A strongly aperiodic SFT in the Grigorchuk group.

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Algorithmic questions in dynamical systems
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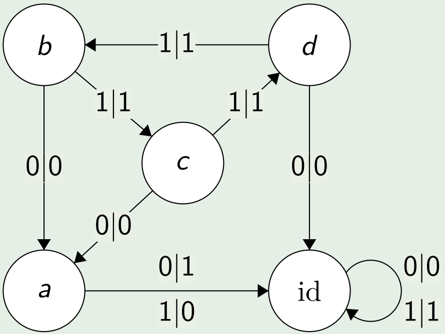
The Grigorchuk group

Generated by a, b, c, d acting over $\{0, 1\}^{\mathbb{N}}$.



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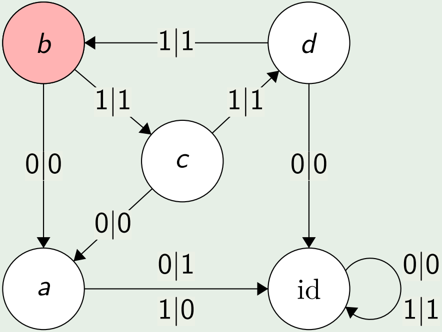

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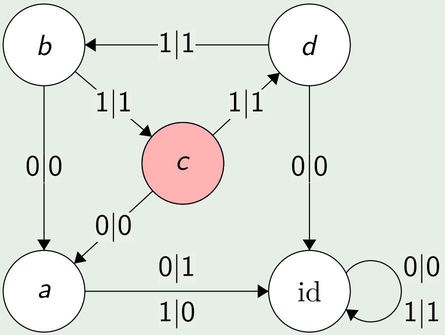

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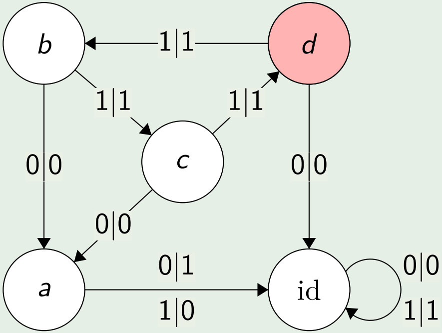

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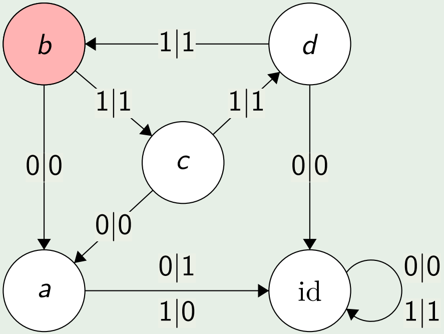

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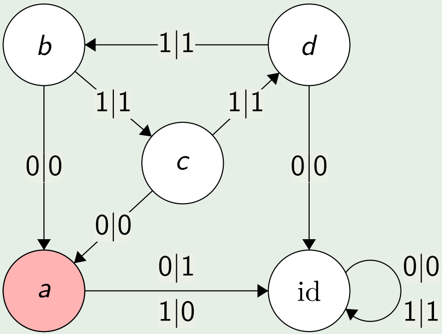

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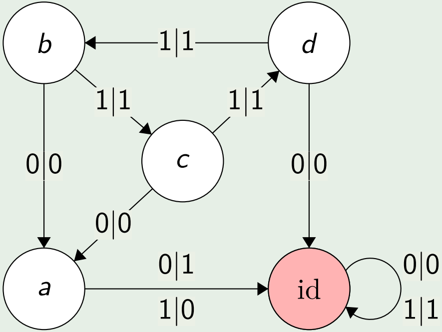

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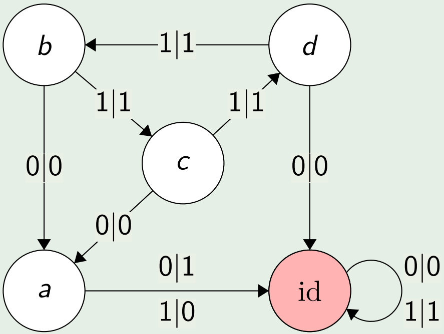

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What about the Grigorchuk group?

