# Self-simulable groups

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**In this talk:** for some groups, these results generalize to a large class of actions if we don't care about the factor being "nice".

## Subshift of finite type

Let $A$ be a finite set and consider $A^\Gamma = \{x \colon \Gamma \to A\}$ with the prodiscrete topology and the action $\Gamma \curvearrowright A^\Gamma$ given by

$$(gx)(h) = x(g^{-1}h) \text{ for every } g, h \in \Gamma.$$

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$Y \subset A^\Gamma$ is a $\Gamma$-**subshift of finite type** (SFT) is there is a finite set $F \subset \Gamma$ and $\mathcal{F} \subset A^F$ such that $y \in Y$ if and only if

$$(gy)|_F \notin \mathcal{F} \text{ for every } g \in \Gamma.$$

A subshift is of finite type if it is the set of configurations $x \in A^\Gamma$ which avoid a finite list of forbidden patterns (represented by $\mathcal{F}$).

**Theorem (B. Sablik, Salo (2021))**

*For a class of groups, called **self-simulable**, every "computable" action $\Gamma \curvearrowright X$ on a compact "computable" space $X$ is the factor of a subshift of finite type.*

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Intuitively:

- **Computable action $\Gamma \curvearrowright X$:** there's an algorithm that from a description of $x \in X$ and $g \in \Gamma$ can compute $gx$.
- **Computable space**: there's an algorithm which can approximate the space $X$.

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Let's look at $X \subset \{0,1\}^{\mathbb{N}}$.

## Computable space

For a word $w = w_0 w_1 \dots w_{n-1} \in \{0,1\}^n$ consider the cylinder set

$$[w] = \{x \in \{0,1\}^{\mathbb{N}} : x|_{\{0,\dots,n-1\}} = w\}.$$

### Effectively closed set

A set $X \subset \{0,1\}^{\mathbb{N}}$ is called **effectively closed** if there is a Turing machine which enumerates a sequence of words $(w_n)_{n\in\mathbb{N}}$ such that

$$X = \{0,1\}^{\mathbb{N}} \setminus \bigcup_{n\in\mathbb{N}} [w_n].$$

# Definitions

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Let $X \subset \{0,1\}^{\mathbb{N}}$. A function $\varphi \colon X \to \{0,1\}^{\mathbb{N}}$ is **computable** if there is a Turing machine which on input

- $n \in \mathbb{N}$
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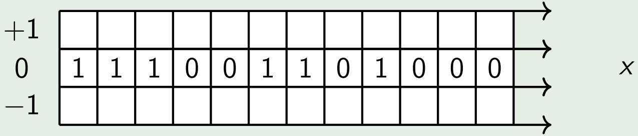
Let us restrict to a finitely generated group $\Gamma = \langle S \rangle$.

An action $\Gamma \curvearrowright X$ is computable, if for every $s \in S$ the map $\varphi_s\colon X \to X$ given by $\varphi_s(x) = sx$ is computable.

Computable action $\Gamma \curvearrowright X$

**Example: $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{N}}$ odometer**

Let $S = \{-1, 0, +1\}$



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| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | $y_7$ | $y_8$ | $y_9$ | $y_{10}$ | $y_{11}$ | $y \in Y$ |

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We want $Y$ to be an effectively closed set!

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$y \in Y$

Let $\Gamma$ be finitely generated by a symmetric set $S \ni 1_\Gamma$ and $X \subset \{0,1\}^{\mathbb{N}}$. Given $\Gamma \curvearrowright X$ consider the set

$$Y = \{y \in (\{0,1\}^S)^{\mathbb{N}} : \pi_s(y) = s \cdot \pi_{1_\Gamma}(y) \in X \text{ for every } s \in S\}.$$

Where $\pi_s(y) \in \{0,1\}^{\mathbb{N}}$ is such that $\pi_s(y)(n) = y(n)(s)$.

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An action $\Gamma \curvearrowright X \subset \{0,1\}^{\mathbb{N}}$ is effectively closed if $Y$ is an effectively closed set.

