Topological entropies of subshifts of finite type in amenable groups.

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Symbolic Dynamical Systems

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At the dawn of time

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… However, they kept a shameful secret!

The shameful secret

Consider

- $\mathbb{Z} \curvearrowright X = \{0,1\}^{\mathbb{Z}}$ with the Bernoulli measure $(\frac{1}{2}, \frac{1}{2})$.
- $\mathbb{Z} \curvearrowright Y = \{0,1,3\}^{\mathbb{Z}}$ with the Bernoulli measure $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

The secret: ergodic theory cannot tell them appart.

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The secret: ergodic theory cannot tell them appart.

Why: the Koopman operator of a system $(G \curvearrowright X, \mu)$ is

$$\kappa_X \colon G \to \mathcal{B}(L^2(X))$$

given by

$$\kappa_X(s)(f) = f \circ s^{-1}$$

And it happens that

$$\kappa_X, \kappa_Y \text{ are equivalent to } 1_{\mathbb{Z}} \oplus \lambda_{\mathbb{Z}}^{\oplus \mathbb{N}}.$$

The age of heroes

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And they showed that it was an invariant under isomorphism.

This was great:

Theorem (Ornstein, 70.)

Two Bernoulli shifts are isomorphic if and only if they have the same entropy.

The age of heroes

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Let G be amenable, $(F_n)_{n\in\mathbb{N}}$ a Følner sequence, $\mathcal{U}$ an open cover of X and $N(\mathcal{U})$.

$$h_{\text{top}}(G \curvearrowright X) = \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{|F_n|} \log N(\mathcal{U}^{F_n}).$$

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- $\mathbb{Z}^d$-SFTs, $d \geq 2$ [Hochman and Meyerovitch, 10] The set of non-negative upper semi-computable numbers.
- Effectively closed $\mathbb{Z}$-subshifts. The set of non-negative upper semi-computable numbers.

Goal of this talk

Even more annoying question

Let G be a countable amenable group. Characterize the set $\mathcal{E}_{\mathsf{SFT}}(G)$ of entropies attainable by G-SFTs

Two simple remarks

Trivial realization result

If G is a countable amenable group and $H \leq G$, then

$$\mathcal{E}_{\mathsf{SFT}}(H) \subseteq \mathcal{E}_{\mathsf{SFT}}(G)$$

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Trivial realization result

If G is a countable amenable group and $H \leq G$, then

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Computability bound

If G is a finitely generated amenable group with decidable word problem, then

$$\mathcal{E}_{\mathsf{SFT}}(G) \subseteq \mathcal{E}_{\mathsf{SFT}}(\mathbb{Z}^2)$$

A simple consequence:

G is polycyclic if there exists a sequence of N_i

$$G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_n \triangleright N_{n+1} = \{1_G\}.$$

such that every quotient N_i/N_{i+1} is cyclic. The Hirsch index of G is the number of infinite quotients in such a series.

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Theorem (B., 19.)

Let G be a polycyclic-by-finite group and denote by $h(G)$ its Hirsch index.

1. *If $h(G) = 0$ then $\mathcal{E}_{SFT}(G) = \{\frac{1}{|G|}\log(n) \mid n \in \mathbb{Z}_+\}$.*
2. *If $h(G) = 1$ then $\mathcal{E}_{SFT}(G) = \mathcal{E}_{SFT}(\mathbb{Z})$, the set of non-negative rational multiples of logarithms of Perron eigenvalues.*
3. *If $h(G) \geq 2$ then $\mathcal{E}_{SFT}(G) = \mathcal{E}_{SFT}(\mathbb{Z}^2)$, the set of non-negative upper semi-computable numbers.*

Something a little bit more interesting

Theorem (B,. 19.)

Let G be a finitely generated amenable group such that

- *G has decidable word problem.*
- *G admits a translation-like action by $\mathbb{Z}^2$.*

Then the set of entropies attainable by G-subshifts of finite type is the set of non-negative upper semi-computable numbers.

$$\mathcal{E}_{SFT}(G) = \mathcal{E}_{SFT}(\mathbb{Z}^2).$$

Translation-like action

$H \curvearrowright G$ is translation-like if

1. $H \curvearrowright G$ is free. $h \cdot g = g \implies h = 1_H$.
2. $H \curvearrowright G$ is bounded.
   $\{(h \cdot g)g^{-1} \mid g \in G\}$ is finite for every $h \in H$.

