Deciding the Sequentiality of a Finitely Ambiguous Max-Plus automaton

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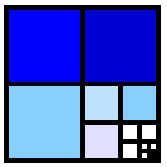
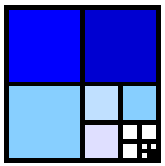


Max-Plus automata

- Automata in which every transition requires some time $\longrightarrow$ weight.



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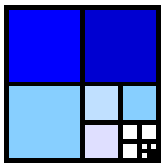

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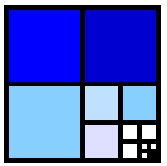

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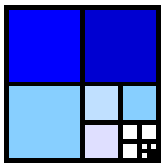

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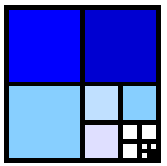
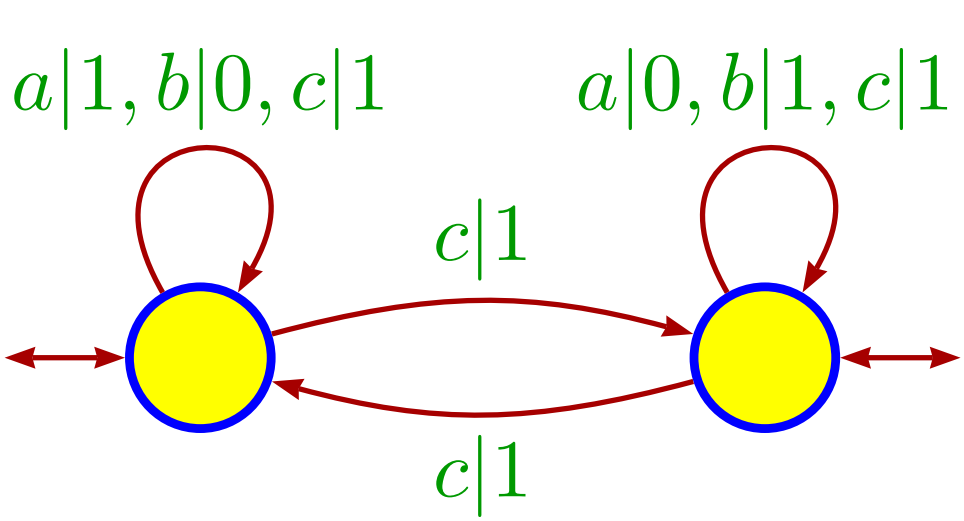
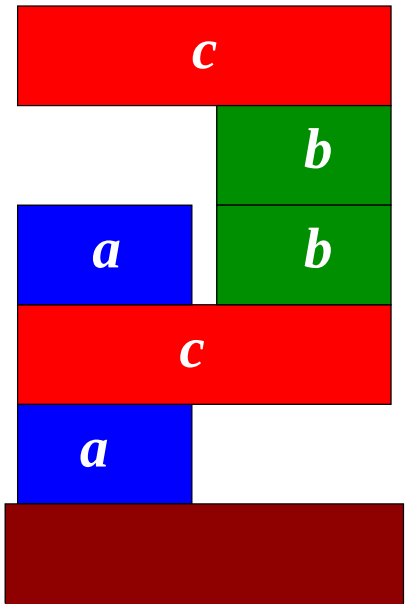
- $\longrightarrow$ Max-plus rational series: $S = \bigoplus \langle S, u \rangle u$
- The “zero” element is $-\infty$

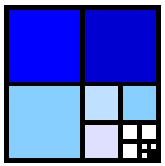


Example



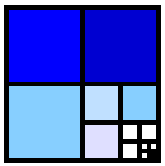
$$\langle S, acbbac \rangle = 5$$





Some bad news

- The equivalence of Max-plus automata over $\mathbb{Z}$ is undecidable (Krob, 94)



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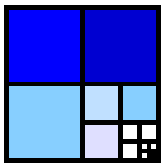
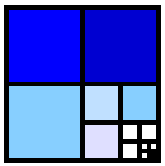


Sub-classes of Max-plus automata and series

- Sequential (deterministic): Seq



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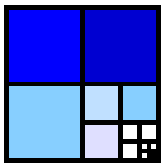
- Unambiguous: at most one successful path for each word: NAmb



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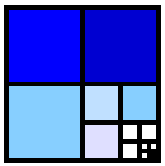

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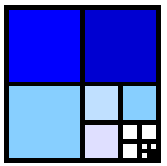
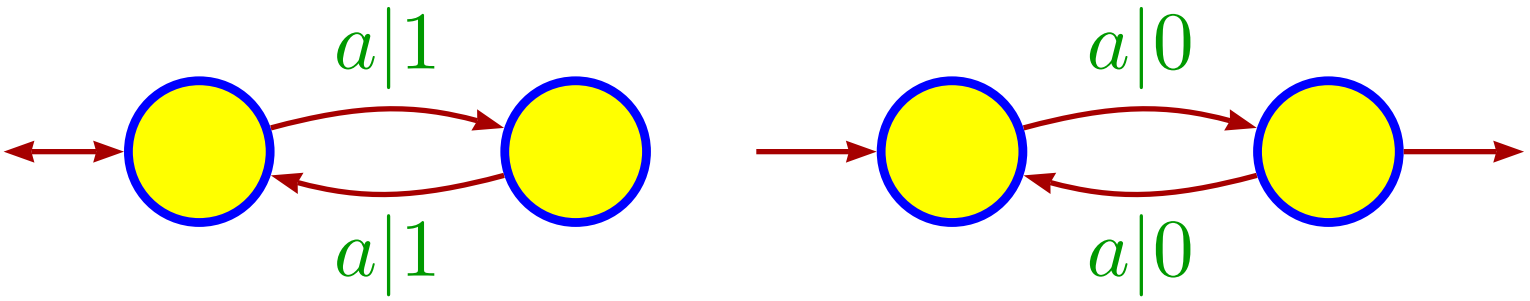
- Unambiguous: at most one successful path for each word: NAmb
- Finite union of sequentials: FSeq
- Finite union of unambiguous: FAmb $\Leftrightarrow$
  Finitely ambiguous: bounded number of successful paths for each word

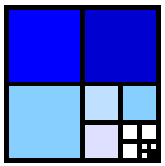
Def: Series $S \in$ class Cls iff $\exists$ automaton $\mathcal{A} \in$ Cls such that $\mathcal{A}$ realizes S.



