



# THÈSE

PRÉSENTÉE À

L'UNIVERSITÉ DE BORDEAUX

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET  
D'INFORMATIQUE

par **Mohammed Senhaji**

POUR OBTENIR LE GRADE DE

**DOCTEUR**

SPÉCIALITÉ : INFORMATIQUE

---

**Neighbour-distinguishing decompositions of graphs**

---

**Date de soutenance :** 14 Septembre 2018

**Devant la commission d'examen composée de :**

Maria AXENOVICH	Professeure, Karlsruhe Institute of Technology	Rapportrice
Alexandre PINLOU	Professeur, Université de Montpellier .....	Rapporteur
Olivier BAUDON ..	Maître de conférences, Université de Bordeaux	Directeur
Aline PARREAU ...	Chargée de Recherche, CNRS .....	Examinatrice
Arnaud PÊCHER ..	Professeur, Université de Bordeaux .....	Examinateur
Éric SOPENA .....	Professeur, Université de Bordeaux .....	Directeur

**Président de jury :** Arnaud Pêcher



---

**Titre** Décompositions de graphes voisins-distinguantes

**Résumé** Pour un entier positif  $k$ , une  $k$ -arête-pondération  $\omega$  d'un graphe  $G$  est une application de  $E(G)$  vers l'ensemble  $\{1, \dots, k\}$ . L'application  $\omega$  est *voisins-somme-distinguante* (vsd en abrégé) si la coloration  $\sigma_\omega$  des sommets de  $G$  induite par  $\omega$  et définie par  $\sigma_\omega(v) = \sum_{e \in E(G), e \ni v} \omega(e)$ , est une coloration propre. Le plus petit entier  $k$ , noté  $\chi_\Sigma^e(G)$ , tel que  $G$  admet une  $k$ -arête-pondération vsd est l'*indice voisins-somme-distinguant* de  $G$ . Comme les extrémités d'une arête isolée ne peuvent pas être distinguées par leurs sommes, les pondérations vsd sont considérées uniquement pour les graphes sans arêtes isolées, appelés *bons* graphes. Avec cette notation, la célèbre 1-2-3 Conjecture peut être formulée comme suit:

**1-2-3 Conjecture** ([KLT04]). *Pour tout bon graphe  $G$ ,  $\chi_\Sigma^e(G) \leq 3$ .*

Notons que si cette conjecture est vraie alors le résultat est optimal car il existe des graphes qui n'admettent pas de 2-arête-pondération voisins-somme-distinguante comme les graphes complets et les cycles de taille non divisible par 4. En 2010 Kalkowski, Karoński et Pfender [KKP10] ont montré que tout bon graphe admet une 5-arête-pondération vsd. Mais avant ce résultat, le meilleur connu à ce jour, plusieurs autres auteurs ont travaillé sur la 1-2-3 Conjecture (voir [ABDM<sup>+</sup>07], [ABDR08] et [WY08]). En 2011, Dudek et Wajc [DW11] ont prouvé que le problème de décider si un graphe admet une 2-arête-pondération vsd est NP-Complet. Cependant, Thomassen, Wu et Zhang [TWZ16] ont montré que le problème est polynomial si on se restreint aux graphes bipartis.

Notons que si un graphe est localement irrégulier, c'est-à-dire qu'il ne contient pas de sommets adjacents de même degré, alors le graphe admet une 1-arête-pondération vsd. De plus, une pondération des arêtes d'un graphe avec des entiers strictement positifs est équivalente, du point de vue des degrés pondérés, à la duplication des arêtes de ce graphe. On peut donc voir une arête-pondération voisins-somme-distinguante d'un graphe  $G$  comme la duplication de certaines de ses arêtes pour le transformer en un mutli-graphe localement irrégulier. Notant cela, Baudon, Bensmail, Przybyło et Woźniak introduisent dans [BBPW15] les décompositions localement irrégulières des graphes, à savoir, des partitions des arêtes d'un graphe telles que chaque partie induisent un sous-graphe localement irrégulier. Cependant, comme pour les arête-pondérations vsd, tous les graphes n'admettent pas une décomposition localement irrégulière, il existe des exceptions caractérisées dans le même article. Les graphes qui ne sont pas des exceptions sont *décomposables*. Les auteurs ont ainsi proposé la conjecture suivante :

**Conjecture 1** ([BBPW15]). *Tout graphe décomposable admet une décomposition localement irrégulière en au plus trois parties.*

Toujours dans l'objectif de résoudre la 1-2-3 Conjecture, Przybyło et Woźniak [PW10] ont introduit les pondérations totales vsd des graphes, qui en plus d'attribuer des poids aux arêtes d'un graphe attribuent aussi des poids aux sommets, la couleur induite d'un

---

sommet étant la somme de son propre poids et de celui des arêtes qui lui sont incidentes. Przybyło et Woźniak ont conjecturé que tout graphe admet une 2-pondération totale vsd. Le meilleur résultat connu à ce jour sur ce problème est dû à Kalkowski [Kal10], qui a prouvé que tout graphe admet une pondération totale vsd avec les poids 1 et 2 sur les sommets et les poids 1, 2 et 3 sur les arêtes.

Dans cette thèse nous explorons plusieurs variantes des pondérations vsd ainsi que leur lien avec les décompositions localement irrégulières. La thèse est organisée en six chapitres, chaque chapitre présente les résultats obtenus sur une variante différente, résultats qui ont fait l'objet, pour chaque chapitre, d'une publication en journal ou en conférence internationale. De plus chaque chapitre possédant une conclusion propre, il n'y a pas de chapitre de conclusion pour cette thèse.

**Pondérations vsd équitables** Pour un entier positif  $k$ , une  $k$ -arête-pondération ou une  $k$ -pondération totale d'un graphe  $G$  est *équitable* si tous les poids de l'ensemble  $\{1, \dots, k\}$  sont utilisés le même nombre de fois (à l'unité près). Le plus petit entier  $k$ , tel que  $G$  admet une  $k$ -arête-pondération (resp.  $k$ -pondération totale) vsd équitable est noté  $\overline{\chi}_{\Sigma}^e(G)$  (resp.  $\overline{\chi}_{\Sigma}^t(G)$ ). Nous explorons, dans le chapitre 2, la valeur de  $\overline{\chi}_{\Sigma}^e$  pour les graphes complets, les graphes bipartis complets, les forêts et plusieurs autres familles de graphes peu denses comme les cycles ou les  $\Theta$ -graphes. L'objectif est de trouver des familles de graphes pour lesquelles  $\overline{\chi}_{\Sigma}^e > \chi_{\Sigma}^e$ . Néanmoins, à part quelques graphes de petite taille, à savoir  $K_{3,3}$ ,  $K_4$  et trois graphes bipartis cubiques sur 10 et 14 sommets, nous n'avons pas d'exemple de familles infinies de tels graphes. De plus, parmi ces graphes, seul  $K_4$  requiert au moins quatre couleurs pour une arête-pondération vsd équitable. Malgré ces résultats, nous nous abstenons de conjecturer que  $\overline{\chi}_{\Sigma}^e(G) \leq 3$  pour tout bon graphe connexe différent de  $K_4$ . Nous pensons qu'il faudrait explorer plus de familles de graphes avant de pouvoir énoncer une telle conjecture. De plus, la meilleure borne supérieure générale que nous avons pour  $\overline{\chi}_{\Sigma}^e$  est de l'ordre de  $2 \cdot |E(G)|$  pour tout bon graphe  $G$ , nous n'avons donc pas réussi à exhiber une borne constante. Les résultats montrant cette borne supérieure sont introduits lors de l'étude d'un cas particulier des pondérations équitables dans le chapitre 3.

La seconde partie de notre travail sur les pondérations équitables consiste en l'étude de  $\overline{\chi}_{\Sigma}^t$  pour différentes familles de graphes, à savoir les graphes complets et les graphes bipartis.

**Arête-pondérations injectives** Les arête-pondérations équitables les plus simples sont les arête-pondérations *injectives*, à savoir des arête-pondérations où les arêtes se voient attribuées des poids distincts. Nous nous sommes donc intéressé à ces pondérations afin de mieux comprendre les pondérations équitables. De plus, dans le cas optimal, à savoir quand on utilise exactement les poids dans  $\{1, \dots, |E(G)|\}$  pour pondérer les arêtes d'un graphe, les arête-pondérations injectives sont une version locale des étiquetages anti-magiques [HR90]. Nous présentons dans le chapitre 3 les résultats obtenus sur les pondérations injectives, notamment des bornes supérieures générales sur le poids maximum nécessaire dans une telle pondération, de la forme  $|E(G)| + 2\Delta(G)$  et  $2 \cdot |E(G)|$ ,

---

ainsi que des bornes supérieures de la forme  $|E(G)| + c$  ou  $c$  est une petite constante, pour les graphes 2-dégénérés et les graphes de degré moyen maximum au plus 3. Nous proposons aussi une variante locale de la conjecture de l'étiquetage anti-magique [HR90]:

**Conjecture 2.** *Tout bon graphe  $G$  admet une arête-pondération injective avec les poids  $\{1, \dots, |E(G)|\}$ .*

Cette conjecture a été prouvée plus tard par Haslegrave dans [Has17].

**Décompositions localement irrégulières** Un graphe  $G$  est *localement irrégulier* si tous les sommets voisins de  $G$  ont des degrés différents. Il est facile de voir qu'un graphe localement irrégulier admet une 1-arête-pondération vsd. Toujours dans l'objectif d'étudier la 1-2-3 Conjecture, nous nous intéressons à la *décomposition localement irrégulière* des graphes, *i.e.*, une partition des arêtes d'un graphe telle que chaque partie induise un sous-graphe localement irrégulier. Pour un graphe décomposable  $G$ , on s'intéresse au plus petit nombre de parties dans une décomposition localement irrégulière de  $G$ . Dans le chapitre 6, nous étudions ce paramètre, ainsi que deux autres variantes, pour les graphes subcubiques.

**Autres variantes** Deux autres variantes des pondérations vsd sont considérées dans les chapitres 4 et 5. La première requiert que les sommets voisins d'un graphe soient non seulement distingués par la somme des poids de leurs arêtes incidentes, mais que ces sommes diffèrent d'au moins 2. Une arête pondération permettant une telle distinction est *voisins-somme-2-distinguante* (vs2d en abrégé). Le plus petit entier  $k$  tel qu'un bon graphe  $G$  admette une  $k$ -arête pondération vs2d est noté  $\chi_{\Sigma > 1}^e(G)$ . Nous pouvons dès à présent remarquer que multiplier par 2 les poids utilisés dans une arête-pondération vsd d'un graphe donne une arête-pondération vs2d de ce graphe, ce qui montre que  $\chi_{\Sigma > 1}^e(G) \leq 2 \cdot \chi_{\Sigma}^e(G)$ , pour tout bon graphe  $G$ . Nous montrons que le problème de déterminer la valeur de  $\chi_{\Sigma > 1}^e$  est NP-complet en général, même en se restreignant aux graphes bipartis, ce qui contraste avec le fait que le problème de déterminer la valeur de  $\chi_{\Sigma}^e$  est polynomial pour les graphes bipartis. Cependant nous montrons que notre problème est polynomial sur les arbres. De plus en analysant les résultats obtenus sur quelques familles de graphes et les liens entre les pondérations vsd et vs2d, nous proposons la conjecture suivante :

**Conjecture 3.** *Tout bon graphe admet une arête-pondération vs2d avec les poids 1, 3 et 5.*

La deuxième variante étudiée est une version jeu des pondérations vsd, à savoir, un jeu à deux joueurs, Alice et Bob, qui jouent alternativement sur un graphe  $G$ , en pondérant une arête non-pondérée de  $G$  à chaque tour. Le jeu se termine quand toutes les arêtes sont pondérées. Alice gagne si à la fin du jeu la pondération obtenue est vsd, sinon Bob gagne. Nous déterminons dans le chapitre 5 quelques conditions suffisantes sur le graphe  $G$ , pour que Alice ou Bob gagnent le jeu, ainsi que l'issue du jeu sur quelques familles de graphes.

---

**Une décomposition générale** Au chapitre 7, nous introduisons une nouvelle décomposition généralisant les pondérations vsd et les décompositions localement irrégulières, l'idée étant d'attribuer à chaque arête d'un graphe  $G$ , une couleur et un poids. Ainsi pour chaque sommet, nous pouvons calculer plusieurs sommes « colorées ». Selon les critères de distinction que nous choisissons, ainsi que le nombre de couleurs et de poids que l'on permet, nous pouvons couvrir soit les arête-pondérations vsd ou les décompositions localement irrégulières. Par exemple, nous définissons une *coloration forte par multi-somme* comme la variante pour laquelle deux sommets voisins doivent avoir des sommes colorées différentes pour chacune des couleurs présentes sur les arêtes qui leur sont incidentes, et *coloration standard par multi-somme* comme celle où il suffit que les sommets voisins diffèrent par la somme de la couleur attribuée à l'arête qui les relie. Si un graphe admet une telle coloration avec  $\ell$  couleurs et  $k$  poids sur les arêtes, alors il est *fortement  $(\ell, k)$ -colorable* ou  *$(\ell, k)$ -standard-colorable*. La 1-2-3 Conjecture est ainsi équivalente à l'existence d'une  $(1, 3)$ -coloration forte par multi-somme pour tout bon graphe. Par contre, si nous permettons l'utilisation de deux couleurs à la place d'une, nous obtenons une version plus faible de la 1-2-3 Conjecture. De la même manière la Conjecture 1 est équivalente à l'existence d'une  $(3, 1)$ -coloration standard par multi-somme pour tout graphe décomposable. Encore une fois, si on permet l'utilisation de deux poids à la place d'un, nous obtenons une version plus faible. Ceci nous a encouragé à formuler la conjecture suivante en tant que fil directeur de notre travail :

**Conjecture 4.** *Tout bon graphe sans triangle isolé est fortement  $(2, 2)$ -colorable.*

Dans le chapitre 7 nous présentons les résultats obtenus sur cette conjecture et ses variantes pour différentes familles de graphes.

**Mots-clés** 1-2-3 Conjecture, décompositions de graphes, arête-pondérations voisins-somme-distinguantes, pondérations totales voisins-somme-distinguantes, décompositions localement irrégulières, étiquetage anti-magique, pondérations équitables.

**Laboratoire d'accueil** Laboratoire Bordelais de Recherche en Informatique, UMR5800, F-33400 Talence, France

---

**Title** Neighbour-distinguishing decompositions of graphs

**Abstract** In this thesis we explore graph decompositions under different constraints. The title of the present thesis is due to the fact that most of these decompositions are neighbour-distinguishing. That is, we can extract from each such decomposition a proper vertex colouring. Moreover, most of the considered decompositions are edge partitions, and therefore can be seen as edge-colourings.

The main question presented in this thesis was introduced by Karoński, Łuczak and Thomason in [KLT04]: Can we weight the edges of a graph  $G$ , with weights 1, 2, and 3, such that any two of adjacent vertices of  $G$  are distinguished by the sum of their incident weights? This question later becomes the famous 1-2-3 Conjecture.

In this thesis we explore several variants of the 1-2-3 Conjecture, and their links with locally irregular decompositions. We are interested in both optimisation results and algorithmic problems. We first introduce an equitable version of the neighbour-sum-distinguishing edge-weightings, that is a variant where we require every edge weight to be used the same number of times up to a difference of 1. Then we explore an injective variant where each edge is assigned a different weight, which yields necessarily an equitable weighting. This gives us first general upper bounds on the equitable version. Moreover, the injective variant is also a local version of the well-known antimagic labelling. After that we explore how neighbour-sum-distinguishing weightings behave if we require sums of neighbouring vertices to differ by at least 2. Namely, we present results on the smallest maximal weight needed to construct such weightings for some classes of graphs, and study some algorithmic aspects of this problem. Due to the links between neighbour-sum-distinguishing edge weightings and locally irregular decompositions, we also explore the locally irregular index of subcubic graphs, along with other variants of the locally irregular decomposition problem. Finally, we present a more general work toward a general theory unifying neighbour-sum-distinguishing edge-weightings and locally irregular decompositions. We also present a 2-player game version of neighbour-sum-distinguishing edge-weightings and exhibit sufficient conditions for each player to win the game.

**Keywords** 1-2-3 Conjecture, neighbour-sum-distinguishing edge weighting, neighbour-sum-distinguishing edge weighting total weighting, locally irregular decomposition, equitable weightings, antimagic labelling.

**Host research institution** Laboratoire Bordelais de Recherche en Informatique, UMR5800, F-33400 Talence, France

---



# Contents

<b>Introduction</b>	<b>1</b>
<b>1 Definitions and notation</b>	<b>5</b>
1.1 General mathematics . . . . .	5
1.1.1 Multisets . . . . .	5
1.1.2 Semigroups and groups . . . . .	6
1.2 Graphs . . . . .	6
1.2.1 General definition . . . . .	6
1.2.2 Graph homomorphisms and subgraphs . . . . .	7
1.2.3 Graph operations . . . . .	9
1.2.4 Neighbouring and degree . . . . .	10
1.2.5 Paths, cycles and cliques . . . . .	11
1.2.6 Connectedness and distance . . . . .	13
1.2.7 Trees . . . . .	14
1.2.8 Matchings . . . . .	15
1.3 Graph colouring . . . . .	15
1.3.1 Proper colourings and chromatic parameters . . . . .	15
1.4 Graph weighting and labelling . . . . .	17
1.4.1 Graph labelling . . . . .	17
1.4.2 Neighbour-sum-distinguishing edge-weightings . . . . .	18
1.4.3 Neighbour-sum-distinguishing total-weighting . . . . .	21
1.4.4 Neighbour-multiset-distinguishing edge-weighting . . . . .	22
1.4.5 Locally irregular decompositions . . . . .	24
<b>2 Equitable neighbour-sum-distinguishing weightings</b>	<b>27</b>
2.1 Definitions and preliminary results . . . . .	28
2.2 Equitable neighbour-sum-distinguishing edge-weighting . . . . .	29
2.2.1 Simple families of graphs . . . . .	29
2.2.2 Complete graphs . . . . .	36
2.2.3 Complete bipartite graphs . . . . .	44
2.2.4 Forests . . . . .	45
2.3 Equitable neighbour-sum-distinguishing total-weighting . . . . .	51
2.3.1 Bipartite graphs . . . . .	52

2.3.2	Complete graphs . . . . .	54
2.4	Discussions and open problems . . . . .	56
<b>3</b>	<b>The 1-2-3 Conjecture and the Antimagic Labelling Conjecture</b>	<b>59</b>
3.1	Definitions and remarks . . . . .	59
3.2	Preliminary results . . . . .	61
3.3	Classes of locally antimagic graphs . . . . .	63
3.3.1	Forests . . . . .	63
3.4	General upper bounds . . . . .	68
3.5	Refined upper bounds for particular classes of sparse graphs . . . . .	72
3.5.1	2-degenerate graphs . . . . .	72
3.5.2	Graphs with maximum average degree at most 3 . . . . .	74
3.6	Discussion . . . . .	78
<b>4</b>	<b>Doubling the 1-2-3 Conjecture</b>	<b>81</b>
4.1	Introduction . . . . .	81
4.2	Preliminaries . . . . .	82
4.3	Conjecture 4.1.1 for bipartite graphs . . . . .	83
4.3.1	Paths . . . . .	85
4.3.2	Cycles . . . . .	86
4.3.3	Odd multicacti . . . . .	86
4.4	Algorithmic aspects . . . . .	88
4.4.1	General case . . . . .	89
4.4.2	Tree case . . . . .	93
4.5	Conclusion . . . . .	95
<b>5</b>	<b>The neighbour-sum-distinguishing edge-weighting game</b>	<b>97</b>
5.1	Introduction . . . . .	97
5.2	General results . . . . .	98
5.3	Simple graph classes . . . . .	102
5.4	Complete graphs . . . . .	103
5.5	Complete bipartite graphs . . . . .	109
<b>6</b>	<b>Locally irregular decompositions of subcubic graphs</b>	<b>111</b>
6.1	Introduction . . . . .	111
6.2	Locally irregular decompositions of bounded-degree graphs . . . . .	113
6.3	Locally irregular decompositions of subcubic graphs . . . . .	114
6.4	Locally irregular decompositions of subcubic graphs with maximum average degree less than $\frac{12}{5}$ . . . . .	120
6.5	$K_2$ -irregular decompositions of subcubic graphs . . . . .	127
6.6	Regular-irregular decompositions of subcubic graphs . . . . .	128
6.7	Conclusion . . . . .	132

---

<b>7</b>	<b>A general decomposition theory for 1-2-3 Conjecture and locally irregular decompositions</b>	<b>133</b>
7.1	Introduction . . . . .	133
7.2	Previous results and connections to $(\ell, k)$ -colourings . . . . .	135
7.2.1	Early observations . . . . .	135
7.2.2	Previous results . . . . .	136
7.3	New problems . . . . .	138
7.4	Strong $(\ell, k)$ -colouring . . . . .	141
7.5	Standard $(\ell, k)$ -colouring . . . . .	143
7.5.1	Standard $(40, 3)$ -colourability . . . . .	143
7.5.2	The Standard Conjecture for 2-degenerate graphs and subcubic graphs	144
7.5.3	Conjecture 7.5.1 for 9-colourable graphs . . . . .	146
7.6	Weak $(\ell, k)$ -colouring . . . . .	150
7.6.1	Weak $(3, 2)$ - and $(2, 4)$ -colourability . . . . .	150
7.6.2	Graphs with $\delta \geq 59$ are weakly $(2, 3)$ -colourable . . . . .	151



# List of Figures

- 1.1 Examples of graphs. . . . . 7
- 1.2 Examples of subgraphs. . . . . 8
- 1.3 Example for vertices degree. . . . . 11
- 1.4 Examples of paths, cycles and cliques. . . . . 12
- 1.5 Example of a tree. . . . . 14
- 1.6 Examples of proper colourings. . . . . 16
- 1.7 An illustration of the two first steps of the complete graph weighting in Proposition 1.4.2. . . . . 19
- 1.8 Examples of neighbour-sum-distinguishing weightings. . . . . 22
- 1.9 An example of an auxiliary graph used in the construction of exceptional graphs. . . . . 25
  
- 2.1 A neighbour-sum-distinguishing 2-edge-weighting of the path  $P_6$ . . . . . 29
- 2.2 The generalised Theta-graph  $\Theta(1, 2, 3, 4, 4)$ . . . . . 31
- 2.3 An equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_5$  and its matrix representation. . . . . 38
- 2.4 Matrix representation of an equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_6$ . . . . . 38
- 2.5 An equitable neighbour-sum-distinguishing 4-edge-weighting of  $K_4$  and its matrix representation. . . . . 39
- 2.6 The 3-edge-weighting  $\omega_0$  of  $K_{n+2}$ . . . . . 40
- 2.7 Re-weighting of type 1. . . . . 41
- 2.8 Re-weighting of type 2. . . . . 42
- 2.9 An equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_{3,3}$ . . . . . 44
- 2.10 Illustrations to Claims 2.2.17, 2.2.18 and 2.2.20. . . . . 46
- 2.11 Subcases of Claim 2.2.21. . . . . 49
- 2.12 Illustrations to cases in Claim 2.2.24. . . . . 50
- 2.13 Equitable nsd total-weightings of  $K_2$ ,  $K_3$  and  $K_4$ . . . . . 55
  
- 3.1 Illustrations to Claim 3.3.6. . . . . 65
- 3.2 Illustration of the three children types mentioned in the proof of Theorem 3.3.3. . . . . 67

3.3	Illustration of the three types of components in the proof of Theorem 3.4.1, each edge or component on the figure is weighted using the weights of the same colour. . . . .	69
4.1	The $(1, \{3, 4\})$ -gadget with root $u_1$ (left), the $(2, \{5\})$ -gadget with root $u_1$ (middle), and the $(2, \{2k + 1\})$ -gadget with root $u_1$ (right). . . . .	90
5.1	Some $R_4$ graphs: (a) $K_3 \cup K_2$ and (b) $T^+$ . The first edge played by Bob is dotted. . . . .	105
5.2	Some $R_6$ graphs. The first edge played by Bob is dotted. . . . .	106
7.1	Three $(2, 2)$ -colourings of $K_4$ . . . . .	134
7.2	Decomposing an exception as described in the proof of Observation 7.3.2. . . . .	139
7.3	A decomposition into two nice bipartite graphs mentioned in the proof of Lemma 7.5.8. Solid edges stand for red edges. Dashed edges stand for blue edges. . . . .	148

# Introduction

In this thesis I present the work accomplished and the results obtained during my PhD years between 2015 and 2018 at the University of Bordeaux, under the supervision of Associate Professor Olivier BAUDON and Professor Éric SOPENA. It is dedicated to the study of graph decompositions that are neighbour distinguishing. The main motivation is the famous 1-2-3 Conjecture and its variants. In this introduction we present the notions we worked on during the last three years. More detailed definitions are gathered in Chapter 1.

For a positive integer  $k$ , a  $k$ -edge-weighting  $\omega$  of a graph  $G$  is a mapping from  $E(G)$  to the set  $\{1, \dots, k\}$ . If the induced vertex sum colouring  $\sigma_\omega$ , defined, for every vertex  $v$  of  $G$ , by  $\sigma_\omega(v) = \sum_{e \in E(G), e \ni v} \omega(e)$ , is a proper colouring, then  $\omega$  is a *neighbour-sum-distinguishing* (or nsd for short) edge-weighting of  $G$ . The smallest integer  $k$ , such that  $G$  admits an nsd  $k$ -edge-weighting is the *neighbour-sum-distinguishing index* of  $G$  denoted by  $\chi_\Sigma^e(G)$ . Note that the ends of an isolated edge cannot be sum-distinguished. Therefore nsd edge weightings are considered only for graphs with no isolated edges. Such graphs are called nice graphs. With this notation the 1-2-3 Conjecture can be formulated as follows:

**1-2-3 Conjecture** ([KLT04]). *For every nice graph  $G$ ,  $\chi_\Sigma^e(G) \leq 3$ .*

Note that a graph  $G$  with  $\chi_\Sigma^e(G) = 1$  cannot have adjacent vertices with the same degree, such graphs are *locally irregular*. As assigning positive weights to the edges of a graph is equivalent to duplicating these edges from the perspective of weighted degree, another way to see nsd edge-weightings is to ask, for a graph  $G$ , how many copies of each edge of  $G$  must be added to  $G$  in order to turn it into a locally irregular multigraph. This motivated Baudon, Bensmail, Przybyło and Woźniak to introduce, in [BBPW15], the locally irregular decompositions, that is, partitions of the edges of a graph into sets, each inducing a locally irregular graph. As for nsd edge-weighting, not every graph admits such a decomposition: there exist exceptions, fully characterised in the same paper. Graphs that are not exceptions are decomposable. Moreover, the authors stated the following conjecture:

**Conjecture 5** ([BBPW15]). *Every decomposable graph  $G$  can be decomposed into at most three locally irregular subgraphs.*

In this thesis, we explore several variants of the 1-2-3 Conjecture, and their links with locally irregular decompositions. Each chapter presents work published in international

---

journals or conferences. Moreover, since each chapter has its own conclusion, no general conclusion was added at the end of this thesis.

## Equitable neighbour-sum-distinguishing weightings

For a positive integer  $k$ , a  $k$ -edge-weighting or total  $k$ -weighting of a graph  $G$  is equitable if every weight in  $\{1, \dots, k\}$  is used on the same number of elements (up to a difference of 1). The smallest integer  $k$ , such that  $G$  admits an nsd  $k$ -edge-weighting (resp. total  $k$ -weighting) is denoted by  $\overline{\chi}_\Sigma^e(G)$  (resp.  $\overline{\chi}_\Sigma^t(G)$ ). We explore the value of  $\overline{\chi}_\Sigma^e$  for complete graphs, complete bipartite graphs, forests and several families of sparse graphs (*e.g.* cycles and  $\Theta$ -graphs). The main goal is to find families of graphs for which  $\overline{\chi}_\Sigma^e > \chi_\Sigma^e$ . However, except for some small graphs, namely  $K_{3,3}$ ,  $K_4$  and three cubic bipartite graphs on 10 and 14 vertices, we did not manage to come up with examples of such graphs. Moreover, among the graphs we study, only  $K_4$  requires weights strictly greater than 3 in an equitable nsd edge-weighting. While this could motivate us to conjecture that  $\overline{\chi}_\Sigma^e(G) \leq 3$  for every connected nice graph different from  $K_4$ , we prefer to wait for further results to propose such a conjecture. Moreover, in the general case, except for an obvious exponential upper bound of  $2^{|E(G)|-1}$  obtained by assigning a distinct power of 2 to each edge, we did not succeed to exhibit a small upper bound on  $\overline{\chi}_\Sigma^e$  in general, and we still do not know if it is bounded. On the other hand, we also determine the value of  $\overline{\chi}_\Sigma^t$  for complete graphs and bipartite graphs.