Corollaries

Products of f.g. groups

If G_1, G_2 are f.g, amenable, and have decidable word problem, then

$$\mathcal{E}_{\mathsf{SFT}}(G_1 \times G_2) = \mathcal{E}_{\mathsf{SFT}}(\mathbb{Z}^2).$$

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Products of countable, non-locally finite groups

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Branch groups

If G is an infinite, f.g, amenable branch group with decidable word problem (ex: Grigorchuk group), then

$$\mathcal{E}_{\mathsf{SFT}}(G) = \mathcal{E}_{\mathsf{SFT}}(\mathbb{Z}^2).$$

Group charts

To prove the main theorem, we need to introduce group charts.

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H-cocycle

Consider $G \curvearrowright X$. A continuous function $\gamma : H \times X \to G$ is an H-**cocycle** if it satisfies

$$\gamma(h_1 h_2, x) = \gamma(h_1, \gamma(h_2, x)x) \cdot \gamma(h_2, x) \text{ for every } h_1, h_2 \text{ in } H.$$

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Remark: every H-cocycle induces a family of left actions $H \overset{x}{\curvearrowright} G$

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G-charts

A pair (X, γ) is called a G-**chart of** H. If every induced action $H \overset{x}{\curvearrowright} G$ is free, it is called a **free** G-chart of H.

Group charts

Example

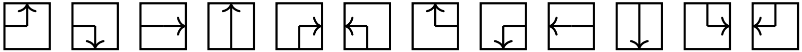
Let $H \leq G$, then for any $G \curvearrowright X$ if we define:

$$\gamma(h, x) = h \text{ for } x \in X, h \in H$$

Then γ is an H-cocycle and (X, γ) is a free G-chart of H.

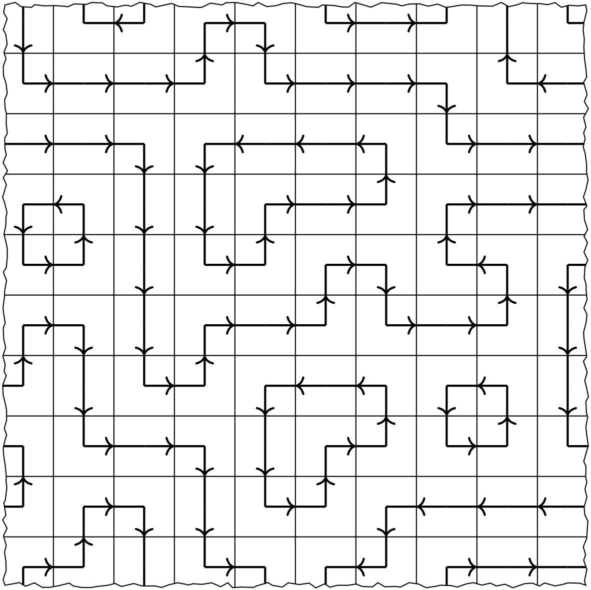
Group charts

Let $G = \mathbb{Z}^2$ and consider the set of tiles Σ:



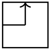
Let $X \subset \Sigma^{\mathbb{Z}^2}$ the set of all configurations such that every outgoing arrow matches with an incoming arrow.

Group charts

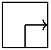


Group charts

Let $\gamma\colon \mathbb{Z} \times X \to \mathbb{Z}^2$ be the cocycle such that: $\gamma(1, x)$ is the unit vector represented by the outgoing arrowhead of $x((0,0))$ and $\gamma(-1, x)$ the vector represented by the incoming arrow.

if $x(0,0) =$ 

$$\gamma(1, x) = (0, 1)$$
$$\gamma(-1, x) = (-1, 0)$$

if $y(0,0) =$ 

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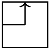
if $x(0,0) =$ 

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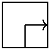
if $y(0,0) =$ 