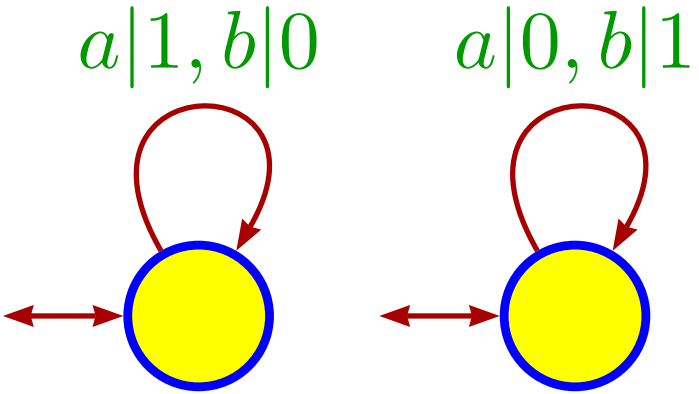
Example: NAmb ∩ FSeq ∩ $\overline{\text{Seq}}$



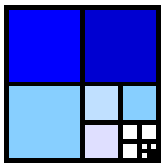
$$\langle S, a^n \rangle = \begin{cases} n & \text{if } n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$



Example: FSeq $\cap$ $\overline{\text{NAmb}}$



$$\langle S, u \rangle = \max(|u|_a, |u|_b).$$



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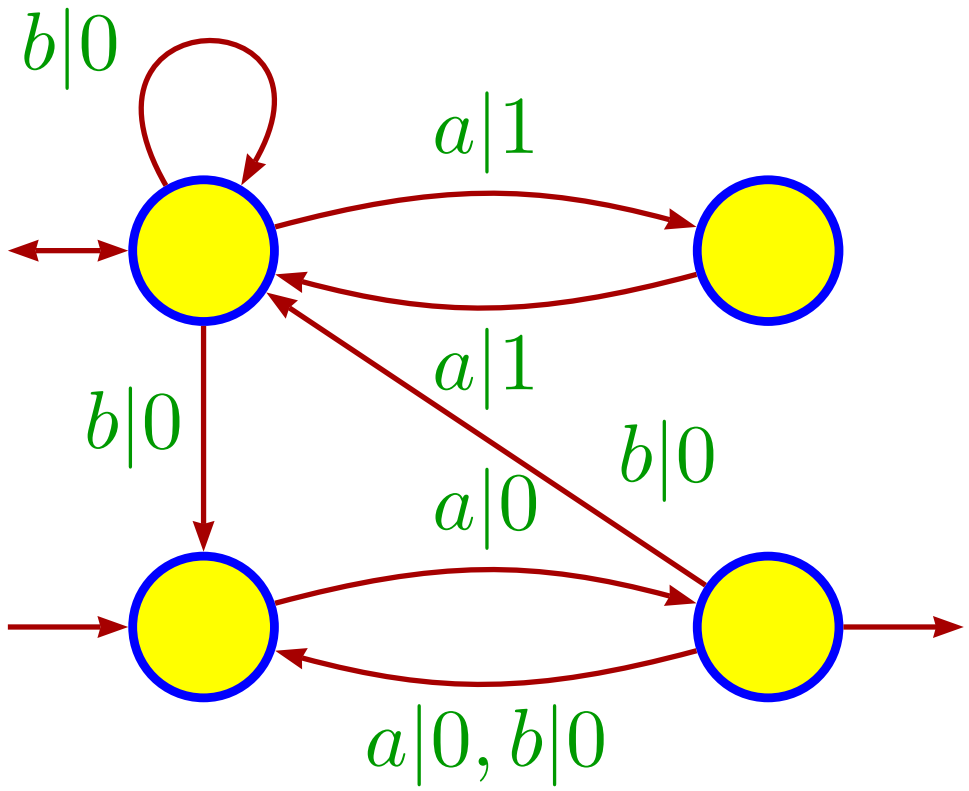
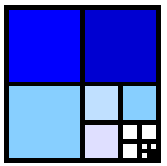
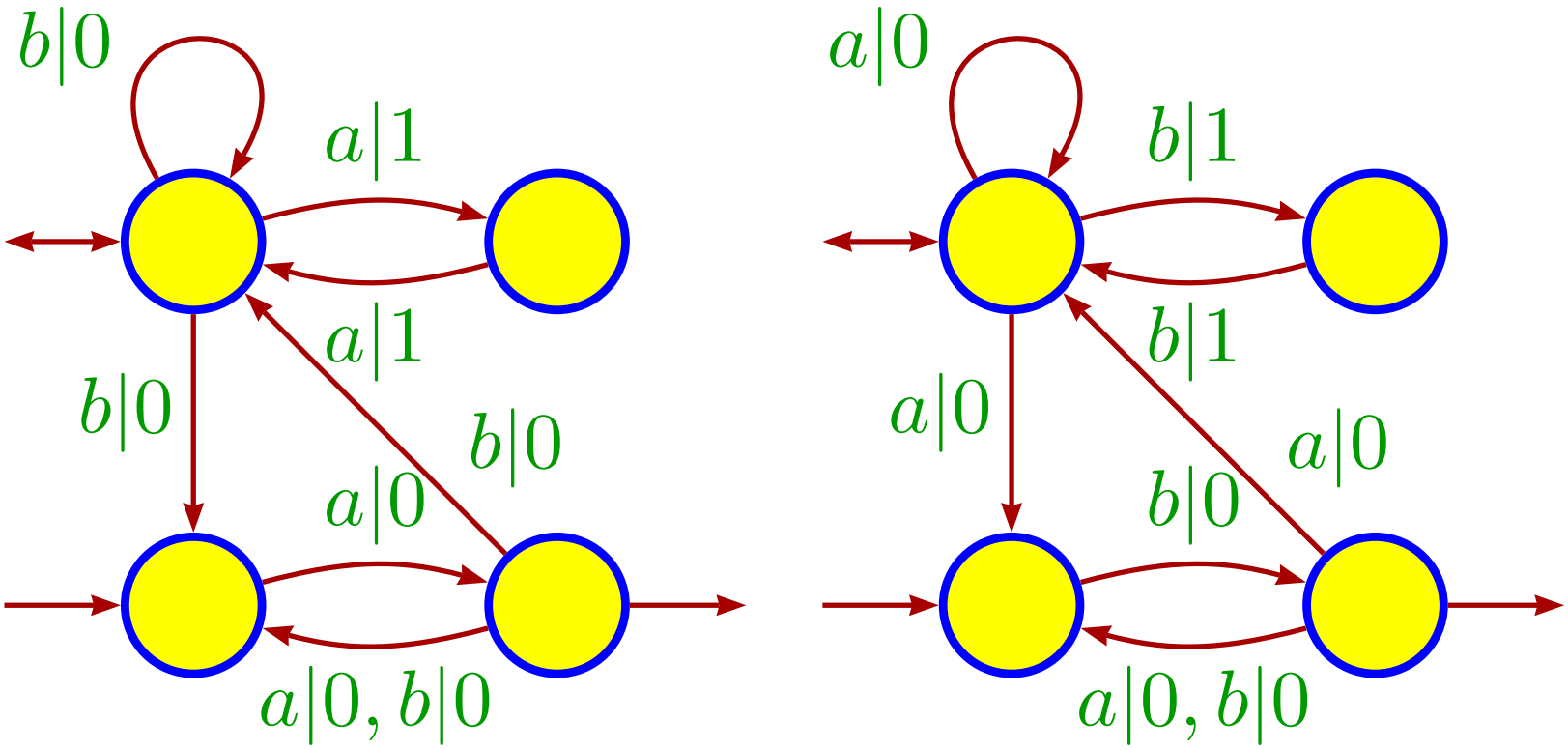


