Insdistinguishability and multidimensional Sturmian configurations

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One world  seminar

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- Let A be a finite set and $d \geq 1$ be an integer.
- A configuration is a map $x\colon \mathbb{Z}^d \to A$.

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- A configuration is a map $x \colon \mathbb{Z}^d \to A$.

For example if $d = 2$ and $A = \{0, 1, 2\}$ a configuration looks like:

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	1	0	2	1	0	2	1	0	2
2	1	0	2	1	0	2	2	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

We say two configurations $x, y \in A^{\mathbb{Z}^d}$ are **asymptotic** if there exists a finite $F \subset \mathbb{Z}^d$ such that $x|_{\mathbb{Z}^d \setminus F} = y|_{\mathbb{Z}^d \setminus F}$.

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Given asymptotic x, y, we call $F = \{n \in \mathbb{Z}^d : x_n \neq y_n\}$ their **difference set**.

x

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	1	0	2	1	0	2	1	0	2
2	1	0	2	1	0	2	2	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

y

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	0	2	2	1	0	2	1	0	2
2	1	0	2	1	0	2	1	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

Why the name "asymptotic"?

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Let σ be the $\mathbb{Z}^d$ action of $A^{\mathbb{Z}^d}$ given by

$$\sigma^n(x)(m) = x(n+m) \text{ for every } n, m \in \mathbb{Z}^d.$$

x, y are asymptotic if and only if for any sequence $(n_k)_{k \in \mathbb{N}}$ in $\mathbb{Z}^d$ with $\|n_k\| \to \infty$ then $d(\sigma^{n_k}(x), \sigma^{n_k}(y)) \to 0$.

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x	$=$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
y	$=$	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0

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$\sigma\ (x) =$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$\sigma\ (y) =$

0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0

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$\sigma^2(x) =$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$\sigma^2(y) =$

0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0

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$\sigma^3(x) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma^3(y) =$	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0

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$\sigma^4(x) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma^4(y) =$	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

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$$\sigma^5(x) = \boxed{0\ 0\ 0\ 0\ {\color{red}0}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0}$$

$$\sigma^5(y) = \boxed{0\ 0\ 0\ 0\ {\color{red}1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0}$$

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$\sigma^6(x) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma^6(y) =$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

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$\sigma^7(x) =$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma^7(y) =$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

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$\sigma^8(x) =$ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

$\sigma^8(y) =$ 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

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$\sigma^9(x) =$ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

$\sigma^9(y) =$ 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

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$\sigma^{10}(x) =$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$\sigma^{10}(y) =$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

- Let $x, y \in A^{\mathbb{Z}^d}$ be asymptotic.
- Given $S \Subset \mathbb{Z}^d$ and a pattern $p \in A^S$ let

$$[p] = \{z \in A^{\mathbb{Z}^d} : z|_S = p\}.$$

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We wish to compute how many times p occurs in x vs how many times it occurs in y.

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We wish to compute how many times p occurs in x vs how many times it occurs in y.

$$\Delta_p(x, y) = \sum_{u \in \mathbb{Z}^d} 1_{[p]}(\sigma^u(y)) - 1_{[p]}(\sigma^u(x)).$$

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We wish to compute how many times p occurs in x vs how many times it occurs in y.

$$\Delta_p(x, y) = \sum_{u \in \boxed{F - S}} 1_{[p]}(\sigma^u(y)) - 1_{[p]}(\sigma^u(x)).$$

We say an asymptotic pair x, y is indistinguishable if $\Delta_p(x, y) = 0$ for every pattern p.

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	1	0	2	1	0	2	1	0	2
2	1	0	2	1	0	2	2	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

y

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	0	2	2	1	0	2	1	0	2
2	1	0	2	1	0	2	1	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

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x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0), (0,1), (1,0), (2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

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x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

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x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

Example: Let $d = 2$ and $S = \{(0,0),(0,1),(1,0),(2,0)\}$.

x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
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x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

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x

0	2	2	1	0	2
2	1	1	0	2	1
1	0	2	2	1	0
0	2	1	0	2	2

y

0	2	2	1	0	2
2	1	0	2	2	1
1	0	2	1	1	0
0	2	1	0	2	2

So for every pattern p with support S, we have $\Delta_p(x, y) = 0$.