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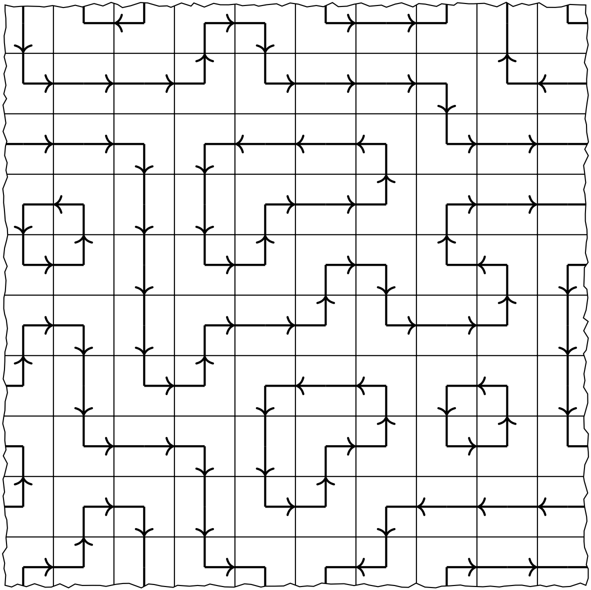
Example

(X,γ) is a $\mathbb{Z}^2$-chart of $\mathbb{Z}$. It is not free.

Group charts



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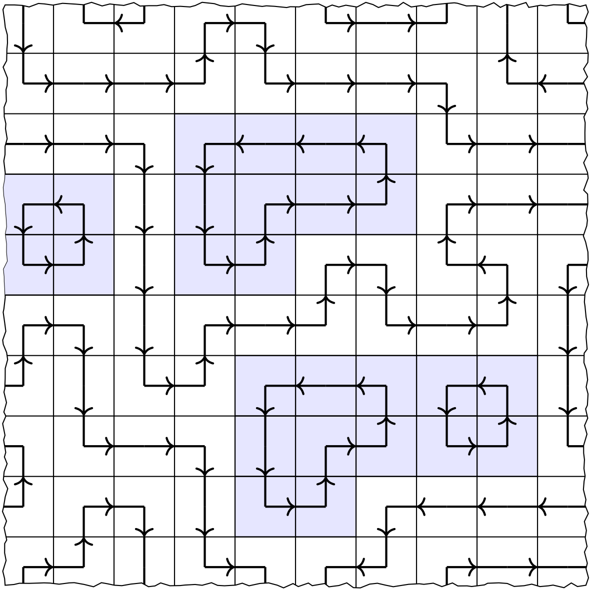
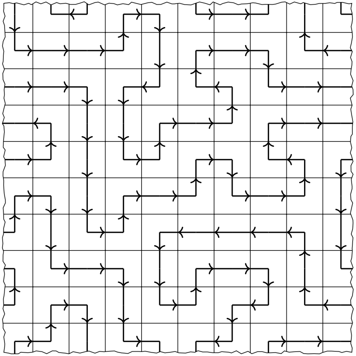


Group charts

Consider $\widehat{X} \subset X$ be the subshift of X such that no cycles appear.



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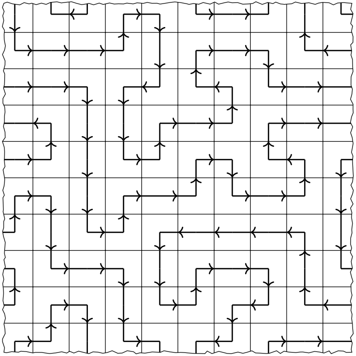

Example

$(\widehat{X}, \gamma|_{\mathbb{Z}\times\widehat{X}})$ is a **free** $\mathbb{Z}^2$-chart of $\mathbb{Z}$.

Embedding H-actions

Consider

- A G-subshift X.
- A G-chart (X, γ) of H.
- An H-subshift Y.

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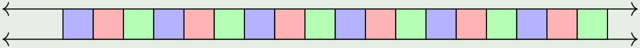
Embedding of $H \curvearrowright Y$ into (X, γ)

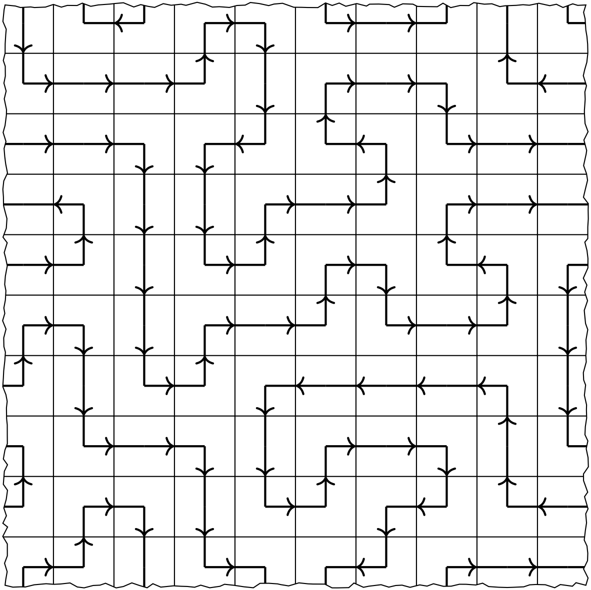
Let $Y_\gamma[X]$ be the set of all $(x, y) \in X \times \mathcal{A}^G$ such that every copy of H induced by the cocycle γ carries a configuration from Y.