## Injective neighbour-sum-distinguishing edge-weightings

An injective edge-weighting of a graph is an edge-weighting where all the edges get distinct weights. We get interested in injective nsd weightings for two main reasons. First, every injective edge-weighting is an equitable one, hence, every upper bound obtained on injective edge-weightings is also an upper bound on  $\overline{\chi}_\Sigma^e$ . Second, in the optimal case, that is when we use weights in  $\{1, \dots, |E(G)|\}$  on the edges of a graph  $G$ , injective nsd edge-weighting is a local variant of the antimagic labelling of graphs [HR90]. We denote by  $\chi_\Sigma^{e,1}(G)$ , the smallest positive integer  $k$  such that  $G$  admits an injective nsd  $k$ -edge-weighting. We prove that  $\chi_\Sigma^{e,1}(G) = |E(G)|$  if  $G$  is a forest and exhibit two upper bounds on  $\chi_\Sigma^{e,1}$ , namely  $|E(G)| + 2\Delta(G)$  and  $2 \cdot |E(G)|$  in general case. Moreover we prove upper bounds of the form  $|E(G)| + c$  for a small constant  $c$  in the cases of 2-degenerate graphs and graphs with maximum average degree at most three. We also propose a local version of the antimagic labelling conjecture [HR90]:

**Conjecture 6.** *For every nice graph  $G$ ,  $\chi_\Sigma^{e,1}(G) = |E(G)|$ .*

This conjecture was later proved by Haslegrave in [Has17].

## Locally irregular decompositions of subcubic graphs

For a decomposable graph  $G$ , we denote by  $\chi'_{\text{irr}}(G)$ , the smallest number of classes in a locally irregular decomposition of  $G$ . Toward Conjecture 5, we study the value of  $\chi'_{\text{irr}}(G)$



for subcubic graphs and prove the following theorem:

**Theorem 7.** *For every decomposable subcubic graph  $G$ ,  $\chi'_{\text{irr}}(G) \leq 5$ . Moreover if  $\text{mad}(G) < \frac{12}{5}$ , then  $\chi'_{\text{irr}}(G) \leq 3$ .*

We also study two relaxations of Conjecture 5. In the first one, we allow decomposition classes to induce a disjoint union of a locally irregular subgraph and a matching (a set of independent edges). And in the second one, we generalize that by allowing each decomposition class to induce a disjoint union of a locally irregular graph and a regular graph. In both cases we completely determine the minimum number of classes in such decompositions for subcubic graphs.

## Other variants

We consider other variants of nsd edge-weightings. The first variant requires that the induced sums on adjacent vertices differ by at least 2. Such a weighting is a neighbour-sum-2-distinguishing (ns2d for short) edge-weighting, and the smallest integer  $k$  such that a graph  $G$  admits an ns2d  $k$ -edge-weighting is denoted by  $\chi_{\Sigma > 1}^e(G)$ . We prove that the problem of determining the value of  $\chi_{\Sigma > 1}^e$  is NP-complete in general, even when restricted to planar bipartite 3-degenerate graphs. This contrasts with the fact that the problem of determining  $\chi_{\Sigma}^e(G)$  is polynomial if  $G$  is bipartite. However, the problem becomes polynomial on trees. Moreover, following some observations on links with the 1-2-3 Conjecture, especially for bipartite graphs, we propose the following conjecture:

**Conjecture 8.** *Every nice graph  $G$  admit an ns2d edge-weighting with the weights 1, 3 and 5, and thus,  $\chi_{\Sigma > 1}^e(G) \leq 5$ .*

We also study a game variant of nsd edge-weightings, that is a game with two players, Alice and Bob, weighting alternately one edge of their choice at each turn, using positive integers. Alice wins if the final weighting is nsd, otherwise Bob wins. We completely determine the game issue on complete graphs and complete bipartite graphs, and define several sufficient conditions for Alice or Bob to win the game.

## A general decomposition theory

Finally we introduce a general theory of graph decomposition, generalising both nsd edge-weighting and locally irregular decompositions. The idea is to assign a colour to each edge together with a weight. This allows us to calculate for each vertex a tuple of induced coloured sums. Formally, if  $\ell, k \geq 1$  are two integers, and  $G$  is a graph, then to each edge  $e$  of  $G$ , we assign, via an edge-colouring  $\omega$ , a pair  $(\alpha, \beta)$ , where  $\alpha \in \{1, \dots, \ell\}$  and  $\beta \in \{1, \dots, k\}$ , which can be regarded as a *coloured weight* (with value  $\beta$  and colour  $\alpha$ ). Now, for every vertex  $v$  of  $G$ , and every colour  $\alpha \in \{1, \dots, \ell\}$ , one can compute the *weighted  $\alpha$ -degree*  $\sigma_{\alpha}(v)$  of  $v$ , being the sum of weights with colour  $\alpha$  incident to  $v$ . So, with every vertex  $v$  is associated a palette  $(\sigma_1(v), \dots, \sigma_{\ell}(v))$  of  $\ell$  coloured sums.

---

When dealing with these new notions, there are many possible ways for asking for distinction, as several coloured sums are available. For example, a strong distinction between two adjacent vertices  $u$  and  $v$  requires that for every colour  $\alpha$ ,

$$\sigma_\alpha(u) = \sigma_\alpha(v) \implies \sigma_\alpha(u) = \sigma_\alpha(v) = 0,$$

while a *standard* distinction requires that  $\sigma_\alpha(u) \neq \sigma_\alpha(v)$ . If pairs of adjacent vertices of a graph are strongly (resp. standardly) distinguished, then the pair  $\omega$  is a strong (resp. standard)  $(\ell, k)$ -edge-colouring and  $G$  is strongly (resp. standardly)  $(\ell, k)$ -edge-coloured.

Rephrased with this new terminology, the 1-2-3 Conjecture asks whether every nice graph is strongly  $(1, 3)$ -edge-colourable. Similarly, Conjecture 5 asks whether every decomposable graph is standardly  $(3, 1)$ -edge-colourable. As a leading objective, we consider the following conjecture, which has flavour of both problems:

**Conjecture 9.** *Every graph with no isolated edge or triangle is strongly  $(2, 2)$ -edge-colourable.*

We explore conjecture 9 and its variants on several graph families, Chapter 7.

# Chapter 1

## Definitions and notation

In this chapter we present several mathematical tools used in this thesis. In Section 1.2, we use terminology inspired by [BM08] and [Die17].

### 1.1 General mathematics

#### 1.1.1 Multisets

A *multiset* is a set in which each element can appear multiple times. Formally a multiset  $M$  is a pair  $(E, f)$ , where  $E$  is a set, and  $f$  is a function  $f : E \rightarrow \mathbb{N}$  giving the multiplicity in  $M$ , of each element of  $E$ . For every  $x \in E$ , we say that  $x \in M$  if  $f(x) > 0$ , otherwise we say that  $x \notin M$ . We also say, for any object  $x$ , that  $x \notin M$  if  $x \notin E$ . The subset  $F$  of  $E$  defined by  $F = \{x \in E \mid f(x) > 0\}$  is the *ground set* of  $M$ . If the ground set of  $M$  is finite, then  $M$  is *finite* and the *size* of  $M$ , denoted by  $|M|$ , is defined by:

$$|M| = \sum_{x \in M} f(x).$$

If  $M = (E, f)$  and  $M' = (E', f')$  are two multisets, we say that  $M$  is included in  $M'$  and write  $M \subseteq M'$ , if for every  $x \in M$ , we have  $x \in M'$  and  $f(x) \leq f'(x)$ , and we say that  $M = M'$  if  $M \subseteq M'$  and  $M' \subseteq M$ . In the context of multisets, the operations of union and intersection must also be defined differently from the case of sets. Namely, If  $M = (E, f)$  and  $M' = (E', f')$  are two multisets we define the *intersection*, denoted by  $M \cap M'$ , as:

$$M \cap M' = (E \cap E', \min(f, f')).$$

And we define the union of  $M$  and  $M'$ , denoted by  $M \cup M'$ , as:

$$M \cup M' = (E \cup E', f + f').$$

### 1.1.2 Semigroups and groups

For a set  $S$ , a *binary operation* on  $S$  is a function  $+^1: S \times S \rightarrow S$ . For  $x, y \in S$ , we usually note  $x + y$  instead of  $+(x, y)$ . The binary operation  $+$  is *associative* if

$$\forall x, y, z \in S, (x + y) + z = x + (y + z).$$

We then simply write  $x + y + z$  for  $(x + y) + z$ . The function  $+$  is *commutative* if for each  $x, y \in S$ ,  $x + y = y + x$ .

A *semigroup* is an ordered pair  $(S, +)$  where  $S$  is a set and  $+$  an associative binary operation on  $S$ . When no confusion is possible we denote by  $S$  the semigroup  $(S, +)$ . The semigroup  $(S, +)$  is *commutative* if  $+$  is commutative. If  $S$  is a finite set, then the group  $(S, +)$  is *finite*, and the *order* of the group  $(S, +)$  is the cardinality of the set  $S$ .

If  $S' = \{x_1, \dots, x_n\}$  is a finite subset of  $S$ , and  $(S, +)$  is commutative, then we denote by  $\sum_{x \in S'} x$  the sum  $x_1 + \dots + x_n$ . By convention, we have  $1 \cdot x = x$ , for each  $x \in S$ , and if  $n \geq 2$  is an integer, then  $n \cdot x = (n - 1) \cdot x + x$ . Moreover, if  $\omega$  is a function from a finite multiset  $M = (E, f)$  to a semigroup  $(S, +)$ , then we write  $\sum_{x \in M} \omega(x)$  to designate

$$\sum_{x \in M} f(x) \cdot \omega(x).$$

A *group* is a semigroup  $(S, +)$  such that:

- $S$  contains a *neutral element* i.e.  $\exists e \in S, \forall x \in S, x + e = e + x = x$ .
- Each element of  $S$  has an *inverse element* in  $S$ , i.e.  $\forall x \in S, \exists y \in S, x + y = y + x = e$ .

It is easy to prove the uniqueness of the neutral element in a group, and the uniqueness of the inverse element of each element of the group. By convention, if  $(S, +)$  is a group, with the neutral element  $e$ , we have  $0 \cdot x = e$ , for each  $x \in S$ .

## 1.2 Graphs

### 1.2.1 General definition

Let  $V$  be a nonempty set. An *undirected graph*  $G$  on  $V$  is an ordered pair  $(V, E)$ , where  $E$  is a multiset of unordered pairs  $\{u, v\}$  where  $u, v \in V$ . The elements of  $V$  are the *vertices* of  $G$ , and the elements of  $E$  are the *edges* of  $G$ . For the sake of conciseness, and if there is no possible confusion, an edge  $\{u, v\} \in E$  will be denoted by  $uv$  or  $vu$ .

A *directed graph*  $G$  on  $V$  is an ordered pair  $(V, E)$ , where  $E$  is a multiset of ordered pairs  $(u, v)$  where  $u, v \in V$ . The elements of  $V$  are the *vertices* of  $G$ , and the elements of  $E$  are the *arcs* of  $G$ . An arc  $(u, v)$  of  $G$  is *directed from  $u$  to  $v$* . For convenience, we abbreviate directed graph to digraph, and if there is no possible confusion, an arc  $(u, v) \in E$  is denoted by  $uv$ , but not  $vu = (v, u) \neq (u, v) = uv$ .

---

<sup>1</sup>Usually the symbol “ $\cdot$ ” is used to designate the operation in a semigroup. But since we use only associative commutative semigroups we use the symbol “ $+$ ” instead.

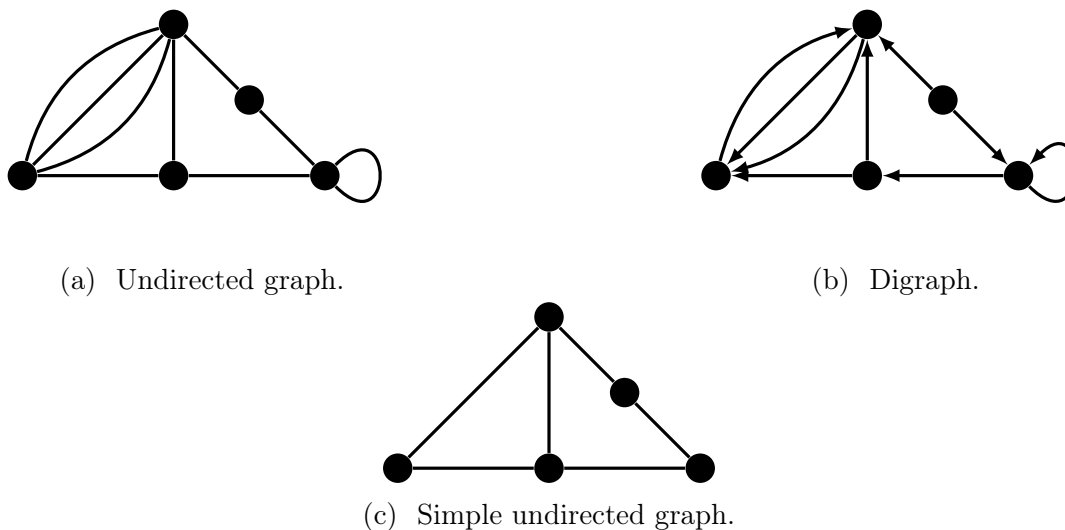


Figure 1.1: Examples of graphs.

A *graph* is either an undirected graph or a digraph.

If  $G$  is a graph, we denote by  $V(G)$  the set of its vertices and by  $E(G)$  the set (or multiset) of its edges (or arcs). For an edge or arc  $e = uv \in E(G)$ ,  $u$  and  $v$  are the *ends* of  $e$ , and  $e$  is *incident* with  $u$  and  $v$ . Moreover,  $u$  and  $v$  are *adjacent*, and  $e$  is *joining*  $u$  and  $v$ . If  $u = v$ , then  $e$  is a *loop*. If neither  $uv$  nor  $vu$  are in  $E(G)$ , then  $u$  and  $v$  are *independent*, sometimes we say that  $uv$  is a *non-edge*. Two edges of  $G$  are *adjacent* if they share at least one end, otherwise they are *independent*. Two adjacent edges of  $G$  are *parallel* if they share all their ends. If  $G$  is a digraph, the ends  $u$  and  $v$  of an arc  $e = uv \in E(G)$  are respectively the *tail* and *head* of  $e$ . An undirected graph (resp. digraph)  $G$  is *simple* if  $E(G)$  is a set, *i.e.*  $G$  does not have parallel edges (resp. arcs), and for every edge (resp. arcs)  $uv \in E(G)$ ,  $u \neq v$ , *i.e.*  $G$  has no loop. Otherwise the graph  $G$  is a *multigraph*.

A graph  $G$  is *finite* if both  $V(G)$  and  $E(G)$  are finite. For a finite graph  $G$ , the *order* of  $G$  is the integer  $|V(G)|$ , and the *size* of  $G$  is the integer  $|E(G)|$ . If the order of  $G$  is 1, then  $G$  is the *trivial* graph.

On Figure 1.1, we can see graphical representations of an undirected graph, a digraph and a simple undirected graph.

All along this thesis, and unless opposite mention, by *graph* we mean a undirected simple finite graph.

## 1.2.2 Graph homomorphisms and subgraphs

Let  $G$  and  $H$  be two graphs.

A *homomorphism* from  $G$  to  $H$  is a mapping  $\phi : V(G) \rightarrow V(H)$ , which preserves the adjacency of vertices:

$$\forall u, v \in V(G), uv \in E(G) \implies \phi(u)\phi(v) \in E(H).$$

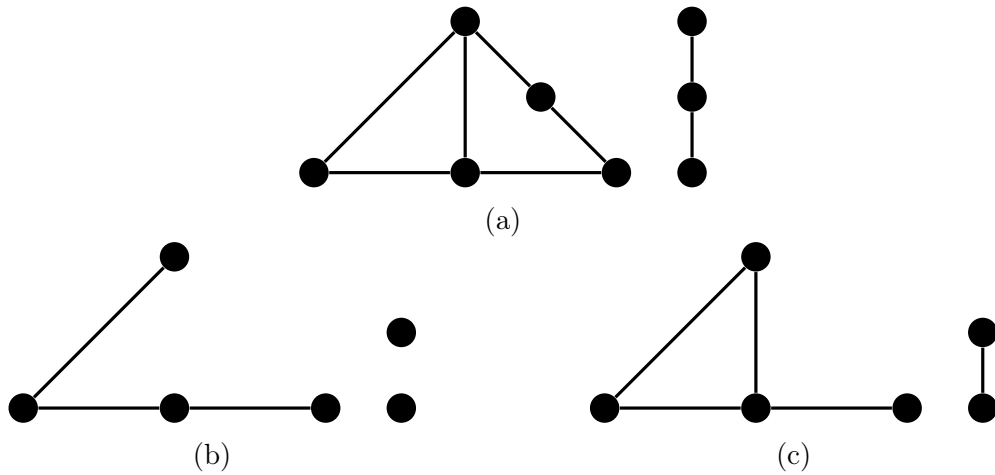


Figure 1.2: Examples of subgraphs.

An *isomorphism* from  $G$  to  $H$  is a bijective homomorphism from  $G$  to  $H$  (and thus preserves the adjacency and the nonadjacency), *i.e.*:

$$\forall u, v \in V(G), uv \in E(G) \iff \phi(u)\phi(v) \in E(H).$$

The graphs  $G$  and  $H$  are then *isomorphic*, we also say that  $G$  is *isomorphic* to  $H$  and vice versa, and write  $G \cong H$ . Note that if  $\phi$  is an isomorphism from  $G$  to  $H$ , then  $\phi^{-1}$  is an isomorphism from  $H$  to  $G$ .

The graph  $G$  is a *subgraph* of  $H$  if the identity is a homomorphism from  $G$  to  $H$ . We then write  $G \subseteq H$ . We also say that there is a  $G$  *in*  $H$ . If  $G$  is not  $H$  itself, then  $G$  is a *proper* subgraph of  $H$ . Note that if  $G \subseteq H$ , we then have  $V(G) \subseteq V(H)$  and  $E(G) \subseteq E(H)$ . If  $V(G) = V(H)$  we say that  $G$  is a *spanning* subgraph of  $H$ . For the sake of shortness, and if there is no possible confusion, we will say that the subgraph  $G$  is spanning. If  $G$  has no vertices with no adjacent vertices, then we say that  $G$  is the subgraph of  $H$  *induced* by  $E(G)$ , and we write  $G = H[E(G)]$ . If  $G$  contains all the edges in  $H$  between its vertices, *i.e.*

$$\forall u, v \in V(G), uv \in E(H) \implies uv \in E(G),$$

then  $G$  is an *induced* subgraph of  $H$ ,  $G$  is induced by  $V(G)$ , and we write  $G = H[V(G)]$ .

An *independent set* in  $H$  is an induced subgraph of  $H$  with no edges. The order of a largest independent set in  $H$  is the *independence number* of  $H$ , denoted by  $\alpha(H)$ .

The graph  $H$  is  $G$ -*free* if  $H$  does not have any induced subgraph isomorphic to  $G$ .

Figure 1.2 depicts a subgraph (1.2b) and an induced subgraph(1.2c) of the graph represented in Figure 1.2a.

### 1.2.3 Graph operations

After we defined graphs, a natural question that can be raised is what useful modifications can we define on graphs? All along this thesis, we will use several operations. This subsection defines and illustrates these operations.

Let  $G$  and  $H$  be two graphs with  $V(G) \cap V(H) = \emptyset$  (and consequently  $E(G) \cap E(H) = \emptyset$ ).

The *complementary graph* of  $G$  is the graph, denoted by  $\overline{G}$ , obtained from  $G$  by replacing all its edges by non-edges and all its non-edges by edges. Formally,

$$V(\overline{G}) = V(G) \text{ and } E(\overline{G}) = \{uv \mid u, v \in V, u \neq v, uv \notin E(G)\}.$$

The *disjoint union* of  $G$  and  $H$ , denoted by  $G + H$ , is the graph  $(V(G) \cup V(H), E(G) \cup E(H))$ . Note that  $G + H = H + G$ , and that the order (resp. size) of  $G + H$  is the sum of the orders (resp. sizes) of  $G$  and  $H$ .

If  $V'$  is a set of vertices, we can *add* the vertices of  $V'$  to  $G$ . This gives a new graph denoted by  $G + V' = (V(G) \cup V', E(G))$ . If  $E'$  is a set of edges with ends in  $V(G)$ , then we can also *add* the edges of  $E'$  to  $G$ . Again, this gives a graph  $G + E' = (V(G), E(G) \cup E')$ . Symmetrically, we can define the deletion of edges or vertices. If  $S$  is a subset of  $V(G)$ , then we can *delete*  $S$  from  $G$ . This gives a graph  $G - S$  whose vertices are all the vertices in  $V(G) \setminus S$ , and whose edges are all the edges of  $G$  which are not incident with a vertex from  $S$ . Formally:

$$G - S = (V(G) \setminus S, E(G) \setminus \{uv \in E \mid u \in S \text{ or } v \in S\}).$$

We can also see  $G - S$  as the subgraph of  $G$  induced by  $V(G) \setminus S$ . If  $F$  is a subset of  $E(G)$ , the *deletion* of  $F$  from  $G$  gives a subgraph of  $G$ , denoted by  $G - F$ , whose vertices are all the vertices of  $G$  (*i.e.*  $G - F$  is spanning), and whose edges are all the edges of  $G$  which does not belong to  $F$ . Formally:

$$G - F = (V(G), E(G) \setminus F).$$

In the graph  $G$ , the *identification* of two of its vertices  $u$  and  $v$ , is the deletion of the vertex  $v$  from  $G$  and the addition of edges joining  $u$  to all vertices adjacent to  $v$  in  $G$ . Identifying  $u$  and  $v$  yields a new graph, denoted by  $G_{u=v}$

$$G - \{v\} + \{uw \mid w \neq u, vw \in E(G)\}.$$

If  $e = uv \in E(G)$ , the *contraction* of the edge  $e$ , denoted by  $G/e$ , is the identification of  $u$  and  $v$ . The obtained graph is then denoted by  $G \cdot e$ . *Subdividing* an edge  $e = uv \in E(G)$  is the addition of a new vertex  $w$ , the deletion of the edge  $e$ , and the addition of the edges  $uw$  and  $wv$ . This yields the new graph:

$$G + \{w\} + \{uw, wv\} - \{uv\}.$$

Different notions of binary operations can be defined on graphs, we mainly use Cartesian product of graphs. The *Cartesian product* of  $G$  and  $H$ , denoted by  $G \square H$ , is the

graph with vertex set  $V(G) \times V(H)$ , and for two vertices  $x = (u, v)$  and  $y = (u', v')$  in  $V(G \square H)$ ,  $xy$  is an edge of  $G \square H$  if  $u = u'$  (resp.  $v = v'$ ) and  $vv'$  (resp.  $uu'$ ) is an edge in  $H$  (resp. in  $G$ ). Formally:

$$\begin{aligned} V(G \square H) &= V(G) \times V(H); \\ E(G \square H) &= \{((u, v), (u', v')) \mid (u = u' \wedge vv' \in E(H)) \vee (v = v' \wedge uu' \in E(G))\}. \end{aligned}$$

### 1.2.4 Neighbouring and degree

#### Digraphs

Let  $G$  be a digraph, and  $u$  and  $v$  two vertices of  $G$ .

If  $uv \in E(G)$ , then  $u$  is an *in-neighbour* of  $v$  and  $v$  is an *out-neighbour* of  $u$ . The *in-neighbourhood* of  $u$  in  $G$ , denoted by  $N_G^-(u)$ , as the set of in-neighbours of  $u$  in  $G$ . Formally,

$$N_G^-(u) = \{v \in V(G) \mid vu \in E(G)\}.$$

The *out-neighbourhood* of  $u$  in  $G$ , denoted by  $N_G^+(u)$ , as the set of out-neighbours of  $u$  in  $G$ . Formally,

$$N_G^+(u) = \{v \in V(G) \mid uv \in E(G)\}.$$

The integer  $|N_G^-(u)|$  is the *in-degree* of  $u$  in  $G$  and is denoted by  $d_G^-(u)$ . The integer  $|N_G^+(u)|$  is the *out-degree* of  $u$  in  $G$  and is denoted by  $d_G^+(u)$ . If there is no possible confusion, we write  $N^-(u)$  (resp.  $N^+(u)$ ) instead of  $N_G^-(u)$  (resp.  $N_G^+(u)$ ), and  $d^-(u)$  (resp.  $d^+(u)$ ) instead of  $d_G^-(u)$  (resp.  $d_G^+(u)$ ).

#### Undirected graphs

Let  $G$  be a graph, and  $u$  and  $v$  two vertices of  $G$ .

If  $u$  and  $v$  are adjacent, then  $u$  is a *neighbour* of  $v$  and vice versa. We define the *neighbourhood* of  $u$  in  $G$ , denoted by  $N_G(u)$ , as the set of vertices of  $G$  which are neighbours of  $u$ . Formally,

$$N_G(u) = \{v \in V(G) \mid uv \in E(G)\}.$$

The integer  $|N_G(u)|$  is the *degree* of  $u$  in  $G$  and is denoted by  $d_G(u)$ . If there is no possible confusion, we write  $N(u)$  instead of  $N_G(u)$ , and  $d(u)$  instead of  $d_G(u)$ . The graph  $G$  is *regular* if all its vertices have the same degree. If this degree is  $k$ , then  $G$  is *k-regular*.

The maximum (resp. minimum) degree of a graph  $G$ , denoted by  $\Delta(G)$  (resp.  $\delta(G)$ ), is the maximum (resp. minimum) degree taken over all vertices of  $G$ . Formally:

$$\Delta(G) = \max_{u \in V(G)} d_G(u) \text{ and } \delta(G) = \min_{u \in V(G)} d_G(u).$$

Note that if  $H$  is a subgraph of  $G$ , then for every vertex  $u$  of  $H$ ,  $d_H(u) \leq d_G(u)$ , and consequently  $\Delta(H) \leq \Delta(G)$ .

For a nonnegative integer  $k$ , if  $d_G(u) = k$ , then  $u$  is a *k-vertex* of  $G$ . A 0-vertex is an *isolated* vertex, a 1-vertex is a *pending* vertex and a  $(|V(G)| - 1)$ -vertex is a *universal*



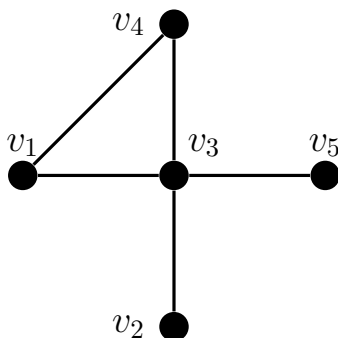


Figure 1.3: Example for vertices degree.

vertex of  $G$ . A  $k^-$ -vertex (resp.  $k^+$ -vertex) of  $G$  is a vertex with degree at most (resp. at least)  $k$ . An edge whose both ends are pendent vertices of  $G$  is an *isolated* edge of  $G$ .

If we sum the degrees of all vertices of  $G$ , we can convince ourselves that we count twice each edge of  $G$ . Hence we have the following equality for every graph  $G$ :

$$\sum_{u \in V(G)} d_G(u) = 2 \cdot |E(G)|.$$

The *average degree* of the graph  $G$ , denoted by  $\text{ad}(G)$ , is the ratio of the sum of the degrees of the vertices of  $G$  to the order of  $G$ . By the previous equality, we then have:

$$\text{ad}(G) = \frac{2 \cdot |E(G)|}{|V(G)|}.$$

The *maximum average degree* of the graph  $G$ , denoted by  $\text{mad}(G)$ , is the maximum average degree taken over all the subgraphs of  $G$ . Formally:

$$\text{mad}(G) := \max \left\{ \frac{2 \cdot |E(H)|}{|V(H)|} \mid H \subseteq G, H \text{ nonempty} \right\}.$$

For a nonnegative integer  $k$ ,  $G$  is  $k$ -degenerate if every subgraph of  $G$  has a  $k^-$ -vertex, *i.e.*

$$\forall H \subseteq G, \delta(H) \leq k.$$

In particular,  $G$  itself has a  $k^-$ -vertex.

