, T) where $T : G \curvearrowright X$ is the group

$$[[T]] = \{\phi \in \text{Homeo}(X) \mid \exists s : X \to G \text{ continuous}, \phi(x) = T^{s(x)}(x)\}.$$

Theorem (B, Kari, Salo)

Let $(A^{\mathbb{Z}^d}, \sigma)$ be a full shift and $|A| \geq 2$. The topological fullgroup $[[\sigma]]$ contains a finitely generated subgroup with undecidable torsion problem if and only if $d \geq 2$.

Background

Recall that a Turing machine is defined by a rule :

$$\delta_T : \Sigma \times Q \to \Sigma \times Q \times \{-1, 0, 1\}$$

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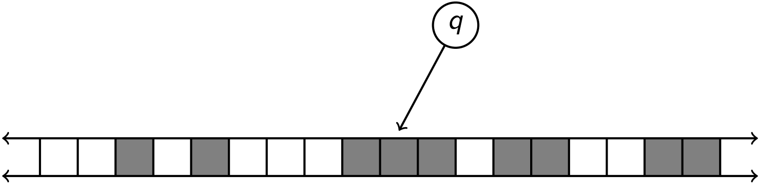

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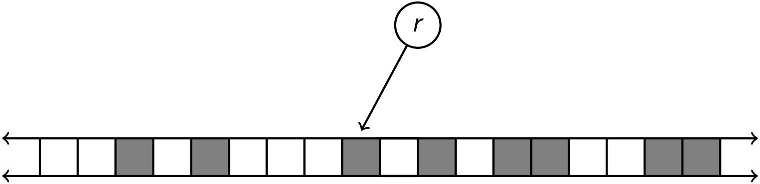

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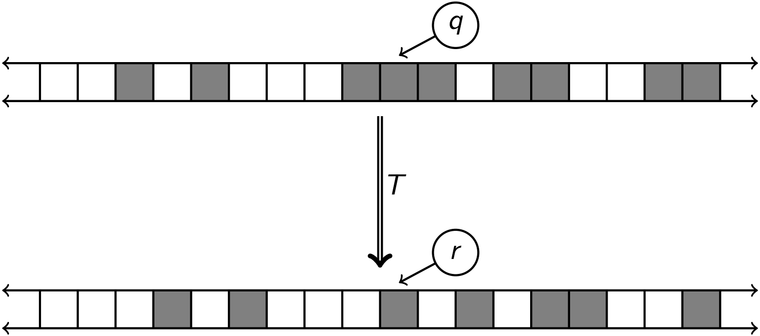
Background

This defines a natural action

$$T : \Sigma^{\mathbb{Z}} \times Q \to \Sigma^{\mathbb{Z}} \times Q$$



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As in cellular automata, the class of CA with radius bounded by some $k \in \mathbb{N}$ is not closed under composition or inverses.

Definition

Let's get rid of these constrains. Given F, F' finite subsets of a group G, consider instead of δ_T a function :

$$f_T : \Sigma^F \times Q \to \Sigma^{F'} \times Q \times G,$$

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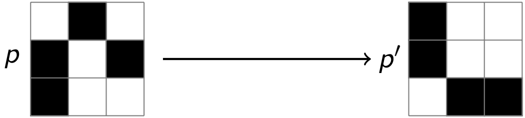
$$f_T : \Sigma^F \times Q \to \Sigma^{F'} \times Q \times G,$$

Let $F = F' = \{0, 1, 2\}^2$, then $f_T(p, q) = (p', q', \vec{d})$ means :

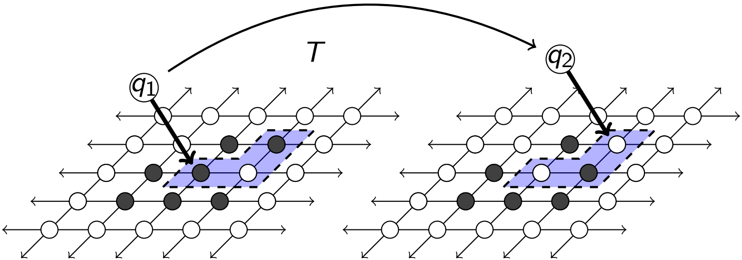
- 

- Turn state q into state q'
- Move head by $\vec{d}$.