Example

- The $\mathbb{Z}^2$-subshift $\widehat{X}$ from the example before.
- The free G-chart $(\widehat{X}, \gamma)$ of H.
- The $\mathbb{Z}$-subshift consisting of the orbit of the periodic configuration:



Embedding H-actions: example



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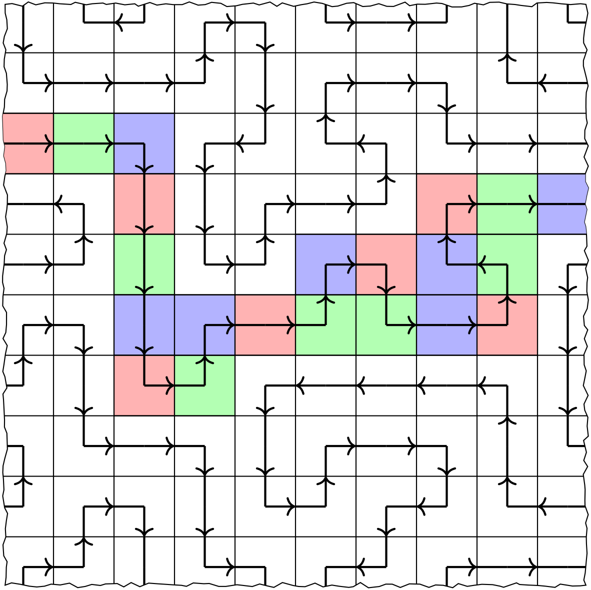

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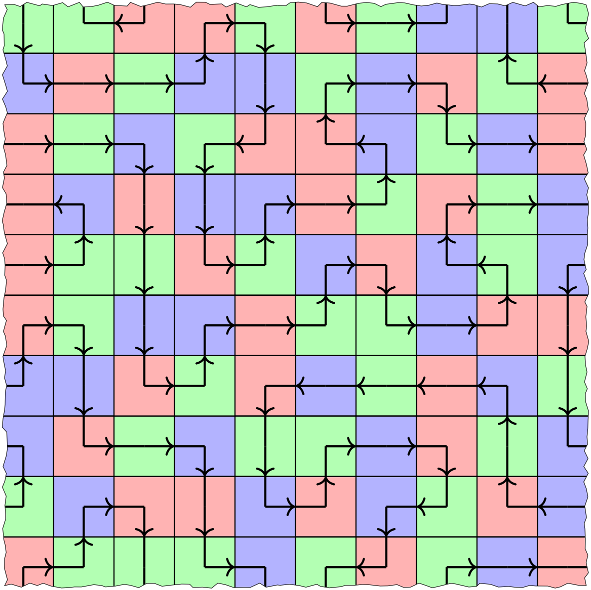

Addition formula

Let G, H be countable amenable groups.

Theorem (B., 19.)

If (X, γ) is a free G-chart of H, then for every H-subshift Y,

$$h_{top}(G \curvearrowright Y_\gamma[X]) = h_{top}(H \curvearrowright Y) + h_{top}(G \curvearrowright X)$$

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Remark: If both X and Y are SFTs, then $Y_\gamma[X]$ is an SFT.

$$h_{\text{top}}(G \curvearrowright X) + \mathcal{E}_{\text{SFT}}(H) \subset \mathcal{E}_{\text{SFT}}(G).$$

Two issues

Goal: Let G be a finitely generated amenable group such that

- G has decidable word problem.
- G admits a translation-like action by $\mathbb{Z}^2$.

Then $\mathcal{E}_{\mathsf{SFT}}(G) = \mathcal{E}_{\mathsf{SFT}}(\mathbb{Z}^2)$.