**Note:** In this talk we will always suppose that $\Gamma$ has decidable word problem to avoid certain technicalities.

"$\Gamma$ has decidable word problem if there's an algorithm that can *draw* arbitrarily large balls of its Cayley graph"

# Example: natural actions of Thompson's groups

Consider $X = \{0,1\}^{\mathbb{N}}$ and let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be non-empty words in $\{0,1\}^*$ such that

$$X = [u_1] \sqcup [u_2] \sqcup \cdots \sqcup [u_n] = [v_1] \sqcup [v_2] \sqcup \cdots \sqcup [v_n].$$

Let $\varphi$ be the homeomorphism of $\{0,1\}^{\mathbb{N}}$ which maps every cylinder $[u_i]$ to $[v_i]$ by replacing prefixes, that is

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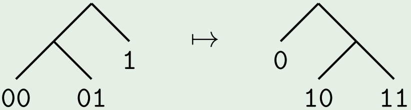
$u_1 = 00, u_2 = 01, u_3 = 1$ and $v_1 = 0, v_2 = 10, v_3 = 11.$

$$\varphi(0101010\dots) = 1001010\dots \qquad \varphi(0000000\dots) = 0000000\dots$$
$$\varphi(1111111\dots) = 1111111\dots \qquad \varphi(0011001\dots) = 011001\dots$$



## Natural action of Thompson's groups

- $F$ is the group of all such homeomorphisms where $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are given in lexicographical order.
- $T$ is the group of all such homeomorphisms where $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are given in lexicographical order up to a cyclic permutation.
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- $T, V$ are nonamenable.
- It is a famous open problem whether $F$ is amenable.

# What results are known?

Let $H \curvearrowright X$ be an effectively closed action and let

$$1 \to N \to \Gamma \to H \to 1$$

The action $H \curvearrowright X$ extends to an action $\Gamma \curvearrowright X$ where $N$ acts trivially.

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3. **Theorem** [B. 2019] $H, G, K$ are three infinite f.g groups

$$1 \to G \times K \to H \times G \times K \to H \to 1.$$

# Question

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**Theorem (B., Sablik, Salo 2021)**

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▷ there are several obstructions to self-simulability.

- Amenable groups cannot be self-simulable.
- Groups with infinitely many ends cannot be self-simulable.
- Some one-ended non-amenable groups are not self-simulable.
  Ex: $F_2 \times \mathbb{Z}$ (multi-ended $\times$ amenable).

# Amenable groups are not self-simulable

If $\Gamma$ is amenable, we can associate to every action $\Gamma \curvearrowright X$ on a compact metrizable space by homeomorphisms a non-negative real number

$$h_{\text{top}}(\Gamma \curvearrowright X) \in [0, +\infty].$$

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2. Topological entropy cannot increase under factors.
3. Conclusion: no action with entropy $+\infty$ can be the factor of a subshift.
4. If $\Gamma$ is recursively presented, there are effectively closed actions $\Gamma \curvearrowright X$ with infinite entropy (the inverse limit of the full $\Gamma$-shifts on $n$ symbols).

# Our main result

**Theorem (B., Sablik, Salo 2021)**

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- No need for self-similar or hierarchical structures as in the other results in the literature.
- Proof based on the existence of paradoxical decompositions.
- The technique is very flexible and allows for many other applications.

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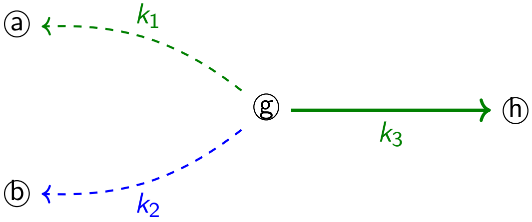
$$\textbf{Alphabet} = K^3 \times \{\mathtt{G}, \mathtt{B}\}.$$

- Three directions $K^3$: one pointing to $\varphi(g)$, the next two pointing to the two preimages
- A color (green or blue) (partitioning the elements of the group into two paradoxical sets).

# The paradoxical subshift

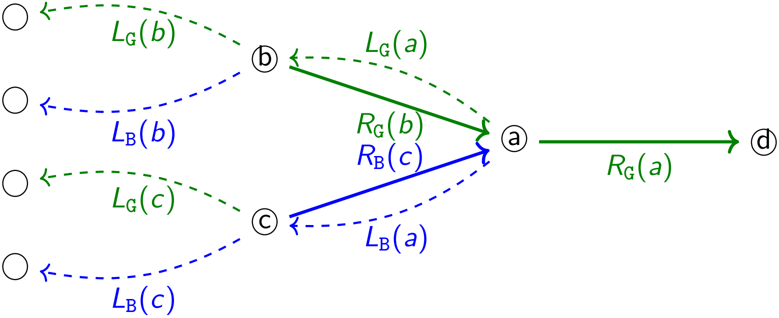
In pictures, the alphabet represents the following structure.

$$(k_1, k_2, k_3, \mathtt{G}) \in K^3 \times \{\mathtt{G}, \mathtt{B}\}$$




- $a \neq b$,
- $\varphi(a) = ak_1^{-1} = g$,
- $\varphi(b) = bk_2^{-1} = g$,
- $\varphi(g) = gk_3 = h$.