Moving head model

f_T defines naturally an action

$$T : \Sigma^G \times Q \times \mathbb{Z}^d \to \Sigma^G \times Q \times \mathbb{Z}^d$$



$f(\substack{\quad\bullet\\ \bullet\ \circ}, q_1) = (\substack{\quad\circ\\ \circ\ \bullet}, q_2, (1,1))$ $\quad F = \{(0,0),(1,0),(1,1)\}$

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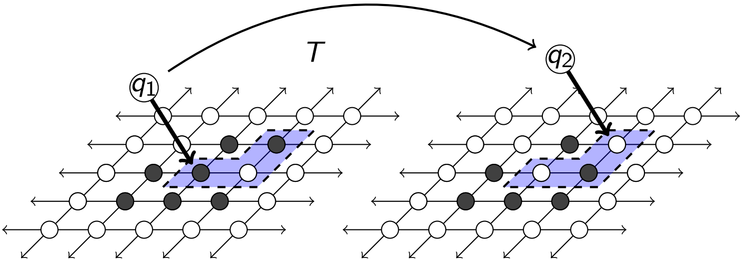

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Let $|\Sigma| = n$ and $|Q| = k$. $(\mathrm{RTM}(G, n, k), \circ)$ is the group of all such T which are bijective.

$\triangleright$ It can be seen as a group of CA over a sofic shift.

Moving tape model

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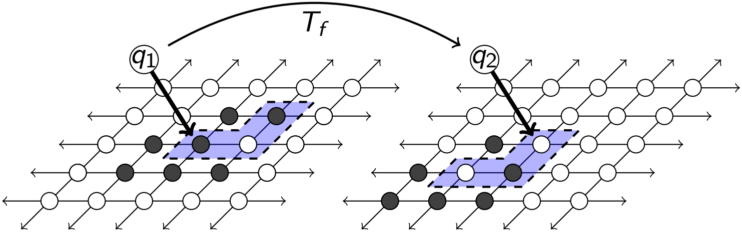
$$T : \Sigma^G \times Q \to \Sigma^G \times Q$$



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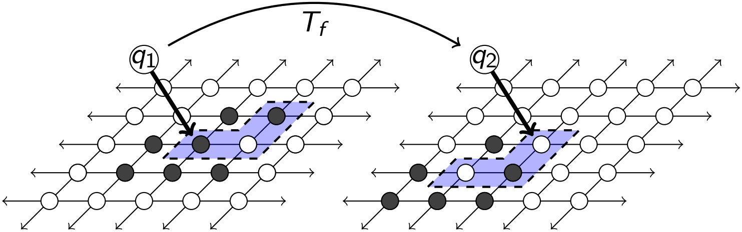

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Let $|\Sigma| = n$ and $|Q| = k$. $(\mathrm{RTM}_{\mathrm{fix}}(G,n,k), \circ)$ is the group of all such T which are bijective.

$\triangleright$ It's like the topological fullgroup but admits local changes.

Equivalence of the models

$\mathrm{RTM}_{\mathrm{fix}}(G, 1, k) \cong S_k$ and $G \hookrightarrow \mathrm{RTM}(G, 1, k)$.

Proposition

If $n \geq 2$ then : $\mathrm{RTM}_{\mathrm{fix}}(G, n, k) \cong \mathrm{RTM}(G, n, k)$.

Properties of RTM

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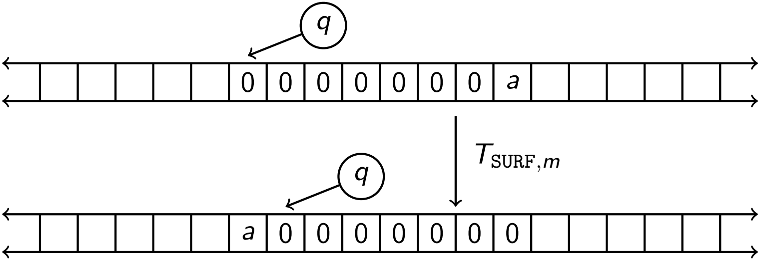
Proof : We find an epimorphism from RTM to a non-finitely generated group.
Let $T \in \mathrm{RTM}_{\mathrm{fix}}(\mathbb{Z}, n, k)$, therefore, it has a cocycle $s : \Sigma^{\mathbb{Z}} \times Q \to \mathbb{Z}$. Define

$$\alpha(T) := \mathrm{E}_{\mu}(s) = \int_{\Sigma^{\mathbb{Z}} \times Q} s(x, q) d\mu,$$

One can check that $\alpha(T_1 \circ T_2) = \alpha(T_1) + \alpha(T_2)$.
Therefore $\alpha : \mathrm{RTM}(\mathbb{Z}, n, k) \to \mathbb{Q}$ is an homomorphism

Properties of RTM

Now consider the machine $T_{\mathtt{SURF},m}$ where for all $a \in \Sigma$ and $q \in Q$:



$f(0^m a, q) = (a0^m, q, 1)$. Otherwise $f(u, q) = (u, q, 0)$.