Examples:

- (x, x) for any $x \in A^{\mathbb{Z}^d}$ is an indistinguishable asymptotic pair. We call it **trivial**.

Examples:

- (x, x) for any $x \in A^{\mathbb{Z}^d}$ is an indistinguishable asymptotic pair. We call it **trivial**.
- If $x, y \in A^{\mathbb{Z}^d}$ are asymptotic and on the same orbit $(\sigma^n(y) = x$ for some $n \in \mathbb{Z}^d)$ then they are indistinguishable.

x

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	1	0	0
0	0	0	0	1	1	1	1	1	0
0	0	0	0	1	1	1	1	1	0
0	0	0	0	0	1	1	1	0	0
0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

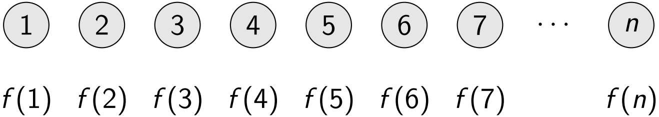
y

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0
0	1	1	1	1	1	0	0	0	0
0	1	1	1	1	1	0	0	0	0
0	0	1	1	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

Does there exist indistinguishable asymptotic pairs which are not on the same orbit?

Motivation

Consider n balls with real weights given by a map f.



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¿What is the probability distribution $\mu = (\mu_1, \ldots, \mu_n)$ on $\{1, \ldots, n\}$ that maximizes entropy plus average weight?

$$\max_{\mu} \left(H(\mu) + \int f \, \mathrm{d}\mu \right) = \max_{\mu} \sum_{i=1}^{n} \left(-\mu_i \log(\mu_i) + f(i)\mu_i \right).$$

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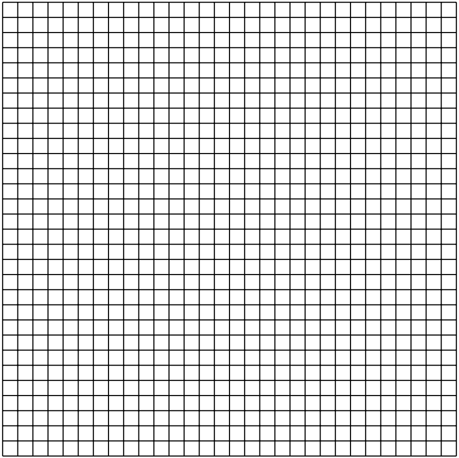
Answer: Boltzmann's distribution.

$$\mu_k = \frac{\exp(f(k))}{\sum_{i=1}^{n} \exp(f(i))}.$$

We can extend this idea to sets of configurations, yielding the notion of **Gibbs measures**.

Gibbs measures



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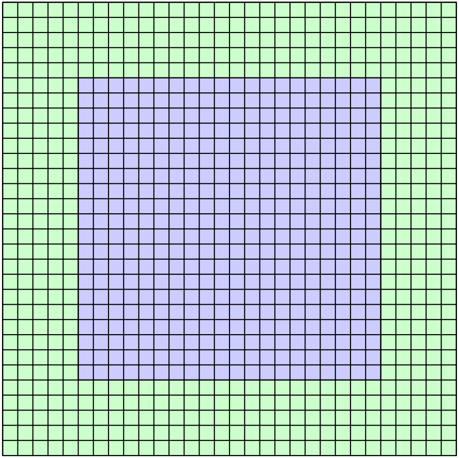