- a, b, c, d are involutions.
- Infinite and finitely generated.
- It contains no copy of $\mathbb{Z}$ as a subgroup. For every $g \in G$, there is $n \in \mathbb{N}$ such that $g^n = 1_G$.
- Decidable word (and conjugacy) problem.
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The goal of this talk is to construct a strongly aperiodic SFT here.

Definitions

- G is a finitely generated group.
- $\mathcal{A}$ is a finite alphabet. Ex: $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of configurations, $x : G \to \mathcal{A}$
- $G \curvearrowright \mathcal{A}^G$ is the left shift action given by:

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A subshift is a set of configurations avoiding patterns from a list $\mathcal{F}$.

$$p \in \mathcal{A}^S, \qquad [p] = \{x \in \mathcal{A}^G \mid x|_S = p\}$$

$$X = X_{\mathcal{F}} = \mathcal{A}^G \setminus \bigcup_{g \in G, p \in \mathcal{F}} g([p])$$

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Strongly aperiodic

A subshift $X \subset \mathcal{A}^G$ is *strongly aperiodic* if the shift action is free.

$$\forall x \in X, gx = x \implies g = 1_G.$$

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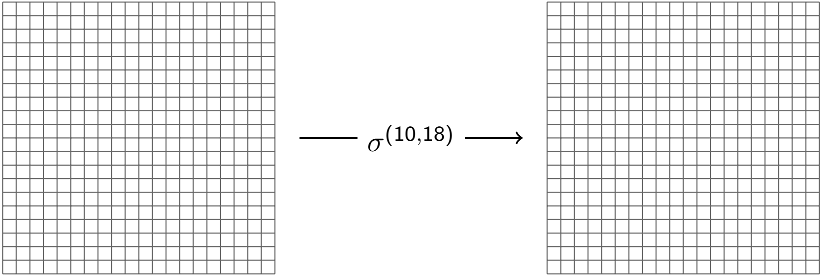
Baby (alpaca) example: Let $G = \mathbb{Z}^2/20\mathbb{Z}^2$



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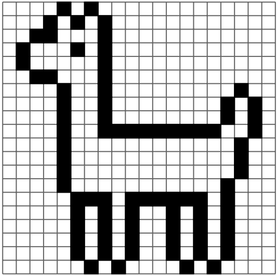
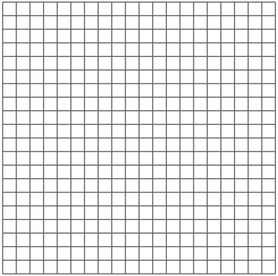


$$\xrightarrow{\quad\sigma^{(10,18)}\quad}$$



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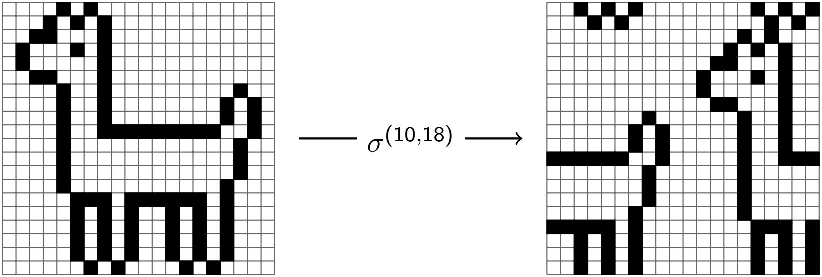

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Proposition

Every non-empty $\mathbb{Z}$-SFT contains a periodic configuration.

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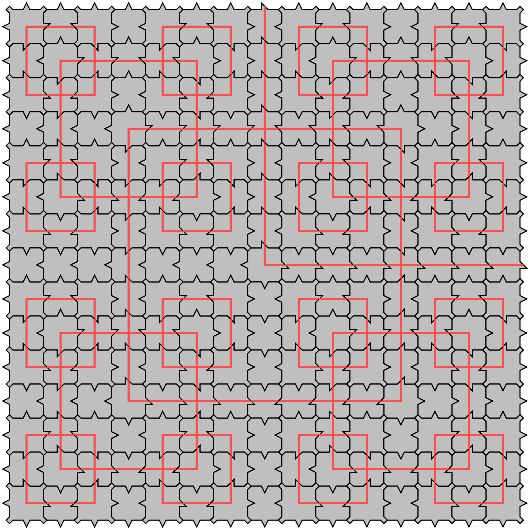
Every non-empty $\mathbb{Z}$-SFT contains a periodic configuration.