Tool: If (X, γ) is a G-chart of H and X is an SFT then,

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1. For which groups H, G are there free G-charts (X, γ) of H such that X is an SFT.
2. How to reduce the entropy of such a chart.

First issue

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Let H, G be finitely generated groups. There exists a free G-chart of H if and only if H admits a translation-like action on G.

Free SFT charts

With the extra hypotheses:

- H is finitely presented.
- There exists a strongly aperiodic H-SFT.

Then the free G-chart of H can be chosen as an SFT.

Note: This is essentially due to Jeandel.

Second issue

Question: How to reduce the entropy of an SFT preserving the cocycle?

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Let G be a countable amenable group and X a G-SFT.
For every $\varepsilon > 0$ there exists a G-SFT Y such that

- $h_{top}(G \curvearrowright Y) \leq \epsilon$,
- *There exists a continuous G-equivariant map $\phi \colon Y \to X$.*

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"Every SFT contains sofic subsystems which admit an SFT extension with arbitrarily low entropy"

Second issue: corollaries

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"If there exists one SFT chart, then one can find another SFT chart with arbitrarily low entropy "

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Corollary: Every G-SFT contains a subshift with zero entropy. In particular, every minimal G-SFT has zero topological entropy.

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Question: Is there a G-SFT which does not contain a zero entropy sub-G-SFT? (I don't know the answer for $G = \mathbb{Z}^2$)

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$$\mathcal{E}_{SFT}(G) = \mathcal{E}_{SFT}(\mathbb{Z}^2).$$

proof:

- $\mathbb{Z}^2$ is finitely presented and admits strongly aperiodic SFTs. Thus there exists an SFT free G-chart (X, γ) of $\mathbb{Z}^2$.
- For arbitrarily $\varepsilon > 0$ we can choose $h_{\text{top}}(G \curvearrowright X) \leq \varepsilon$.
- $h_{\text{top}}(G \curvearrowright X) + \mathcal{E}_{\text{SFT}}(\mathbb{Z}^2) \subseteq \mathcal{E}_{\text{SFT}}(G) \subseteq \mathcal{E}_{\text{SFT}}(\mathbb{Z}^2)$.
- Done. $\square$

Proving the entropy reduction theorem (in $\mathbb{Z}^2$)

0	0	1	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
0	0	0	1	0	1	0	0	1	0	0	0	1	0	0	1	0	1	0	1	0	0	0	0	1	0	0	0	1	0	1	0	0	1	0	0	0
1	0	1	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	1	0	0	1	0
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1	0	1	0	0	1	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0
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Proving the entropy reduction theorem (in $\mathbb{Z}^2$)

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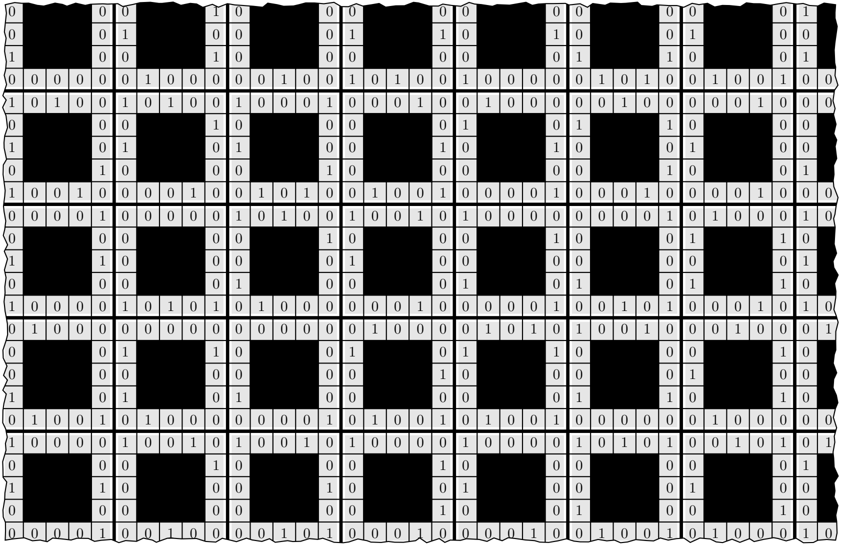
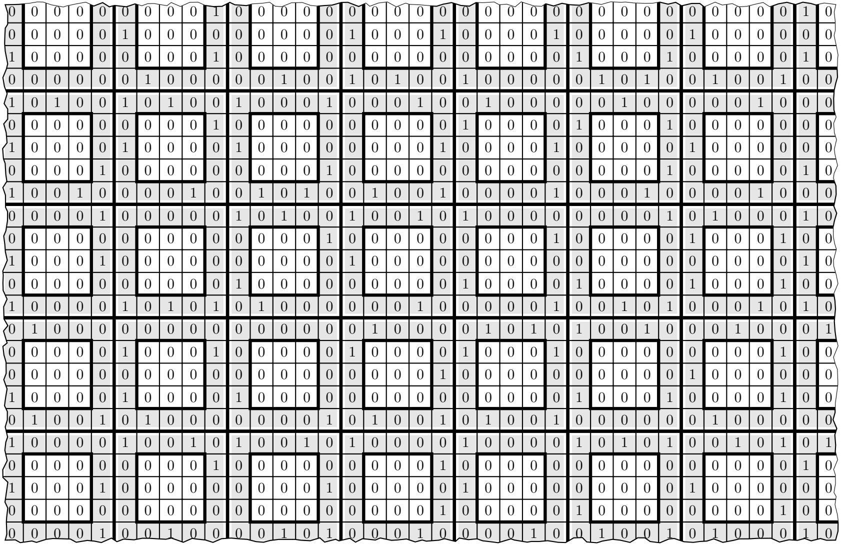
Proving the entropy reduction theorem (in $\mathbb{Z}^2$)