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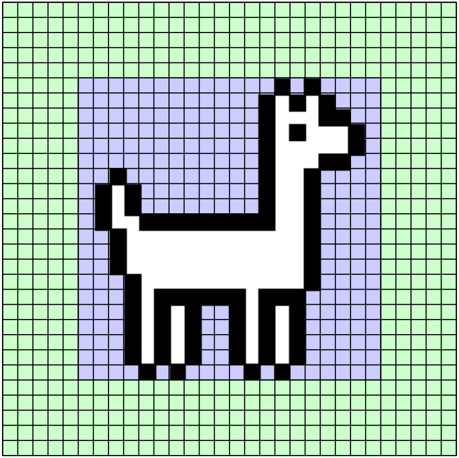

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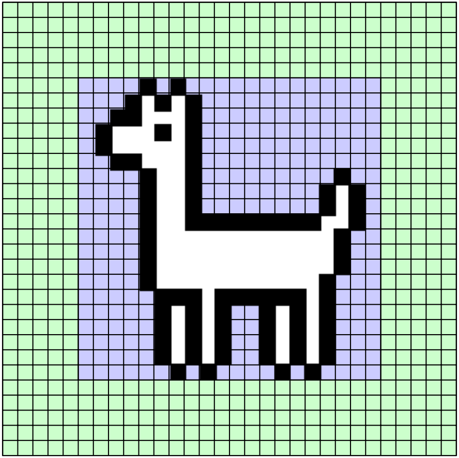

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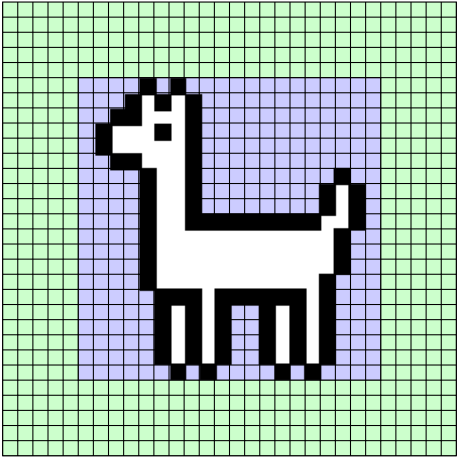

$\mu($$|$$)$ and $\mu($$|$$)$ follow Boltzmann's distribution for some f.

Gibbs Measures

Denote the set of all asymptotic pairs (x, y) by $\mathcal{A}$. The Boltzmann distribution of a Gibbs measure is determined by a **cocycle** $\Psi\colon \mathcal{A} \to \mathbb{R}$, that is, a map which satisfies:

$$\Psi(x, y) = \Psi(x, z) + \Psi(z, y) \text{ for all } (x, y), (y, z) \in \mathcal{A}.$$

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Example: if all configurations all equally likely (that is, there is no associated weight) we obtain the cocycle $\Psi = 0$ and the sole Gibbs measure for Ψ is the uniform Bernoulli measure on $A^{\mathbb{Z}^d}$.

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3. It can be shown that the strong norm on $\mathcal{B}^*$ is given by

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4. **An asymptotic pair gives the trivial linear functional if and only if it is indistinguishable.**

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Theorem (SB + SL + ŠS, 2021)

Yes. We completely characterize them on $\mathbb{Z}$. They are closely connected to Sturmian codings of irrational rotations.

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Indistinguishable asymptotic pairs are invariant under actions of the affine group of $\mathbb{Z}^d$.

In particular, they are invariant under the shift map.

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If $(x_n, y_n)_{n \in \mathbb{N}}$ converges in the asymptotic relation to (x, y) and every pair (x_n, y_n) is indistinguishable, then (x, y) is indistinguishable.

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4. As $\bigcap_{n\in\mathbb{N}}[p_n] = \sigma^k(x)$, we conclude that $\sigma^k(x) = \sigma^m(y)$.

The case of $\mathbb{Z}$

On $\mathbb{Z}$ life is easier (as opposed to $\mathbb{Z}^d$ with $d \geq 2$):

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Corollary: If x, y are indistinguishable with difference set $F = [\![0, k-1]\!]$ then their word complexity satisfies

$$|\mathcal{L}_n(x)| \leq k + n - 1.$$

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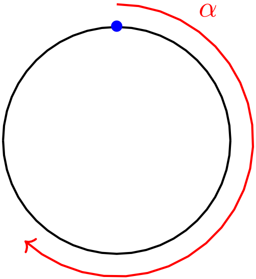
Thus x, y must be Sturmian configurations!