In the graph depicted in Figure 1.3 the neighbourhood of the vertex  $v_1$  is  $\{v_3, v_4\}$ , and its degree is 2. The vertex  $v_3$  is a universal vertex and  $v_2$  and  $v_5$  are pendent vertices. The maximum degree of the graph is 4 and its minimum degree is 1. The average degree is 2, and the maximum average degree is 2.

### 1.2.5 Paths, cycles and cliques

Let  $n$  be a positive integer.

---

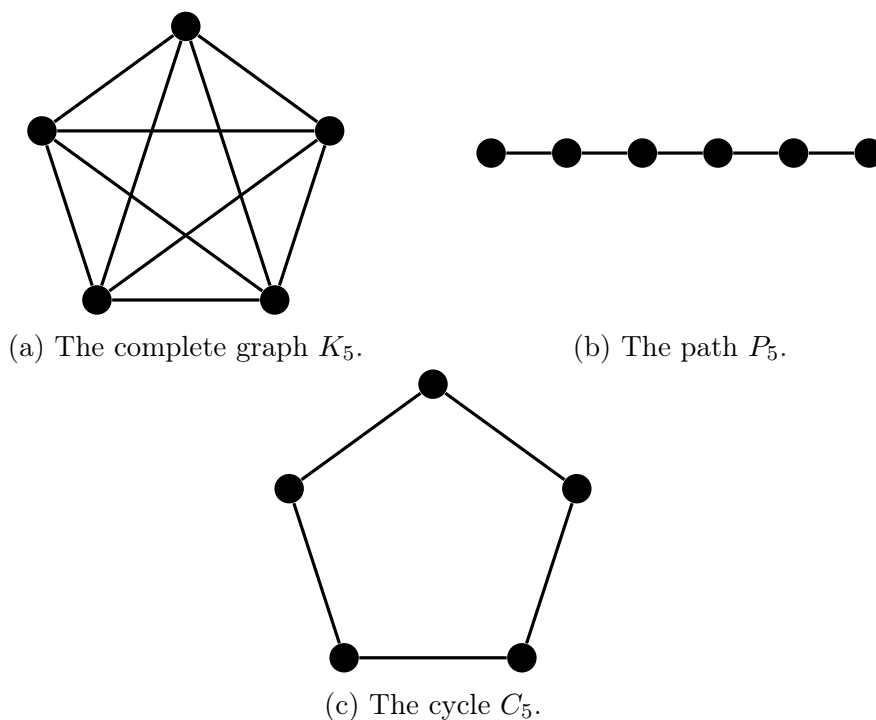


Figure 1.4: Examples of paths, cycles and cliques.

The *path* of length  $n$ , denoted by  $P_n$ , is the graph on the set of  $n + 1$  vertices  $\{v_0, \dots, v_n\}$ , with the edge set  $E = \{v_i v_{i+1} \mid 0 \leq i \leq n - 1\}$ . For simplicity, such a path will be denoted by  $v_0 \dots v_n$ . The integer  $n$  is the *length* of the path. Note that the order of  $P_n$  is  $n + 1$  and the size of  $P_n$  is  $n$ . Moreover all the vertices of a path  $P_n = v_0 v_1 \dots v_n$  are of degree 2 except  $v_0$  and  $v_n$ , which are 1-vertices. The vertices  $v_0$  and  $v_n$  are the *ends* of the path  $P_n$ , and  $P_n$  is a path *between*  $v_0$  and  $v_n$  or a  $\{v_0, v_n\}$ -path, the other vertices are *inner vertices*. The *end edges* of  $P_n$  are  $v_0 v_1$  and  $v_{n-1} v_n$ , the other edges are *inner edges*. If  $P_n$  is an induced subgraph of a graph  $G$ , such that  $d_G(v_0) \geq 2$ ,  $d_G(v_n) = 1$ , and  $d_G(v_i) = 2$ , for every  $i \in \{1, \dots, n - 1\}$ , then  $P_n$  is a *pendent path* of  $G$ .

If  $n \geq 3$ , then a *cycle* on  $n$  vertices, denoted by  $C_n$ , is a graph such that  $V(C_n) = \{v_0, \dots, v_{n-1}\}$ , and  $E(C_n) = \{v_i v_{(i+1) \bmod n} \mid 0 \leq i \leq n - 1\}$ . For simplicity, such a cycle will be denoted by  $v_0 v_1 \dots v_{n-1} v_0$ . The *length*, order and size of the cycle  $C_n$  is  $n$ . Note that all the vertices of a cycle are of degree 2. The cycle  $C_3$  is called a *triangle*. For a graph  $G$ , the *girth* of  $G$ , denoted by  $g(G)$ , is the length of a shortest induced cycle in  $G$ . If  $G$  has no cycles, then  $g(G) = \infty$ . A *forest* or *acyclic* graph is a graph with no induced cycles.

The *complete graph* on  $n$  vertices, denoted by  $K_n$ , is the graph of order  $n$  such that  $E(K_n)$  is the set of all the edges between two distinct vertices of  $K_n$ . All the vertices of a complete graph  $K_n$  are of degree  $n - 1$ , and its size is  $\frac{n(n-1)}{2}$ . For a graph  $G$ , a *clique* in  $G$  is a subgraph of  $G$  which is complete. The order of a largest clique in  $G$  is the *clique number* of  $G$ , denoted by  $\omega(G)$ .

Figure 1.4 depicts the path  $P_5$ , the cycle  $C_5$  and the complete graph  $K_5$ .

## 1.2.6 Connectedness and distance

Let  $G$  be a graph, and  $u$  and  $v$  two vertices of  $G$ .

A *walk*  $W$  in  $G$  between  $u$  and  $v$  is a sequence of vertices of  $G$ ,  $W = u_0 \dots u_n$ , with  $n$  being a positive integer, such that  $u_0 = u$ ,  $u_n = v$  and  $u_{i+1}$  is adjacent to  $u_i$ , for every  $i \in \{0, \dots, n-1\}$ . The vertices  $u$  and  $v$  are the *ends* of  $W$ , and  $W$  is a  $\{u, v\}$ -walk in  $G$ . The walk  $W$  *visits*  $u_i$ , for every  $i \in \{0, \dots, n-1\}$ . We write  $u_i \in W$  for every node  $u_i$  visited by  $W$ .

The vertex  $v$  is *reachable* from  $u$  in  $G$  if there exists a  $\{u, v\}$ -walk in  $G$ . By convention, a vertex is always reachable from itself. Note that  $v$  is reachable from  $u$  if and only if  $u$  is reachable from  $v$ . If  $u$  and  $v$  are distinct, and  $u$  is reachable from  $v$ , then we can find a walk between the two vertices which is a path. The distance in  $G$  between  $u$  and  $v$ , denoted  $\text{dist}_G(u, v)$ , is the length of a shortest path in  $G$  between  $u$  and  $v$ . By convention if  $u$  is not reachable from  $v$ ,  $\text{dist}_G(u, v) = \infty$ . Moreover we clearly have  $\text{dist}_G(x, y) = \text{dist}_G(y, x)$ , for every vertices  $x$  and  $y$  of  $G$ . If  $uv$  and  $u'v'$  are two edges of  $G$ , then we define the distance in  $G$  between  $uv$  and  $u'v'$ , denoted by  $\text{dist}_G(uv, u'v')$ , as :

$$\text{dist}_G(uv, u'v') = \min\{\text{dist}_G(u, u'), \text{dist}_G(v, v'), \text{dist}_G(u, v'), \text{dist}_G(u', v)\};$$

which is the minimal distance between an end of  $uv$  and an end of  $u'v'$ .

The graph  $G$  is *connected* if for every two vertices  $x, y \in V(G)$ ,  $x$  is reachable from  $y$ . By convention a trivial graph (*i.e.* of order 1) is connected. In a connected graph all the distances between its vertices are finite. The maximum distance between two vertices of  $G$  is called the *diameter* of  $G$ , denoted by  $\text{diam}(G)$ . If  $H$  is a subgraph of  $G$ , then for every vertices  $x$  and  $y$  of  $H$ ,  $\text{dist}_H(x, y) \geq \text{dist}_G(x, y)$ , and if  $H$  is spanning then  $\text{diam}(H) \geq \text{diam}(G)$ .

A *component* of  $G$  is an induced subgraph  $H$  of  $G$  which is connected, and whose vertices are not reachable from vertices in  $V(G) \setminus V(H)$ . Note that  $G$  is connected if and only if  $G$  has exactly one component, and  $u$  is reachable from  $v$  if and only if  $u$  and  $v$  belong to the same component of  $G$ .

If  $G$  is connected, nontrivial and non-complete, the *connectivity* of  $G$ , denoted by  $\kappa(G)$ , is the minimum size of a subset  $S$  of  $V(G)$ , such that  $G - S$  is not connected. This means that we cannot “disconnect” the graph  $G$  by deleting less than  $\kappa(G)$  of its vertices. By convention,  $\kappa(K_n) = n - 1$  for  $n \geq 2$  and  $\kappa(K_1) = 1$ . For a positive integer  $k$ , we say that  $G$  is  $k$ -connected if  $k \leq \kappa(G)$ . Note that a graph is connected if and only if it is 1-connected. Another way to see the connectivity is through the characterization given by Menger’s theorem [Men27]. It states that in order to “disconnect” two distinct nonadjacent vertices  $x$  and  $y$  of a graph  $G$ , one must delete at least  $k$  vertices of  $G$ , where  $k$  is the number of pairwise vertex-disjoint  $\{x, y\}$ -paths in  $G$ . The consequence of this result is that in a  $k$ -connected graph, there exist at least  $k$  pairwise vertex-disjoint paths between every distinct non adjacent vertices of the graph.

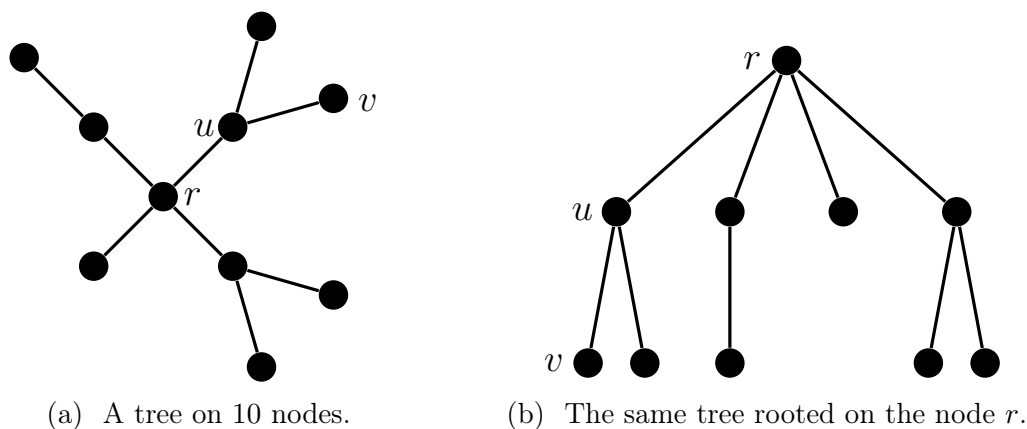


Figure 1.5: Example of a tree.

### 1.2.7 Trees

A tree is a connected graph with no cycle. Sometimes the vertices of a tree are called *nodes*. A 1-node in a tree is a *leaf* while a  $2^+$ -node is an *internal* node. If a tree  $T$  is nontrivial, then it has at least two leaves. Particularly, this means that trees are 1-degenerate. The following theorem gives five characterizations of trees, it is not difficult to prove that they are equivalent.

**Theorem 1.2.1** ([BM08]). *If  $T$  is a graph then the following assertions are equivalent:*

1.  $T$  is a tree.
2. For any two vertices  $u$  and  $v$  of  $T$ , there exists a unique  $\{u, v\}$ -path.
3.  $T$  is minimally connected, i.e.  $T - e$  is not connected for every  $e \in E(T)$ .
4.  $T$  is connected and  $|E(T)| = |V(T)| - 1$ .
5.  $T$  is maximally acyclic, i.e.  $T + \{xy\}$  is not acyclic for every non adjacent nodes  $x$  and  $y$  of  $T$ .
6.  $T$  is acyclic and  $|E(T)| = |V(T)| - 1$ .

If  $G$  is a connected graph, Theorem 1.2.1 implies that  $G$  has a spanning subgraph  $T$  which is a tree. Indeed, if  $G$  is a tree then we take  $T = G$ , otherwise since  $G$  is connected, then by Theorem 1.2.1 there exists an edge  $e \in E(G)$ , such that  $G - \{e\}$  is connected. By iterating this argument on  $G - \{e\}$ , we end up with a spanning subgraph of  $G$ , which is a tree. Such a tree is called a *spanning tree* of  $G$ . This notion reveals itself very useful in several proofs in graph theory.

If we distinguish one node  $r$  of a tree  $T$  as the *root* of  $T$ , we say that  $T$  is *rooted at  $r$* . This naturally defines a binary relation *descendant of*, on the nodes of  $T$ . The node  $v$  is

a descendant of  $u$ , if the unique  $\{v, r\}$ -path in  $T$  visits  $u$ . If  $v$  is a descendant of  $u$  and  $uv \in E(T)$ , we say that  $u$  is the *father* or *parent* of  $v$  and that  $v$  is a *child* of  $u$ . Note that every node of  $T$  has exactly one parent except for  $r$  which does not have a parent. Moreover, if  $T$  is not a trivial graph, the leaves different from  $r$  are the only nodes of  $T$  that have no children.

If  $T$  is a tree rooted at a node  $r$ , the *depth* of a node  $u \in V(T)$  is the distance between  $u$  and  $r$ . Note that if  $u$  is the parent of  $v$ , then  $\text{dist}_T(v, r) = \text{dist}_T(u, r) + 1$ . The *height* of a tree is the maximum depth of its nodes. A node with maximum depth is necessarily a leaf.

On Figure 1.5b we can see a rooted tree on the node  $r$ , the nodes  $u$  and  $v$  are descendants of  $r$ . Moreover,  $u$  is the parent of  $v$  which is a leaf.

### 1.2.8 Matchings

A *matching* in  $G$  is a set of pairwise independent edges of  $G$ . If  $U$  is the set of the end vertices of the edges of a matching  $M$ , we say that  $M$  is a matching of  $U$ . If  $uv \in M$ ,  $u$  and  $v$  are *matched under*  $M$ . The graph  $(U, M)$  is a 1-regular subgraph of  $G$ , with only isolated edges. We usually identify a matching  $M$  with the subgraph  $(U, M)$ . A matching  $M$  in  $G$  is *perfect* if for every vertex  $v$  of  $G$ , there is an edge  $e$  in  $M$ , such that  $e$  is incident with  $v$ , *i.e.*  $M$  is a matching of  $V(G)$ . A *maximum matching* of  $G$  is a matching with maximum size over all the matchings in  $G$ . Note that a perfect matching is a maximum matching. The opposite implication is generally not true, a counter-example is any graph with an odd order.

## 1.3 Graph colouring

Map colouring, task scheduling, wedding table planning or timetables optimization are problems that can easily be modelled with graphs. Indeed, the set of edges of a graph can be seen as a binary relation on the vertices. Hence any instance of a problem with an underlying binary relation  $R$  (dependency, incompatibility ...) on a set  $S$  can be represented by a graph  $G = (S, R)$ . As in the cited examples, the binary relation is used to define some constraints on the attribution of some resources to the elements of  $S$ . The map colouring problem, for example, asks to attribute colours to the regions of a map such that no two adjacent regions get the same colour. One can ask what is the least number of colours needed to fulfil this condition?

### 1.3.1 Proper colourings and chromatic parameters

A *vertex-colouring* of a graph  $G$  is a mapping  $f : V(G) \rightarrow S$ , where  $S$  is a nonempty set. If  $|S| = k \geq 1$ ,  $f$  is then a *vertex  $k$ -colouring* of  $G$ . The colouring  $f$  is *proper* if it assigns different colours to adjacent vertices, *i.e.* for every  $uv \in E(G)$  we have  $f(u) \neq f(v)$ . If a graph  $G$  admits a proper vertex  $k$ -colouring, then  $G$  is  *$k$ -colourable*. The smallest integer

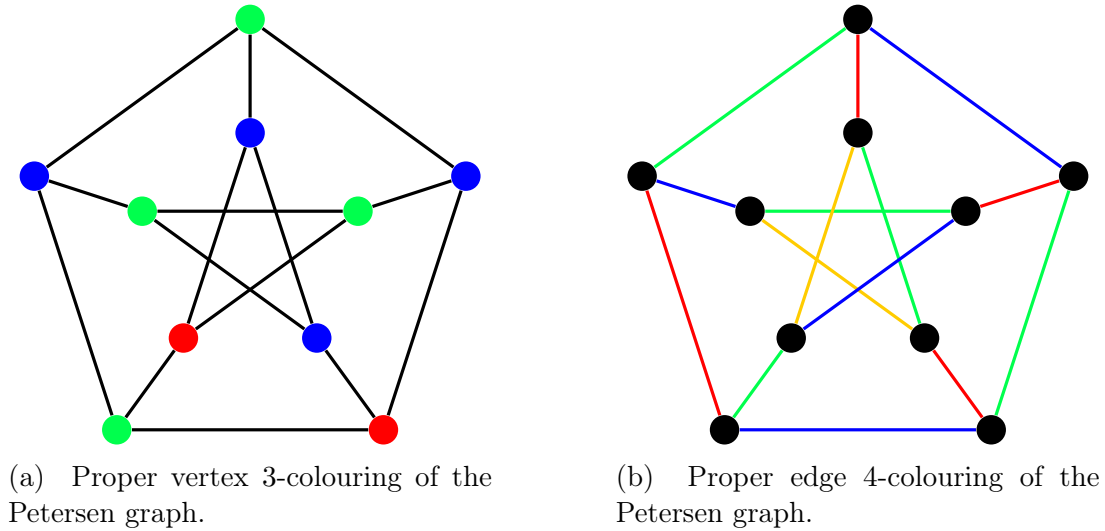


Figure 1.6: Examples of proper colourings.

$k$  such that a graph  $G$  is  $k$ -colourable is the *chromatic number* of  $G$ , and is denoted by  $\chi(G)$ . If  $\chi(G) = k$ , then  $G$  is  *$k$ -chromatic*.

Note that by attributing a different colour to each vertex of a graph  $G$  we produce a proper vertex-colouring, hence  $|V(G)|$  is an obvious upper bound of  $\chi(G)$ . If  $G$  is a complete graph of order  $n$ , then  $\chi(G) = n = \Delta(G) + 1$  since we can not assign the same colour to any distinct vertices of  $G$ . It is not difficult to prove that if  $G$  is an odd cycle, then we also have  $\chi(G) = 3 = \Delta(G) + 1$ . Moreover,  $\Delta(G) + 1$  colours is enough to colour any graph  $G$ . Indeed greedy first-fit algorithm can produce a proper vertex  $(\Delta(G) + 1)$ -colouring of  $G$ . At each step we have at most  $\Delta(G)$  forbidden colours and since we use  $\Delta(G) + 1$  colours there is always at least one available colour. Hence  $\chi(G) \leq \Delta(G) + 1$ , for every graph  $G$ . This bound is sharp for complete graphs and odd cycles, but it is not sharp for the other graphs. The main general upper bound on the chromatic number is given by the celebrated Brooks' theorem.

**Theorem 1.3.1** ([Bro41]). *Let  $G$  be a graph with  $\Delta(G) \geq 2$ . If none of the components of  $G$  is a clique nor a cycle with odd order, then  $\chi(G) \leq \Delta(G)$ .*

A proper vertex-colouring of a graph  $G$  can be seen as a partition of  $V(G)$  into independent sets, since every colour class (*i.e.* monochromatic subset of  $V(G)$ ) is an independent set. This gives a lower bound for the chromatic number of a graph  $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$ . Moreover it is easy to see that if  $G$  has a clique of order  $k$  as subgraph, then the chromatic number of  $G$  is at least  $k$ , since we need  $k$  different colours in order to colour the clique. This gives another lower bound for the chromatic number  $\chi(G) \geq \omega(G)$ .

Similarly, an *edge colouring* of a graph  $G$  is a mapping  $\gamma : E(G) \rightarrow S$ , where  $S$  is a nonempty set. If  $|S| = k \geq 1$ ,  $\gamma$  is then an *edge  $k$ -colouring* of  $G$ . The colouring  $\gamma$  is *proper* if it assigns different colours to adjacent edges, *i.e.* for every vertex  $u \in V(G)$  and  $v, v' \in N_G(u)$  with  $v \neq v'$  we have  $\gamma(uv) \neq \gamma(uv')$ . If a graph  $G$  admits a proper edge

$k$ -colouring, then  $G$  is  $k$ -edge colourable. The smallest integer  $k$  such that a graph  $G$  is  $k$ -edge colourable is the *chromatic index* of  $G$ , and is denoted by  $\chi'(G)$ . If  $\chi'(G) = k$ , then  $G$  is  $k$ -edge chromatic.

As in the vertex-colouring case, we can find a loose upper bound for  $\chi'$  by noting that assigning a different colour to each edge of a graph  $G$  yields a proper edge colouring of  $G$ . Hence  $\chi'(G) \leq |E(G)|$ . This bound is reached for star graph (*i.e.* graphs with one universal vertex and then only pending vertices). Another upper bound can be found using a greedy algorithm to properly colour the edges of  $G$ , at each step we have at most  $2 \cdot \Delta(G)$  forbidden colours, hence if we use  $2 \cdot \Delta(G) + 1$  colours we can guarantee that we always have at least one available colour. This algorithm hence proves that  $\chi'(G) \leq 2 \cdot \Delta(G) + 1$ . The main upper bound on  $\chi'(G)$  is given by the also celebrated Vizing's theorem.

**Theorem 1.3.2** ([Viz64]). *For every graph  $G$ ,  $\chi'(G) \leq \Delta(G) + 1$ .*

By definition, in a proper edge colouring of a graph  $G$ , for every vertex  $v$  of  $G$ , all the incident edges with  $v$  are of distinct colours, hence the incident edges with  $v$  use  $d_G(v)$  different colours. This gives us a lower bound  $\chi'(G) \geq \Delta(G)$ . Moreover each colour class (*i.e.* monochromatic set of edges) is a matching. Hence if  $k$  is the size of a maximum matching in  $G$ , then  $\chi'(G) \geq \frac{|E(G)|}{k}$ .

On Figure 1.6 we can see examples of proper vertex and edge colourings of the Petersen graph.

## 1.4 Graph weighting and labelling

Let  $G$  be a graph, and  $(S, +)$  a commutative semigroup. An  $S$ -edge-weighting (resp.  $S$ -total-weighting)  $\omega$  of  $G$  is a mapping from  $E(G)$  (resp.  $E(G) \cup V(G)$ ) to the commutative semigroup  $(S, +)$ . When no confusion is possible we will simply talk about *edge-weighting* (resp. *total-weighting*). For each element  $x \in S$ , we denote by  $E_\omega(x)$  the set of edges  $e \in E(G)$  such that  $\omega(e) = x$ , and if  $\omega$  is a total-weighting we also denote by  $V_\omega(x)$ , the set of vertices  $v \in V(G)$  such that  $\omega(v) = x$ . Moreover, for each vertex  $v$  of  $G$ , and for each  $i \in S$ , we denote by  $D_{\omega,i}(v)$  the set of edges  $e$  incident with  $v$  such that  $\omega(e) = i$ , and we denote by  $d_{\omega,i}(v)$  the cardinality of  $D_{\omega,i}(v)$ . The *palette* of  $v$ , denoted by  $P_\omega(v)$ , is the multiset formed by the weights of the edges incident with  $v$ . The vertex  $v$  has a *monochromatic palette of  $i$ 's*, for  $i \in S$ , if  $d_{\omega,i}(v) = |P_\omega(v)| = d_G(v)$ , and, if  $\omega$  is a total-weighting,  $\omega(v) = i$ . If  $S = \mathbb{N}^*$ , then the edge-weighting (resp. total-weighting)  $\omega$  is a  $k$ -edge-weighting (resp.  $k$ -total-weighting) where  $k$  is the largest integer in  $\omega(E(G))$  (resp.  $\omega(V(G) \cup E(G))$ ).

### 1.4.1 Graph labelling

Given an edge- (total-) weighting  $\omega$  of a graph  $G$ , we can induce a vertex-colouring  $c_\omega$  from  $\omega$  in several ways. The link between graph weightings and vertex-colourings is a part of the more general graph labelling theory, where labels on the vertices or edges of a

graph are used to induce colourings on its vertices or its edges. For example, in a *graceful labelling* of a graph  $G$ , we ask to assign a unique label  $\omega(v) \in \{0, \dots, |E(G)|\}$  to each vertex  $v$  of  $G$ , such that if we assign to each edge  $xy$  of  $G$  the value  $|\omega(x) - \omega(y)|$ , we obtain a one-to-one mapping between  $E(G)$  and  $\{1, \dots, |E(G)|\}$ . A major conjecture on graceful labellings is the graceful tree conjecture stated by Ringel and Kotzig, in which they claim that every tree admits a graceful labelling. Other examples are the *magic* and *antimagic* labellings, where we use distinct weights on the edges and induce vertex-colourings by summing for each vertex the labels of the edges it is incident with, and ask for all vertices to have the same colour in the case of supermagic labelling, and for vertices colours to be pairwise distinct in the case of antimagic labelling. Another labelling that is closely related to the weightings we will consider in this thesis is the *irregular labelling*. An edge-weighting  $\omega$  of a graph  $G$  is an irregular labelling of  $G$  if for each pair of distinct vertices  $u, v \in V(G)$ , we have :

$$\sum_{e \in E(G), e \ni u} \omega(e) \neq \sum_{e \in E(G), e \ni v} \omega(e).$$

The smallest integer  $k$  such that  $G$  admits a  $k$ -irregular labelling is the *irregularity strength* of  $G$ , denoted by  $s(G)$ . For a more detailed description of graph labellings please refer to the comprehensive survey by Gallian [Gal17].

### 1.4.2 Neighbour-sum-distinguishing edge-weightings

Note that in the context of graph weighting, we mainly use the operation defined on the semigroup  $S$  to induce vertex-colourings. In 2004, Karoński, Łuczak and Thomason introduced in [KLT04] a weaker version of irregular labelling. Namely, from an  $S$ -edge-weighting  $\omega$  of a graph  $G$ , where  $(S, +)$  is a commutative semigroup, we induce a vertex-colouring  $\sigma_\omega$  defined by :

$$\sigma_\omega(u) = \sum_{e \in E(G), e \ni u} \omega(e).$$

The weighting  $\omega$  is a *neighbour-sum-distinguishing* (*nsd* for short) edge-weighting of  $G$ , if  $\sigma_\omega$  is a proper vertex-colouring of  $G$ . In this context, two vertices of  $G$  are *sum-distinguished* if  $\sigma_\omega(u) \neq \sigma_\omega(v)$ , otherwise if  $uv \in E(G)$  and  $\sigma_\omega(u) = \sigma_\omega(v)$  we say that  $u$  and  $v$  are in *conflict* and that the edge  $uv$  is a *conflict*. Note that if  $G$  contains an isolated edge, then we cannot sum-distinguish the ends of such an edge. Hence whenever dealing with neighbour-sum-distinguishing edge-weightings, we will only consider graphs with no isolated edge. Such graphs are said to be *nice*. In their paper, Karoński, Łuczak and Thomason define neighbour-sum-distinguishing edge-weightings in special cases where  $S = \mathbb{N}^*$  or  $\mathbb{R}$  or where  $S$  is a finite group. They asked a question which later became the famous 1-2-3 Conjecture :

**Question** (1-2-3 Conjecture). *Is it possible to weight the edges of any nice graph with the integers  $\{1, 2, 3\}$  in such a way that the induced vertex-colouring is a proper colouring?*

---



1. Definitions and notation

---

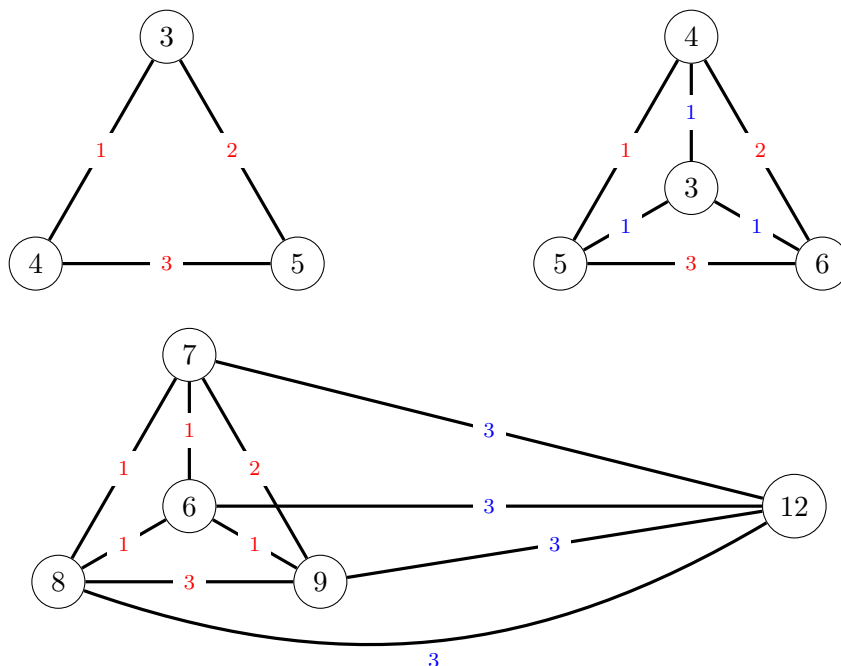


Figure 1.7: An illustration of the two first steps of the complete graph weighting in Proposition 1.4.2.