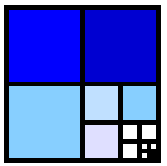
$$\langle S, a^{n_1}b^{m_1}a^{n_2}b^{m_2}\ldots\rangle = \sum_{n_i \text{ even}} n_i$$



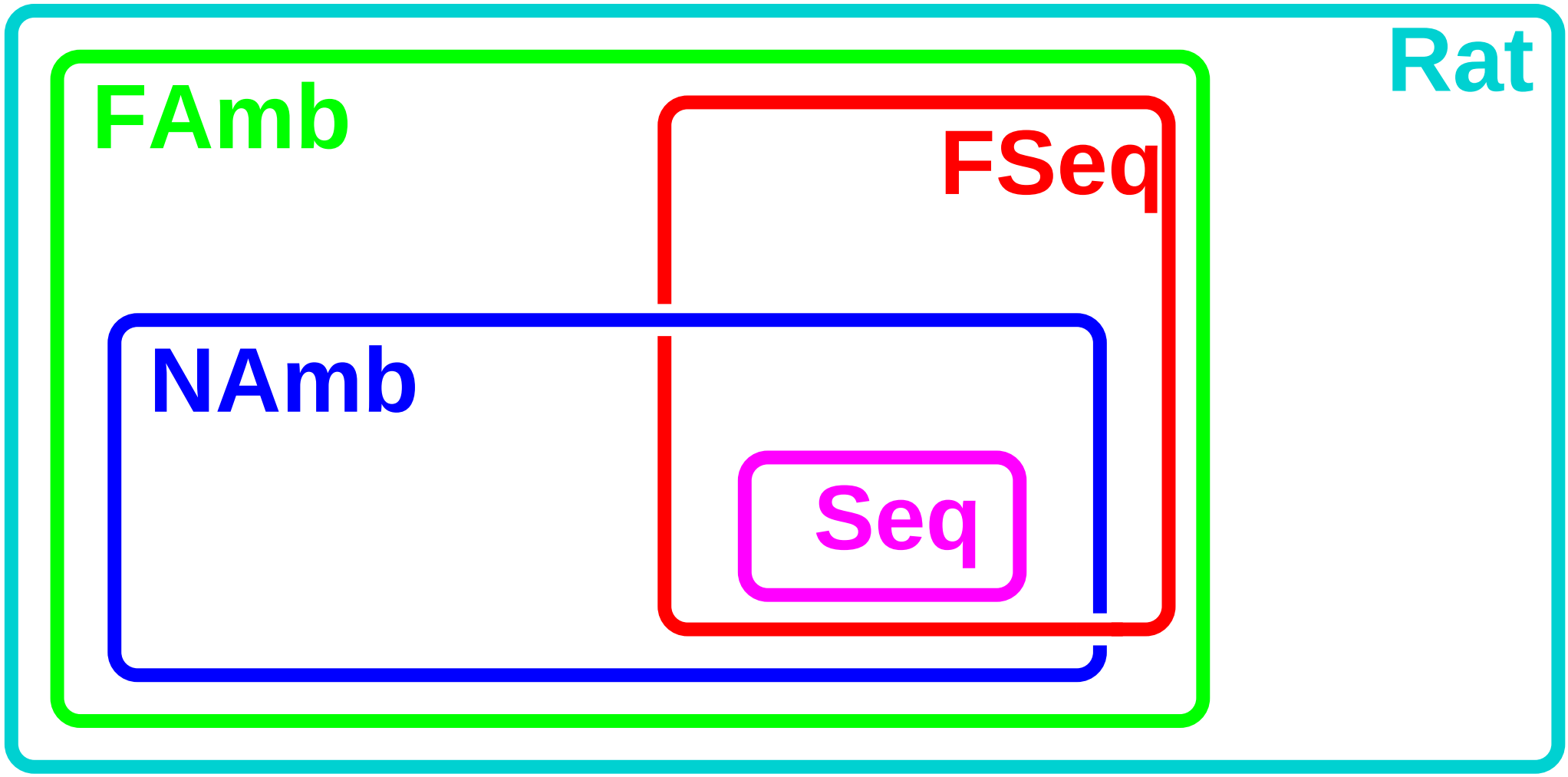
Example: FAmb $\cap$ $\overline{\text{FSeq} \cup \text{NAmb}}$

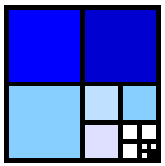


$$\langle S, a^{n_1} b^{m_1} a^{n_2} b^{m_2} \ldots \rangle = \max\Big(\sum_{n_i \text{ even}} n_i, \sum_{m_j \text{ even}} m_j \Big)$$



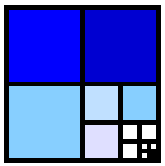
Hierarchy





Some good news

- If $S \in$ NAmb, it is decidable whether $S \in$ Seq. (Mohri 97, Choffrut 77)



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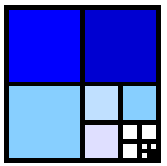
- NEW: If $S \in$ FAmb, it is decidable whether $S \in$ NAmb (and therefore, whether $S \in$ Seq).



