

Midterm 1 May 30, 2018 Duration: 50 minutes*This test has 4 questions on 6 pages, for a total of 40 points.*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations. Answers without justifications will not be marked, unless otherwise indicated.
- Continue on the closest blank page if you run out of space, and **indicate this clearly on the original page**.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **No aids of any kind are allowed**, including: documents, cheat sheets, electronic devices of any kind (including calculators, phones, etc.)

First Name: Solutions Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
- Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - speaking or communicating with other examination candidates, unless otherwise authorized;
 - purposely exposing written papers to the view of other examination candidates or imaging devices;
 - purposely viewing the written papers of other examination candidates;
 - using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

4 marks

1. (a) Compute the equation of the plane which contains the points $A = (2, 1, 1)$, $B = (2, 2, 3)$ and $C = (1, -1, -2)$ and write it in the form $ax + by + cz = d$.

Solution: Use A as a base point and form the vectors $\vec{AB} = \langle 2, 2, 3 \rangle - \langle 2, 1, 1 \rangle = \langle 0, 1, 2 \rangle$ and $\vec{AC} = \langle 1, -1, -2 \rangle - \langle 2, 1, 1 \rangle = \langle -1, -2, -3 \rangle$. Use both vectors to compute a normal vector to the plane using the cross product $\vec{AB} \times \vec{AC}$. Now, $\vec{AB} \times \vec{AC} = \langle 1, -2, 1 \rangle$, so the solution is of the form $x - 2y + z = d$. Plugging any of the three points yields that $d = 1$. Note that the solution $-x + 2y - z = -1$ is also correct.

Answer: $x - 2y + z = 1$.

2 marks

- (b) Consider the point $D = (0, 1, t)$ where t is a value such that D is in the plane which contains A, B and C . Determine the value of t .

Solution: Replacing $(0, 1, t)$ in the plane equation yields $0 - 2 + t = 1$, therefore $t = 3$.

Answer: $t = 3$.

4 marks

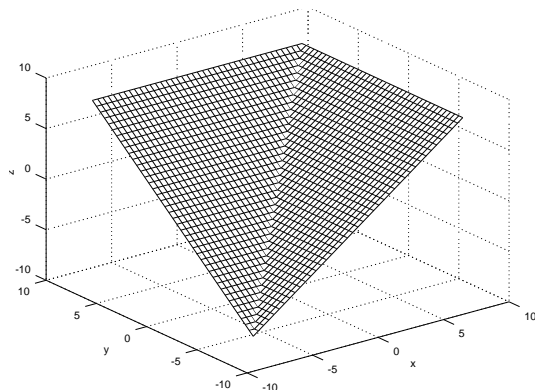
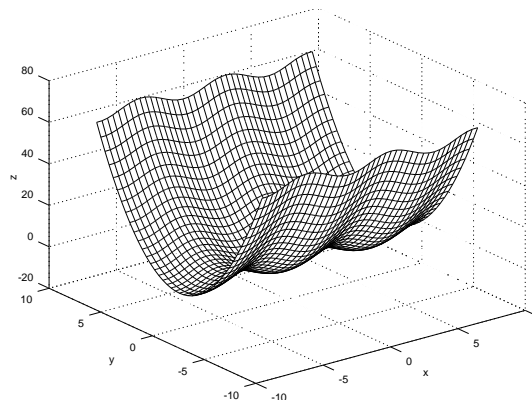
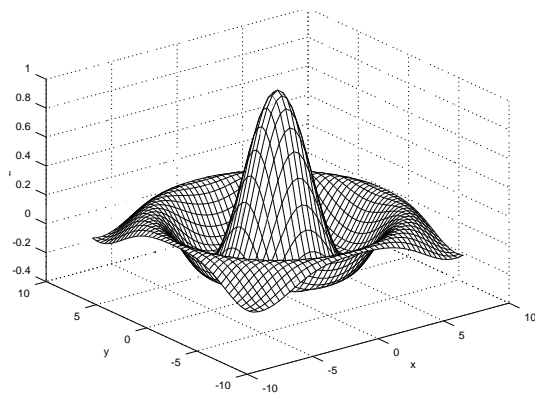
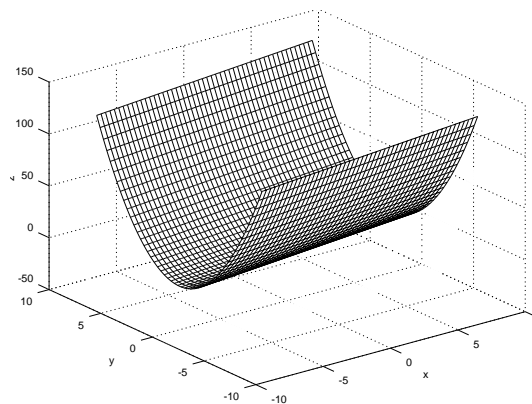
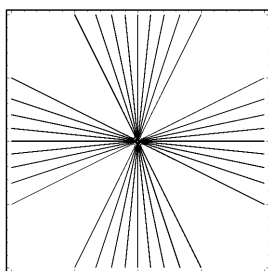
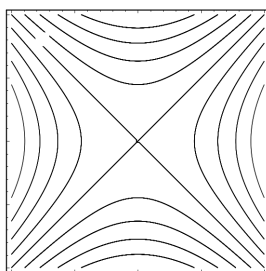
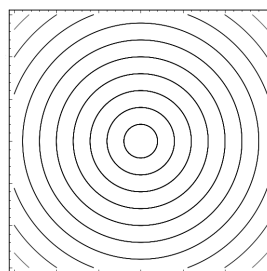
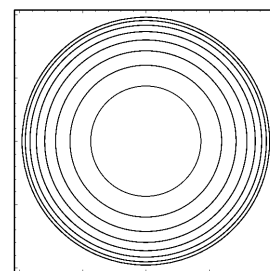
- (c) What angle does the plane from (a) form with the plane given by the equation $x - y = 0$?