They answered by the affirmative to their question for 3-colourable nice graphs by proving the following stronger result :

**Theorem 1.4.1** ([KLT04]). *Let  $\Gamma$  be a finite commutative group of odd order and let  $G$  be a nice  $|\Gamma|$ -colourable graph. Then  $G$  admits a neighbour-sum-distinguishing  $\Gamma$ -edge-weighting.*

They also proved that if we allow real weights on the edges, that is  $S = \mathbb{R}$ , then there is a finite set of weights which can be used to weight the edges of any nice graph in a neighbour-sum-distinguishing way.

For a graph  $G$ , if  $\omega$  is an edge-weighting of  $G$  with values in  $\mathbb{N}^*$ , then  $\omega$  is a  $k$ -edge-weighting of  $G$ , if  $k \geq \max_{e \in E(G)} \{\omega(e)\}$ , i.e.  $\omega : E(G) \rightarrow \{1, \dots, k\}$ . The smallest integer  $k$  such that a graph  $G$  admits a neighbour-sum-distinguishing  $k$ -edge-weighting is the *neighbour-sum-distinguishing index* of  $G$ , denoted by  $\chi_{\Sigma}^e(G)$ . Moreover, the smallest integer  $k$  for which there exists a commutative group  $\Gamma$  with  $|\Gamma| = k$ , such that  $G$  admits a neighbour-sum-distinguishing  $\Gamma$ -edge-weighting is denoted by  $\chi_g^e(G)$ . Rephrased with these new notations, the 1-2-3 Conjecture asks whether  $\chi_{\Sigma}^e(G) \leq 3$  holds for every nice graph  $G$ , and Theorem 1.4.1 states that  $\chi_g^e(G) \leq k$ , if  $k$  is odd, for every nice  $k$ -colourable graph.

The 1-2-3 Conjecture can be easily checked for simple families of graphs such as nice trees and nice complete graphs.

**Proposition 1.4.2.** *Let  $n \geq 3$  be a positive integer,*

1. For every nice tree  $T$ ,  $\chi_{\Sigma}^e(T) \leq 2$ ;
2.  $\chi_{\Sigma}^e(K_n) = 3$ .

*Proof.* 1. Let  $T$  be a nice tree, and  $r$  be an internal node of  $T$ . We root  $T$  at  $r$ , then arbitrarily weight all edges of  $T$  incident with  $r$  using weights 1 and 2. Note that for every neighbour  $v$  of  $r$ , either  $v$  is a leaf, which implies that  $\sigma_{\omega}(v) = \omega(rv) < \sigma_{\omega}(r)$  (because  $d_T(r) \geq 2$ ), or  $v$  is an internal vertex, which means that  $v$  has at least one child  $w$ , in which case we can use a weight (1 or 2) on the edge  $vw$  to ensure that  $\sigma_{\omega}(r)$  and  $\sigma_{\omega}(v)$  have different parity. In both cases we have  $\sigma_{\omega}(r) \neq \sigma_{\omega}(v)$ . By inductively applying the same procedure for each child of  $r$  we can ensure that every node in  $T$  is sum-distinguished from its father, hence we obtain an nsd 2-edge-weighting of  $T$ .

2. Let  $K_n$  be the complete graph on vertices  $v_1, \dots, v_n$ . Assume that  $\omega$  is an nsd 2-edge-weighting of  $K_n$  and let  $\sigma_{\omega}$  be the vertex-colouring induced by  $\omega$ . We then have  $n - 1 \leq \sigma(v) \leq 2(n - 1)$  for every vertex  $v$  of  $K_n$ , hence, since we need  $n$  distinct values for  $n$  vertices, there exist two vertices  $u$  and  $v$  with  $\sigma(u) = n - 1$  and  $\sigma(v) = 2(n - 1)$ , in contradiction with the colour of the edge  $uv$ . Hence  $\chi_{\Sigma}^e(K_n) \geq 3$ .

Now in order to prove that  $\chi_{\Sigma}^e(K_n) \leq 3$ , we construct an nsd 3-edge-weighting of  $K_n$ . We start by weighting the edges  $v_1v_2$ ,  $v_2v_3$  and  $v_1v_3$  using weights 1, 2 and 3, respectively. If  $n = 3$  then we are done. If  $n \geq 4$ , in order to weight the remaining “un-weighted” edges we process the vertices from  $v_4$  to  $v_n$  as follows (see Figure 1.7). For  $i \in \{4, n\}$ :

- If  $i = 4$  or  $\omega(v_{i-1}v_j) = 3$  for every  $j < i - 1$ , then we choose  $\omega(v_iv_j) = 1$  for every  $j < i$ ;
- otherwise we choose  $\omega(v_iv_j) = 3$  for every  $j < i$ ;

At each step,  $v_i$  is the only vertex whose incident edges are all weighted 3 or all weighted 1. Hence  $\sigma_{\omega}(v_i)$  is either smaller than all  $\sigma_{\omega}(v_j)$  for  $j < i$  or greater than all  $\sigma_{\omega}(v_j)$  for  $j < i$ . Which means that we create no conflict. Therefore the procedure yields an nsd 3-edge-weighting, and we have  $\chi_{\Sigma}^e(K_n) \leq 3$ . □

Note that in an edge-weighting  $\omega$  of a graph  $G$ , if  $uv \in E(G)$ , then the sum-distinction between  $u$  and  $v$  does not depend on the value of  $\omega(uv)$ , because this value appears in both sums of  $u$  and  $v$ . Hence, if  $u$  and  $v$  are sum-distinguished by  $\omega$ , they will stay sum-distinguished if we modify the value of  $\omega(uv)$ .

First efforts to tackle the 1-2-3 Conjecture focused on finding an upper bound for  $\chi_{\Sigma}^e$  in term of the chromatic number  $\chi$ :

- if  $G$  is  $k$ -colourable for  $k$  odd, then  $\chi_{\Sigma}^e(G) \leq k$  [KLT04];
- if  $G$  is  $k$ -colourable for  $k \equiv 0 \pmod{4}$ , then  $\chi_{\Sigma}^e(G) \leq k$  [DLY12];

- if  $G$  is  $k$ -colourable and  $\delta(G) \leq k - 2$ , then  $\chi_{\Sigma}^e(G) \leq k$  [DLY12];
- if  $G$  is  $k$ -colourable and  $|V(G)|$  is odd, then  $\chi_{\Sigma}^e(G) \leq k$  [DLY12];
- if  $G$  is 2-connected,  $k$ -colourable, and has  $\delta(G) \geq k + 1$  for  $k \equiv 2 \pmod{4}$ , then  $\chi_{\Sigma}^e(G) \leq k$  [LYY09].

Meanwhile, other authors concentrated on finding general upper bounds on  $\chi_{\Sigma}^e$ . Addario-Berry *et al.* established the first constant general upper bound in [ABDM<sup>+</sup>07], by proving that every nice graph admits a neighbour-sum-distinguishing 30-edge-weighting, then Addario-Berry, Dalal and Reed decreased this bound to 16 in [ABDR08]. In the same year Wang and Yu proved that  $\chi_{\Sigma}^e(G) \leq 13$  for every nice graph  $G$ . In 2009 Kalkowski, Karoński and Pfender [KKP09] came with an algorithm inspired by Kalkowski's work on neighbour-sum-distinguishing total-weightings in [Kal10]. This algorithm provides, for every nice graph  $G$ , a neighbour-sum-distinguishing 6-edge-weighting of  $G$ . In 2010, Kalkowski, Karoński and Pfender [KKP10] modified their algorithm to provide neighbour-sum-distinguishing 5-edge-weightings, giving, at the same time, the best known upper bound on  $\chi_{\Sigma}^e$ .

**Theorem 1.4.3** (Kalkowski, Karoński and Pfender [KKP10]). *If  $G$  is a nice graph, then  $\chi_{\Sigma}^e(G) \leq 5$ .*

Since  $\chi_{\Sigma}^e(G) = 1$ , if and only if  $G$  has no adjacent vertices with the same degree, which is easy to characterise, another direction toward the 1-2-3 Conjecture is characterising the graphs that satisfy  $\chi_{\Sigma}^e(G) \leq 2$ . Chang, Lu, Wu, and Yu [CLWY11] showed that  $\chi_{\Sigma}^e(G) \leq 2$  if  $G$  is bipartite and  $d$ -regular for  $d \geq 3$ . Lu, Yu, and Zhang [LYZ11] proved that if  $G$  is a nice graph which is either 3-connected and bipartite or has minimum degree  $\delta(G) \geq 8\chi(G)$ , then  $\chi_{\Sigma}^e(G) \leq 2$ . Davoodi and Omooni [DO15] explored the neighbour-sum-distinguishing edge-weighting for Cartesian products of graphs proving that  $\chi_{\Sigma}^e(G \square H) \leq 2$ , for every two bipartite graphs  $G$  and  $H$  with  $G \square H \not\cong K_2$ , and more generally, that  $\chi_{\Sigma}^e(G \square H) \leq \max\{\chi_{\Sigma}^e(G), \chi_{\Sigma}^e(H)\}$  for every two nice graphs  $G$  and  $H$  with  $G \square H \not\cong K_2$ . Khatirinejad, Naserasr, Newman, Seamone, and Stevens [KNN<sup>+</sup>12] proved that  $\chi_{\Sigma}^e(G) \leq 2$  if all cycles of  $G$  have length divisible by 4, and that nsd 2-edge weighting is equivalent to nsd  $\{a, b\}$ -edge weighting, for every two integers  $a$  and  $b$  with  $\gcd(a, b) = 1$ , where  $\gcd(a, b)$  is the greatest common divisor of  $a$  and  $b$ . Finally, Dudek and Wajc proved in [DW11] that given a nice graph  $G$ , it is NP-complete to decide whether  $\chi_{\Sigma}^e(G) \leq 2$ . However, since most of the graphs that are known to admit neighbour-sum-distinguishing 2-edge-weightings are bipartite, one can ask if we can characterise all such bipartite graphs. Thomassen, Wu and Zhang [TWZ16] gave a full characterisation of such bipartite graphs, and by the same occasion, proved that the problem of deciding whether  $\chi_{\Sigma}^e(G) \leq 2$  can be solved in polynomial time if  $G$  is bipartite.

### 1.4.3 Neighbour-sum-distinguishing total-weighting

In 2010, Przybyło and Woźniak introduced graph total-weightings, where we assign positive integer weights to the edges and the vertices of a graph, and ask to sum-distinguish

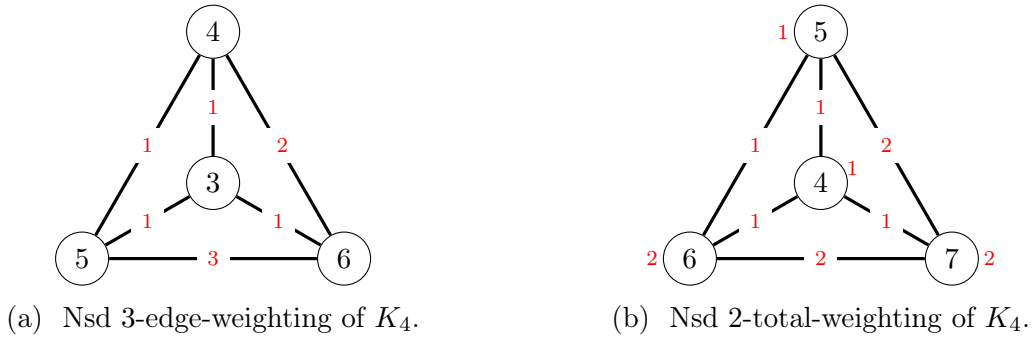


Figure 1.8: Examples of neighbour-sum-distinguishing weightings.

each pair of adjacent vertices. Namely, a *neighbour-sum-distinguishing total-weighting*  $\omega$  of a graph  $G$  is a total-weighting of  $G$  with values in  $\mathbb{N}^*$  such that the induced vertex-colouring  $\sigma_\omega^t$  of  $G$ , defined for each vertex  $v$  of  $G$  by :

$$\sigma_\omega^t(v) = \omega(v) + \sum_{e \in E(G), e \ni v} \omega(e),$$

is a proper vertex-colouring of  $G$ . The smallest integer  $k$  such that  $G$  admits a neighbour-sum-distinguishing  $k$ -total-weighting is the *neighbour-sum-distinguishing total index* of  $G$ , denoted by  $\chi_\Sigma^t(G)$ . Note that in the context of total-weightings, we can sum-distinguish the ends of any isolated edge, hence  $\chi_\Sigma^t(G)$  is defined even if  $G$  is not nice. Motivated by the 1-2-3 Conjecture and by a result from [ABDR08] stating that for each real  $p \in ]0, 1[$ , the random graph  $G_{n,p}$  (asymptotically) almost surely admits an neighbour-sum-distinguishing 2-edge-weighting, Przybyło and Woźniak stated the following conjecture :

**Conjecture 1.4.4** ([PW10]). *For every graph  $G$ ,  $\chi_\Sigma^t(G) \leq 2$ .*

They proved the conjecture for bipartite graphs, complete graphs, 3-colourable graphs and 4-regular graphs. They also proved that weights in  $\{1, \dots, 11\}$  are sufficient to get an nsd total-weighting for any graph. The best known bound toward Conjecture 1.4.4, given in the following theorem, was proved by Kalkowski in his PhD Thesis [Kal10], using an early version of the algorithm used later by himself, Karoński and Pfender to prove that  $\chi_\Sigma^e(G) \leq 5$  for every nice graph  $G$ .

**Theorem 1.4.5** ([Kal10]). *For every graph  $G$ , we can find a neighbour-sum-distinguishing total-weighting of  $G$  using weights  $\{1, 2, 3\}$  on the edges and  $\{1, 2\}$  on the vertices.*

As a consequence of Theorem 1.4.5 we get  $\chi_\Sigma^t(G) \leq 3$  for every graph  $G$ .

### 1.4.4 Neighbour-multiset-distinguishing edge-weighting

Other variants of neighbour-distinguishing edge-weightings were considered in the last decade. For example, when the edge weights are positive integers, we can define the

## 1. Definitions and notation

---

induced colour of each vertex as the multiset formed by the weights of the edges it is incident with. Namely, if  $\omega$  is an edge-weighting of  $G$ , then we define the induced colour  $m_\omega(v)$  on a vertex  $v$  by :

$$m_\omega(v) = \bigcup_{e \in E(G), e \ni v} \{\omega(e)\},$$

where the union is to be considered as a multiset union. An edge-weighting  $\omega$  is a *neighbour-multiset-distinguishing* (nmd for short) edge-weighting, if  $m_\omega$  is a proper vertex-colouring of  $G$ . In this context, two vertices  $u$  and  $v$  of  $G$  are *multiset-distinguished* if  $m_\omega(u) \neq m_\omega(v)$ . For a graph  $G$ , the smallest integer  $k$  such that  $G$  admits a neighbour-multiset-distinguishing  $k$ -edge-weighting is denoted by  $\chi_m^e(G)$ . In [ABADR05], Addario-Berry, Aldred, Dalal and Reed proved that  $\chi_m^e(G) \leq 4$  for every nice graph  $G$ , and that  $\chi_m^e(G) \leq 3$  if  $\delta(G) \geq 1000$ .

Neighbour-multiset-distinguishing edge-weighting is a weaker version of neighbour-sum-distinguishing edge-weighting, *i.e.* every nsd edge-weighting is an nmd edge-weighting. Indeed, if the sums of elements of two multisets are distinct, then the two multisets are clearly distinct. The following proposition establishes another link between nmd and nsd edges-weightings, and will be very useful in some of our proofs.

**Proposition 1.4.6.** *For every nice regular graph  $G$ , and every 2-edge-weighting  $\omega$  of  $G$ ,  $\omega$  is nmd if and only if  $\omega$  is nsd.*

*Proof.* Let  $d$  be a positive integer and  $G$  a nice  $d$ -regular graph. Since every nsd edge-weighting of  $G$  is also an nmd edge-weighting, we only have to prove that every nmd edge-weighting of  $G$  is an nsd edge-weighting of  $G$ .

Let  $\omega$  be an nmd 2-edge-weighting of  $G$ , and  $uv$  an edge of  $G$ . Suppose that  $m_\omega(u) = (\{1, 2\}, f_u)$  and  $m_\omega(v) = (\{1, 2\}, f_v)$ . Since  $\omega$  is nmd we have  $m_\omega(u) \neq m_\omega(v)$ . Hence we have  $(f_u(1), f_u(2)) \neq (f_v(1), f_v(2))$ . Moreover, since  $G$  is  $d$ -regular we have  $f_u(1) + f_u(2) = f_v(1) + f_v(2) = d$ . Hence, we have  $f_u(1) \neq f_v(1)$  and  $f_u(2) \neq f_v(2)$ , otherwise we would have  $(f_u(1), f_u(2)) = (f_v(1), f_v(2))$  which is a contradiction because  $\omega$  is nmd. Therefore,

$$\begin{aligned} \sigma_\omega(u) - \sigma_\omega(v) &= f_u(1) + 2f_u(2) - (f_v(1) + 2f_v(2)) \\ &= f_u(2) - f_v(2) \\ &\neq 0. \end{aligned}$$

This means that  $\sigma_\omega(u) \neq \sigma_\omega(v)$ , and that  $u$  and  $v$  are sum-distinguished. Hence  $\omega$  sum-distinguishes the ends of any edge of  $G$ , which means that  $\omega$  is an nsd 2-edge-weighting of  $G$ .  $\square$

Note that in an nmd edge-weighting, the order of the weights does not influence the distinction between vertices. Hence if  $\omega$  is an nsd 2-edge-weighting of a regular graph  $G$ , by Proposition 1.4.6, we can replace all the weights 1 by 2, and all the weights 2 by 1 and have another nsd 2-edge-weighting of  $G$ . This result can be easily generalised to sum-distinguishing any vertices of the same degree in a graph  $G$ , with the same arguments as in the proof of Proposition 1.4.6, leading to the following observation:

**Observation 1.4.7.** *If  $u$  and  $v$  are two vertices of  $G$  with the same degree, and if  $u$  and  $v$  are sum-distinguished by a 2-edge-weighting  $\omega$  of  $G$ , then  $u$  and  $v$  are sum-distinguished by any edge-weighting  $\omega'$  such that for every edge  $e \in E(G)$  incident with  $u$  or with  $v$ ,  $\omega'(e) = 3 - \omega(e)$ , which replaces the weights 1 by 2, and all the weights 2 by 1.*

### 1.4.5 Locally irregular decompositions

Another way to formulate the problem of neighbour-sum-distinguishing  $k$ -edge-weighting in a graph  $G$ , is to ask whether we can transform  $G$  by adding at most  $k - 1$  copies of each of its edges in such a way that no two adjacent vertices of  $G$  have the same degree. Graphs in which every two adjacent vertices have different degree are *locally irregular* graphs. Note that  $G$  is a locally irregular graph if and only if  $G$  admits an nsd 1-edge-weighting. Moreover, using arguments similar to the ones used in the proof of Proposition 1.4.6, we can easily prove that in any neighbour-sum-distinguishing 2-edge-weighting  $\omega$  of a regular graph  $G$ ,  $d_{\omega,1}(u) \neq d_{\omega,1}(v)$  and  $d_{\omega,2}(u) \neq d_{\omega,2}(v)$  for every edge  $uv \in E(G)$ . Hence the subgraphs of  $G$  induced, respectively, by  $E_{\omega}(1)$  and by  $E_{\omega}(2)$ , are locally irregular graphs. Therefore, we can see an nsd 2-edge-weighting of a regular graph  $G$  as a partition of its edge set into two parts, each inducing a locally irregular graph. Generally, a *locally irregular decomposition* of a graph  $G$  is a partition  $E_1, \dots, E_k$  of  $E(G)$  such that each  $G[E_i]$  is locally irregular. We also say that  $G$  *decomposes* into  $k$  locally irregular subgraphs. However, there exist graphs that do not admit locally irregular decompositions. If  $G$  is such a graph, then it is *exceptional*, otherwise  $G$  is *decomposable*. For a decomposable graph  $G$ , the smallest integer  $k$  such that  $G$  decomposes into  $k$  locally irregular subgraphs is the *locally irregular chromatic index* of  $G$  and is denoted by  $\chi'_{\text{irr}}(G)$ .

One first important result in the study of locally irregular decompositions is the full characterization of exceptional graphs, due to Baudon, Bensmail, Przybyło and Woźniak [BBPW15]. In order to state this characterization, we first need to formally define the following family  $\mathfrak{T}$  of graphs. The definition is recursive:

- The triangle  $K_3$  belongs to  $\mathfrak{T}$ .
- Every other graph in  $\mathfrak{T}$  can be constructed by:
  1. taking an auxiliary graph  $H$  being either an even-length path or a triangle glued on one end of an odd-length path (Figure 1.9);
  2. choosing a graph  $G \in \mathfrak{T}$  containing a triangle with at least one vertex, say  $v$ , of degree 2 in  $G$ ;
  3. identifying  $v$  with a vertex of degree 1 of  $H$ .

The full characterization of exceptional graphs is then the following:

**Theorem 1.4.8** ([BBPW15]). *A connected graph  $G$  is exceptional if and only if  $G$  is either*

- *an odd-length path;*

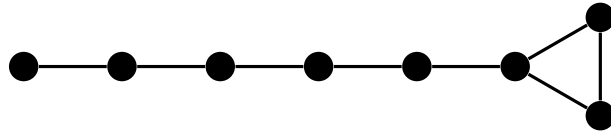


Figure 1.9: An example of an auxiliary graph used in the construction of exceptional graphs.

- *an odd-length cycle;*
- *a member of  $\mathfrak{T}$ .*

The link between locally irregular decomposition and nsd edge-weighting motivated several authors, in the last years, to explore the properties of locally irregular decomposition. It was conjectured by Baudon, Bensmail, Przybyło and Woźniak [BBPW15] that every decomposable graph decomposes into at most 3 locally irregular graphs. Towards that conjecture, it was recently proved by Bensmail, Merker and Thomassen [BMT17] that every decomposable graph decomposes into at most 328 locally irregular graphs. This bound was improved to 220 by Lužar, Przybyło and Soták in [LPS16].





## Chapter 2

# Equitable neighbour-sum-distinguishing weightings

As seen in Chapter 1, if  $k$  is a positive integer, and  $G$  a graph, we can associate with any  $k$ -edge-weighting  $\omega$  of  $G$  a vertex-colouring  $\sigma_\omega$  given by  $\sigma_\omega(v) = \sum_{e \ni v} \omega(e)$ , for each  $v \in V(G)$ . A neighbour-sum-distinguishing  $k$ -edge-weighting is an edge  $k$ -weighting whose associated vertex-colouring is proper. The neighbour-sum-distinguishing index of a graph  $G$  is then the smallest  $k$  for which  $G$  admits a neighbour-sum-distinguishing edge  $k$ -weighting. These notions naturally extend to total weightings of graphs that assign colours to both vertices and edges. In this chapter we will present the obtained results on *equitable* neighbour-sum-distinguishing edge-weightings and equitable neighbour-sum-distinguishing total-weightings, that is weightings  $\omega$  for which the number of elements in any two weight classes of  $\omega$  differ by at most one. In this chapter, we determine or provide optimal upper bounds for the equitable neighbour-sum-distinguishing index or the equitable neighbour-sum-distinguishing total index of complete graphs, complete bipartite graphs, forests, theta-graphs and, in case of total-weightings, for bipartite graphs in general.

The results presented in sections 2.2 and 2.3 were published in [BPP<sup>+</sup>17].

## 2.1 Definitions and preliminary results

Let  $G$  be a graph,  $k$  a positive integer and  $\omega$  a  $k$ -weighting of  $G$ . The weighting  $\omega$  is *equitable* if for every  $i, j \in \{1, \dots, k\}$  we have  $||\omega^{-1}(i)| - |\omega^{-1}(j)|| \leq 1$ . The smallest integer  $k$  such that the graph  $G$  admits an equitable neighbour-sum-distinguishing  $k$ -edge-weighting is the *equitable neighbour-sum-distinguishing index* of  $G$ , and is denoted by  $\overline{\chi}_{\Sigma}^e(G)$ . The smallest integer  $k$  such that  $G$  admits an equitable neighbour-sum-distinguishing  $k$ -total-weighting is the *equitable neighbour-sum-distinguishing total index* of  $G$ , and is denoted by  $\overline{\chi}_{\Sigma}^t(G)$ . If  $\omega$  is an equitable neighbour-sum-distinguishing  $k$ -weighting of a nice graph  $G$ , with the definition above, we can skip using some weights in  $\{1, \dots, k\}$ , but this implies to use every weight at most once, and that  $k \geq |E(G)|$ . To avoid this constraint, we can define the equitability differently by ignoring the unused weights, and requiring to use the other weights the same number of times up to a difference of 1. This would be a weaker version of equitable nsd weightings. Particularly, in this version, every equitable nsd  $k$ -weighting of  $G$  is also an equitable nsd  $\ell$ -weighting of  $G$  for every  $\ell \geq k$ . This seems to us to artificially bias the notion of equitability and to alter the importance of the maximum weight used in an equitable nsd weighting. Therefore, in this thesis we only use the first version defined in the previous paragraph.

A legitimate question that can be raised about equitable nsd edge-weightings is to ask if every nice graph admits an equitable nsd edge-weighting, and if we can exhibit a general bound on  $\overline{\chi}_{\Sigma}^e(G)$ , for every nice graph  $G$ . It is easy to answer in the affirmative to this question noticing that every two vertices in a nice graph have different sets of incident edges. Then by indexing the edges with integer in  $\{1, \dots, |E(G)|\}$ , and weighting each edge  $e$  with  $2^{i(e)}$ , where  $i(e)$  is the index of the edge  $e$ , we ensure that the binary representation of the sum at each vertex  $v$  represents the indicator function of the set of edges incident with  $v$ . Hence, the obtained weighting is neighbour-sum-distinguishing. Moreover, it uses each weight in  $\{1, \dots, 2^{|E(G)|}\}$  at most once. Therefore, we obtain an equitable nsd  $2^{|E(G)|}$ -edge-weighting of  $G$ . This gives the first general bound on  $\overline{\chi}_{\Sigma}^e(G)$  for every nice graph  $G$ .

Note that if  $\omega$  is a neighbour-sum-distinguishing  $k$ -edge-weighting of a graph  $G$ , then the mapping  $\omega_t$  defined by  $\omega_t(v) = 1$  for every  $v \in V(G)$  and  $\omega_t(e) = \omega(e)$  for every  $e \in E(G)$  is clearly a neighbour-sum-distinguishing  $k$ -total-weighting of  $G$  since  $\sigma_{\omega_t}(v) = \sigma_{\omega}(v) + 1$  for every vertex  $v \in V(G)$ . Hence,  $\overline{\chi}_{\Sigma}^t(G) \leq \overline{\chi}_{\Sigma}^e(G)$  for every graph  $G$ . The same relation holds for equitable weightings:

**Proposition 2.1.1.** *For every graph  $G$  without isolated edges,  $\overline{\chi}_{\Sigma}^t(G) \leq \overline{\chi}_{\Sigma}^e(G)$ .*

*Proof.* Let  $\omega$  be an equitable neighbour-sum-distinguishing  $k$ -edge-weighting of  $G$ . We will extend  $\omega$  to an equitable neighbour-sum-distinguishing  $k$ -total-weighting  $\omega_t$  of  $G$  with  $\omega_t(e) = \omega(e)$  for every edge  $e \in E(G)$ . We thus need to extend  $\omega$  to vertices in such a way that no two adjacent vertices are in conflict and the weighting remains equitable. We first order the vertices of  $G$  as  $v_1, v_2, \dots, v_n$ ,  $n = |V(G)|$ , in such a way that

$$\sigma_{\omega}(v_1) \leq \sigma_{\omega}(v_2) \leq \dots \leq \sigma_{\omega}(v_n). \quad (2.1)$$

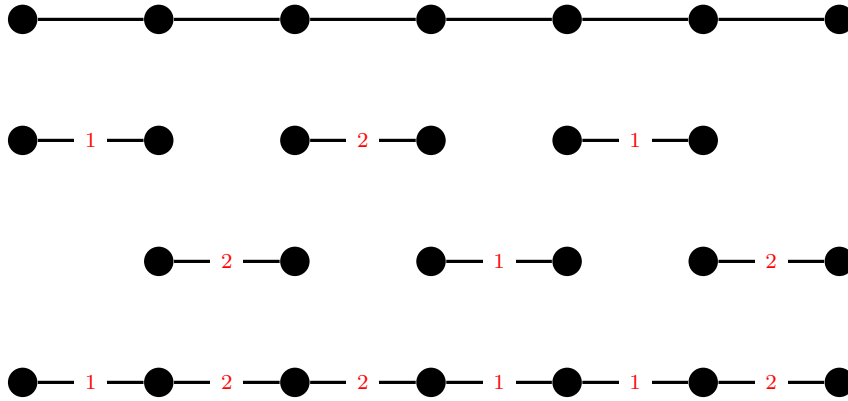


Figure 2.1: A neighbour-sum-distinguishing 2-edge-weighting of the path  $P_6$ .