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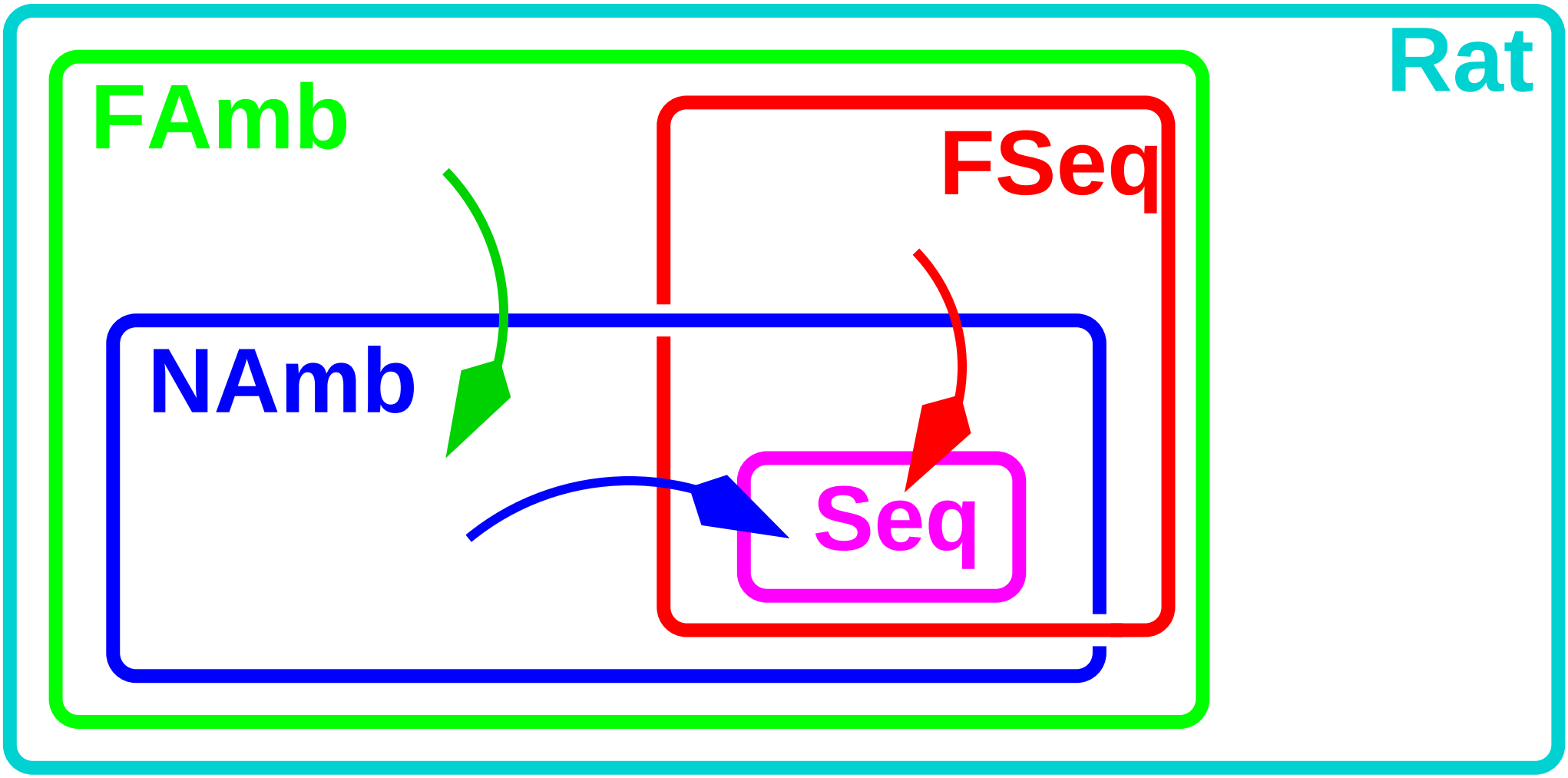
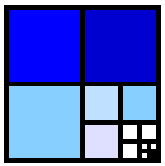
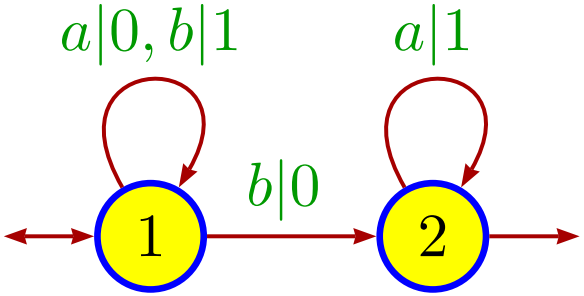
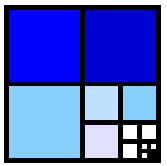




FAmb $\longmapsto$ $\cup$ NAmb (Weber, 94)

New method:





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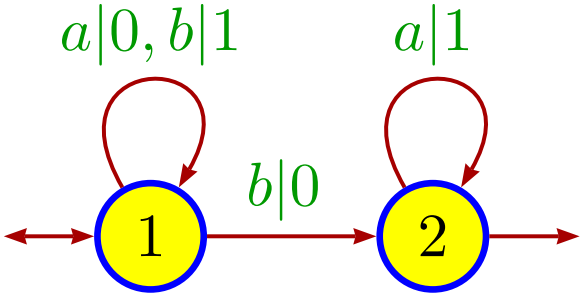

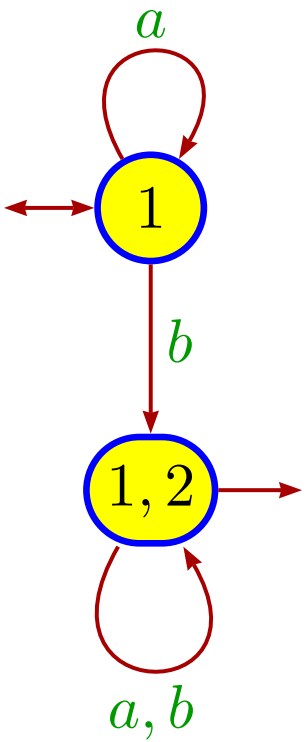

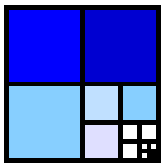

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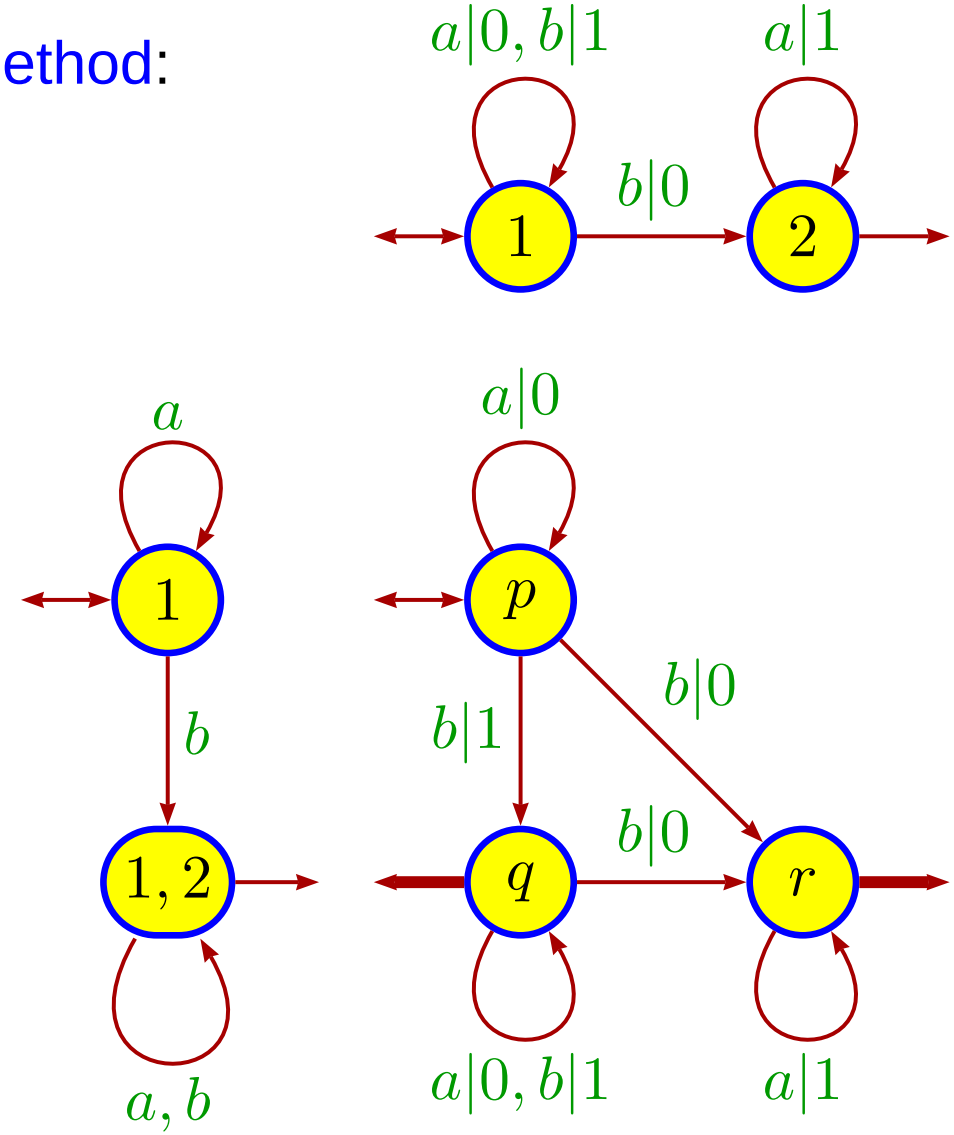