Proving the entropy reduction theorem (in $\mathbb{Z}^2$)



Remark: This idea fails in a countable amenable group.

(No sequence of subgroups of finite index isomorphic to the original group such that a choice of coset representatives forms a Følner sequence)

Proving the entropy reduction theorem

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Proving the entropy reduction theorem

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Idea: Replace the square by a finite set of sufficiently invariant finite sets which **tile** the group.

Tilings of a group (Introduced by Ornstein and Weiss, 87.)

Let $\mathcal{T} = \{T_1, \ldots, T_n\}$ be a set of finite subsets of G. A tiling of the group is a function $\tau \colon G \to \mathcal{T} \cup \{\varnothing\}$ such that:

1. (τ is pairwise-disjoint) For every $g, h \in G$, if $g \neq h$ then $\tau(g)g \cap \tau(h)h = \varnothing$.
2. (τ covers G) For every $g \in G$ there exists $h \in G$ such that $g \in \tau(h)h$.

Remark: Given $\mathcal{T} = \{T_1, \dots, T_n\}$. The set of all tilings of G by $\mathcal{T}$ is a G-SFT.

Proving the entropy reduction theorem

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Theorem (Downarowicz, Huczek and Zhang, 19.)

Let G be a countable amenable group. For any $F \Subset G$ and $\varepsilon > 0$ there exists a tile set $\mathcal{T}$ such that:

- *Every $T \in \mathcal{T}$ is (F, ε)-invariant,*
- *There exists a tiling τ by $\mathcal{T}$ such that the topological entropy of its orbit closure is zero.*

Proving the entropy reduction theorem

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To prove the entropy reduction Theorem, use the set of tiles from the Theorem above instead of the squares.

Closing remarks

Goal: Study the possible sets $\mathcal{E}_{\mathsf{SFT}}(G)$ for an amenable group.

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In this talk:

- Full classification for Polycyclic groups.
- Tools for embedding entropies of one group into another.
- A class of groups with a full classification

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In this talk:

- Full classification for Polycyclic groups.
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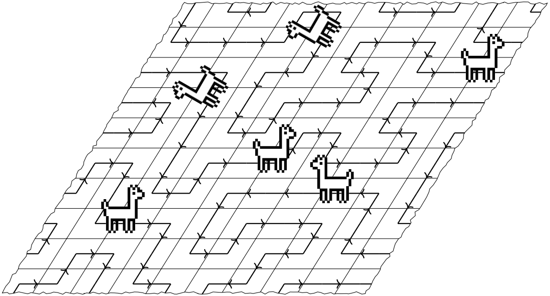
- A class of groups with a full classification

To keep in mind:

- This is far from a full characterization.
- Does not cover many solvable groups with decidable word problem. Baumslag-Solitar groups, Lamplighter, etc.

Thank you for your attention!



Entropies of subshifts of finite type on countable amenable groups

Draft available on request.