Solution: A vector which is normal to the plane $x - y = 0$ is $\vec{n} = (1, -1, 0)$. A normal vector to the first plane is $\vec{m} = (1, -2, 1)$. We can use the dot product formula to obtain that $\cos \theta = \frac{\vec{n} \cdot \vec{m}}{\|\vec{n}\| \|\vec{m}\|} = \frac{3}{\sqrt{2}\sqrt{6}}$.

Answer: $\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

10 marks

2. In the box below each plot, place a function from the list below which could generate that plot. There are two functions which are not associated to any plot. You do not need to justify your answers.

Answer: *A*Answer: *G*Answer: *J*Answer: *E*Answer: *D*Answer: *I*Answer: *F*Answer: *C*

$$A(x, y) = \max(x, y),$$

$$B(x, y) = 3y + x^2,$$

$$C(x, y) = \sqrt{1 - x^2 - y^2},$$

$$D(x, y) = \frac{xy}{x^2 + y^2},$$

$$E(x, y) = x + 2y^2,$$

$$F(x, y) = \sqrt{x^2 + y^2},$$

$$G(x, y) = 3 \sin(x) + y^2,$$

$$H(x, y) = x^2 + y^2,$$

$$I(x, y) = x^2 - y^2,$$

$$J(x, y) = \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}.$$

6 marks

3. (a) Let $u(x, y) = \sin(x + 3y) + (x - 3y)^3$. Compute the partial derivatives u_x, u_y, u_{xx} and u_{yy} .

Answer: $u_x = \cos(x + 3y) + 3(x - 3y)^2$.

Answer: $u_y = 3 \cos(x + 3y) - 9(x - 3y)^2$

Answer: $u_{xx} = -\sin(x + 3y) + 6(x - 3y)$

Answer: $u_{yy} = -9 \sin(x + 3y) + 54(x - 3y)$

4 marks

- (b) Determine for which values $a \in \mathbb{R}$ is $u(x, y)$ a solution of the equation

$$\frac{\partial^2 u}{\partial y^2} + a^3 \left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x} \right) = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Answer: $a = -3$ and $a = 3$.