Let  $t_i = |\omega^{-1}(i)|$  be the number of edges with weight  $i$ . Moreover, let  $t$  and  $r$  be non-negative integers such that  $r < k$  and  $|V(G)| + |E(G)| = tk + r$ . For  $\omega_t$  to be equitable, we then must have  $r$  weight classes of order  $t + 1$  and  $k - r$  weight classes of order  $t$ . Let  $C_t$  be any subset of  $k - r$  weights from  $\{1, \dots, k\}$  such that  $t_i \leq t$  for every  $i \in C_t$ , and  $C_{t+1} = \{1, \dots, k\} \setminus C_t$ . We will then weight  $t'_i = t - t_i$  vertices with weight  $i$  for each  $i \in C_t$  and  $t'_j = t + 1 - t_j$  vertices with weight  $j$  for each  $j \in C_{t+1}$ . In order to produce a neighbour-sum-distinguishing total-weighting, we will weight the vertices according to the above defined order, and assign the weight 1 to the first  $t'_1$  vertices, then weight 2 to the next  $t'_2$  vertices and so on. More formally, we let

$$\omega_t(v_i) = \min_{\sum_{p=1}^j t'_p \geq i} \{j\}. \quad (2.2)$$

For every edge  $v_i v_j$ ,  $i < j$ , we then have

$$\sigma_{\omega_t}(v_i) = \sigma_\omega(v_i) + \omega_t(v_i) < \sigma_\omega(v_j) + \omega_t(v_j) = \sigma_{\omega_t}(v_j)$$

since  $\sigma_\omega(v_i) < \sigma_\omega(v_j)$  (by (2.1), as  $\omega$  is neighbour-sum-distinguishing) and  $\omega_t(v_i) \leq \omega_t(v_j)$  (by (2.2)). The total-weighting  $\omega_t$  is therefore an equitable neighbour-sum-distinguishing total-weighting, and thus  $\overline{\chi}_\Sigma^t(G) \leq \overline{\chi}_\Sigma^e(G)$ .  $\square$

## 2.2 Equitable neighbour-sum-distinguishing edge-weighting

### 2.2.1 Simple families of graphs

One can naturally wonder what is the value of the new parameter  $\overline{\chi}_\Sigma^e$  on simple families of graphs. For paths, by Proposition 1.4.2, we know that they admit nsd 2-edge-weightings. However, because of the small degree of vertices, we have limited number of ways to

---

## 2.2. Equitable neighbour-sum-distinguishing edge-weighting

construct nsd 2-edge-weightings. Indeed if  $n \geq 3$  and  $P_n = v_0 \dots v_n$ , then for every nsd 2-edge-weighting  $\omega$  of  $P_n$  and every  $i \in \{0, \dots, n-3\}$ , we have

$$\omega(v_i v_{i+1}) \neq \omega(v_{i+2} v_{i+3}), \quad (2.3)$$

since otherwise, we would have a conflict between  $v_{i+1}$  and  $v_{i+2}$ . Hence, once we choose the weight of an edge  $e$  of  $P_n$ , the weights of all edges at odd distance from  $e$  are determined. Moreover, no conflict can involve a pendent vertex. Therefore, any nsd 2-edge-weighting of  $P_n$  can be obtained with the following procedure (see Figure 2.1):

1. Take a maximum matching  $M_1$  in  $P_n$ .
2. Weight the edges of  $M_1$ , following the increasing order of their ends indices, with weights 1 and 2, alternately.
3. The remaining edges also induce a matching  $M_2$ , we use the same procedure to weight  $M_2$  with weights 1 and 2, alternately.

Since for each matching we have at most two choices for the 2-edge-weighting, we have at most four possible nsd 2-edge-weighting of  $P_n$ . The two matchings  $M_1$  and  $M_2$  are of sizes  $\lceil \frac{n}{2} \rceil$  and  $\lfloor \frac{n}{2} \rfloor$ , respectively. The so obtained nsd 2-edge-weighting is equitable unless  $M_1$  and  $M_2$  are of odd size, *i.e.*  $n \equiv 2 \pmod{4}$ , in which case we can modify it to be equitable by starting the weightings of  $M_1$  and  $M_2$  with different weights. Hence we get  $\overline{\chi}_\Sigma^e(P_n) = 2$ , for every  $n \geq 3$ . Moreover, using Inequality 2.3 and the fact that we are using only two weights, we can also state that every nsd 2-edge-weighting of the path  $P_n$  is 4-periodic if  $n \geq 5$ . We summarize all this in the following observation.

**Observation 2.2.1.** *Let  $n$  be a positive integer with  $n \geq 3$ , and  $P_n = v_0 v_1 \dots v_n$ . If  $n \not\equiv 2 \pmod{4}$ , then every nsd 2-edge-weighting of  $P_n$  is equitable, otherwise, every nsd 2-edge-weighting  $\omega$  of  $P_n$  with  $\omega(v_0 v_1) \neq \omega(v_1 v_2)$  is equitable. Moreover, if  $n \geq 5$ , then every nsd 2-edge-weighting  $\omega$  of  $P_n$  is 4-periodic, that is,  $\omega(v_{i-1} v_i) = \omega(v_{i+3} v_{i+4})$  for every  $i \in \{1, \dots, n-4\}$ .*

The following observation extends some of the observations above to the case of  $k$ -edge-weightings, with  $k \geq 3$ .

**Observation 2.2.2.** *Let  $n$  be a integer with  $n \geq 3$ ,  $P_n = v_0 v_1 \dots v_n$  and  $k \geq 3$ . The  $k$ -periodic  $k$ -edge-weighting  $\omega$  of  $P_n$ , defined, for  $i \in \{0, \dots, n-1\}$ , by:*

$$\omega(v_i v_{i+1}) = (i \pmod{k}) + 1,$$

*is an equitable nsd  $k$ -edge-weighting of  $P_n$ . Moreover we have  $E_\omega(k) \leq \dots \leq E_\omega(2) \leq E_\omega(1)$ .*

In the following lemma we give an inequality on the values of  $\sigma_\omega$  for adjacent vertices. This inequality is very useful when treating graphs having vertices of high degree adjacent to vertices of low degree.

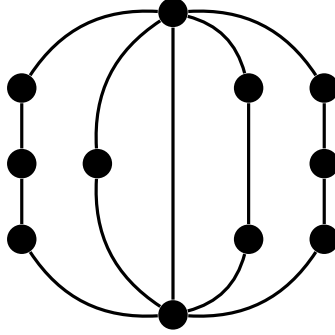


Figure 2.2: The generalised Theta-graph  $\Theta(1, 2, 3, 4, 4)$ .

**Lemma 2.2.3.** *If  $\omega$  is a  $k$ -edge-weighting of a graph  $G$ , for a positive integer  $k$ , and  $uv$  an edge of  $G$ , then*

$$\sigma_\omega(u) \geq \sigma_\omega(v) - k(d_G(v) - 1) + \max_{e \in E(G) \setminus \{uv\}, e \ni u} \{\omega(e)\} + d(u) - 2.$$

*Proof.* A lower bound for  $\sigma_\omega(u)$  is its eventual value if we set the weights of all the edges incident with  $u$  to 1, except  $uv$  and the edge with maximum weight. Formally:

$$\sigma_\omega(u) \geq \omega(uv) + \max_{e \in E(G) \setminus \{uv\}, e \ni u} \{\omega(e)\} - 1 + d_G(u) - 1.$$

While an upper bound for  $\sigma_\omega(v)$  is its eventual value if we set to  $k$  the weights of all the edges incident with  $v$ , except  $uv$ . Formally:

$$\sigma_\omega(v) \leq \omega(uv) + k(d_G(v) - 1).$$

Combining those two bounds yields the desired result. □

The following proposition gives the exact value of  $\overline{\chi}_\Sigma^e$  for cycles and generalised theta-graphs. For  $n \geq 3$  and  $d_1, d_2, \dots, d_n \in \mathbb{N}^*$ , the *generalised Theta-graph*  $\Theta_{d_1, d_2, \dots, d_n}$ ,  $d_1 \leq d_2 \leq \dots \leq d_n$ , is the graph obtained by joining two vertices  $u$  and  $v$  by  $n$  internal-vertex-disjoint paths of respective lengths  $d_1, d_2, \dots, d_n$ . In a generalised theta-graph  $\Theta_{d_1, d_2, \dots, d_n}$  all vertices are of degree 2, except  $u$  and  $v$  which are of degree  $n$ .

**Proposition 2.2.4.**

1. For every  $n \geq 3$ ,  $\overline{\chi}_\Sigma^e(C_n) = 2$  if  $n \equiv 0 \pmod{4}$  and  $\overline{\chi}_\Sigma^e(C_n) = 3$  otherwise;
2. for every  $n \geq 3$ ,  $\overline{\chi}_\Sigma^e(\Theta_{d_1, d_2, \dots, d_n}) = 3$  if  $d_1 = 1$  and  $d_i \equiv 1 \pmod{4}$  for every  $i$ ,  $2 \leq i \leq n$ , and  $\overline{\chi}_\Sigma^e(\Theta_{d_1, d_2, \dots, d_n}) = 2$  otherwise.

*Proof.* We prove the two items separately.

1. We can easily check that  $\overline{\chi}_\Sigma^e(C_3) = 3$  and  $\overline{\chi}_\Sigma^e(C_4) = 2$ . Suppose that  $n \geq 5$  and let  $C_n = v_1 \dots v_n v_1$ . The cycle  $C_n$  can be obtained by identifying the end vertices of  $P_n = v_0 v_1 \dots v_n$ . If  $n \equiv 0 \pmod{4}$ , then construct a 2-edge-weighting  $\omega$  of  $P_n$  as described in the procedure depicted in Figure 2.1, such that  $\omega(v_0 v_1) = 1$  and  $\omega(v_1 v_2) = 2$ . By Observation 2.2.1 we have  $\omega(v_{n-2} v_{n-1}) = 2$  and  $\omega(v_{n-1} v_n) = 1$ , hence  $\sigma_\omega(v_{n-1}) = 3$  and  $\sigma_\omega(v_1) = 3$ . Now we identify  $v_0$  and  $v_n$  by creating a new vertex that we call  $v_n$ . We then have  $\omega(v_n v_1) = 1$ , so that  $\sigma_\omega(v_n) = 2 \neq 3$ , which means that  $v_n$  is sum-distinguished from  $v_1$  and  $v_{n-1}$ , and since we did not modify the neighbourhood of any vertex  $v_i$  with  $i \in \{2, n-1\}$ , the obtained weighting is an nsd 2-edge-weighting of  $C_n$ . Moreover, by Observation 2.2.1  $\omega$  is equitable, hence  $\omega$  is an nsd 2-edge-weighting of  $C_n$  and  $\overline{\chi}_\Sigma^e(C_n) \leq 2$ . Since  $\overline{\chi}_\Sigma^e(C_n) \geq \chi_\Sigma^e(C_n) \geq 2$ , we get  $\overline{\chi}_\Sigma^e(C_n) = 2$ . Suppose now that  $n \not\equiv 0 \pmod{4}$ . We know from [KNN<sup>+</sup>12] that  $\overline{\chi}_\Sigma^e(C_n) \geq \chi_\Sigma^e(C_n) \geq 3$ , hence we can use the technique as in the previous case to build an equitable nsd 3-edge-weighting  $\omega$  of  $C_n$ . We will discuss the two following cases:

- if  $n \equiv 0$  or  $1 \pmod{3}$ , then weight the path  $P_n$  by repeating the pattern 1–2–3 starting from  $v_0 v_1$ ;
- if  $n \equiv 2 \pmod{3}$ , then use the same pattern from  $v_0 v_1$  to  $v_{n-3} v_{n-2}$ , then set  $\omega(v_{n-2} v_{n-1}) = 3$  and  $\omega(v_{n-1} v_n) = 1$ .

The weighting  $\omega$  is an nsd 3-edge-weighting of  $P_n$  since  $\omega(v_i v_{i+1}) \neq \omega(v_{i+2} v_{i+3})$  for every  $i \in \{0, \dots, n-3\}$ . Moreover  $\omega$  is equitable since it is just the repetition of the pattern 1–2–3. When we identify  $v_0$  and  $v_n$  we can easily check that in each case we do not create any conflict between  $v_n$  and its neighbours. Hence  $\omega$  is an equitable nsd 3-edge-weighting. Therefore  $\overline{\chi}_\Sigma^e(C_n) = 3$ .

2. Khatirinejad, Naserasr, Newman, Seamone, and Stevens [KNN<sup>+</sup>12] and Lu, Yang, and Zhang [LYZ11] showed that for every integer  $n \geq 3$ ,  $\chi_\Sigma^e(\Theta_{d_1, \dots, d_n}) = 3$  if  $d_1 = 1$  and  $d_i \equiv 1 \pmod{4}$  for every  $i \in \{2, \dots, n\}$ , and  $\chi_\Sigma^e(\Theta_{d_1, \dots, d_n}) = 2$  otherwise. Let  $G = \Theta_{d_1, \dots, d_n}$  for some set of positive integers  $\{d_1, \dots, d_n\}$ , and let  $u$  and  $v$  be the two vertices of degree  $n$  in  $G$ . Moreover, let  $Q_1, \dots, Q_n$  be the paths joining  $u$  and  $v$  in  $G$  of length  $d_1, \dots, d_n$ , respectively,  $u_1, \dots, u_n$  be the neighbours of  $u$  on the paths  $Q_1, \dots, Q_n$ , respectively, and  $v_1, \dots, v_n$  be the neighbours of  $v$  on the paths  $Q_1, \dots, Q_n$ , respectively. Note that we have  $u_1 = v$  and  $v_1 = u$  if  $d_1 = 1$ . We consider three cases, depending on the values of  $d_1, \dots, d_n$ .

- **Case 1:** Suppose that  $d_1 = 1$  and  $d_i \equiv 1 \pmod{4}$  for every  $i$ ,  $2 \leq i \leq n$ . We construct an equitable nsd 3-edge-weighting of  $G$ . First we set  $\omega(uv) = 3$ . Note that by doing this we ensure that  $u$  is sum-distinguished from  $u_2, \dots, u_n$  and that  $v$  is sum distinguished from  $v_2, \dots, v_n$ . This is true because  $d_G(u) = d_G(v) \geq 3$ , and for every  $i$ ,  $2 \leq i \leq n$ ,  $d_G(u_i) = d_G(v_i) = 2$  and  $d_i \geq 2$ . Hence

$$\sigma_\omega(u) = 3 + \sum_{i=2}^n \omega(uu_i) > \omega(uu_j) + 3 \geq \sigma_\omega(u_j) \text{ for every } j, 2 \leq j \leq n,$$

## 2. Equitable neighbour-sum-distinguishing weightings

---

and

$$\sigma_\omega(v) = 3 + \sum_{i=2}^n \omega(vv_i) > \omega(vv_j) + 3 \geq \sigma_\omega(v_j) \text{ for every } j, 2 \leq j \leq n.$$

Moreover we have  $u_1 = v$  and  $v_1 = u$ . For  $k \in \{0, 1, 2\}$ , let  $I_k := \{Q_i | 2 \leq i \leq n, d_i \equiv k \pmod{3}\}$ . If  $|I_1 \cup I_2| < |I_0|$ , then we weight the edges of each path in  $I_0$  with the pattern 1 – 2 – 3, starting from  $uu_i$ . This will guarantee that  $\sigma_\omega(u) < \sigma_\omega(v)$ , independently from how we weight the edges of the paths in  $I_1 \cup I_2$ . Indeed, every edge incident to  $u$  (resp.  $v$ ) in a path in  $I_0$  is weighted 1 (resp. 3). Hence,

$$\sigma_\omega(u) \leq 3 + |I_0| + 3|I_1 \cup I_2| < 3 + 3|I_0| + |I_1 \cup I_2| \leq \sigma_\omega(v).$$

Moreover, every two adjacent internal vertices of each path  $Q_i \in I_0$  are sum-distinguished because no two edges at distance 2 in  $Q_i$  have the same weight. Note that if  $|I_1 \cup I_2| = 0$ , then we are done. Now suppose that  $|I_1 \cup I_2| > 0$ . We consider the walk  $W$  starting from  $u$  and ending at  $u$  or  $v$  and going once through every path in  $I_1 \cup I_2$ . Then we use the pattern 1 – 2 – 3 to weight periodically the edges of  $W$ . By Observation 2.2.2 we create no conflict between the 2-vertices of  $W$ . This means that the obtained 3-edge-weighting  $\omega$  is neighbour-sum-distinguishing. Moreover, by Observation 2.2.2, the restriction of  $\omega$  to  $W$ ,  $\omega|_W$ , is equitable and  $|E_{\omega|_W}(3)| \leq |E_{\omega|_W}(2)| \leq |E_{\omega|_W}(1)|$ , while for the restriction  $\omega|_{E(I_0) \cup \{uv\}}$  of  $\omega$  to the edges in  $E(I_0) \cup \{uv\}$  we have  $|E_{\omega|_{E(I_0) \cup \{uv\}}}(1)| = |E_{\omega|_{E(I_0) \cup \{uv\}}}(2)| = |E_{\omega|_{E(I_0) \cup \{uv\}}}(3)| - 1$ . Hence  $\omega$  is equitable.

- **Case 2:** Suppose that  $d_1 = 1$  and there exists  $i \in \{2, \dots, n\}$  such that  $d_i \not\equiv 1 \pmod{4}$ . Note that if  $\omega$  is a 2-edge-weighting of  $\Theta_{d_1, d_2, \dots, d_n}$ , then for every  $j \in \{1, \dots, n\}$ ,  $\sigma_\omega(u) \geq \sigma_\omega(u_j) + n - 3$  and  $\sigma_\omega(v) \geq \sigma_\omega(v_j) + n - 3$ . We prove by induction on  $n$  that  $G$  admits an equitable nsd 2-edge-weighting  $\omega$  of  $\Theta_{d_1, d_2, \dots, d_n}$ .

- If  $n = 3$ , then we set  $\omega(uv) = 2$ . By Lemma 2.2.3, this guarantees that  $u$  and  $v$  are distinguished from their respective neighbours on the paths  $Q_2$  and  $Q_3$ . If  $d_2$  or  $d_3$  is even, let  $i \in \{2, 3\}$  such that  $d_i$  is even, and  $j = 5 - i$ . Particularly, this means that  $d_i \not\equiv 1 \pmod{4}$ . Now we set  $\omega(uu_i) = 2$ , and weight the edges of  $Q_i$  in order to have  $\omega(vv_i) = 1$  and  $\omega|_{E(Q_i)}$  is equitable. This is always possible using the weighting depicted in Figure 2.1 since  $d_i \not\equiv 1 \pmod{4}$ . Moreover, since  $d_i$  is even  $\omega|_{uv \cup E(Q_i)}$  is equitable. Then we weight  $Q_j$  with weights 1 and 2 in such a way that  $\omega(uu_j) \geq \omega(vv_j)$  and  $\omega$  is equitable. This is always possible because we can exchange the 1's and 2's on  $Q_j$  in order to guarantee that  $\omega$  is equitable, and we can take the symmetric weighting of  $Q_j$  in order to get  $\omega(uu_j) \geq \omega(vv_j)$ . Finally,  $\sigma_\omega(u) > \sigma_\omega(v)$ . Hence,  $\omega$  is an equitable nsd 2-edge-weighting of  $\Theta_{d_1, d_2, \dots, d_n}$ . Now suppose that  $d_2$  and  $d_3$  are odd, then we

set  $\omega(uu_i) = 2$ , and weight the edges of  $Q_i$  in order to have  $\omega(vv_i) = 1$  and  $\omega|_{E(Q_i)}$  is equitable. If  $\omega|_{uv \cup E(Q_i)}$  is not equitable, then we exchange the 1's and 2's in the weighting of  $Q_i$ . Since  $d_i$  is odd, this will make  $\omega|_{uv \cup E(Q_i)}$  equitable. Now we use the same procedure as above to weight  $Q_j$ , but this time ensuring that  $\omega(uu_j) \geq \omega(vv_j)$  if and only if  $\omega(uu_i) \geq \omega(vv_i)$ . Hence, by the same argument as above  $\omega$  is an equitable nsd 2-edge-weighting of  $\Theta_{d_1, d_2, \dots, d_n}$ .

- Let  $n \geq 3$  be an integer. Suppose that for any positive integers  $d_1 \leq d_2 \leq \dots \leq d_n$ , with  $d_1 = 1$  and  $d_2 \geq 2$  and  $d_i \not\equiv 1 \pmod{4}$  for some  $i \in \{2, \dots, n\}$ , there exists an equitable nsd 2-edge-weighting  $\omega$  of  $\Theta_{d_1, \dots, d_n}$  with  $\omega(uv) = 2$ . Now let  $d_1 \leq d_2 \leq \dots \leq d_{n+1}$  be  $n + 1$  positive integers with  $d_1 = 1$  and  $d_2 \geq 2$  and  $d_i \not\equiv 1 \pmod{4}$  for some  $i \in \{2, \dots, n + 1\}$ , and let  $G = \Theta_{d_1, \dots, d_{n+1}}$ . Now let  $j \in \{2, \dots, n\} \setminus \{i\}$ , and  $G' = G - Q_j$ . By induction hypothesis  $G'$  admits an equitable nsd 2-edge-weighting  $\omega'$  with  $\omega'(uv) = 2$ . By Lemma 2.2.3,  $u$  and  $v$  are sum-distinguished from their respective neighbours in  $P_{d_{n+1}}$ , no matter how we extend  $\omega'$  to  $P_{d_{n+1}}$ . Moreover,  $\omega'$  is an nsd edge-weighting, hence,  $\sigma'_{\omega}(u) \neq \sigma'_{\omega}(v)$ . Now we weight  $Q_j$  with weights 1 and 2 in such a way that  $\omega(uu_j) \geq \omega(vv_j)$  if  $\sigma'_{\omega}(u) > \sigma'_{\omega}(v)$ , and  $\omega(uu_j) \leq \omega(vv_j)$  if  $\sigma'_{\omega}(u) < \sigma'_{\omega}(v)$ , and thus  $\omega$  the extension of  $\omega'$  to  $G$ , is equitable. This is possible because we can exchange the 1's and 2's on  $Q_j$  in order to guarantee that  $\omega$  is equitable, and we can take the symmetric weighting of  $Q_j$  in order to get  $\omega(uu_j) \geq \omega(vv_j)$ . Hence,  $\omega$  is an equitable nsd 2-edge-weighting of  $G$ .

- **Case 3 :** Now suppose that  $d_1 > 1$ . We construct an equitable nsd 2-edge-weighting  $\omega$  of  $\Theta_{d_1, \dots, d_n}$ . Since  $u$  and  $v$  are not adjacent, we do not need to sum-distinguish  $u$  and  $v$ . We separate two sub-cases.

- If there exists  $i \in \{1, \dots, n\}$  such that  $d_i$  is even, then we first weight the edges of  $Q_i$  repeating the pattern  $2 - 1 - 1 - 2$  starting from the edge  $uu_i$ . Then we weight the edges of  $Q_j$ , for some  $j \neq i$ , repeating the pattern  $2 - 1 - 1 - 2$  starting from the edge  $vv_j$ . This guarantees that  $u$  and  $v$  are sum-distinguished from their respective neighbours, no matter how we complete the weighting  $\omega$ . Indeed, for every  $k \in \{1, \dots, n\}$ ,  $k \neq i$ , we have  $\sigma_{\omega}(u) > \sigma_{\omega}(u_k)$  by Lemma 2.2.3 and

$$\sigma_{\omega}(u) \geq \omega(uu_i) + n - 1 > \omega(uu_i) + 1 = \sigma_{\omega}(u_i),$$

and similarly for  $v$ . Moreover,

$$|E_{\omega|_{E(Q_i)}}(1)| = |E_{\omega|_{E(Q_i)}}(2)|$$

and  $\omega|_{E(Q_j)}$  is equitable. Hence  $\omega|_{E(Q_i) \cup E(Q_j)}$  is equitable. Now consider a walk  $W$  starting from  $u$  and ending at  $u$  or  $v$  and going once through every edge of  $\Theta_{d_1, \dots, d_n} - (E(Q_i) \cup E(Q_j))$ . Then we use the pattern  $1 - 2 - 2 - 1$



## 2. Equitable neighbour-sum-distinguishing weightings

---

to weight periodically the edges of  $W$ . The so obtained weighting  $\omega$  is clearly an nsd 2-edge-weighting of  $\Theta_{d_1, \dots, d_n}$ . Moreover,  $\omega|_W$  is equitable by Observation 2.2.1. Hence  $\omega$  is equitable, unless  $\omega|_W$  and  $\omega|_{E(Q_j) \cup E(Q_j)}$  use the same weight, say 1, once more than the weight 2, in this case we can modify the weighting of  $W$ , exchanging the 1's and 2's. The so obtained 2-edge-weighting is still neighbour-sum-distinguishing, and this time is equitable.

- If  $d_i$  is odd for every  $i, 1 \leq i \leq n$ , then there exist  $i$  and  $j$  in  $\{1, \dots, n\}$ , with  $i \neq j$  and  $d_i \equiv d_j \pmod{4}$ . Now we use the pattern  $2 - 1 - 1 - 2$  to weight periodically the edges of  $Q_i$  starting from  $uu_i$ , and we use the same pattern to weight the edges of  $Q_j$  starting from  $vv_j$ . Using the same argument as above we can say that  $u$  and  $v$  are sum-distinguished from their respective neighbours no matter how we complete the 2-edge-weighting. Now we weight the edges of each path  $Q_k$ , with  $k \notin \{i, j\}$ , using the pattern  $1 - 2 - 2 - 1$  periodically. The so obtained 2-edge-weighting is clearly nsd. However, it is not necessarily equitable. In order to have an equitable nsd 2-edge-weighting, we can modify the weighting of the edges of a path  $Q_k$ , with  $k \notin \{i, j\}$ , exchanging the 1's and 2's. This modification keeps the weighting nsd and, since  $d_k$  is odd, it necessarily decrease or increase by 1 the value of  $||E_\omega(1)| - |E_\omega(2)||$ . Since  $||E_\omega(1)| - |E_\omega(2)|| \leq 2$  after the weighting of  $Q_i$  and  $Q_j$ , and  $n \geq 3$ , we can ensure at the end that  $||E_\omega(1)| - |E_\omega(2)|| \leq 1$ .

□

Davoodi and Omooni proved in [DO15] that  $\chi_\Sigma^e(G \square H) \leq \max\{\chi_\Sigma^e(G), \chi_\Sigma^e(H)\}$  for every nice graphs  $G$  and  $H$ . The next proposition is a weaker version of this result for the equitable edge-weightings. Moreover, their proof also works for our result.