Theorem (Berger 1966, Robinson 1971, Kari 1996, Jeandel & Rao 2015)

There exist strongly aperiodic SFTs on $\mathbb{Z}^2$.

Abelian case



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AS 2010, DRS 2010

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ABT 2018+ (2015)

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Commensurability

We say that two groups G_1, G_2 are *commensurable* if they contain finite index subgroups H_1, H_2 such that $H_1 \cong H_2$.

$$G_1 \hookleftarrow H_1 \cong H_2 \hookrightarrow G_2$$

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Theorem (Carroll-Penland, 2015)

Admitting a strongly aperiodic SFT is a commensurability invariant.

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$$\downarrow$$

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In fact, the same result can be extended to **branch groups**.

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In fact, the same result can be extended to **branch groups**.

Theorem

Let G be a finitely generated and recursively presented branch group. Then G has decidable word problem if and only if there exists a non-empty strongly aperiodic G-SFT.

We want to show next:

Every EC G-subshift is a subaction of a $G \times H_1 \times H_2$-sofic

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Square-free vertex coloring

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Let $G = (V, E)$ be a graph. A vertex coloring is a function $x : V \to \mathcal{A}$. We say it is square-free if for every odd-length path $p = v_1 \dots v_{2n}$ then there exists $1 \le j \le n$ such that $x(v_j) \ne x(v_{j+n})$.

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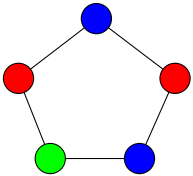

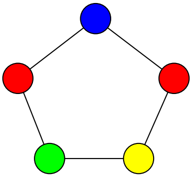

C_5 has a square-free vertex coloring with 4 colors, but not with 3.

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Let

$$\Gamma(G,S) = (G, \{\{g, gs\}, g \in G, s \in S\})$$

be the undirected right Cayley graph of G with respect to $S \Subset G$.
A compactness argument shows:

Theorem

$\Gamma(G,S)$ can be square-free vertex colored with $2^{19}|S|^2$ colors.

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- π is a path and $x_{v_i} = x_{v_{i+n}}$. $\rightarrow\leftarrow$

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Let $|\mathcal{A}| \geq 2^{19}|S|^2$ and $X \subset \mathcal{A}^G$ be the subshift such that every square in $\Gamma(G, S)$ is forbidden.

- $X \neq \emptyset$.
- Let $g \in G$ such that $gx = x$ for some $x \in X$.
- Factorize g as uwv with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$.
- If not, let $w = w_1 \dots w_n$ and consider the odd length walk $\pi = v_0 v_1 \dots v_{2n-1}$ on $\Gamma(G, S)$ defined by:

$$v_i = \begin{cases} 1_G & \text{if } i = 0 \\ w_1 \dots w_i & \text{if } i \in \{1, \dots, n\} \\ w w_1 \dots w_{i-n} & \text{if } i \in \{n+1, \dots, 2n-1\} \end{cases}$$

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If G has decidable word problem, then X is effectively closed.

We want to show next:

Every EC G-subshift is a subaction of a $G \times H_1 \times H_2$-sofic

$\exists$ SA EC subshifts in G

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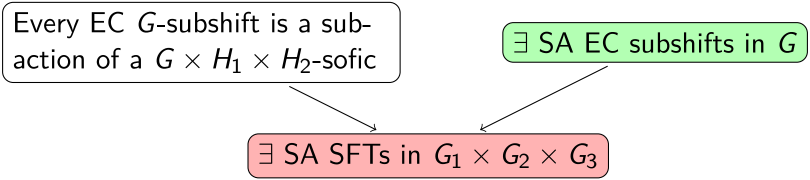

The philosophy behind it

Finitely presented group

A group G is finitely presented if $G \cong \langle S|R\rangle$ where both S and $R \subset (S \cup S^{-1})^*$ are finite.

$$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1}\rangle$$

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Recursively presented group

A group G is recursively presented if $G \cong \langle S|R\rangle$ where $S \subset \mathbb{N}$ and $R \subset (S \cup S^{-1})^*$ are recursively enumerable sets.

$$L = \langle a, t \mid (at^n at^{-n})^2, n \in \mathbb{N}\rangle$$

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Theorem (Higman, 1961)

For every recursively presented group H there exists a finitely presented group G such that H embeds into G.