The local rules of the subshift impose that every node has two preimages of distinct color, and left arrows must match with right arrows.



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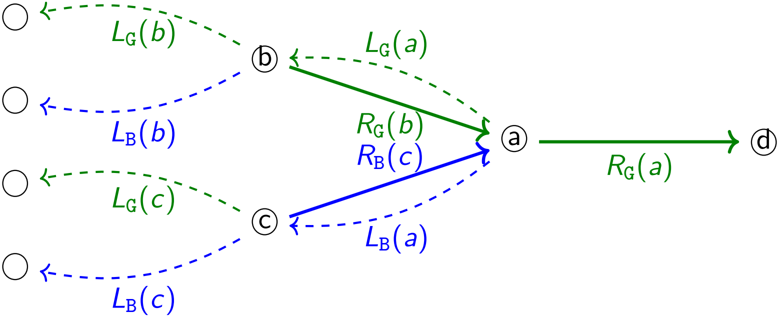
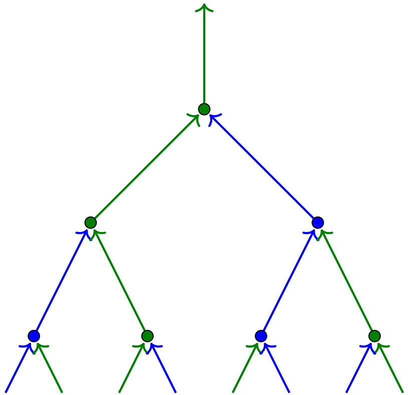



▷ This induces a binary tree-like structure (except for loops).

▷ **Key observation**: In a bi-colored infinite binary tree, there is a canonical way to assign one-sided infinite paths to every node.



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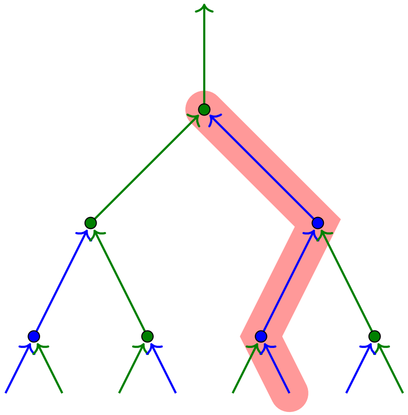

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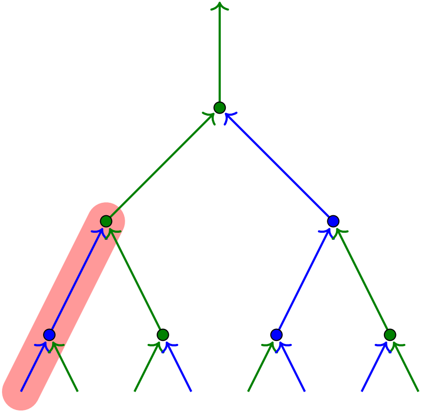

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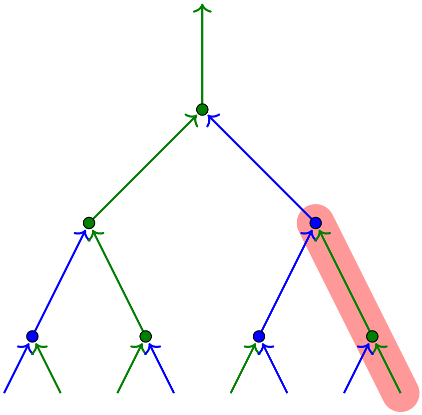

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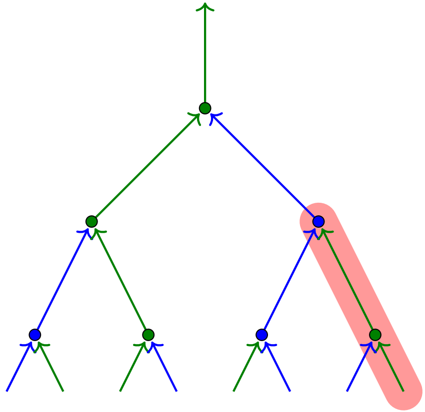

Follow the arrow tails of the opposite color!
The paths do not intersect.