Let $\alpha \in [0,1] \setminus \mathbb{Q}$. Consider the rotation $R_\alpha \colon S^1 \to S^1$ given by $R_\alpha(x) = x + \alpha$.
Consider the partition $\mathcal{P} = \{P_0 = [0, 1-\alpha), [1-\alpha, 1)\}$.



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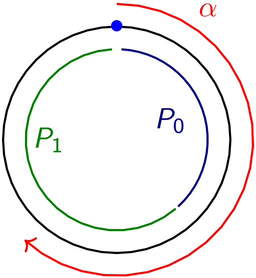

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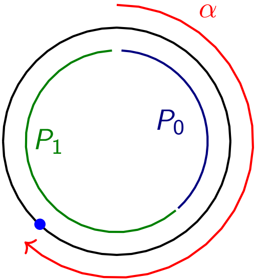

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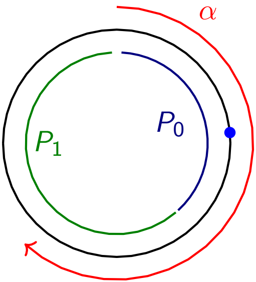

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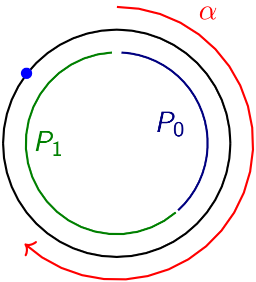

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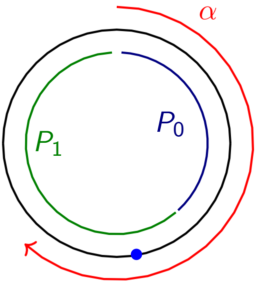

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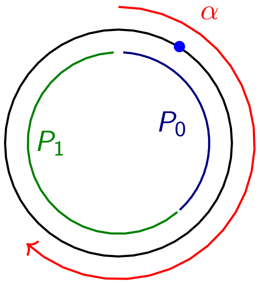

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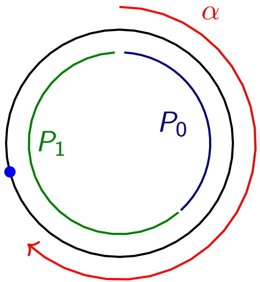

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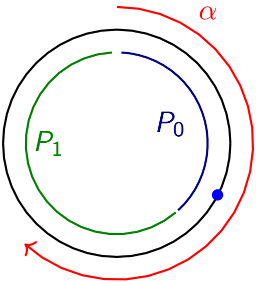

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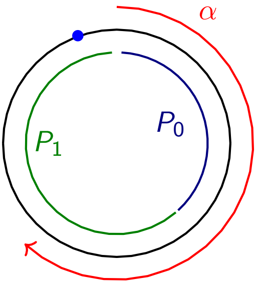

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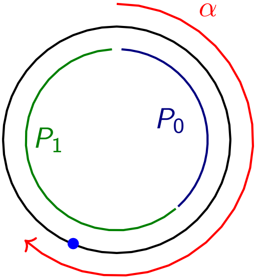

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Formally, given $\alpha \in [0,1] \setminus \mathbb{Q}$ let $c_\alpha \in \{0,1\}^{\mathbb{Z}}$ be given by

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Choosing instead the partition $\mathcal{P}' = \{P_0 = (0, 1-\alpha], (1-\alpha, 1]\}$ gives

$$c'_\alpha(n) = \lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil.$$

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The pair (c_α, c'_α) is indistinguishable. In fact, every pattern in their language occurs exactly once intersecting each of their difference sets.