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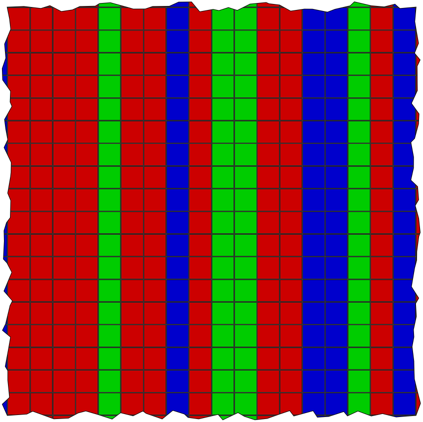
Corollary [Theorem: Novikov 1955, Boone 1958]

There are finitely presented groups with undecidable word problem

Just apply Higman's theorem to
$G = \langle a, b, c, d \mid b^{-n}ab^n = c^{-n}dc^n, n \in \mathtt{HALT}\rangle$... done!

The case of subshifts

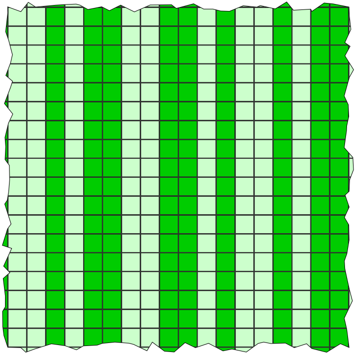
Every EC $\mathbb{Z}$-subshift X is a subaction of a $\mathbb{Z}^2$-sofic Y

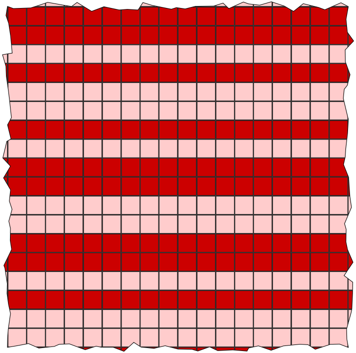


$\in Y$

$\in X$

The case of subshifts





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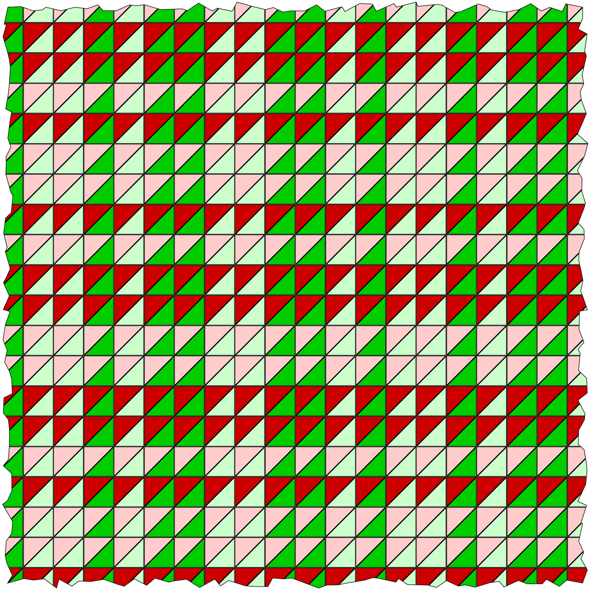

In our case

proof

- Take G_1 EC SA subshift. Use simulation to obtain a $G_1 \times G_2 \times G_3$-sofic subshift Y_1 such that $G_2 \times G_3$ act trivially and G_1 acts freely.
- Do the same for G_2, G_3 to get Y_2, Y_3.
- $Y_1 \times Y_2 \times Y_3$ is a SA sofic subshift.
- Any SFT extension $X \twoheadrightarrow Y_1 \times Y_2 \times Y_3$ works.