**Proposition 2.2.5.** *If  $G$  and  $H$  admit equitable nsd  $k$ -edge-weightings and  $|E(G)| \equiv |E(H)| \equiv 0 \pmod{k}$ , then  $G \square H$  also admits an nsd  $k$ -edge-weighting.*

*Proof.* [DO15]. Let  $\omega_G$  and  $\omega_H$  be two equitable nsd  $k$ -edge-weightings of  $G$  and  $H$ , respectively. For  $(u, v)(u', v') \in E(G \square H)$ , we define  $\omega((u, v)(u', v')) = \omega_G(uu')$  if  $v = v'$  and  $\omega((u, v)(u', v')) = \omega_H(vv')$  if  $u = u'$ . The edge-weighting  $\omega$  is equitable because  $\omega_G$  and  $\omega_H$  are equitable and  $|E(G)| \equiv |E(H)| \equiv 0 \pmod{k}$ . Moreover, if  $(u, v) \in V(G \square H)$ , then  $\sigma_\omega((u, v)) = \sigma_{\omega_G}(u) + \sigma_{\omega_H}(v)$ . Hence, for any neighbour  $(u, v')$  or  $(u', v)$  of  $(u, v)$  we have :

$$\sigma_\omega((u', v)) = \sigma_{\omega_G}(u') + \sigma_{\omega_H}(v) \neq \sigma_{\omega_G}(u) + \sigma_{\omega_H}(v) = \sigma_\omega((u, v))$$

or

$$\sigma_\omega((u, v')) = \sigma_{\omega_G}(u) + \sigma_{\omega_H}(v') \neq \sigma_{\omega_G}(u) + \sigma_{\omega_H}(v) = \sigma_\omega((u, v)).$$

Therefore  $\omega$  is an equitable nsd  $k$ -edge-weighting of  $G \square H$ . □

It is known that for every graph  $G$  with no isolated edge, the neighbour-sum-distinguishing index of  $G$  is 3 if  $G$  is a complete graph, see e.g. [CLWY11], 2 if  $G$  is a complete bipartite graph [LYZ11] and at most 2 if  $G$  is a forest [KNN<sup>+</sup>12]. In the next three subsections we explore the equitable neighbour-sum-distinguishing index of these graphs.

### 2.2.2 Complete graphs

The most popular proof of the fact that  $\chi_{\Sigma}^e(K_n) = 3$  for  $n \geq 3$  is the construction of a neighbour-sum-distinguishing 3-edge-weighting of  $K_n$  in the following way:

1. Choose any triangle of  $K_n$  and weight its edges with weights 1, 2 and 3.
2. Then process the other vertices in an arbitrary order, and at each step use the same weight (alternately 1 or 3) on all the edges joining the current vertex and the previously processed ones.

The resulting edge weighting is neighbour-sum-distinguishing because the initial triangle is neighbour-sum-distinguished, and at each step we add the same weight to all processed vertex, creating no conflict between them, and the new processed vertex get a sum which is either strictly greater or strictly smaller than all previously processed vertices. Hence it is not in conflict with previously processed vertices. By induction we can conclude the proof.

The main observation on this construction is that it uses the weight 2 only once, hence the resulting weighting is somehow very “non-equitable”. In the following theorem we explore the equitable neighbour-sum-distinguishing edge-weightings of complete graphs.

**Theorem 2.2.6.** *For every complete graph  $K_n$  with  $n \geq 3$ ,  $\overline{\chi}_{\Sigma}^e(K_n) = 3$  except for the case of  $n = 4$ , for which we have  $\overline{\chi}_{\Sigma}^e(K_4) = 4$ .*

In order to prove Theorem 2.2.6, we first introduce some definitions and preliminary results.

Let  $\omega$  be an edge-weighting of a graph  $G$ , and let  $\sigma = \sigma_{\omega}$  be the vertex-colouring of  $G$  induced by  $\omega$ , and  $\bar{\sigma}$  be the mean value of  $\sigma$  on  $V(G)$ , that is

$$\bar{\sigma} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \sigma(v).$$

The  $\sigma$ -deviation (or deviation if there is no possible confusion) of a vertex  $v$  is the value  $\mu_{\sigma}(v) = \sigma(v) - \bar{\sigma}$ . If  $G$  is regular graphs, and  $\omega$  is a 3-edge-weighting of  $G$ , the deviation can be computed using the following two lemmas:

**Lemma 2.2.7.** *Let  $G$  be a  $d$ -regular graph,  $\omega$  be a 3-edge-weighting of  $G$  and  $\sigma$  be the vertex-colouring of  $G$  induced by  $\omega$ . We then have  $\sigma(v) = d_{\omega,3}(v) - d_{\omega,1}(v) + 2d$  for every vertex  $v$  in  $G$ .*

## 2. Equitable neighbour-sum-distinguishing weightings

---

*Proof.* We have

$$\begin{aligned}\sigma(v) &= d_{\omega,1}(v) + 2d_{\omega,2}(v) + 3d_{\omega,3}(v) \\ &= d_{\omega,3}(v) - d_{\omega,1}(v) + 2(d_{\omega,1}(v) + d_{\omega,2}(v) + d_{\omega,3}(v)) \\ &= d_{\omega,3}(v) - d_{\omega,1}(v) + 2d.\end{aligned}$$

and the result follows.  $\square$

**Lemma 2.2.8.** *Let  $G$  be a  $d$ -regular graph,  $\omega$  be a 3-edge-weighting of  $G$  such that  $|E_{\omega}(1)| = |E_{\omega}(3)|$ , and  $\sigma$  be the vertex-colouring of  $G$  induced by  $\omega$ .*

*We then have  $\bar{\sigma} = 2d$  and  $\mu_{\sigma}(v) = d_{\omega,3}(v) - d_{\omega,1}(v)$  for every vertex  $v$  in  $G$ .*

*Proof.* Using Lemma 2.2.7, we get

$$\begin{aligned}\sum_{v \in V(G)} \sigma(v) &= \sum_{v \in V(G)} (d_{\omega,3}(v) - d_{\omega,1}(v) + 2d_G(v)) \\ &= 2|E_{\omega}(3)| - 2|E_{\omega}(1)| + 2d|V(G)| \\ &= 2d|V(G)|.\end{aligned}$$

This gives  $\bar{\sigma} = 2d$ . Moreover, since  $\sigma(v) = d_{\omega,3}(v) - d_{\omega,1}(v) + 2d$ , we get  $\mu_{\sigma}(v) = \sigma(v) - \bar{\sigma} = d_{\omega,3}(v) - d_{\omega,1}(v)$ .  $\square$

We say that a 3-edge-weighting  $\omega$  of  $G$  is *good* if the following conditions hold:

1.  $|E_{\omega}(1)| = |E_{\omega}(3)|$ ,
2. for every vertex  $v$  in  $G$ ,

$$-\left\lfloor \frac{|V(G)|}{2} \right\rfloor \leq \mu_{\sigma}(v) \leq \left\lceil \frac{|V(G)|}{2} \right\rceil,$$

3. there exist two vertices  $w_{min}^{\omega}$  and  $w_{max}^{\omega}$  in  $G$  such that

$$\mu_{\sigma}(w_{min}^{\omega}) = -\left\lfloor \frac{|V(G)|}{2} \right\rfloor$$

and

$$\mu_{\sigma}(w_{max}^{\omega}) = \left\lceil \frac{|V(G)|}{2} \right\rceil.$$

where  $\sigma$  denotes the vertex-colouring of  $G$  induced by  $\omega$ .

**Lemma 2.2.9.** *For every integer  $n \geq 3$ , if  $\omega$  is an equitable 3-edge-weighting of the complete graph  $K_n$  such that  $|E_{\omega}(1)| = |E_{\omega}(3)|$ , then either*

$$|E_{\omega}(2)| = |E_{\omega}(1)| = |E_{\omega}(3)|$$

or

$$|E_{\omega}(2)| = |E_{\omega}(1)| + 1 = |E_{\omega}(3)| + 1.$$

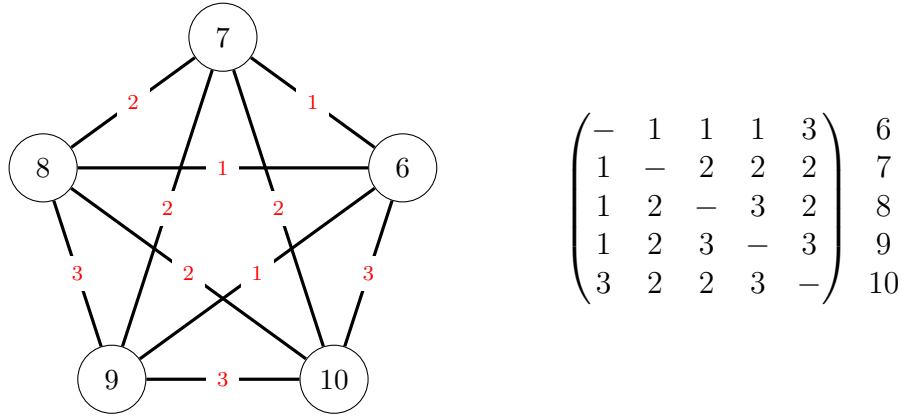


Figure 2.3: An equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_5$  and its matrix representation.

$$\begin{pmatrix} - & 1 & 1 & 1 & 1 & 3 \\ 1 & - & 1 & 2 & 2 & 2 \\ 1 & 1 & - & 2 & 3 & 2 \\ 1 & 2 & 2 & - & 3 & 3 \\ 1 & 2 & 3 & 3 & - & 3 \\ 3 & 2 & 2 & 3 & 3 & - \end{pmatrix} \begin{matrix} 7 \\ 8 \\ 9 \\ 11 \\ 12 \\ 13 \end{matrix}$$

Figure 2.4: Matrix representation of an equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_6$ .

*Proof.* Since  $|E(K_n)| = \frac{n(n-1)}{2}$ , if  $n \equiv 0$  or  $1 \pmod 3$ , then  $|E(K_n)| \equiv 0 \pmod 3$ , which implies  $|E_\omega(2)| = |E_\omega(1)| = |E_\omega(3)|$ . On the other hand, if  $n \equiv 2 \pmod 3$ , then  $|E(K_n)| \equiv 1 \pmod 3$ , which implies  $|E_\omega(2)| = |E_\omega(1)| + 1 = |E_\omega(3)| + 1$ .  $\square$

We are now able to prove Theorem 2.2.6. We first prove that  $\overline{\chi}_\Sigma^e(K_n) \geq 3$  for every  $n \geq 3$ , then we prove a series of claims concerning equitable neighbour-sum-distinguishing edge-weightings of  $K_n$  for small values of  $n$ .

**Claim 2.2.10.** *For every integer  $n \geq 3$ ,  $\overline{\chi}_\Sigma^e(K_n) \geq \chi_\Sigma^e(K_n) \geq 3$ .*

*Proof.* Let  $n \geq 3$  be an integer. Obviously, we have  $\overline{\chi}_\Sigma^e(K_n) \geq \chi_\Sigma^e(K_n)$ . Now assume to the contrary that there exist an equitable neighbour-sum-distinguishing 2-edge-weighting  $\omega$  of  $K_n$  and let  $\sigma$  be the vertex-colouring induced by  $\omega$ . We then have  $n - 1 \leq \sigma(v) \leq 2(n - 1)$  for every vertex  $v$  of  $K_n$ , hence, since we need  $n$  distinct values for  $n$  vertices, there exist two vertices  $u$  and  $v$  with  $\sigma(u) = n - 1$  and  $\sigma(v) = 2(n - 1)$ , in contradiction with the weight of the edge  $uv$ .  $\square$

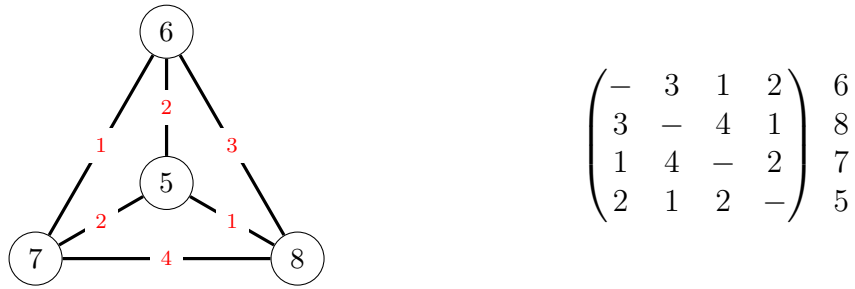


Figure 2.5: An equitable neighbour-sum-distinguishing 4-edge-weighting of  $K_4$  and its matrix representation.

**Claim 2.2.11.** *For every integer  $n \in \{3, 5, 6\}$ , there exists a good equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_n$ .*

*Proof.* A good equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_3$  is obtained by weighting the edges of  $K_3$  with weights 1, 2 and 3, respectively.

A good equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_5$  is depicted in Figure 2.3, together with its matrix representation. The value in row  $i$  and column  $j$  is the weight of the edge  $ij$ . The sum at vertex  $i$  is given at the end of row  $i$ .

The matrix representation of a good equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_6$  is given in Figure 2.4. □

**Claim 2.2.12.**  $\overline{\chi}_\Sigma^e(K_4) = 4$ .

*Proof.* Assume first that  $\omega$  is an equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_4$  and let  $\sigma$  be the vertex-colouring induced by  $\omega$ . Since  $|E(K_4)| = 6$ , we necessarily have  $|E_\omega(1)| = |E_\omega(2)| = |E_\omega(3)| = 2$  which implies  $d_{\omega,i}(v) \leq 2$  for every  $v \in V(K_4)$  and every  $i$ ,  $1 \leq i \leq 3$ . Hence,  $4 \leq \sigma(v) \leq 8$  for every  $v \in V(K_4)$ . On the other hand, we have  $\sum_{v \in V(K_4)} \sigma(v) = 2 \sum_{e \in E(K_4)} \omega(e) = 24$ .

Let  $V(K_4) = \{v_1, v_2, v_3, v_4\}$ . The only way to partition 24 into 4 distinct integers belonging to  $[3, 9]$  is  $24 = 4 + 5 + 7 + 8$ . Hence, we can suppose, without loss of generality, that  $\sigma(v_1) = 4$ ,  $\sigma(v_2) = 5$ ,  $\sigma(v_3) = 7$  and  $\sigma(v_4) = 8$ . The edges incident with  $v_1$  are thus weighted 1, 1 and 2, while the edges incident with  $v_4$  are weighted 3, 3 and 2. This implies  $\omega(v_1v_4) = 2$  and thus  $\omega(v_1v_2) = 1$ ,  $\omega(v_1v_3) = 1$ ,  $\omega(v_2v_4) = 3$  and  $\omega(v_3v_4) = 3$ , hence  $\omega(v_2v_3) = 2$ . Thus, we finally get  $\sigma(v_2) = \sigma(v_3) = 6$ , a contradiction.

Hence,  $\overline{\chi}_\Sigma^e(K_4) > 3$ . An equitable neighbour-sum-distinguishing 4-edge-weighting of  $K_4$  is given in Figure 2.5, together with its matrix representation. □

We will now prove that  $\overline{\chi}_\Sigma^e(K_n) \leq 3$  for every  $n \geq 7$ , by induction on  $n$ . More precisely, we will show that for every  $n \geq 5$  any good equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_n$  can be extended to a good equitable neighbour-sum-distinguishing

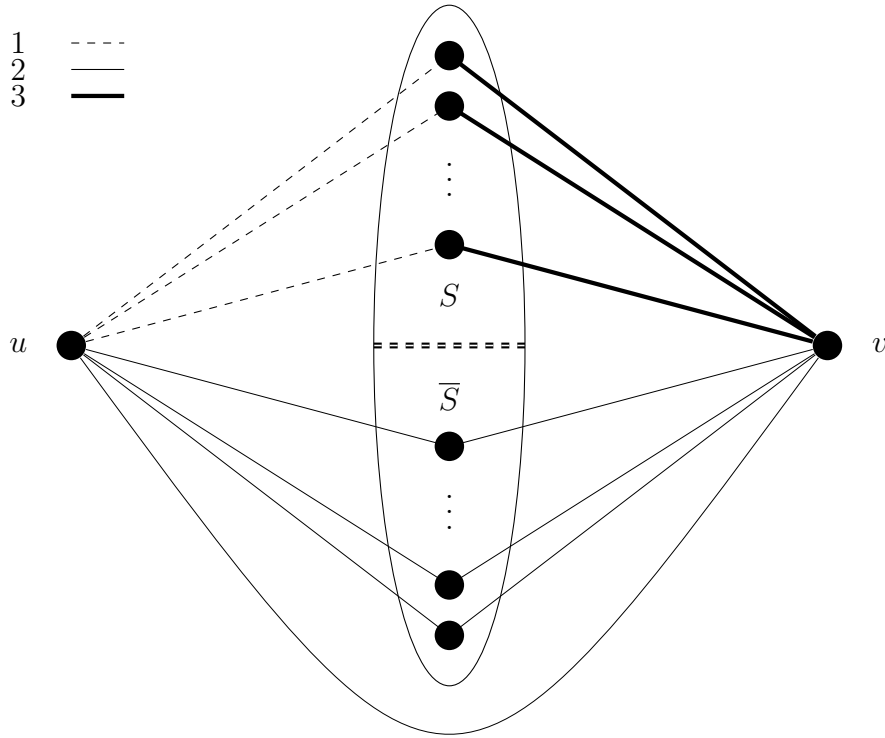


Figure 2.6: The 3-edge-weighting  $\omega_0$  of  $K_{n+2}$ .

3-edge-weighting of  $K_{n+2}$ . Together with Claims 2.2.10 to 2.2.12, this will complete the proof.

Let  $\omega$  be a good equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_n$ , with  $n \geq 5$ . Suppose that  $K_{n+2}$  is obtained from  $K_n$  by adding two new vertices  $u$  and  $v$ . Let  $S \subseteq V(K_n)$  be any fixed set of  $\lfloor \frac{n}{2} \rfloor + 1 = \lfloor \frac{n+2}{2} \rfloor$  vertices and  $\bar{S} = V(K_n) \setminus S$  (we thus have  $V(K_{n+2}) = S \cup \bar{S} \cup \{u, v\}$ ). We first define a 3-edge-weighting  $\omega_0$  of  $K_{n+2}$  as follows:

1.  $\omega_0(xy) = \omega(xy)$  for every edge  $xy$  with  $x, y \in S \cup \bar{S}$ ,
2.  $\omega_0(uv) = 2$ ,
3. for every vertex  $x \in S$ ,  $\omega_0(ux) = 1$  and  $\omega_0(vx) = 3$ ,
4. for every vertex  $y \in \bar{S}$ ,  $\omega_0(uy) = \omega_0(vy) = 2$ .

The 3-edge-weighting  $\omega_0$  is depicted on Figure 2.6 (dashed, thin and thick edges represent edges with weights 1, 2 and 3, respectively). This weighting is not equitable but is indeed good and neighbour-sum-distinguishing:

## 2. Equitable neighbour-sum-distinguishing weightings

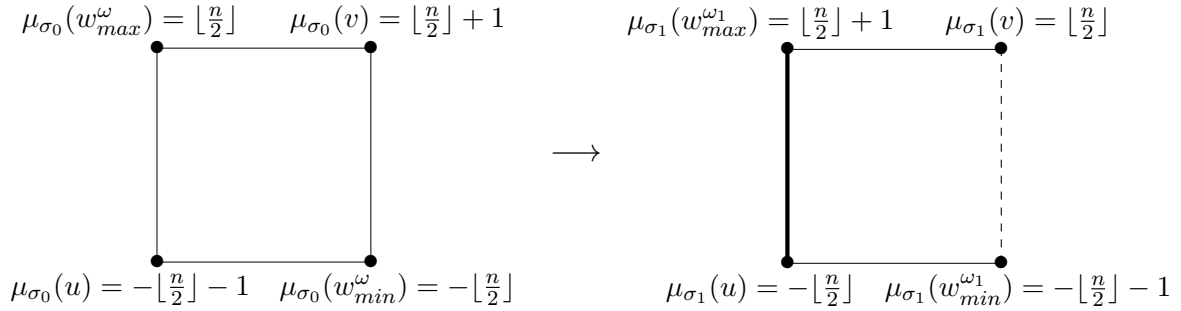


Figure 2.7: Re-weighting of type 1.

**Claim 2.2.13.** *The 3-edge-weighting  $\omega_0$  is a good neighbour-sum-distinguishing 3-edge-weighting of  $K_{n+2}$ .*

*Proof.* Recall that  $\omega$  is a good equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_n$ . We denote by  $\sigma$  and  $\sigma_0$  the vertex-colourings induced by  $\omega$  and  $\omega_0$ , respectively.

For each vertex  $x \in S \cup \bar{S}$ , the two edges  $ux$  and  $vx$  are either assigned weights 1 and 3 or both assigned weight 2 by  $\omega_0$ . Therefore, for every vertex  $x \in S \cup \bar{S}$ ,  $\mu_{\sigma_0}(x) = \mu_{\sigma}(x)$ , which implies  $-\lfloor \frac{n}{2} \rfloor \leq \mu_{\sigma_0}(x) \leq \lfloor \frac{n}{2} \rfloor$ . On the other hand,  $\mu_{\sigma_0}(u) = -(\lfloor \frac{n}{2} \rfloor + 1) = -\lfloor \frac{n+2}{2} \rfloor$  and  $\mu_{\sigma_0}(v) = \lfloor \frac{n}{2} \rfloor + 1 = \lfloor \frac{n+2}{2} \rfloor$ . Hence, all vertices of  $K_{n+2}$  are assigned distinct values by  $\mu_{\sigma_0}$  which means by Lemmas 2.2.7 and 2.2.8 that  $\omega_0$  is a neighbour-sum-distinguishing 3-edge-weighting.

It remains to show that the edge-weighting  $\omega_0$  is good. Since  $|E_{\omega_0}(1)| = |E_{\omega}(1)| + \lfloor \frac{n+2}{2} \rfloor$  and  $|E_{\omega_0}(3)| = |E_{\omega}(3)| + \lfloor \frac{n+2}{2} \rfloor$ , we get  $|E_{\omega_0}(1)| = |E_{\omega_0}(3)|$ . Finally, we already observed that  $\mu_{\sigma_0}(u) = -\lfloor \frac{n+2}{2} \rfloor$  and  $\mu_{\sigma_0}(v) = \lfloor \frac{n+2}{2} \rfloor$ , and that  $-\lfloor \frac{n}{2} \rfloor \leq \mu_{\sigma_0}(x) \leq \lfloor \frac{n}{2} \rfloor$  for every vertex  $x \in S \cup \bar{S}$ . This completes the proof.  $\square$

While constructing  $\omega_0$  from  $\omega$ , we added  $\lfloor \frac{n+2}{2} \rfloor$  edges with weight 1,  $\lfloor \frac{n+2}{2} \rfloor$  edges with weight 3 and  $2n + 1 - 2\lfloor \frac{n+2}{2} \rfloor$  edges with weight 2. The edge weighting  $\omega_0$  is thus (almost always) not equitable. We will then modify the edge weighting  $\omega_0$  in order to obtain an equitable edge weighting  $\omega_1$ . In order to do that, we need to re-weight with weights 1 or 3 some edges which are weighted with the weight 2, say  $p = 2q$  such edges, leading to an edge weighting  $\omega_1$  (with induced vertex-colouring  $\sigma_1$ ) such that  $|E_{\omega_1}(1)| = |E_{\omega_0}(1)| + q$ ,  $|E_{\omega_1}(3)| = |E_{\omega_0}(3)| + q$  and  $|E_{\omega_1}(2)| = |E_{\omega_0}(2)| - 2q$  with either  $|E_{\omega_1}(2)| = |E_{\omega_1}(1)|$  or  $|E_{\omega_1}(2)| = |E_{\omega_1}(1)| + 1$  by Lemma 2.2.9.

The edge-weighting  $\omega_1$  will be produced using two types of re-weightings, both involving edges incident with  $u$  or  $v$ , described as follows:

- *Re-weighting of type 1:* Let  $w_{min}^{\omega}$  and  $w_{max}^{\omega}$  denote the (unique) two vertices such that  $\mu_{\sigma_0}(w_{min}^{\omega}) = -\lfloor \frac{n}{2} \rfloor$  ( $= \mu_{\sigma}(w_{min}^{\omega})$ ) and  $\mu_{\sigma_0}(w_{max}^{\omega}) = \lfloor \frac{n}{2} \rfloor$  ( $= \mu_{\sigma}(w_{max}^{\omega})$ ). If  $uw_{min}^{\omega}vw_{max}^{\omega}$  is a 2-monochromatic 4-cycle, then re-weight with 1 the edge  $vw_{min}^{\omega}$  and re-weight with 3 the edge  $uw_{max}^{\omega}$  (see Figure 2.7). Note that the deviations

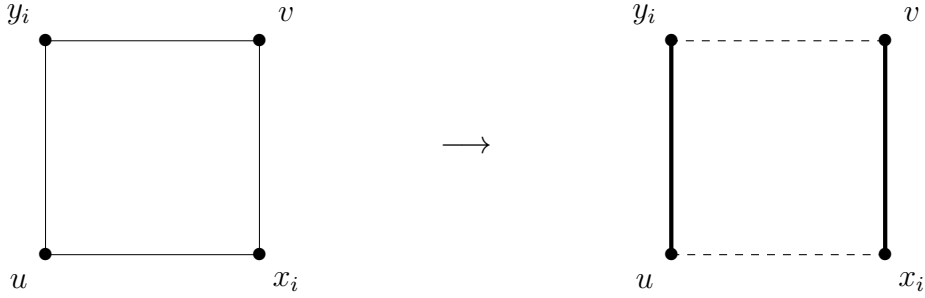


Figure 2.8: Re-weighting of type 2.

of  $u$  and  $w_{min}^\omega$ , and of  $v$  and  $w_{max}^\omega$  have been switched, so that  $w_{min}^{\omega_1} = w_{min}^\omega$  and  $w_{max}^{\omega_1} = w_{max}^\omega$ .

- *Re-weighting of type 2:* If the set of pairs of vertices  $\{(x_i, y_i)\}_{1 \leq i \leq k}$ ,  $k \geq 1$ , is such that  $ux_ivy_i$  is a 2-monochromatic 4-cycle for every  $i$ ,  $1 \leq i \leq k$ , then re-weight with 1 all edges  $ux_i$  and  $vy_i$  and re-weight with 3 all edges  $uy_i$  and  $vx_i$  (see Figure 2.8). Note that the deviation of any of these  $2k + 2$  vertices remains unchanged.

Recall that we need to re-weight  $p = 2q$  edges which are weighted with 2,  $q$  of them with weight 1 and the  $q$  others with weight 3. If  $q = 1$ , since  $n \geq 5$ , we can ensure that the chosen set  $S$  contains none of the vertices  $w_{min}^\omega$  and  $w_{max}^\omega$ . By doing so and then applying the re-weighting of type 1, we obtain edge-weighting  $\omega_1$  such that:

1.  $|E_{\omega_1}(1)| = |E_{\omega_1}(3)|$  and either  $|E_{\omega_1}(2)| = |E_{\omega_1}(1)|$  or  $|E_{\omega_1}(2)| = |E_{\omega_1}(1)| + 1$ ,
2. for every vertex  $x \in V(K_{n+2})$ ,  $-\lfloor \frac{n+2}{2} \rfloor \leq \sigma_1(x) \leq \lfloor \frac{n+2}{2} \rfloor$ ,
3.  $w_{min}^{\omega_1} = w_{min}^\omega$  and  $w_{max}^{\omega_1} = w_{max}^\omega$ .

Hence,  $\omega_1$  is an equitable good neighbour-sum-distinguishing 3-edge-weighting of  $K_{n+2}$  and we are done.