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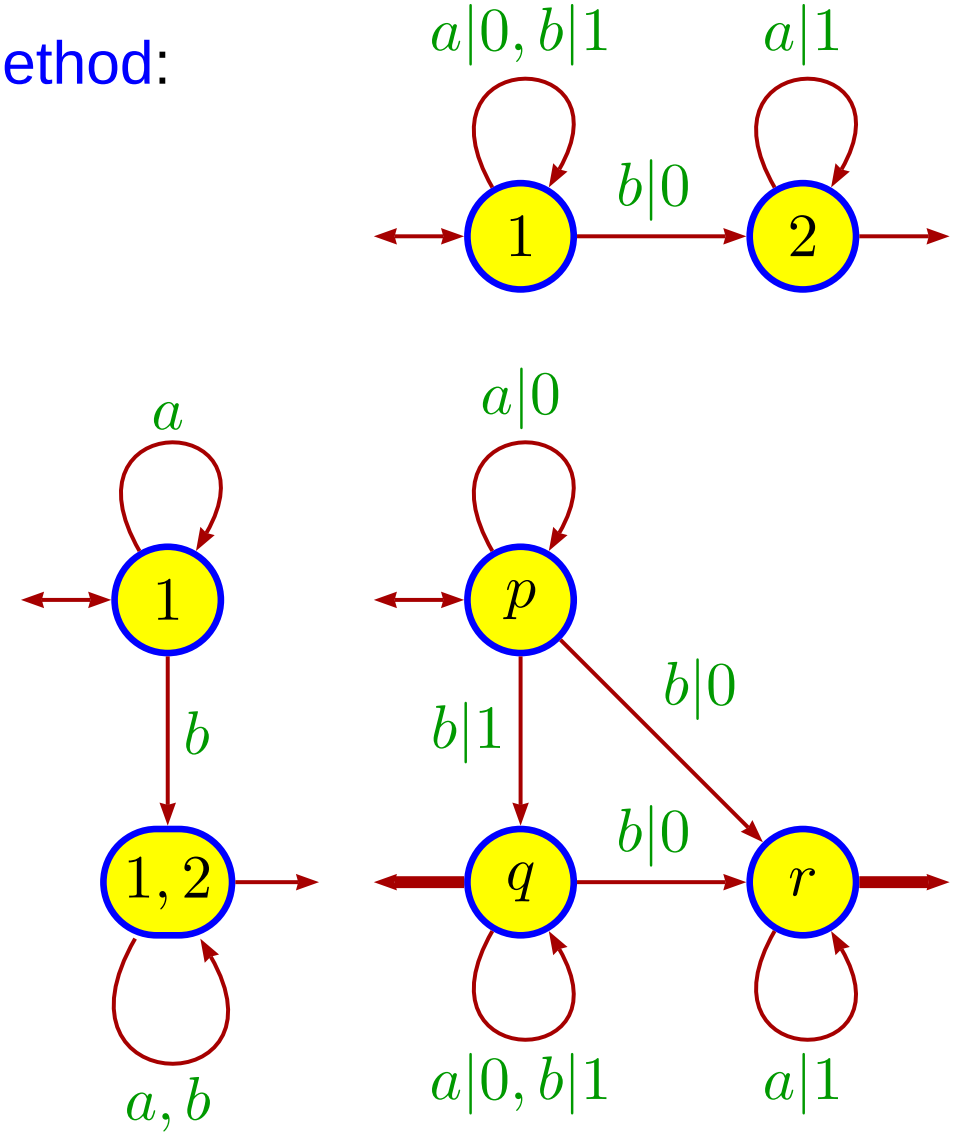
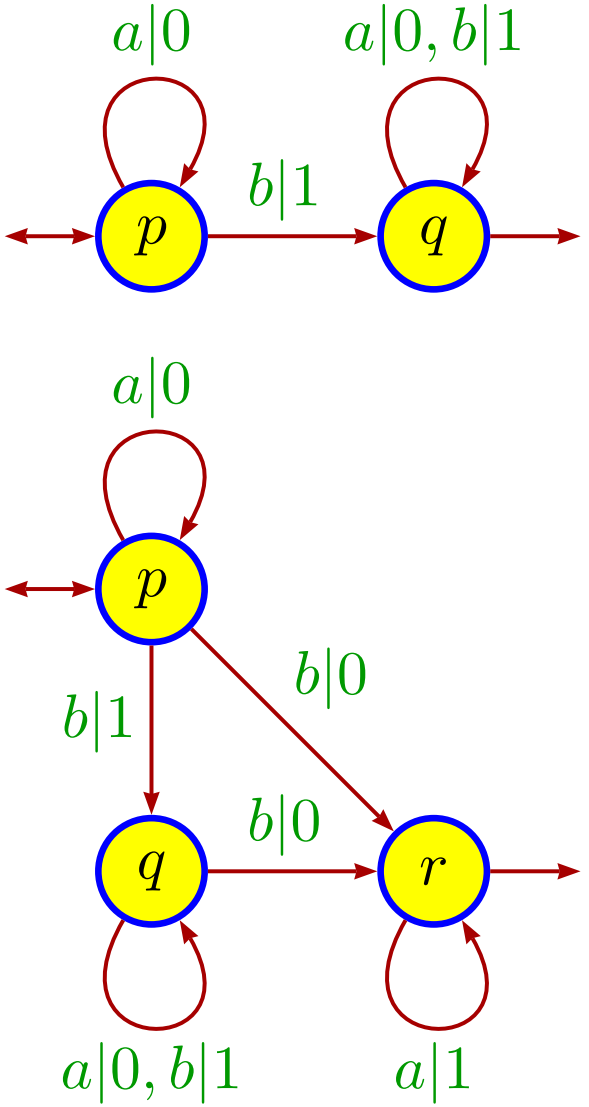


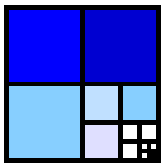


Dominance property

Finite union of unambiguous automata:

$$\mathcal{A} = \bigcup_{i=1}^{n} \mathcal{A}_i$$

Let $\mathcal{P}$ be the product of all $\mathcal{A}_i$ (with coefficients in $\mathbb{Z}^n$).