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$$
\begin{array}{ccc}
\boxed{\text{Every EC } G\text{-subshift is a sub-action of a } G \times H_1 \times H_2\text{-sofic}} & & \boxed{\exists \text{ SA EC subshifts in } G} \\
& \searrow \quad \swarrow & \\
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\end{array}
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How does one prove such a thing?

Every EC $G \curvearrowright \{0,1\}^{\mathbb{N}}$ is a factor of a subaction of a $G \times H_1 \times H_2$-SFT

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Two ingredients:

- A Toeplitz coding of EC actions from a work of me and M. Sablik.
- A coding of E. Jeandel of a theorem of Seward on translation-like actions.

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Let $\Psi : \{0,1\}^{\mathbb{N}} \to \{0,1,\$\}^{\mathbb{Z}}$ be given by:

$$\Psi(x)_j = \begin{cases} x_n & \text{if } j = 3^n \mod 3^{n+1} \\ \$ & \text{in the contrary case.} \end{cases}$$

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Example

If we write $x = x_0x_1x_2x_3\dots$ we obtain,

$$\Psi(x) = \dots \$x_0\$x_1x_0\$\$x_0\$x_2x_0\$x_1x_0\$\$x_0\$\$x_0\$x_1x_0\$\$x_0\$x_3x_0\dots$$

How does one prove such a thing?

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$$\downarrow$$

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How does one prove such a thing?

▷ pick a finite set of generators S of G.
▷ construct a subshift Π where every configuration is (up to small details) an S-tuple of configurations of the previous form.

$$S = \{1_G, s_1, \dots s_n\}$$

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Claim

If $G \curvearrowright X$ is an effectively closed action, Π is an effectively closed subshift.

How does one prove such a thing?

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$\longleftarrow$

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$\triangleright$ There exists a sofic $\mathbb{Z}^2$-subshift $\widetilde{\Pi}$ having Π in every horizontal row.

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$\triangleright$ Put in every G-coset of $G \times \mathbb{Z}^2$ a configuration of $\widetilde{\Pi}$. Tie them using local rules.

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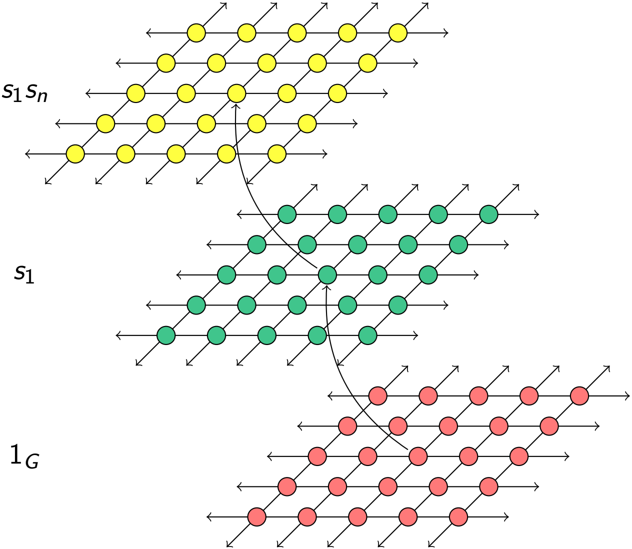

$$\begin{pmatrix} \Psi(z) \\ \Psi(s_1(z)) \\ \vdots \\ \Psi(s_n(z)) \end{pmatrix} \in \widetilde{\Pi}$$

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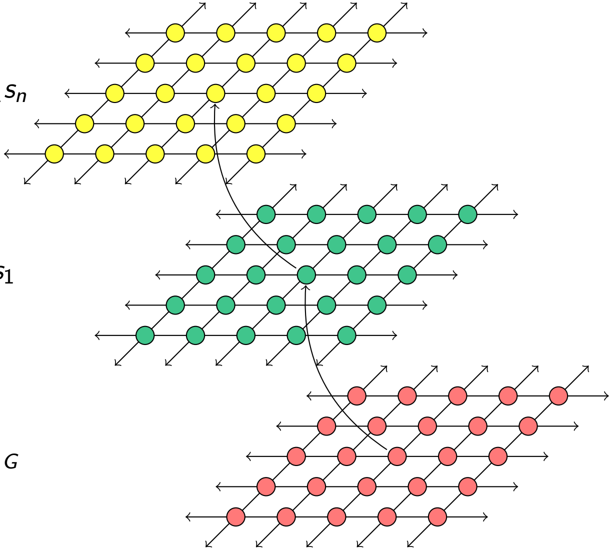