Assume from now on that  $q \geq 2$ . Since  $\omega$  is an equitable good neighbour-sum-distinguishing 3-edge-weighting of  $K_n$ , we know by Lemma 2.2.9 that

$$|E_\omega(1)| = |E_\omega(3)| = r, \text{ and } r \leq |E_\omega(2)| \leq r + 1,$$

with  $r = \lfloor \frac{n(n-1)}{6} \rfloor$ . As observed before, considering the way the edge-weighting  $\omega_0$  has been constructed (see Figure 2.6), we also have

$$|E_{\omega_0}(1)| = |E_{\omega_0}(3)| = r + \left\lfloor \frac{n+2}{2} \right\rfloor,$$

and

$$r + 2n + 1 - 2 \left\lfloor \frac{n+2}{2} \right\rfloor \leq |E_{\omega_0}(2)| \leq r + 2n + 1 - 2 \left\lfloor \frac{n+2}{2} \right\rfloor + 1.$$



## 2. Equitable neighbour-sum-distinguishing weightings

---

Again by Lemma 2.2.9, in order to be an equitable good neighbour-sum-distinguishing 3-edge-weighting of  $K_{n+2}$ , the edge-weighting  $\omega_1$  must be such that

$$|E_{\omega_1}(1)| = |E_{\omega_1}(3)| = r + \left\lfloor \frac{n+2}{2} \right\rfloor + q,$$

and

$$r + \left\lfloor \frac{n+2}{2} \right\rfloor + q \leq |E_{\omega_1}(2)| \leq r + \left\lfloor \frac{n+2}{2} \right\rfloor + q + 1. \quad (2.4)$$

On the other hand, since  $\omega_1$  has been obtained by re-weighting  $2q$  edges which were weighted with weight 2 by  $\omega_0$ , we also have

$$r + 2n + 1 - 2 \left\lfloor \frac{n+2}{2} \right\rfloor - 2q \leq |E_{\omega_1}(2)| \leq r + 2n + 1 - 2 \left\lfloor \frac{n+2}{2} \right\rfloor - 2q + 1. \quad (2.5)$$

Combining (2.4) and (2.5), we get

$$r + \left\lfloor \frac{n+2}{2} \right\rfloor + q \leq r + 2n + 1 - 2 \left\lfloor \frac{n+2}{2} \right\rfloor - 2q + 1,$$

which gives

$$3q \leq 2n + 1 - 3 \left\lfloor \frac{n+2}{2} \right\rfloor + 1.$$

Since  $q \geq 2$ , we thus necessarily have  $n = 11$  or  $n \geq 13$ . Moreover, we also get

$$|\bar{S}| = n - \left\lfloor \frac{n+2}{2} \right\rfloor \geq 3q - n + 2 \left\lfloor \frac{n+2}{2} \right\rfloor - 2 \geq 3q - 1 \geq q + 3 \geq 5. \quad (2.6)$$

The edge-weighting  $\omega_1$  is then obtained as follows, depending on the parity of  $q$ .

1.  $q = 2t, t \geq 1$ .

We choose any set of  $t$  pairs of vertices  $X = \{(x_i, y_i)\}_{1 \leq i \leq t}$  in  $\bar{S}$ . Since  $ux_ivy_i$  is a 2-monochromatic 4-cycle for every  $i, 1 \leq i \leq t$ , we can apply the re-weighting of type 2 to the set  $X$ , so that  $|E_{\omega_1}(1)| = |E_{\omega_1}(3)| = |E_{\omega_0}(1)| + q$  and  $|E_{\omega_1}(2)| = |E_{\omega_0}(2)| - 2q$ . The so-obtained edge-weighting  $\omega_1$  is thus an equitable good neighbour-sum-distinguishing 3-edge-weighting of  $K_{n+2}$ .

2.  $q = 2t + 1, t \geq 1$ .

We first choose any set of  $t$  pairs of vertices  $X = \{(x_i, y_i)\}_{1 \leq i \leq t-1}$  in  $\bar{S} \setminus \{w_{min}^\omega, w_{max}^\omega\}$  (this is possible since, by (2.6),  $|\bar{S}| \geq q + 3$ ). Since  $ux_ivy_i$  is a 2-monochromatic 4-cycle for every  $i, 1 \leq i \leq t$ , we can apply the re-weighting of type 2 to the set  $X$ , so that we have  $|E_{\omega_0}(1)| + q - 1$  vertices weighted with 1 (resp. with 3). We then apply the re-weighting of type 1, since  $uw_{min}^\omega vw_{max}^\omega$  is still a 2-monochromatic 4-cycle, so that  $|E_{\omega_1}(1)| = |E_{\omega_1}(3)| = |E_{\omega_0}(1)| + q$  and  $|E_{\omega_1}(2)| = |E_{\omega_0}(2)| - 2q$ . Therefore, the so-obtained edge-weighting  $\omega_1$  is an equitable good neighbour-sum-distinguishing 3-edge-weighting of  $K_{n+2}$ .

This concludes the proof of Theorem 2.2.6.

---

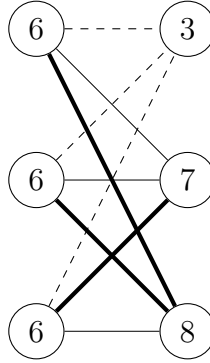


Figure 2.9: An equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_{3,3}$ .

### 2.2.3 Complete bipartite graphs

Let  $K_{m,n}$ , with  $n, m \geq 2$ , be a complete bipartite graph with parts  $A$  and  $B$  of order  $m$  and  $n$ , respectively. If  $m \neq n$  then  $K_{m,n}$  is locally irregular and we have  $\overline{\chi}_{\Sigma}^e(K_{m,n}) = \chi_{\Sigma}^e(K_{m,n}) = 1$ . Otherwise, we can easily obtain an nsd 2-edge-weighting of  $G$  by choosing one vertex  $v$  in  $A$ , and weighting all the edges incident with  $v$  with the weight 2. Then weighting all the other edge with the weight 1. The so obtained 2-edge-weighting  $\omega$  is neighbour-sum-distinguishing because  $\sigma_{\omega}(v) = 2n$  and for every  $u \in A \setminus \{v\}$ ,  $\sigma_{\omega}(u) = n$ , while for every  $w \in B$ ,  $\sigma_{\omega}(w) = n + 1$ . However,  $\omega$  is far from being equitable. In the next theorem we explore equitable nsd 2-edge-weighting for complete bipartite graphs.

**Theorem 2.2.14.**  $\overline{\chi}_{\Sigma}^e(K_{m,n}) = 2$  whenever  $m = n = 2$  or  $m = n \geq 4$ ,  $\overline{\chi}_{\Sigma}^e(K_{3,3}) = 3$  and  $\chi_{\Sigma}^e(K_{m,n}) = 1$  if  $1 \leq m < n$ .

*Proof.* If  $1 \leq m < n$  then adjacent vertices have distinct degrees, and hence weighting all edges with weight 1 gives an equitable nsd 1-edge-weighting of  $K_{m,n}$ .

If  $m = n$  then  $K_{m,n}$  is regular, and therefore,  $\overline{\chi}_{\Sigma}^e(K_{m,n}) \geq 2$ . Suppose first that  $m = n = 3$  and let  $V \cup V'$  denotes the bipartition of  $V(K_{3,3})$ , with  $V = \{v_1, v_2, v_3\}$  and  $V' = \{v'_1, v'_2, v'_3\}$ . We first claim that  $\overline{\chi}_{\Sigma}^e(K_{3,3}) > 2$ . Assume to the contrary that  $\omega$  is an equitable neighbour-sum-distinguishing 2-edge-weighting of  $K_{3,3}$  and let  $\sigma$  denote the vertex-colouring induced by  $\omega$ . Since  $|E(K_{3,3})| = 9$ , we necessarily have

$$\{|E_{\omega}(1)|, |E_{\omega}(2)|\} = \{4, 5\}. \quad (2.7)$$

Moreover, since  $3 \leq \sigma(v) \leq 6$  for every vertex  $v \in V \cup V'$ , we get without loss of generality either  $\sigma(v_1) = \sigma(v_2) = \sigma(v_3)$ , or  $\sigma(v_1) = \sigma(v_2) \neq \sigma(v'_1) = \sigma(v'_2)$ . In the first case, we get  $|E_{\omega}(1)| + 2|E_{\omega}(2)| = 3\sigma(v_1) \equiv 0 \pmod{3}$ , in contradiction with (2.7). In the latter case, we necessarily have  $\{\sigma(v_1), \sigma(v'_1)\} = \{4, 5\}$ , since otherwise we would have six edges with the same weight. Assume without loss of generality that the edges incident with  $v_1$  and  $v_2$  are weighted 1, 1 and 2. Since  $\omega$  is an equitable edge-weighting, the edges incident with  $v_3$  are necessarily weighted 2, 2, and 2, but then  $\sigma(v'_3) = 4 = \sigma(v_1)$ , or 2, 2

## 2. Equitable neighbour-sum-distinguishing weightings

---

and 1, again a contradiction since this would in turn imply  $\sigma(v_3) = \sigma(v'_1) = 5$ . Taking into account the equitable neighbour-sum-distinguishing 3-edge-weighting of  $K_{3,3}$  depicted in Figure 2.9, we get  $\overline{\chi}_\Sigma^e(K_{3,3}) = 3$ .

Finally, suppose that  $m = n = 2$  or  $m = n \geq 4$  and let  $V \cup V'$  denote the bipartition of  $V(K_{n,n})$ , with  $V = \{v_1, \dots, v_n\}$  and  $V' = \{v'_1, \dots, v'_n\}$ . We consider two cases, depending on the parity of  $n$ .

1.  $n = 2t$ ,  $t \geq 1$ .

Let  $\omega$  be the 2-edge-weighting of  $K_{n,n}$  defined as follows. For every edge  $v_i v'_j \in E(K_{n,n})$ , let  $\sigma(v_i v'_j) = 1$  if  $i$  is odd and  $\sigma(v_i v'_j) = 2$  otherwise. Since  $n$  is even,  $\omega$  is an equitable 2-edge-weighting. To see that  $\omega$  is neighbour-sum-distinguishing, observe that for every  $i$ ,  $1 \leq i \leq n$ ,  $\sigma(v_i) = 2t$  if  $i$  is odd,  $\sigma(v_i) = 4t$  if  $i$  is even, while  $\sigma(v'_j) = 3t$  for every  $j$ ,  $1 \leq j \leq n$ .

2.  $n = 2t + 1$ ,  $t \geq 2$ .

Let  $\omega$  be the 2-edge-weighting of  $K_{n,n}$  defined as follows. For the subgraph of  $K_{n,n}$  induced by  $\{v_1, \dots, v_{n-1}\} \cup \{v'_1, \dots, v'_{n-1}\}$ ,  $\omega$  is defined as in the previous case. We then set  $\omega(v_n v'_j) = 1$  for every  $j$ ,  $1 \leq j \leq n-1$ ,  $\omega(v_i v'_n) = 2$  for every  $i$ ,  $1 \leq i \leq n-1$ , and  $\sigma(v_n v'_n) = 1$ . The edge-weighting  $\omega$  thus obtained is clearly an equitable 2-edge-weighting. To see that  $\omega$  is neighbour-sum-distinguishing, observe that for every  $i$ ,  $1 \leq i \leq n-1$ ,  $\sigma(v_i) = 2t+2$  if  $i$  is odd,  $\sigma(v_i) = 4t+2$  if  $i$  is even, while  $\sigma(v'_j) = 3t+1$  for every  $j$ ,  $1 \leq j \leq n-1$ ,  $\sigma(v_n) = 2t+1$  and  $\sigma(v'_n) = 4t+1$ .

This concludes the proof of Theorem 2.2.14. □

### 2.2.4 Forests

By Proposition 1.4.2 we can easily prove that for every nice forest  $F$ ,  $\chi_\Sigma^e(F) \leq 2$ . In this subsection we prove that we do not need more weights to get an equitable nsd edge-weighting for nice forests.

**Theorem 2.2.15.** *For every forest  $F$  with no isolated edge,  $\overline{\chi}_\Sigma^e(F) \leq 2$ .*

*Proof.* Suppose a forest  $F$  is a *minimal counterexample*, i.e. a counterexample with minimal number of edges, to Theorem 2.2.15.

**Claim 2.2.16.** *No component of  $F$  is a path.*

*Proof.* Suppose  $P$  is a component of  $F$  which is a path. Then we weight the forest  $F'$  obtained of  $F$  by removing all vertices of  $P$  by the *minimality of  $F$* , what will mean here and in all further claims that we fix some equitable neighbour-sum-distinguishing 2-edge-weighting of  $F'$ , which exists due to the fact that  $F$  is a minimal counterexample to Theorem 2.2.15 (in cases where we will be left with components  $K_2$  in  $F'$ , what does not take place in this claim, we will mean that we weight the forest formed by the remaining

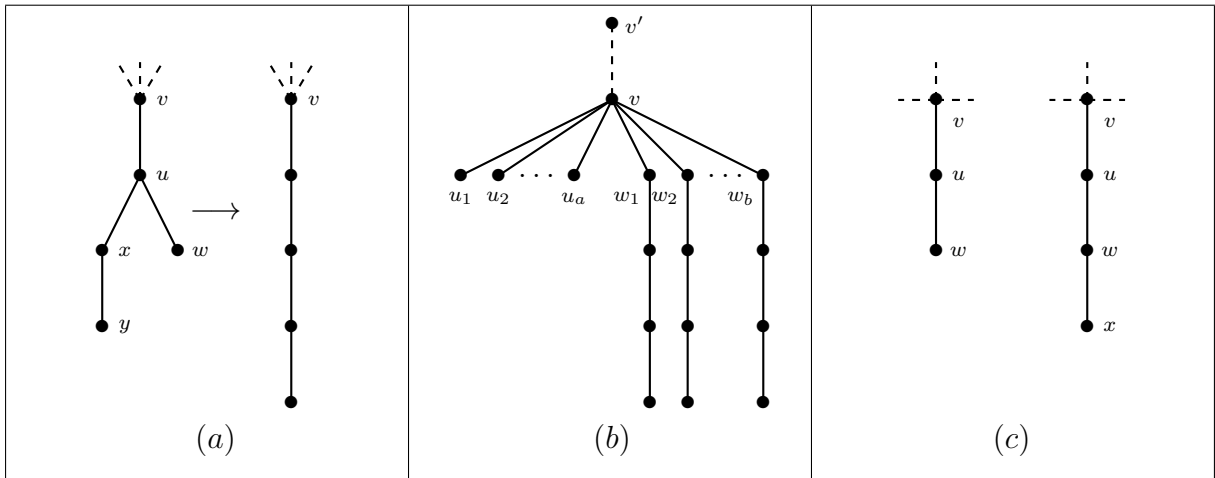


Figure 2.10: Illustrations to Claims 2.2.17, 2.2.18 and 2.2.20.

components of  $F'$  by the minimality of  $F$  and then we put 1's or 2's on the isolated edges of  $F'$  so that the edge-weighting is equitable).

Now it is sufficient to weight the path  $P$  equitably so that its neighbours are sum-distinguished and the edge-weighting of entire  $F$  is equitable in order to obtain an equitable neighbour-sum-distinguishing edge-weighting of  $F$ , a contradiction with the fact that  $F$  is a counterexample to Theorem 2.2.15. In case when  $P$  is of even length it suffices to use the same number of 1's and 2's, while for odd path  $P$  we might be forced to use one more 1 or 2 (and we do not control which one). As leaves are always sum-distinguished from their neighbours, this can however be always easily achieved, as we only need to weight every second edge of the path differently, *i.e.* it is always sufficient to weight appropriately the first two edges of the path – the rest of the weights on the path are the consequence of these two (note also here a fact useful in further reasoning that if in a graph  $G$  we have a pendent path of length 4, then its edges must be weighted with two 1's and two 2's in any neighbour-sum-distinguishing 2-edge-weighting of  $G$ ).  $\square$

Since we want to prove that no counterexample to Theorem 2.2.15 exists, *i.e.*, that in fact  $\overline{\chi}_{\Sigma}^e(F) \leq 2$ , we may make use of the following reduction.

**Claim 2.2.17.** *We may assume that  $F$  contains no vertex  $u$  of degree 3 adjacent to a leaf  $w$  and a vertex  $x$  of degree 2 whose other neighbour  $y$  is a leaf.*

*Proof.* Suppose there is such a vertex  $u$ , and let  $v$  be its remaining neighbour in  $F$  ( $v \notin \{w, x\}$ ), see Figure 2.10(a).

Then  $\overline{\chi}_{\Sigma}^e(F) \leq 2$  only if  $\overline{\chi}_{\Sigma}^e(F') \leq 2$  where  $F'$  is the forest obtained from  $F$  by deleting the vertices  $u, w, x, y$  (together with their four incident edges) and appending a pendent path of length 4 at  $v$ , *i.e.* identifying one end of this path with  $v$  (so that the numbers of edges in  $F$  and  $F'$  are equal). Suppose that there is an equitable nsd 2-edge-weighting of  $F'$  (obviously,  $\overline{\chi}_{\Sigma}^e(F') \neq 1$ ). Then use the same weighting on all edges of  $F$  that appear

## 2. Equitable neighbour-sum-distinguishing weightings

---

also in  $F'$ , so that four edges of  $F$  remain un-weighted. Note that in order to be certain that the edge-weighting of  $F$  is equitable we must use weights 1, 1, 2, 2 on these four remaining edges, as exactly these four weights must have been used on the pendent path of length 4 in  $F'$  (cf. the argument for paths above). First we copy on  $uv$  the weight of the edge incident with  $v$  in  $F'$  from the mentioned path of length 4 (in order to avoid sum conflicts between  $v$  and its neighbours other than  $u$ ). Then we put a weight on  $uw$  so that  $uv$  and  $uw$  have distinct weights, hence we are left with 1 and 2 to use. We choose one of these weights for  $ux$  so that there is no conflict between  $u$  and  $v$ , and we use the remaining weight on  $xy$ . Note that by our construction the sum at  $x$  will always be smaller than the sum at  $u$ .

Hence, as  $F$  is a minimal (i.e., with minimum number of edges) counterexample to Theorem 2.2.15, then so does  $F'$ . We may thus perform the operation described above repeatedly until there are no configurations from the thesis in our forest left.  $\square$

Let us root every tree (component) of  $F$  at any leaf. A vertex  $v$  of degree at least 3 with all descendants of degree at most 2 will be called a *last multifather* (this is just a vertex which induces with its descendants only pendent paths incident with this vertex). First we present a few observations implying that all descendants of any last multifather must in fact be leaves (in other words, all pendent paths incident with such a vertex and containing its descendants are in fact pendent edges), see Claim 2.2.22 below. Some of these observations will be also useful in the further part of the argument, e.g. the following seemingly very specific claim.

**Claim 2.2.18.** *There is no vertex  $v$  of degree at most  $a + b + 1$  incident with  $a \geq 0$  pendent edges and  $b \geq 1$  pendent paths of length 4 in  $F$ .*

*Proof.* Suppose to the contrary that  $u_1, \dots, u_a$  are  $a \geq 0$  leaves adjacent with  $v$ , while  $w_1, \dots, w_b$  are  $b \geq 1$  neighbours of  $v$  such that  $vw_i$  is the first of four edges of a pendent path of length 4 incident with  $v$ ,  $i = 1, \dots, b$ , see Figure 2.10(b). Denote by  $v'$  the remaining neighbour of  $v$  (if there is any).

Let  $F'$  be the forest obtained of  $F$  by deleting  $u_1, \dots, u_a$  and all  $4b$  vertices (except  $v$ ) from the  $b$  pendent paths of length 4 incident with  $v$  (including  $w_1, \dots, w_b$ , resp.). By the minimality of  $F$ ,  $F'$  admits an equitable 2-edge-weighting. It suffices then to complete the weighting using an appropriate equitable number of 1's and 2's.

If  $a = 0$ , we may use the same number of 1's and 2's. Otherwise, we first greedily choose weights for  $vu_1, \dots, vu_a$  so that we obtain a partial equitable edge-weighting of  $F$  (i.e., the number of 1's and 2's used so far on  $F$  is as equal as possible). We will then use  $2b$  1's and  $2b$  2's on the remaining edges. For this goal we first choose any weights for  $vw_1, \dots, vw_b$  so that there is no conflict between  $v$  and  $v'$ . Finally, for each of the pendent paths of length 4 incident with  $v$ , we complete its weighting (similarly as in the case of a path itself above) first using a weight on yet non-weighted edge incident with  $w_i$  to avoid conflict between  $w_i$  and  $v$ , and so on, in order to obtain a nsd 2-edge-weighting of  $F$ . Note that on each such path we will use two 1's and two 2's, thus the edge-weighting will also be equitable, a contradiction with the minimality of  $F$ .  $\square$

As a consequence of the claim above, for  $a = 0$  and  $b = 1$ , we obtain the following:

**Claim 2.2.19.** *There are no pendent paths of length (at least) 5 in  $F$ .*

We supplement this observation with the two following ones.

**Claim 2.2.20.** *There is no vertex  $v$  with  $d(v) \geq 4$  incident with a pendent path of length 2 or 3 in  $F$ .*

*Proof.* Suppose there is such a vertex  $v$  in  $F$ , and let  $v, u, w$  or  $v, u, w, x$  be the consecutive vertices of the corresponding path, see Figure 2.10(c). By the minimality of  $F$ , we may weight  $F - \{uw\}$  or  $F - \{uw, wx\}$ , respectively. In the first case, we conclude by using on  $uw$  any of the available at most 2 weights – note that as  $d(v) \geq 4$  and  $d(u) = 2$ , the sum at  $v$  will always be greater than the one at  $u$ . In the second case, we use 1 or 2 on  $wx$  so that there is no conflict between  $u$  and  $w$ , and put a different weight on  $uw$ , a contradiction.  $\square$

**Claim 2.2.21.** *A vertex  $v$  of degree 3 cannot be incident with two pendent paths of length at least 1 one of which has length at least 2 in  $F$ .*

*Proof.* Suppose there is such a vertex  $v$  in  $F$ . Note that by Claim 2.2.19, both paths have to be of length at most 4, and one of them has to have length at most 3 by Claim 2.2.18 (with  $a = 0$  and  $b = 2$ ). Additionally, if one of the paths is just a pendent edge, then the other cannot have length 2, by Claim 2.2.17, nor 4, by Claim 2.2.18.

We thus are left with 6 cases. In each of these cases, we first weight by the minimality of  $F$  the forest  $F'$  obtained from  $F$  by removing all edges of the two pendent paths. If there is an even number of such edges we then use an even number of 1's and 2's to complete the weighting. Otherwise we might be forced to use one more 1 or 2 so that the weighting of  $F$  is equitable at the end. Thus, for each such case, we analyse the two corresponding sub-cases. For each of these cases (and sub-cases) we start by choosing the weights for the first edges of the two paths (those incident with  $v$ ) appropriately so that  $v$  is not in conflict with its neighbour  $v'$  from  $F'$ . In Figure 2.11 we show that in all (sub)cases there are always two possible choices (with different sums, one of which must be appropriate) for these two edges, which then can be extended without conflicts, regardless of the weight of  $vv'$  (marked thus by “?”), on the remaining yet un-weighted edges (one choice is presented above the edges, while the alternative is presented below). We thus obtain a contradiction with the minimality of  $F$  as a counterexample to Theorem 2.2.15.  $\square$

**Claim 2.2.22.** *Every vertex  $v$  of degree at least 3 with all descendants of degree at most 2 in  $F$  has only descendants of degree 1.*

*Proof.* For vertices of degree at least 4 this follows by Claims 2.2.19, 2.2.20 and 2.2.18, while for vertices of degree 3 this is a consequence of Claim 2.2.21.  $\square$

**Claim 2.2.23.** *Every vertex  $v$  of degree at least 3 in  $F$  with all descendants of degree 1 has a father of degree at least 3.*

2. Equitable neighbour-sum-distinguishing weightings

---

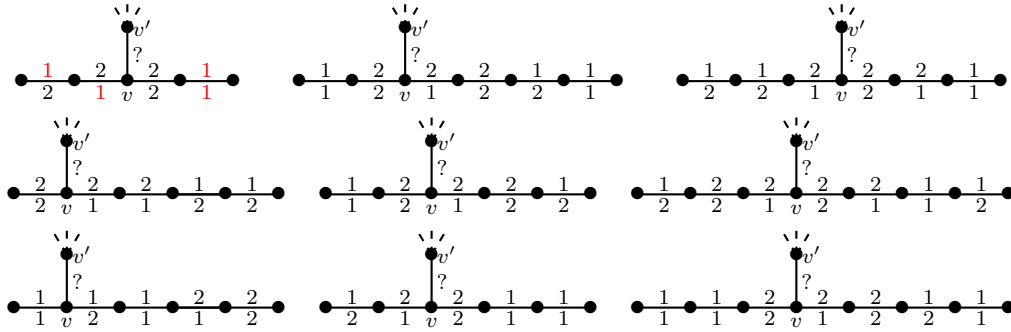


Figure 2.11: Subcases of Claim 2.2.21.

*Proof.* Suppose there is a vertex  $v$  in  $F$  of degree at least 3 with descendants being leaves and a father  $v'$  of degree at most 2. Then we delete all edges joining  $v$  with its descendants in  $F$  and weight the obtained forest by the minimality of  $F$ . To conclude it is sufficient to weight the pendent edges incident with  $v$  using at least one 2 (this will be possible, and even almost always necessary) so that the weighting of  $F$  is equitable. This way  $v$  has certainly a greater sum than  $v'$ , a contradiction.  $\square$

Note that by Claims 2.2.16, 2.2.22 and 2.2.23, every component of  $F$  must in particular have at least two vertices of degree greater than 3. Thus each such component  $T$  (previously rooted at some leaf) must contain at least one vertex which we will call a *last multigrandfather*, that is a vertex  $v$  of degree at least 3 in  $T$  which has at least one descendant of degree at least 3 but none of the descendants of degree at least 3 of  $v$  has further descendants of degree at least 3 (i.e., all descendants of degree at least 3 of  $v$  are last multifathers). Note that such a  $v$  is adjacent with its father, due to Claims 2.2.22, 2.2.23 and 2.2.19, while every child of  $v$  either has degree at least 3 and all children being leaves, or has degree 2 and at most 3 descendants – all of degree at most 2. Moreover, by Claim 2.2.20, if  $d(v) \geq 4$ , every pendent path incident with  $v$  must have length exactly 1 or 4.

Below, see Claim 2.2.24, we obtain a contradiction with the statement above that every component of  $F$  contains a last multigrandfather. Consequently, we will prove that no counterexample to Theorem 2.2.15 may exist, thus concluding its proof.

**Claim 2.2.24.** *No component  $T$  of  $F$  contains a last multigrandfather.*

*Proof.* Assume to the contrary that  $v$  is a last multigrandfather in a component  $T$  of  $F$ .

Suppose first that  $v$  has two children  $u$  and  $w$  of degree at least 3. Then we may delete two pendent edges incident with  $u$ , say  $e_u, e'_u$  and two pendent edges incident with  $w$ , say  $e_w, e'_w$  (recall that all descendants of  $u$  and  $w$  must be leaves), and weight the remaining forest by the minimality of  $F$ . It is then sufficient to weight the remaining four edges with two 1's and two 2's so that there is no conflict between  $v$  and its children  $u, w$  in order to get a contradiction with the minimality of  $F$ . We however have three essentially different ways of extending our weighting, i.e. assigning 1, 1 to  $e_u, e'_u$  (and hence 2, 2 to  $e_w, e'_w$ ) or

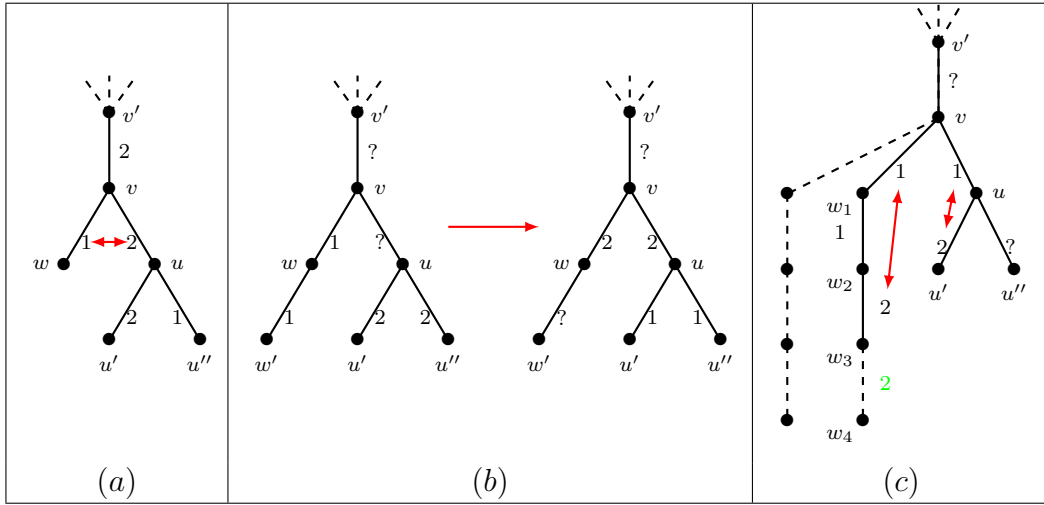


Figure 2.12: Illustrations to cases in Claim 2.2.24.

assigning 1, 2 or 2, 2 to them. Hence, one of these options must fulfil our requirements (as the sum at  $v$  “forbids” only one potential sum at each of  $u$  and  $w$ ).

Hence we may assume that  $v$  has exactly one child who is a last multifather, say  $u$  (if it had no such child it could not be a last multigrandfather by definition). Denote by  $u', u''$  any two leaves adjacent with  $u$ .