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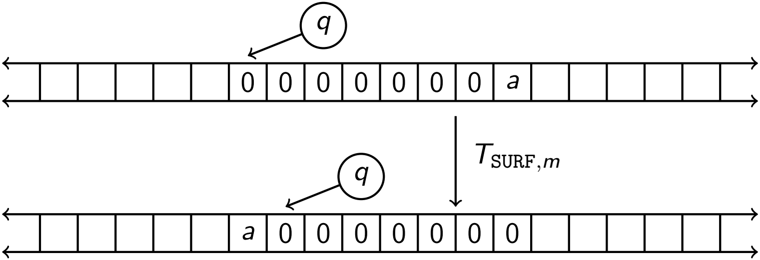

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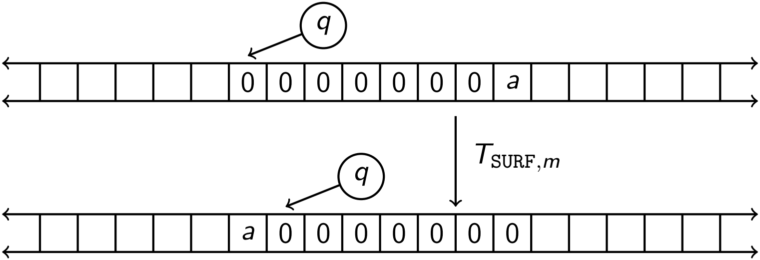

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$\langle (1/n^m)_{m\in\mathbb{N}} \rangle \subset \alpha(\mathrm{RTM}(\mathbb{Z}, n, k))$ which is thus a non-finitely generated subgroup of $\mathbb{Q}$.

Computability properties

Given a finite rules : f, f' :

- It is decidable (in any model) whether $T_f = T_{f'}$.
- We can effectively calculate a rule for $T_f \circ T_{f'}$.
- It is decidable whether T_f is reversible.
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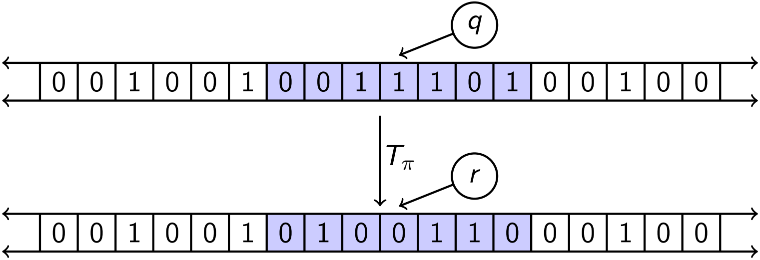
$\mathrm{RTM}(\mathbb{Z}^d, n, k)$ is a recursively presented group with decidable word problem. (Unlike $\mathrm{Aut}(\mathcal{A}^{\mathbb{Z}})$)

Interesting subgroups of RTM

▷ $\mathrm{LP}(G, n, k) \longrightarrow$ Local permutations.



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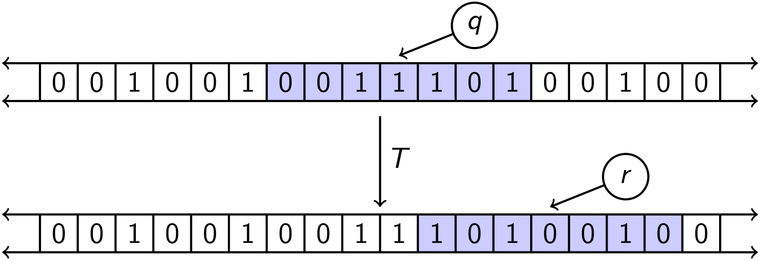
- $\mathrm{RFA}(G, n, k) \longrightarrow$ Reversible finite-state automata.



Note that for $(\{0, \ldots, n-1\}^G, \sigma)$ then $[[\sigma]] = \mathrm{RFA}(G, n, 1)$.

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- $\mathrm{OB}(G, n, k) \longrightarrow$ Oblivous machines $\langle \mathrm{LP}, \sigma \rangle$.
- $\mathrm{EL}(G, n, k) \longrightarrow$ Elementary machines $\langle \mathrm{LP}, \mathrm{RFA} \rangle$.

The torsion problem for $[[\sigma]]$

The torsion problem for RFA

$\mathrm{RFA}(\mathbb{Z}, n, k)$ has decidable torsion problem.

Proof idea : As $\mathbb{Z}$ is two-ended, any non-torsion machine must shift to the left or right in at least a periodic configuration.