Theorem: B, Labbé and Starosta

Let $x, y \in \{0,1\}^{\mathbb{Z}}$ and assume that x is recurrent. The following are equivalent:

- (x, y) is an indistinguishable asymptotic pair with difference set $F = \{-1, 0\}$ such that $x_{-1}x_0 = 10$ and $y_{-1}y_0 = 01$
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The non-recurrent case is an asymptotic limit of Sturmians.

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But there is more...

The general case can be obtained from Sturmians using shifts and substitutions.

Theorem: B, Labbé and Starosta

Let A be a finite alphabet and $x, y \in A^{\mathbb{Z}}$ a non-trivial asymptotic pair. Then x, y is indistinguishable if and only if either

- x is recurrent and there exists $\alpha \in [0,1] \setminus \mathbb{Q}$, a substitution $\varphi\colon \{0,1\} \to A^{+}$ and an integer $m \in \mathbb{Z}$ such that

$$\{x, y\} = \{\sigma^m \varphi(\sigma(c_\alpha)), \sigma^m \varphi(\sigma(c'_\alpha))\},$$

- x is not recurrent and there exists a substitution $\varphi\colon \{0,1\} \to A^{+}$ and an integer $m \in \mathbb{Z}$ such that

$$\{x, y\} = \{\sigma^m \varphi({}^{\infty}0.10^{\infty}), \sigma^m \varphi({}^{\infty}0.010^{\infty})\}.$$

What about $d \geq 2$?

Things are much harder:

- Patterns may occur without intersecting the difference set.
- recurrent indistinguishable pairs may not be uniformly recurrent.
- Substitutions do not help reduce the problem to a small size difference set (no good notion of derived sequences).
- In general, there is no complexity bound.

Example:

x

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	0	1	0	1	0	0	1
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

y

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	1	0	0	1	0	0	1
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

The horizontal configuration is a 1-dimensional indistinguishable pair, everything else is the symbol 2.

Example:

x

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	0	1	0	1	0	0	1
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

y

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	1	0	1	0	0	1	0	0	1
2	2	2	2	2	2	2	2	2	2	2	2
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2	2	2	2	2	2	2	2	2	2	2	2

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- Recurrent but not uniformly recurrent.
- Some patterns do not occur in the difference set.

Theorem: B and Labbé.

Let $d \geq 1$ and $x, y \in \{0, \ldots, \mathtt{d}\}^{\mathbb{Z}^d}$ be an asymptotic pair with difference set $F = \{0, -e_1, \ldots, -e_d\}$. TFAE:

1. The asymptotic pair (x, y) is indistinguishable, satisfies the **flip condition** and x is uniformly recurrent.
2. There exists a totally irrational vector $\alpha \in [0, 1)^d$ such that $x = c_\alpha$ and $y = c'_\alpha$ are the **characteristic multidimensional Sturmian configurations** of slope α.

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1. The asymptotic pair (x, y) is indistinguishable.
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Let $(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ and consider the associated rotations $R_{\alpha_1}, \ldots, R_{\alpha_d}$.

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- Respectively, let $\mathcal{W}'$ be the partition of S^1 generated by refining the Sturmian partitions $\mathcal{P}'_i = \{(0, 1-\alpha_i], (1-\alpha_i, 1]\}$ for every $1 \leq i \leq d$.

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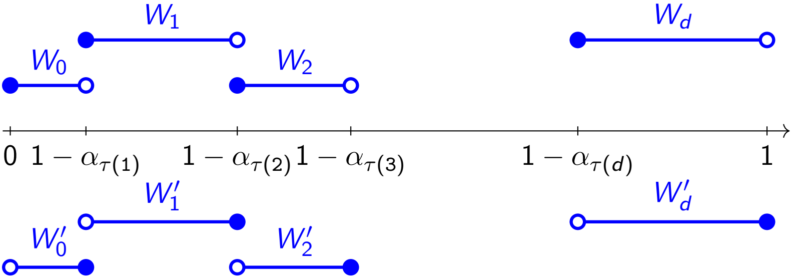
The characteristic Sturmian configurations c_α, c_α' of slope α are the codings of 0 under the $\mathbb{Z}^d$-orbit generated by the rotations R_{α_i} and the partitions $\mathcal{W}$ and $\mathcal{W}'$ respectively.

Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in [0,1]^d$, let $\tau \in S_d$ such that

$$1 \geq \alpha_{\tau(1)} \geq \alpha_{\tau(2)} \geq \cdots \geq \alpha_{\tau(d)} \geq 0.$$

Then the partitions $\mathcal{W}$ and $\mathcal{W}'$ are given by:



Explicitly, given $\alpha = (\alpha_1, \ldots, \alpha_d)$ we have

$$\begin{array}{rccl} c_\alpha : & \mathbb{Z}^d & \to & \{0, \ldots, \mathtt{d}\} \\ & n & \mapsto & \sum_{i=1}^{d} \left(\lfloor \alpha_i + n \cdot \alpha \rfloor - \lfloor n \cdot \alpha \rfloor \right), \end{array}$$

and

$$\begin{array}{rccl} c'_\alpha : & \mathbb{Z}^d & \to & \{0, \ldots, \mathtt{d}\} \\ & n & \mapsto & \sum_{i=1}^{d} \left(\lceil \alpha_i + n \cdot \alpha \rceil - \lceil n \cdot \alpha \rceil \right). \end{array}$$

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The configurations c_α, c'_α are asymptotic with difference set $F = \{0, -e_1, \ldots, -e_d\}$.

Recall the picture from the beginning:

x

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	1	0	2	1	0	2	1	0	2
2	1	0	2	1	0	2	2	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

y

1	0	2	2	1	0	2	1	0	2	1	1	0	2	1
0	2	1	1	0	2	1	0	2	1	0	2	2	1	0
2	1	0	2	2	1	0	2	1	0	2	1	1	0	2
1	0	2	1	1	0	2	1	0	2	1	0	2	2	1
0	2	1	0	2	2	1	0	2	1	0	2	1	1	0
2	1	0	2	1	1	0	2	1	0	2	1	0	2	2
1	0	2	1	0	2	2	1	0	2	1	0	2	1	1
0	2	1	0	2	1	0	2	2	1	0	2	1	0	2
2	1	0	2	1	0	2	1	1	0	2	1	0	2	1
1	0	2	1	0	2	1	0	2	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	1	0	2	1	0	2
1	1	0	2	1	0	2	1	0	2	2	1	0	2	1
0	2	2	1	0	2	1	0	2	1	1	0	2	1	0
2	1	1	0	2	1	0	2	1	0	2	2	1	0	2
1	0	2	2	1	0	2	1	0	2	1	1	0	2	1

We have $x = c_\alpha$ and $y = c'_\alpha$ respectively for

$$\alpha = \left(\frac{\sqrt{2}}{2}, \sqrt{19} - 4 \right).$$

Flip Condition

Let $x, y \in \{0, \ldots, \mathtt{d}\}^{\mathbb{Z}^d}$ be an asymptotic pair. We say it satisfies the **flip condition** if:

1. the difference set of x and y is $F = \{0, -e_1, \ldots, -e_d\}$,
2. the restriction $x|_F$ is a bijection $F \to \{0, \ldots, \mathtt{d}\}$ such that $x_0 = 0$,
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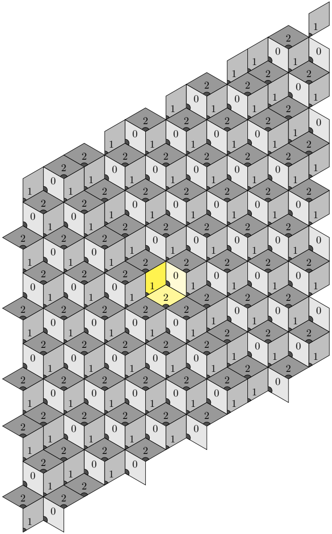
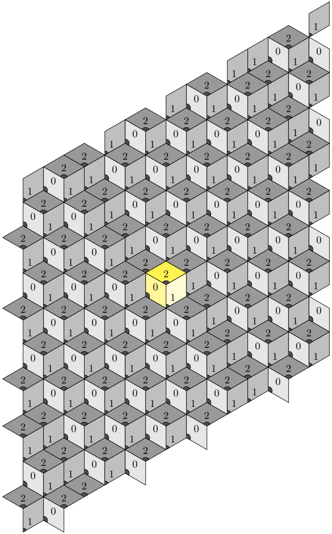
The conditions above induce a permutation on $\{0, \dots, \mathtt{d}\}$ defined by $y_n \mapsto x_n$ for every $n \in F$, which is the cyclic permutation $(0, 1, \dots, d)$ of the alphabet.

The flip condition can be interpreted as flipping the unit hypercube on a co-dimension 1 discrete subspace.

x



y



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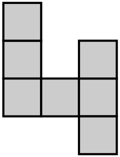
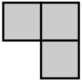
Say $c_\alpha \in \{0, 1, 2\}^{\mathbb{Z}^d}$ and you need to know how many patterns with support $S \Subset \mathbb{Z}^2$ there are.

$S =$  $F =$  $F - S =$

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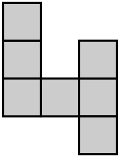
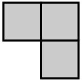
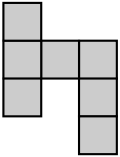
$S =$ 

$F =$ 

$F - S =$ 

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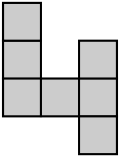
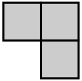
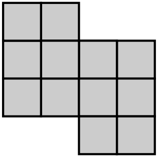
$S =$ 

$F =$ 

$F - S =$ 

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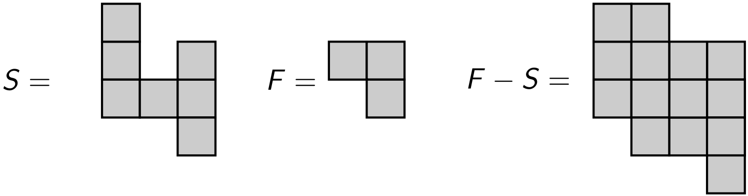

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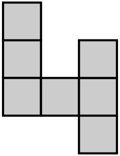
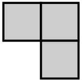
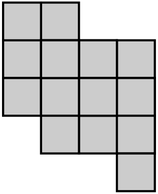
$S =$  $F =$  $F - S =$ 

There are exactly 14 patterns with support S on a 2-dimensional Sturmian configuration.

Let $(m_1, \dots, m_d) \in \mathbb{N}^d$ and consider the rectangle

$$R = \prod_{i=1}^{d} [\![0, m_i - 1]\!].$$

In this case we get a beautiful formula for the rectangular complexity of a multidimensional Sturmian configuration x:

$$|\mathcal{L}_R(x)| = |\mathcal{L}_{(m_1,\dots,m_d)}(x)| = m_1 \cdots m_d \left(1 + \frac{1}{m_1} + \cdots + \frac{1}{m_d}\right).$$

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We can interpret it as $|F - R|$, which is the volume of R, plus the volume of each of the $d-1$ dimensional faces.

Let $(m_1, \ldots, m_d) \in \mathbb{N}^d$ and consider the rectangle

$$R = \prod_{i=1}^{d} [\![0, m_i - 1]\!].$$

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$\triangleright$ For $d = 1$ we recover $L_n(x) = n + 1$.

Thanks!

Indistinguishable asymptotic pairs and multidimensional Sturmian configurations.
S. Barbieri, S. Labbé
https://arxiv.org/abs/2204.06413

A characterization of Sturmian sequences by indistinguishable asymptotic pairs
S. Barbieri, S. Labbé, Š. Starosta
https://doi.org/10.1016/j.ejc.2021.103318