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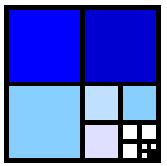
- i is a victorious coordinate of a circuit θ iff it has a maximal weight on θ.



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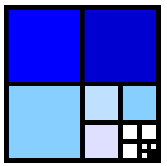

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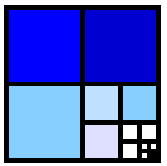
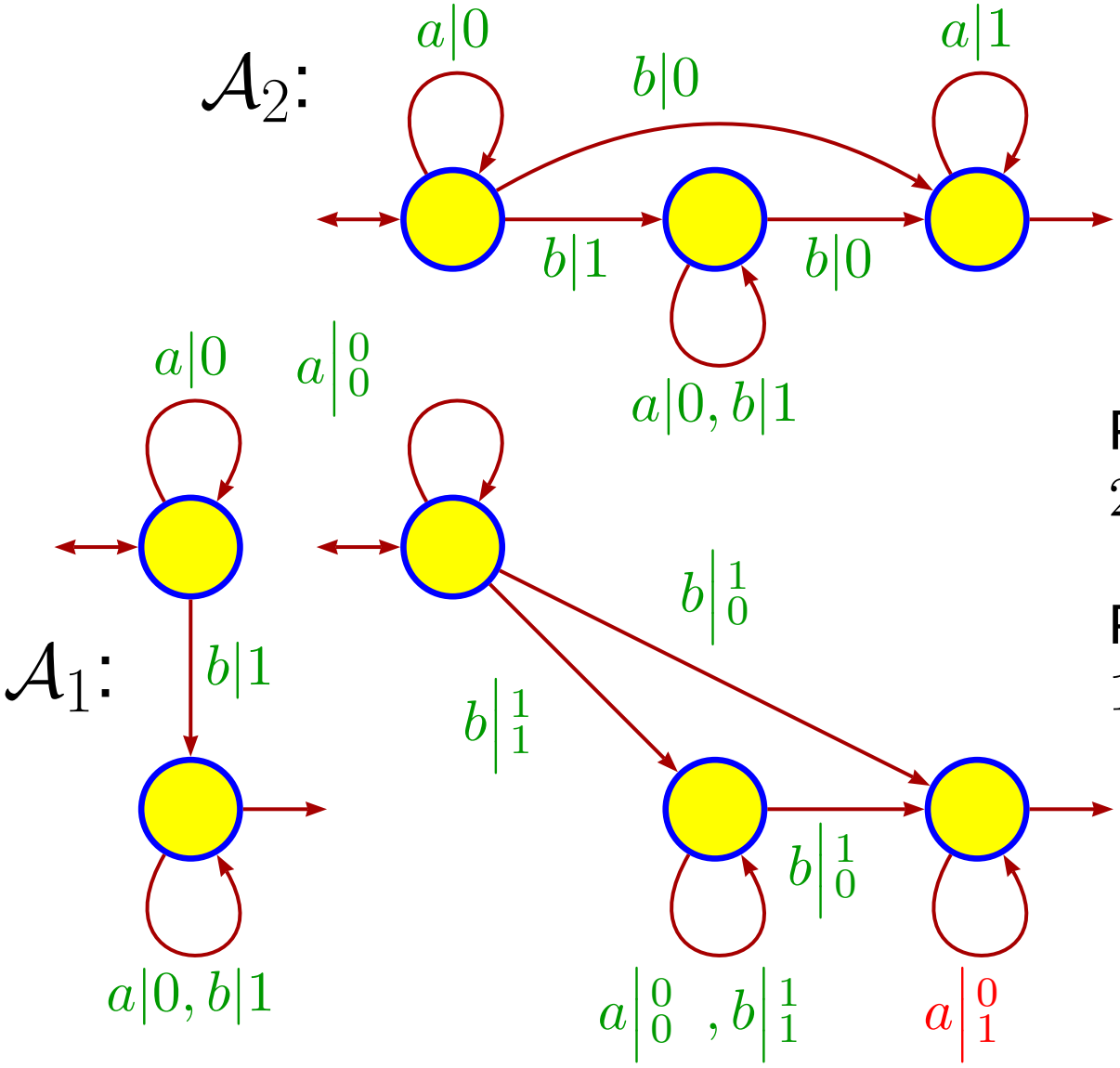
- i is a victorious on a path π iff i is victorious on every sub-circuit of π.
- $\bigcup \mathcal{A}_i$ verifies the dominance property iff $\forall\ \pi$ of $\mathcal{P}$, Vict(π)$\neq \emptyset$.

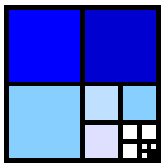


Dominance property: Example



Path labeled by ba:
2 is dominating.

Path labeled by bab:
1 and 2 are dominating.

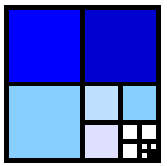


Characterization

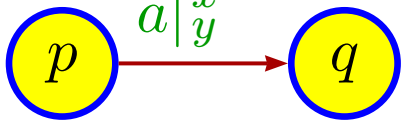
Theorem: Let S realized by $\bigcup \mathcal{A}_i$.

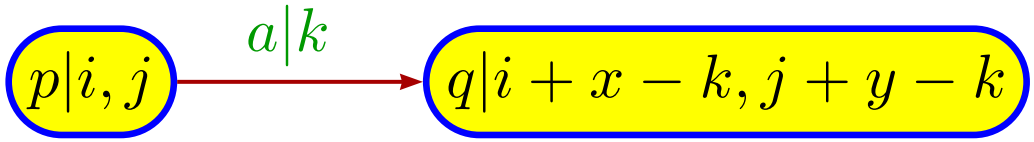
$S \in$NAmb iff dominance property holds on $\bigcup \mathcal{A}_i$.

Then, an equivalent unambiguous automaton can effectively be computed.