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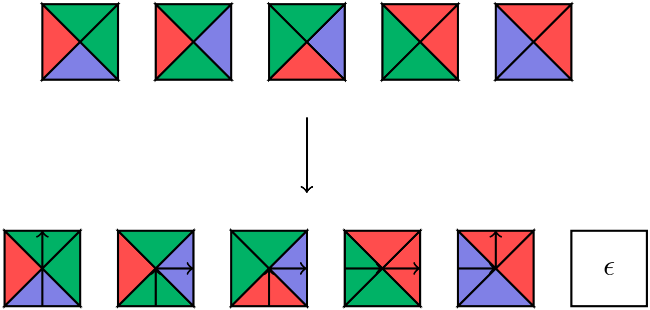
The snake problem



Can we tile the plane in a way which produces a bi-infinite path ?

The snake problem

Theorem (Kari)

The snake tiling problem is undecidable.

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For every instance of the snake tiling problem, one can construct $T \in \mathrm{RFA}$ which walks the path of the snake, and turns back if it encounters a problem.

The torsion problem for RFA : Cheating version

We'll first do it by cheating : Arbitrary alphabet τ as an instance of the snake tiling problem and at least two states L, R.

- Let t be the tile at $(0, 0)$. If $t = \epsilon$, do nothing.
- Otherwise :
  - If the state is L. Check the tile in the direction $\texttt{left}(t)$. If it matches correctly with t move the head to that position, otherwise switch the state to R.
  - If the state is R. Check the tile in the direction $\texttt{right}(t)$. If it matches correctly with t move the head to that position, otherwise switch the state to L

We are going to code everything in a binary alphabet and use no states.

1	1	1	1	1	1	1
1	0	0	0	0	0	1
1	0	b_1	b_2	b_3	0	1
1	0	r_1	r_2	b_4	0	1
1	0	l_1	l_2	b_5	0	1
1	0	0	0	0	0	1
1	1	1	1	1	1	1

Let C be the set of all patterns of this form.

The torsion problem for RFA : The real deal

Consider the group spanned by the following machines :

1. $\{T_{\vec{v}}\}_{v \in D}$ that shifts by v
2. $T_{\texttt{walk}}$ that walks along the direction codified by l_1, l_2 or r_1, r_2 depending on the direction bit.
3. $\{g_c\}_{c \in C}$ that flips the direction bit if the current pattern is $c \in C$,
4. $\{h_c\}_{c \in C}$ that flips the auxiliary bit if the current pattern is $c \in C$,
5. $\{g_{+,c}\}_{c \in C}$ that adds the auxiliary bit to the direction bit if the current pattern is $c \in C$, and
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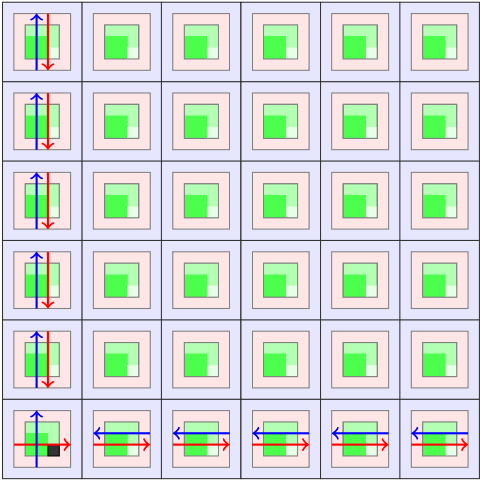
$$g_{p^*} = (T_{-7\vec{v}} \circ g_{+,c} \circ T_{7\vec{v}} \circ h_{p^*_{F\setminus\{\vec{v}\}}})^2.$$

$$h_{p^*} = (T_{-7\vec{v}} \circ h_{+,c} \circ T_{7\vec{v}} \circ g_{p^*_{F\setminus\{\vec{v}\}}})^2.$$

Finally, we use these machines to code the first ones.

The torsion problem for RFA : The real deal

$\mathcal{M}(t) =$ 

The torsion problem for RFA : The real deal

$$T^* = (T_{\texttt{walk}})^M \circ \prod_{p^* \in \mathcal{M}} g_{p^*} \circ \prod_{c \in C} g_c$$

Acts as the first machine, but using these coded macrotiles.

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Corollary

Let $d \geq 2$ and σ be the shift action of $\mathbb{Z}^d$ over a full shift $\mathcal{A}^{\mathbb{Z}^d}$ where $|\mathcal{A}| \geq 2$. Then the full group $[[\sigma]]$ contains a finitely generated subgroup with undecidable torsion problem.