## The paradoxical shift

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- a $\mathbb{N}^2$-grid with moves in a finite set $K \subset \Gamma$ for every $g \in \Gamma$.
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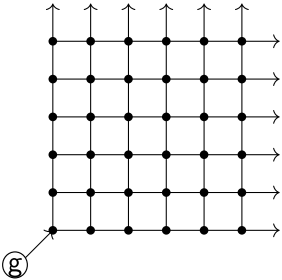

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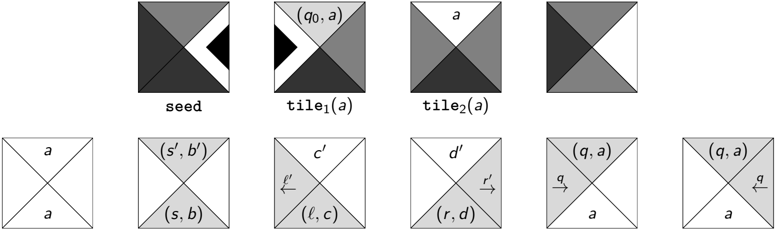


Where $\delta(s, b) = (s', b', 0)$, $\delta(\ell, c) = (\ell', c', -1)$ and $\delta(r, d) = (r', d', 1)$.

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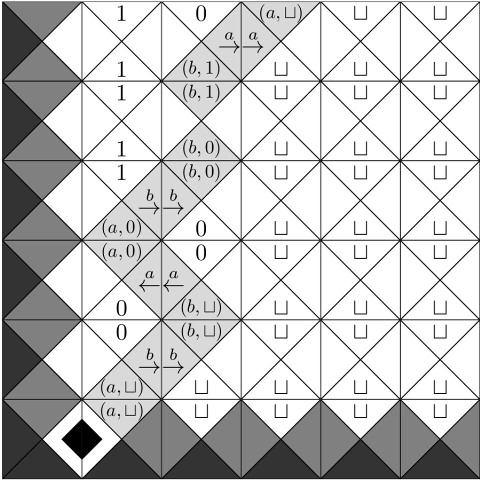

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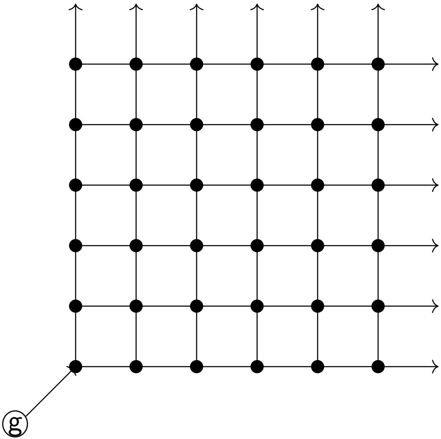
- Take out the tiles containing the accepting state.

Result: The only remaining configurations are the ones in the set representation.

Start with $x = x_0x_1x_2x_3\cdots \in A^{\mathbb{N}}$



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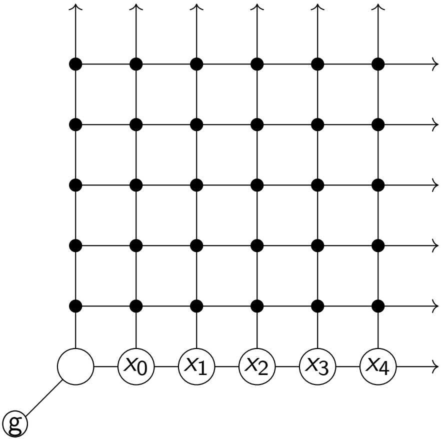

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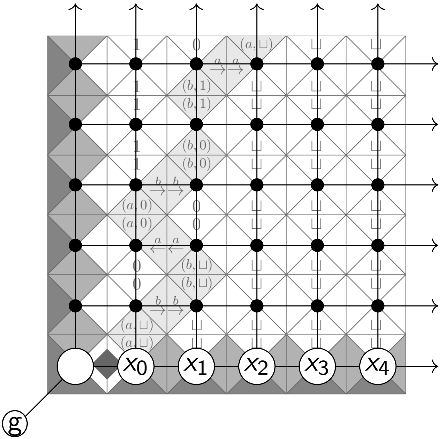

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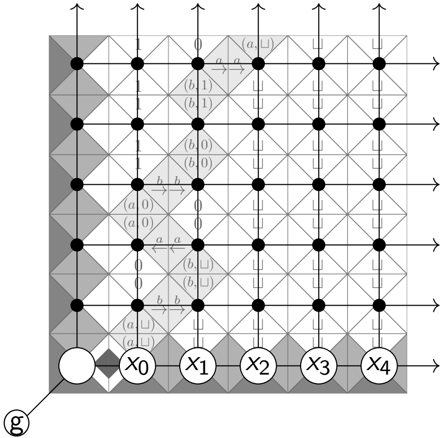


If the configuration survives (i.e. If the Turing machine does not stop), then $x$ is in the set representation of $\Gamma \curvearrowright X$.