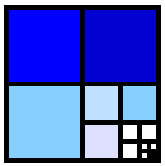


Construction of the unambiguous automaton

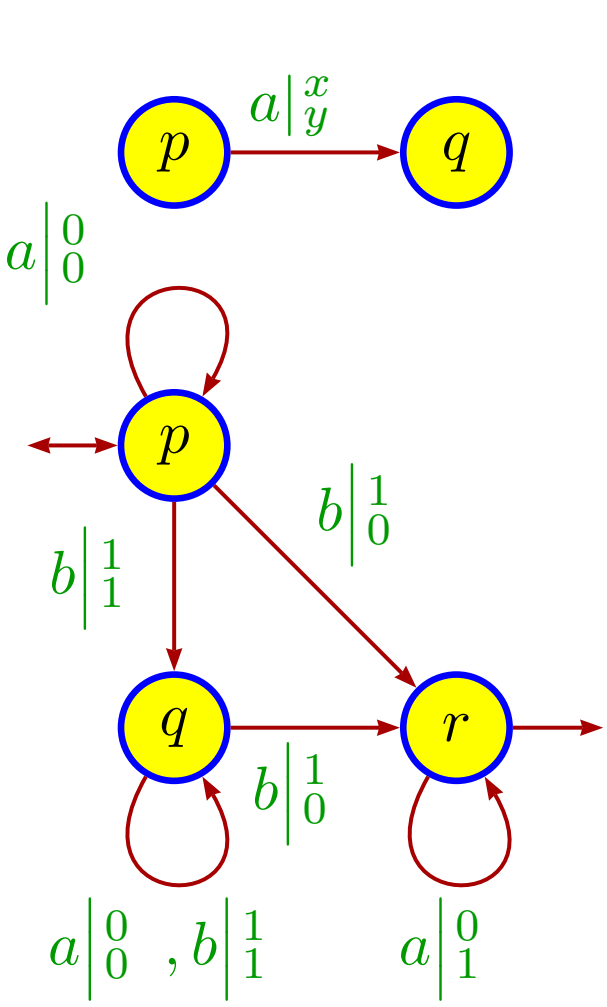


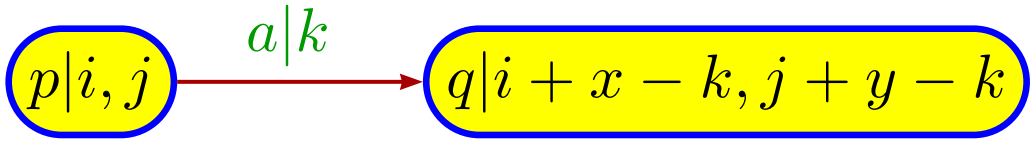


$$k = min(i + x, j + y)$$



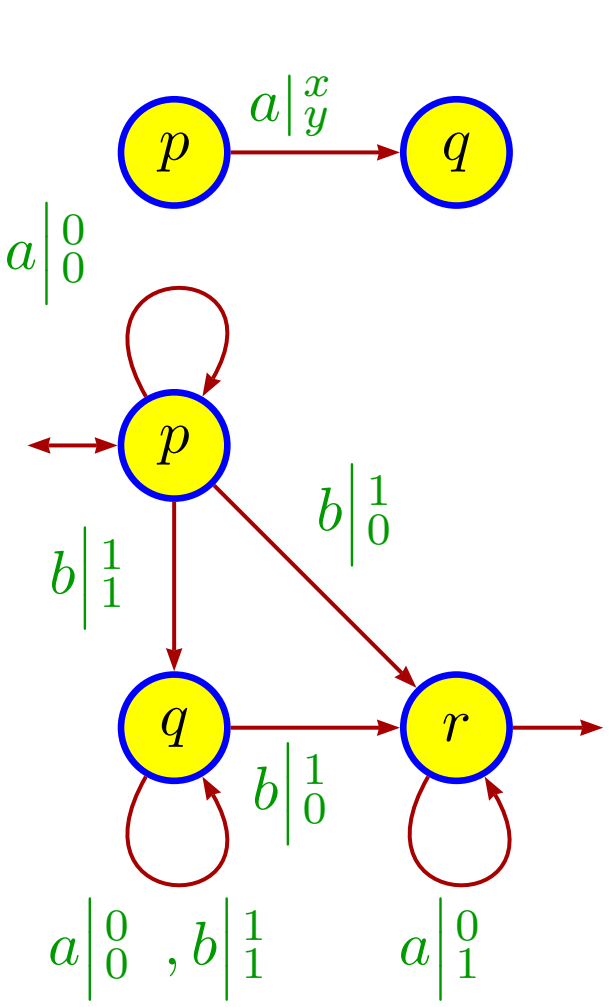
Construction of the unambiguous automaton