$s_1 s_n$
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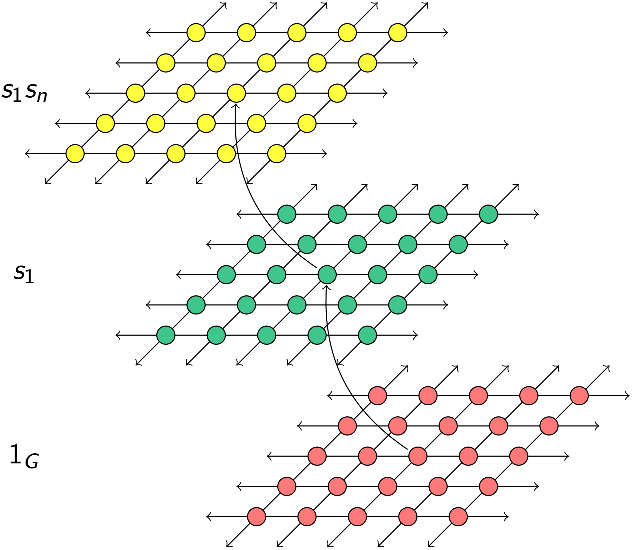

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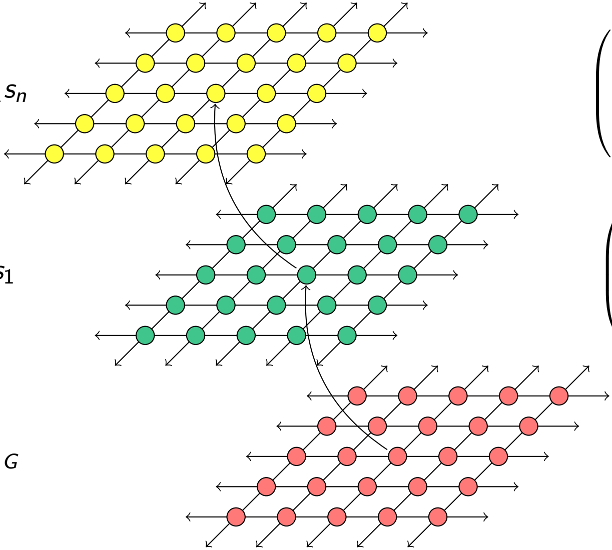

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From $\mathbb{Z}^2$ to $H_1 \times H_2$

How to go from $\mathbb{Z}^2$ to $H_1 \times H_2$?

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an action $G \curvearrowright (X, d)$ is *translation-like* if:

- G acts freely
- For each $g \in G$, $\sup_{x \in X} d(x, gx) < \infty$.

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Each infinite and f.g. group admits a translation-like action of $\mathbb{Z}$.

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This means that each infinite and f.g. group admits a Cayley graph that can be partitioned into disjoint bi-infinite paths.

From $\mathbb{Z}^2$ to $H_1 \times H_2$

[Jeandel] Use the set of generators of the Cayley graph to define an SFT which codes the translation-like action.

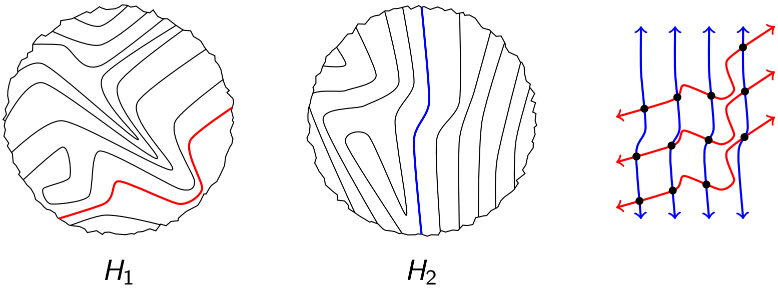

Figure: Finding a grid in $H_1 \times H_2$

Fin

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$\exists$ SA SFTs in the Grigorchuk group

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Thank you for your attention!