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Thus we obtain a natural factor map from this subshift of finite type to $\Gamma \curvearrowright X$.

# Our main result

**Theorem (B., Sablik, Salo 2021)**

The direct product $\Gamma = \Gamma_1 \times \Gamma_2$ of any pair of non-amenable finitely generated groups is self-simulable.

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- Self-simulable groups are stable under quasi-isometries of finitely presented groups.
- Any group which has a normal self-simulable subgroup is self-simulable.
- Any group $\Gamma$ generated by $S$ which has a self-simulable subgroup $\Delta$ with the property that $\Delta \cap s\Delta s^{-1}$ is non-amenable for every $s \in S$ is self-simulable.

# Applications

Mixing the stability properties of this class, we obtain handy ways to show self-simulability:

**Lemma**

Let $\Gamma$ be a finitely generated group which acts faithfully on $X = \{0,1\}^{\mathbb{N}}$ such that for any non-empty open set $U$ the subgroup $\Gamma_U$ which fixes every element of $X \setminus U$ is non-amenable. Then $\Gamma$ is self-simulable.

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**Theorem:** Thompson's $V$ is self-simulable

**Proof:** Consider the natural action $V \curvearrowright \{0,1\}^{\mathbb{N}}$ of Thompson's $V$. For any non-trivial word $w \in \{0,1\}^*$ the subgroup of $V$ which fixes $X \setminus [w]$ is isomorphic to $V$ (which is non-amenable).

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**Proof:** Amenable recursively presented groups are never self-simulable.
Consider the natural action $F \curvearrowright \{0,1\}^{\mathbb{N}}$ of Thompson's $F$. For any non-trivial word $w \in \{0,1\}^*$ the subgroup of $F$ which fixes $X \setminus [w]$ is isomorphic to $F$. As we suppose that $F$ is non-amenable, the lemma holds and we get that $F$ is self-simulable.

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By a similar argument, if $F$ is non-amenable then $T$ is self-simulable.

# Applications

**The following groups are self-simulable:**

- Finitely generated non-amenable branch groups.
- The finitely presented simple groups of Burger and Mozes.
- Thompson's group $V$ and higher-dimensional Brin-Thompson's groups $nV$.
- $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 5$.
- $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ for $n \geq 5$.
- Braid groups $B_n$ on at least $n \geq 7$ strands.
- Right-angled Artin groups associated to the complement of a finite connected graph for which there are two edges at distance at least 3.

# Applications

▷ Suppose $\Gamma \curvearrowright X$ admits a free effectively closed action (for every $x \in X$ then $gx = x$ implies that $g = 1_\Gamma$)

$$(\text{SFT})\ \Gamma \curvearrowright Z \xrightarrow[\text{factor}]{} \Gamma \curvearrowright X$$

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**Proof.**

- Let $\phi \colon Z \to X$ be the factor map, and let $x \in Z$ and $g \in \Gamma$ such that $gx = x$.
- Then $g\phi(x) = \phi(gx) = \phi(x)$.
- As $\Gamma \curvearrowright X$ is free, we have $g = 1_\Gamma$. Thus $\Gamma \curvearrowright Z$ is free.

$\square$

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Every finitely generated group with decidable word problem $\Gamma$ admits an effectively closed $\Gamma$-subshift on which $\Gamma$ acts freely.

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**Corollary**

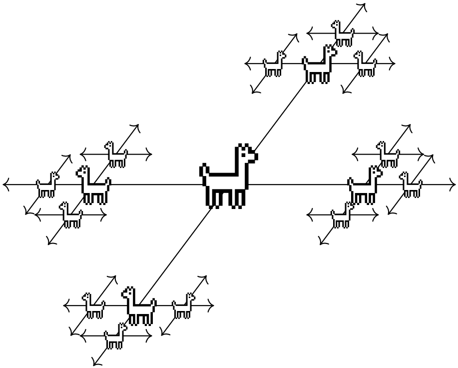
Every self-simulable group $\Gamma$ with decidable word problem admits a $\Gamma$-SFT on which $\Gamma$ acts freely.

Examples:

- $\Gamma = F_n \times F_n$.
- Thompson's $V$.
- Braid groups $B_n$, $n \geq 7$ strands.
- $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 5$.

**Note:** If $\Gamma$ is finitely generated, recursively presented and has undecidable word problem, there are no free effectively closed actions.

# Thank you for your attention!



**Groups with self-simulable zero-dimensional dynamics**
S. Barbieri, M. Sablik and V. Salo
https://arxiv.org/abs/2104.05141