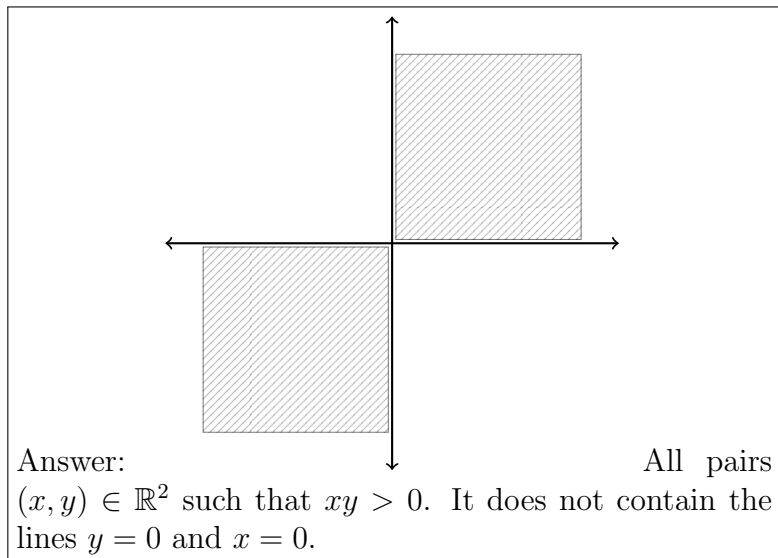
Solution: As $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ the equation above simplifies to

$$\frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Replacing the solutions obtained above we obtain that $a^2 = 9$, and thus the only solutions are $a = -3$ and $a = 3$.

2 marks

4. (a) Consider the function $f(x, y) = \ln(xy)$. Draw a sketch of largest possible domain $D \subset \mathbb{R}^2$ where f is well defined. Does it contain the lines defined by $y = 0$ and $x = 0$?



4 marks

- (b) What is the equation of the tangent plane to $f(x, y) = \ln(xy)$ passing through $(x_0, y_0) = (1, 1)$?

Answer: $z = x + y - 2$

Solution: The equation of the tangent plane to the surface at the point (x_0, y_0) is $z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$.

The partial derivatives are $\frac{\partial f}{\partial x} = \frac{1}{x}$ and $\frac{\partial f}{\partial y} = \frac{1}{y}$. Replacing the values $x_0 = 1, y_0 = 1$ yields $f(1, 1) = 0$, $\frac{\partial f}{\partial x}(1, 1) = 1$ and $\frac{\partial f}{\partial y}(1, 1) = 1$, therefore the equation is $z = (x - 1) + (y - 1)$.

3 marks

- (c) The volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is given by the expression $V = \frac{4\pi}{3}abc$. Suppose that the values of a, b, c are known up to an error $\Delta a = \Delta b = \Delta c = \frac{3}{400\pi}$. Give an estimation of the error ΔV on the volume of the ellipsoid *using total differentials*. Leave your answer in expressed in terms of a, b, c .

$$\text{Answer: } dV = \frac{1}{100} (bc + ac + ab)$$

Solution: In general if $z = g(x, y, z)$ we have $dz = \frac{\partial g}{\partial x}(x, y, z)dx + \frac{\partial g}{\partial y}(x, y, z)dy + \frac{\partial g}{\partial z}(x, y, z)dz$. Computing these partial derivatives and replacing the differentials by the given value yields the result.

1 mark

- (d) Find the estimation on the error ΔV above when $a = 50, b = 40$ and $c = 30$.

$$\text{Answer: } \Delta V = 47$$

Solution: Replacing we get

$$\begin{aligned} dV &= \frac{1}{100} (40 \cdot 30 + 50 \cdot 30 + 50 \cdot 40) \\ &= \frac{1}{100} (1200 + 1500 + 2000) \\ &= 47 \end{aligned}$$