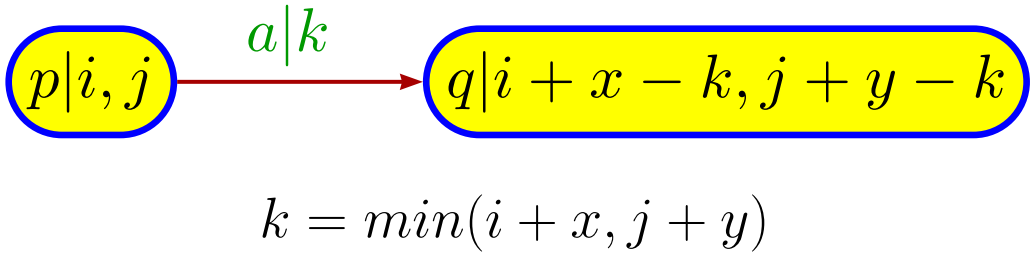




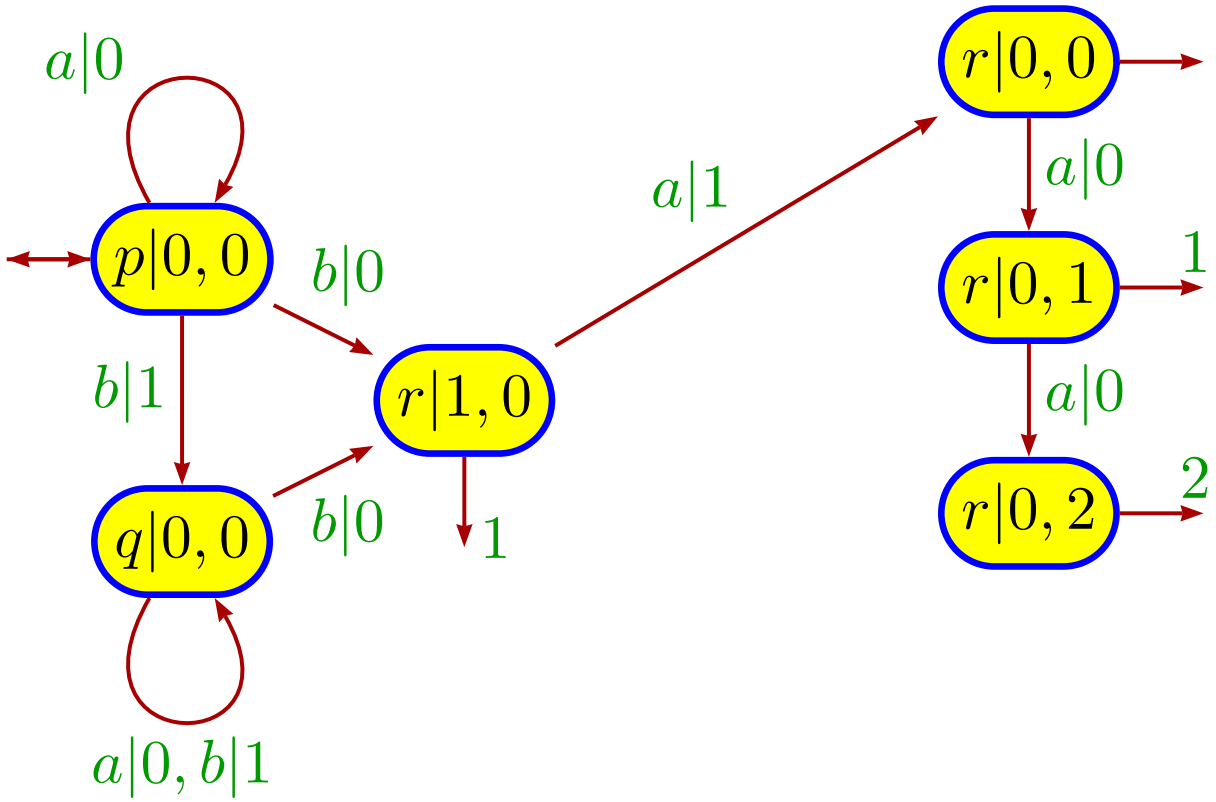
$$k = min(i + x, j + y)$$

Construction of the unambiguous automaton

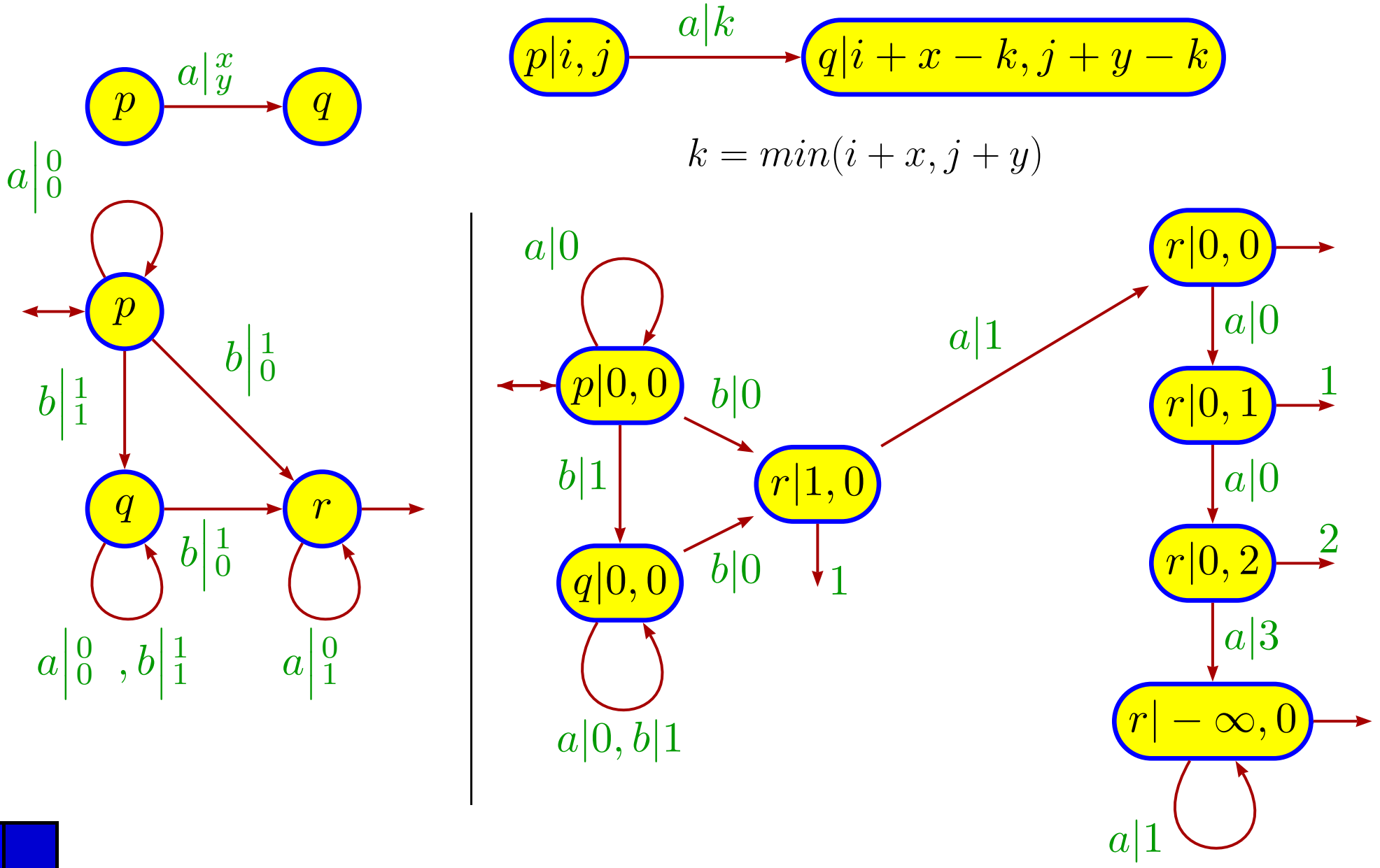




$$k = min(i + x, j + y)$$



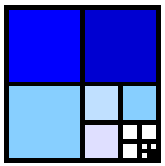
Construction of the unambiguous automaton



Conclusion

Results:

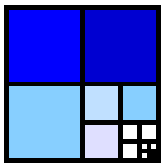
- Classification of rational series



Conclusion

Results:

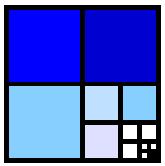
- Classification of rational series
- New proof of the decomposition of finitely ambiguous automata



Conclusion

Results:

- Classification of rational series
- New proof of the decomposition of finitely ambiguous automata
- Sequentialization on a larger family of series



Conclusion

Results:

- Classification of rational series
- New proof of the decomposition of finitely ambiguous automata
- Sequentialization on a larger family of series

Open questions:

- Sequentialization on any rational max-plus series
- Decidability of NAmb
- $etc...$