The torsion problem for $\mathrm{Aut}(A^{\mathbb{Z}})$

$TP(\mathrm{Aut}(A^{\mathbb{Z}}))$ is undecidable.

We want to prove that the torsion problem is undecidable for a f.g. subgroup of $\mathrm{Aut}(A^{\mathbb{Z}})$. The sketch is as follows :

1. The torsion problem for reversible classical Turing machines is undecidable [Kari, Ollinger 2008].

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2. Classical Turing machines embed into $\mathrm{EL}(\mathbb{Z}, n, k)$.
3. $\mathrm{EL}(\mathbb{Z}, n, k)$ is finitely generated.
4. There exists a "torsion preserving function" from $\mathrm{EL}(\mathbb{Z}, n, k)$ to $\mathrm{Aut}(A^{\mathbb{Z}})$

$$\text{Classical} \hookrightarrow \text{EL"} \hookrightarrow \text{"}\mathrm{Aut}(A^{\mathbb{Z}})$$

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This proof is inspired both on the existence of strongly universal reversible gates for permutations of Σ^m and the Juschenko Monod proof for the fullgroup of minimal actions.

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The set $A_1 \cup A_2 \cup A_3$ generate $\mathrm{OB}(\mathbb{Z}^d, n, k) = \langle \mathrm{LP}(\mathbb{Z}^d, n, k), \sigma \rangle$.

- $A_1 =$ Shifts T_{e_i} for $\{e_i\}_{i \leq d}$ a base of $\mathbb{Z}^d$.
- $A_2 =$All $T_\pi \in \mathrm{LP}(\mathbb{Z}^d, n, k)$ of fixed support $E \subset \mathbb{Z}^d$ of size 4.
- $A_3 =$ The swaps of symbols in positions $(\vec{0}, e_i)$.

$\mathsf{EL}(\mathbb{Z}, n, k)$ is finitely generated.

$$\mathsf{EL}(\mathbb{Z}, n, k) = \langle \mathrm{OB}(\mathbb{Z}, n, k), \mathrm{RFA}(\mathbb{Z}, n, k) \rangle$$

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We can show that $\mathrm{RFA}(\mathbb{Z}, n, k)$ is generated by shifts and controlled position swaps.

- f is controlled position swap if for some $u, v \in \Sigma^*$, $f(xu.avy) = xua.vy$ and $f(xua.vy) = xu.avy$.

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We only need to implement controlled position swaps [technical].

Definition

Let G and H be groups. We say a function $\phi : G \to H$ is finiteness preserving (FP) if the following holds : If $F \subset G^*$ is finite, then the group $\langle w \mid w \in F \rangle \leq G$ is infinite if and only if the group $\langle \phi(w_1)\phi(w_2)\cdots\phi(w_{|w|}) \mid w \in F \rangle$ is infinite.

From $\mathrm{EL}(\mathbb{Z}, n, k)$ to $\mathrm{Aut}(A^{\mathbb{Z}})$

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Lemma

An FP function from a f.g. H to G forces the torsion problem of G to be harder than the one of H.

Construction of the FP function

- Let $A = \{\Sigma^2 \times (\{\leftarrow, \rightarrow\} \cup (Q \times \{\uparrow, \downarrow\}))\}$.

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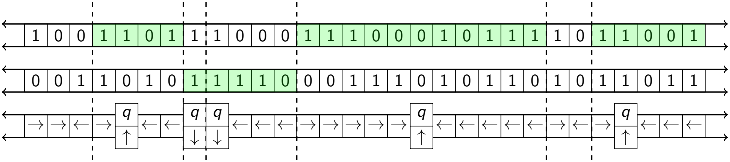

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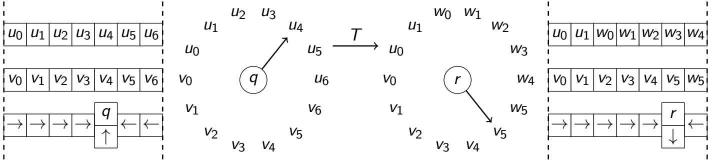

$$f_T(u_2 u_3 . u_4 u_5 u_6 v_6, q) = (w_0 w_1 . w_2 w_3 w_4 w_5, r, 4)$$

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$\triangleright$ There is a finitely generated subgroup of $\mathrm{Aut}(A^{\mathbb{Z}})$ with undecidable torsion problem.
$\triangleright$ As $\mathrm{Aut}(A^{\mathbb{Z}}) \hookrightarrow \mathrm{Aut}(\{0,1\}^{\mathbb{Z}})$ the same is valid for any full shift, mixing SFTs, sofic shift of positive entropy, etc.

Thank you for your attention !