Let us consider first the case when all the remaining children of  $v$  are leaves. If  $d(v) \geq 4$ , then delete two pendent edges incident with  $u$ , say  $e_u, e'_u$  and two pendent edges incident with  $v$ , say  $e_v, e'_v$ , and weight the remaining forest by the minimality of  $F$ . Then, analogously as above, it is sufficient to weight the remaining four edges with two 1’s and two 2’s so that there is no conflict between  $u$  and  $v$  and between  $v$  and its father. As we have three essentially different ways of extending the weighting, at most two of which can be “forbidden” by our requirements on lack of conflicts, we certainly may extend the weighting to  $F$ . If however  $d(v) = 3$ , let  $w$  be the child of  $v$  which is a leaf. Delete from  $F$  all edges induced by  $v$  and its descendants and weight the remaining forest by the minimality of  $F$ . As we have removed at least four edges, we certainly may use at least two 1’s and at least two 2’s while equitably extending the weighting to  $F$ . Assign 1 to  $wv$  and 2 to  $uu'$ . Then choose  $\omega(uv) \in \{1, 2\}$ , where  $\omega(uv)$  denotes the weight of the edge  $uv$ , so that there is no conflict between  $v$  and its father  $v'$ , set  $\omega(uu'') = 3 - \omega(uv)$ , and assign weights to the remaining un-weighted edges (if there are any) so that the obtained weighting of  $F$  is equitable. If there is no conflict between  $u$  and  $v$ , we are done. On the other hand, the only situation in which we may have such a conflict is when  $u', u''$  are the only children of  $u$ ,  $\omega(vv') = 2$ , and  $\omega(uu'') = 1$  (in all other cases the sum at  $u$  would be larger than the one at  $v$ ), and hence  $\omega(uv) = 2$ , see Figure 2.12(a).

But then we may switch the weights of  $uv$  and  $wv$ , decreasing the sum at  $u$  but not changing the sum at  $v$ . In all cases we thus obtain a desired equitable neighbour-sum-distinguishing weighting of  $F$ , a contradiction.

Suppose now that  $v$  has a son, say  $w$ , adjacent with only one leaf  $w'$  (i.e.,  $v, w, w'$  form



a pendent path of length 2 incident with  $v$ ). By Claim 2.2.20,  $d(v) = 3$ , see Figure 2.12(b). Then delete all descendants of  $v$  and weight the forest obtained by the minimality of  $F$ . As we have removed at least five edges from  $F$ , we still may use at least two 1's and two 2's. Set  $\omega(uw') = 1 = \omega(uv)$ ,  $\omega(uu') = 2 = \omega(uu'')$  and if we still have any choice (which will not prevent us from completing the weighting of  $F$  equitably) choose a weight for  $uv$  so that there is no conflict between  $v$  and its father. Finally, weight the remaining un-weighted edges equitably (i.e. so that the weighting of  $F$  is equitable). This way, no conflict is possible, except a potential conflict between  $v$  and its father. In such a case, we must however have had no choice while weighting  $uv$ , and hence  $d(u) = 3$ . Then we weight all previously removed edges once more differently, setting  $\omega(vw) = 2 = \omega(vu)$  (and hence increasing the sum at  $v$ ),  $\omega(uu') = 1 = \omega(uu'')$ , and completing the weighting equitably, see Figure 2.12(b). As no conflict is then possible, we obtain a contradiction.

Suppose finally that there is a pendent path of length 3 or 4 incident with  $v$  and induced by  $v$  and its descendants. Delete the edges of this path and all edges incident with  $u$  and weight the obtained forest by the minimality of  $F$ . Let  $w_1, w_2, \dots, w_j$ ,  $j \in \{3, 4\}$  be the consecutive vertices of this path with  $w_1$  being a child of  $v$ . Set  $\omega(vw_1) = 1 = \omega(w_1w_2)$ ,  $\omega(w_2w_3) = 2$  and  $\omega(w_3w_4) = 2$  (if there is such an edge),  $\omega(vu) = 1$ ,  $\omega(uu') = 2$ , see Figure 2.12(c), and weight the remaining edges so that we obtain an equitable weighting of  $F$ . Note that as  $\omega(w_1w_2) = 1$  (what will not be changed), no conflict is possible between  $w_1$  and  $v$ . We may however have potential conflicts between  $v$  and its remaining neighbours. If there is a conflict between  $v$  and  $u$  or  $v'$ , where  $v'$  is the father of  $v$ , we exchange the weights of  $vu$  and  $uv'$ , increasing the sum at  $v$  (and not changing the sum at  $u$  nor at  $v'$ ). After such a switch, since hitherto there was a conflict,  $v$  must have a greater sum than  $u$  or  $v'$ . If  $v$  is still in conflict with the remaining one of these two, we raise the sum at  $v$  once more (not changing the sums at  $u$  and  $v'$ ) by switching the weights of  $vw_1$  and  $w_2w_3$ . Then if there was still some conflict in  $F$ , it would have to be between  $v$  and its neighbour, say  $x$ , other than  $v', u$  and  $w_1$ , but then  $d(v) \geq 4$  and  $d(x) \leq 2$  (as we have assumed that  $v$  has only one child of degree at least 3), hence  $v$  and  $x$  could not be in conflict.

Thus, in all cases we have been able to obtain a desired equitable neighbour-sum-distinguishing 2-edge-weighting of  $F$ , a contradiction.  $\square$

As a consequence of Claims 2.2.24, 2.2.22 and 2.2.16,  $F$  is a union of stars, *i.e.* each component of  $F$  has exactly one vertex of degree at least 3, and all its other vertices are of degree 1. Then it is easy to check that  $F$  admits an equitable nsd 2-edge-weighting. A contradiction. This proves Theorem 2.2.15.  $\square$

## 2.3 Equitable neighbour-sum-distinguishing total-weighting

In this section we explore the equitable version of neighbour-sum-distinguishing total-weightings of graphs. By Proposition 2.1.1, we have  $\overline{\chi}_{\Sigma}^t(G) \leq \overline{\chi}_{\Sigma}^e(G)$  for every nice graph

$G$ . Moreover, note that for every nice graph  $G$ ,  $\overline{\chi}_{\Sigma}^t(G) = 1$  if and only if  $\overline{\chi}_{\Sigma}^e(G) = 1$ , because we can simply ignore the weights of vertices, which are all equal to 1. These observations mean that if  $\overline{\chi}_{\Sigma}^e(G) = 2$ , then  $\overline{\chi}_{\Sigma}^t(G) = 2$ . Thus in this section we will study graphs for which we do not know if  $\overline{\chi}_{\Sigma}^e(G) \leq 2$ . Namely, we study complete graphs, for which we know that  $\overline{\chi}_{\Sigma}^e \geq 3$ , and bipartite graphs, for which we only know that  $\overline{\chi}_{\Sigma}^e \leq 2$  in the case of forests and complete bipartite graphs.

### 2.3.1 Bipartite graphs

We prove in this subsection that  $\overline{\chi}_{\Sigma}^t(G) \leq 2$  for every bipartite graph  $G$ .

**Theorem 2.3.1.** *For every bipartite graph  $G$ ,  $\overline{\chi}_{\Sigma}^t(G) \leq 2$ .*

*Proof.* In order to prove the result in the non-connected case, we will in fact prove a stronger thesis but in the case of connected bipartite graphs. We will not only prove that such graphs admit neighbour-sum-distinguishing equitable total-weightings using 1 and 2, but also that, if the sum of the numbers of vertices and edges is odd, then there are two such weightings – one with a majority of 1’s and the second with a majority of 2’s. This immediately implies the statement of Theorem 2.3.1 (as we may first weight the components of a non-connected bipartite graph with even sums of numbers of vertices and edges, and then the remaining ones, using alternately a majority of 1’s and a majority of 2’s).

Note that this strengthened thesis for connected bipartite graphs is straightforward in the case of a star (even with no edges) – e.g., if a star has at least two edges, it is sufficient to put 1’s on all its edges and 2’s on the vertices or the other way round. For the remaining connected bipartite graphs it follows by first using Observation 2.3.2, and then sequentially repeating application of Lemma 2.3.3 below until we achieve one or two desired total-weightings.

**Observation 2.3.2.** *Every connected bipartite graph  $G = (X, Y; E)$  with at least one edge admits a 2-total-weighting so that the vertices in one set of the bipartition have even sums and the vertices in the second set of the bipartition have odd sums and so that the number of 1’s used exceeds the non-zero number of 2’s.*

*Proof.* First weight all the edges of  $G$  with 1. Then weight one vertex in  $X$  with 2. Next subsequently weight all the remaining vertices in  $G$ , each with 1 or 2, so that the parities of sums at all the vertices in  $X$  are the same and different from those in  $Y$ . If the number of 1’s used on  $G$  does not exceed the number of 2’s, it means that all vertices are weighted with 2 (note that the number of vertices may exceed the number of edges by at most one in a connected graph). Then choose any edge  $uv \in E$  and change the weights of its end-vertices from 2 to 1 and the weight of the edge from 1 to 2. Note that this will not influence the sums at any vertex in  $G$ , but the number of 1’s used will exceed the number of 2’s afterwards (while at least one 2 will remain on  $G$ ).

□

## 2. Equitable neighbour-sum-distinguishing weightings

---

Note that neighbours are certainly sum-distinguished under the weighting from Observation 2.3.2 above. Now we will show that given such a weighting we can repeatedly increase the number of 2's used (at each step only by one) not spoiling at the same time the neighbour distinction in  $G$  until we achieve our goal (or goals).

**Lemma 2.3.3.** *Given any 2-total-weighting  $\omega$  of a connected bipartite graph  $G = (V, E)$  which is not a star and with  $|E_\omega(1)| \geq |E_\omega(2)| > 0$ , such that the vertices in one set of the bipartition of  $G$  have even sums and the vertices in the second set of the bipartition have odd sums, we may construct a new 2-total-weighting  $\omega'$  of  $G$  complying with the second feature of the given one (concerning the parities of the sums at vertices) with  $|E_{\omega'}(2)| = |E_\omega(2)| + 1$ .*

*Proof.* Suppose we are given a graph and an initial 2-total-weighting as claimed. The proof will be based on the fact that the parities of the sums in  $G$  do not change if we make a negative of any edge  $uv$ , *i.e.* after changing every 1 to 2 and every 2 to 1 used on  $u$ ,  $v$  and  $uv$ .

We will show that starting from our initial weighting we may always subsequently make negatives of a few edges to obtain a total-weighting consistent with the thesis. We will write that an edge  $uv$  is of type  $abc$ , where  $a, b, c \in \{1, 2\}$  if  $u, uv, v$  (or  $v, uv, u$ ) are weighted  $a, b, c$ , respectively, in the initial total-weighting. Note first that we may assume that there are no edges of types 121 and 112 (nor 211) in  $G$ , as we could make a negative of any such edge and immediately achieve our goal.

- Suppose there is no edge of type 111 in  $G$  either, hence there are only edges of types 222, 212 and 221 (or equivalently 122).

Then the graph  $H$  induced in  $G$  by the edges of type 212 cannot be a forest, as otherwise there would be more 2's than 1's on  $G$ . Indeed, if  $H$  is a forest, *i.e.* has more vertices than edges, then more 2's appear on vertices than 1's on edges in  $G$ . On the other hand, there are at least as many edges weighted 2 as there are vertices weighted 1 in  $G$ , as every such vertex must be an end of an edge of type 221 (or 122), and since the other end of such an edge is weighted 2, we may easily define an injective mapping from the set of vertices weighted 1 to the set of edges weighted 2 (by assigning to such a vertex any of its incident edges). Hence, as  $H$  is not a forest and must be bipartite, it contains a cycle of length at least 4, and hence also a path of length 3. It is then sufficient to subsequently make negatives of all these three edges (after which the consecutive edges of this path of types 212, 212, 212 will become edges of types 122, 222, 221 respectively – note in particular that the vertices of the middle edge will switch weights twice, hence in fact will return to their initial values) to obtain a required total-weighting of  $G$ .

- Suppose then to the contrary that there is an edge of type 111 in  $G$ . As the parities of the sums at the ends of such an edge  $e$  must differ (by the definition of our initial weighting),  $e$  must be adjacent with at least one edge weighted 1, say  $f$ , thus  $f$  is also an edge of type 111 (as there are no edges of type 112 nor 211 in  $G$ ). Note

that we may then assume that there is no edge of type 222, as otherwise it could not be adjacent with any edge of type 111 and thus we could make negatives of such an edge of type 222 and two adjacent edges of type 111, and obtain a required weighting of  $G$ . Thus in  $G$  there are only edges of type 111, which we will call edges of type  $A$ , and edges of types 212 and 122 (or 221), which we will all call edges of type  $B$ . Analogously as above, one may verify that if there is a path of length 3 in  $G$  with two consecutive edges of type  $A$  and one of type  $B$  or two consecutive edges of type  $B$  and one of type  $A$ , then by making consecutively negatives of all edges of one such path, we will always obtain a desired total-weighting of  $G$ . We will show that such a path must exist in  $G$ . Let  $H'$  be any component of the graph induced in  $G$  by the edges of type  $A$ . (Recall that each such component must have diameter at least 2.)

Suppose that  $H'$  is not a star. As  $G$  is connected and at least one 2 is used as a weight on it, at least one vertex, say  $v$ , (weighted 1) in  $H'$  must be incident with an edge, say  $e'$  of type  $B$ . Note that the other (different from  $v$ ) end of  $e'$ , say  $u$ , must be weighted 2, hence does not belong to  $H'$ . On the other hand, in  $H'$  there must be a vertex, say  $w$ , at distance 2 from  $v$ , as otherwise  $H'$  would be a star. Hence there is a path of length 3 (starting at  $u$  and ending at  $w$ ) in  $G$  with two consecutive edges of type  $A$  and one of type  $B$ , as claimed.

We thus may finally assume that  $H'$  is a star (with at least two edges). If at least one of its leaves is incident with an edge of type  $B$ , then we obtain a path as above. Otherwise, as  $G$  is connected,  $G$  is not a star and at least one 2 was used on it, the center of the star making up  $H'$  must be incident with one end of a path  $P$  of length two, whose first edge (incident with the center of the star), say  $e'$ , is of type  $B$ . However, as the other end of  $e'$  must be weighted with 2, the second edge of this path must also be of type  $B$ , thus we obtain a path of length 3 with two consecutive edges of type  $B$  (from  $P$ ) and one edge of type  $A$  (incident with the center of  $H'$ ), as claimed.

□

As mentioned before, by Observation 2.3.2 and by repeating the construction of Lemma 2.3.3 we can construct an equitable nsd 2-total-weighting of any bipartite graph  $G$ . This completes the proof of Theorem 2.3.1. □

### 2.3.2 Complete graphs

Finally, we prove in this subsection that  $\overline{\chi}_{\Sigma}^t(K_2) = 2$  and  $\overline{\chi}_{\Sigma}^t(K_n) = 3$  for every  $n \geq 3$ .

**Theorem 2.3.4.** *For every complete graph  $K_n$  with  $n \geq 3$ ,  $\overline{\chi}_{\Sigma}^t(K_n) = 3$ , while  $\overline{\chi}_{\Sigma}^t(K_2) = 2$ .*

*Proof.* The total-weightings of small cases (for  $n \leq 4$ ) are depicted in Figure 2.13. By Theorem 2.2.6 and Proposition 2.1.1, it is enough to show that  $\overline{\chi}_{\Sigma}^t(K_n) > 2$  for  $n \geq 3$ .

## 2. Equitable neighbour-sum-distinguishing weightings

---

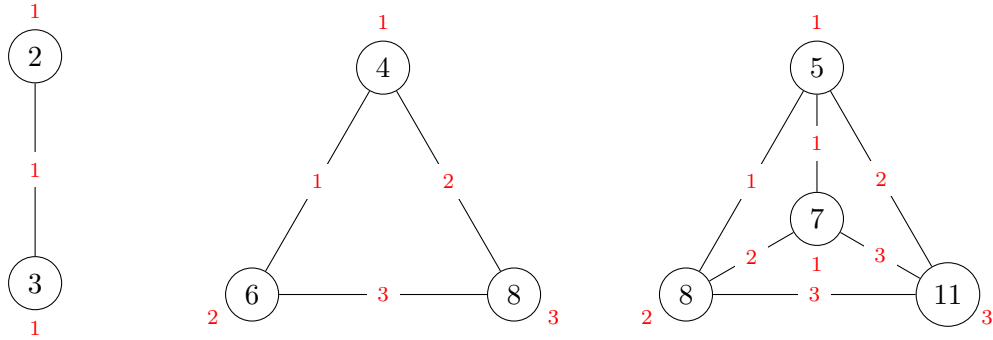


Figure 2.13: Equitable nsd total-weightings of  $K_2$ ,  $K_3$  and  $K_4$ .

First we observe that for any positive integer  $n$ , there exist (essentially) exactly two possible nsd 2-total-weightings of  $K_n$ ; in one of them there is a vertex with a monochromatic palette of 1's, and in the other one there is a vertex with a monochromatic palette of 2's, since we add  $n$  numbers (weights) at every vertex and we obtain sums from the interval  $\{n, \dots, 2n\}$  but not  $n$  and  $2n$  in the same total-weighting. Moreover, it is easy to observe that for any such total-weighting of  $K_n$ ,  $n \geq 2$ , with a monochromatic palette of  $a$ 's,  $a \in \{1, 2\}$ , say at a vertex  $v$ , after deleting  $v$  we obtain an nsd 2-total-weighting of  $K_{n-1}$  with a vertex with a monochromatic palette of  $(3 - a)$ 's. Let  $\omega$  be such an nsd 2-total-weighting of  $K_n$ ,  $n \geq 2$ , with a vertex having a monochromatic palette of 1's - for a monochromatic palette of 2's the reasoning and calculations are the same. Now for every positive integer  $k$  we prove by induction that:

- If  $n = 2k$ , then there exist  $k(k + 1)$  elements (vertices and edges) weighted with 1 and  $k^2$  elements weighted with 2.
- If  $n = 2k + 1$ , then there exist  $(k + 1)^2$  elements weighted with 1 and  $k(k + 1)$  elements weighted with 2.

For  $n = 2$  we have both vertices with distinct weights and an edge is weighted with 1, since there exists a monochromatic palette of 1's. For  $n = 2k$ ,  $k > 1$ , let  $v$  be a vertex with a monochromatic palette of 1's. Then there exists a vertex  $u$  in  $K_n - v \cong K_{n-1}$  having a monochromatic palette of 2's. Consider the graph  $K_{2k-2}$  obtained from  $K_{2k}$  by removing vertices  $u$  and  $v$ . Then, by induction, there exist  $k(k - 1)$  elements weighted with 1 and  $(k - 1)^2$  elements weighted with 2 in  $K_{2k-2}$ . So, there exist  $k(k - 1) + 2k$  elements weighted with 1 (since  $v$  adds  $n$  elements weighted with 1) and  $(k - 1)^2 + 2k - 1$  elements weighted with 2 (since  $u$  adds  $n - 1$  elements weighted with 2) in  $K_{2k}$ .

Similarly, for  $n = 3$  we have four elements weighted with 1 and two elements weighted with 2, since there exists a monochromatic palette of 1's and there exist exactly one such nsd 2-total-weighting of  $K_3$ . For  $n = 2k + 1$ ,  $k > 1$ , let  $v$  be a vertex with a monochromatic palette of 1's. Then there exists a vertex  $u$  in  $K_n - v \cong K_{n-1}$  having a monochromatic palette of 2's. Consider the graph  $K_{2k-1}$  obtained from  $K_{2k+1}$  by removing vertices  $u$  and

$v$ . Then, by induction, there exist  $k^2$  elements weighted with 1 and  $k(k-1)$  elements weighted with 2 in  $K_{2k-1}$ . Hence, there exist  $k^2 + 2k + 1$  elements weighted with 1 (since  $v$  adds  $n$  elements weighted with 1) and  $k(k-1) + 2k$  elements weighted with 2 (since  $u$  adds  $n-1$  elements weighted with 2) in  $K_{2k+1}$ . Now observe that if  $n = 2k$ , then the difference between numbers of 1's and 2's, in the weighting  $\omega$ , is  $k$ , and if  $n = 2k+1$ , then the difference is  $k+1$ . Hence,  $\omega$  is equitable only for  $n = 2$ . This completes the proof, as the same reasoning applies in the case of a monochromatic palette of 2's in  $\omega$  (with 1's and 2's switched).  $\square$

## 2.4 Discussions and open problems

In this chapter, we studied, for some families of nice graphs, the existence of equitable neighbour-sum-distinguishing edge-weightings *i.e.* in which any two distinct edge weights are used about the same number of times (up to a difference of 1). In particular, we introduced, for any given graph  $G$ , the equitable nsd index denoted by  $\overline{\chi}_\Sigma^e(G)$  being the smallest maximal weight in an equitable neighbour-sum-distinguishing edge-weighting of  $G$ . One side of our work was the research of graphs with  $\overline{\chi}_\Sigma^e > \chi_\Sigma^e$  and more generally the research of bounds on  $\overline{\chi}_\Sigma^e$ . This is the reason why we started our investigation with complete graphs. However, we only found two graphs with  $\overline{\chi}_\Sigma^e > \chi_\Sigma^e$ ,  $K_4$  and  $K_{3,3}$ , and only  $K_4$  has an equitable nsd index of 4, which is greater than the bound of the 1-2-3 Conjecture. The problem, presented in the following question, is hence wide open.

**Question.** *Is there an infinite family of graphs  $G$  with  $\overline{\chi}_\Sigma^e(G) > \chi_\Sigma^e(G)$ ?*

In fact we found three more examples of such graphs while exploring the equitable nsd index of bipartite cubic graphs. Using the work of Meringer on regular graphs generation [Mer99], we checked bipartite cubic graphs of orders in  $\{6, \dots, 20\}$ . The two cubic bipartite graphs on 10 vertices and one of the 13 cubic bipartite graphs on 14 vertices do not admit an equitable nsd 2-edge-weightings, while admitting (non-equitable) nsd 2-edge-weightings. However,  $\overline{\chi}_\Sigma^e = 3$  for the three exceptions. Hence we still have only one example,  $K_4$ , with  $\overline{\chi}_\Sigma^e > 3$ .

If a graph  $G$  is a Hamiltonian cubic bipartite graph on  $4k$  vertices for  $k \geq 2$ , then we can construct an equitable nsd 2-edge-weighting as follows :

1. Construct an equitable nsd 2-edge-weighting  $\omega_0$  of the Hamiltonian cycle of length  $4k$  as in an equitable nsd 2-edge-weighting of  $C_{4k}$  (see Proposition 2.2.4). The remaining edges induces a perfect matching of even size of  $G$ , and since the graph is bipartite, those edges never join a vertex  $u$  with  $\sigma_{\omega_0}(u) = 2$  to a vertex  $v$  with  $\sigma_{\omega_0}(v) = 4$ . Hence each edge in the matching has one end  $w$  with  $\sigma_{\omega_0}(w) = 3$ ;
2. Now for each edge  $e$  in the matching with an end  $u$  with  $\sigma_{\omega_0}(u) = 2$ , set  $\omega(e) := 1$  otherwise, set  $\omega(e) := 2$ .

The obtained edge-weighting  $\omega$  is clearly neighbour-sum-distinguishing, since every edge of  $G$  joins a vertex with  $\sigma_\omega = 3$  or 6 to a vertex with  $\sigma_\omega = 4$  or 5. Moreover,  $\omega$  is equitable

## 2. *Equitable neighbour-sum-distinguishing weightings*

---

because  $\omega_0$  is equitable and one half of edges in the matching is weighted with 1 and the other half are weighted with 2.

The condition of the Hamiltonicity can be weakened to the existence of disjoint cycle cover of  $G$ , with cycles of lengths multiples of 4. Moreover, this construction can be generalized to Hamiltonian regular bipartite graphs of any degree greater than 2.





# Chapter 3

## The 1-2-3 Conjecture and the Antimagic Labelling Conjecture

This chapter is dedicated to the following question: is it always possible to injectively assign the weights  $1, \dots, |E(G)|$  to the edges of any given graph  $G$  with no component isomorphic to  $K_2$ , so that every two adjacent vertices of  $G$  are sum-distinguished? One may see this question as a combination of the well-known 1-2-3 Conjecture and the Antimagic Labelling Conjecture.

Throughout this chapter, we exhibit evidence that the answer to that question is affirmative. We prove upper bounds on the number of distinct positive weights that are necessary to get an injective nsd edge-weighting, first by using results on the 1-2-3 Conjecture from [KKP10], then by making an observation on the extension of partial edge-weightings. Then, benefiting from the investigations on the Antimagic Labelling Conjecture, we point out that several classes of graphs, such as regular graphs, indeed admit such assignments. We then show that trees also do, answering a recent conjecture of Arumugam, Premalatha, Bača and Semaničová-Feňovčíková [APBSF17]. Towards a general answer to the question above, we then prove that claimed weightings can be constructed for some classes of sparse graphs, namely 2-degenerate graphs and graphs with maximum average degree 3, provided we are allowed to use a small constant number of additional edge weights.

The results presented in this chapter are the result of a collaboration with J. Bensmail and K. Szabo Lyngsie from DTU (Copenhagen, Denmark) and were published in [BSSL17].

### 3.1 Definitions and remarks

Throughout this chapter, we deal with edge-weightings that are not only neighbour-sum-distinguishing but also do not assign any edge weight more than once. We say that such edge-weightings are *injective*. Still under the assumption that  $G$  is a nice graph, we denote by  $\chi_{\Sigma}^{e,1}(G)$  the smallest  $k$  such that  $G$  admits an injective neighbour-sum-distinguishing  $k$ -edge-weighting.

The main guideline for our work is the following conjecture.

**Conjecture 3.1.1** ([BSSL17]). *For every nice graph  $G$ , we have  $\chi_{\Sigma}^{e,1}(G) = |E(G)|$ .*

By the injectivity property, we note that  $|E(G)|$  is a lower bound on  $\chi_{\Sigma}^{e,1}(G)$  for every nice graph  $G$ . Conjecture 3.1.1, in brief words, hence asks whether, for every nice graph  $G$ , we can bijectively assign weights  $1, \dots, |E(G)|$  to the edges of  $G$  so that no two adjacent vertices of  $G$  get the same sum.

Despite our results in Chapter 2, it does not seem obvious how big  $\overline{\chi}_{\Sigma}^e(G)$  can be, neither whether this parameter can be arbitrarily large compared to  $\chi_{\Sigma}^e(G)$ . This is one of our motivations for studying injective neighbour-sum-distinguishing edge-weightings, as an injective edge-weighting is always equitable. Thus,  $\overline{\chi}_{\Sigma}^e(G) \leq \chi_{\Sigma}^{e,1}(G)$  holds for every nice graph  $G$ . Hence, attacking Conjecture 3.1.1 can be seen as a way to make progress towards those two questions.

Our second motivation for considering Conjecture 3.1.1 is that injective neighbour-sum-distinguishing edge-weightings can be regarded as a weaker notion of well-known *antimagic labellings*. Formally, using our own terminology, an antimagic labelling  $\omega$  of a graph  $G$  is an injective  $|E(G)|$ -edge-weighting of  $G$  for which  $\sigma_{\omega}$  is injective, *i.e.* all vertices of  $G$  get a distinct sum of incident weights by  $\omega$ . We say that  $G$  is *antimagic* if it admits an antimagic labelling. Many lines of research concerning antimagic labellings can be found in the literature, most of which are related to the following conjecture addressed by Hartsfield and Ringel [HR90].

**Antimagic Labelling Conjecture.** *Every nice connected graph is antimagic.*

Despite lots of efforts (refer to the dynamic survey by Gallien [Gal17] for an in-depth summary of the vast and rich literature on this topic), the Antimagic Labelling Conjecture is still open in general, even for common classes of graphs such as nice trees. Conjecture 3.1.1, which is clearly much weaker than the Antimagic Labelling Conjecture, as the distinction condition here only concerns the adjacent vertices, hence sounds as a much easier challenge to us, in particular concerning classes of nice graphs that are not known to be antimagic. Hence, every antimagic graph  $G$  agrees with Conjecture 3.1.1, implying, as described earlier, that

$$\overline{\chi}_{\Sigma}^e(G) \leq \chi_{\Sigma}^{e,1}(G) = |E(G)|$$

holds, thus providing an upper bound on  $\overline{\chi}_{\Sigma}^e(G)$ . This is of interest as several classes of graphs, such as nice regular graphs and nice complete partite graphs, are known to be antimagic, as reported by Gallien [Gal17]. Let us here further mention the works of Bérczi, Bernáth and Vizer [BBV15], and of Cranston, Liang and Zhu [CLZ15], who led to the verification of the Antimagic Labelling Conjecture for nice regular graphs, and contain some proof techniques that partly inspired our own proofs. Conversely, proving that a graph  $G$  verifies  $\chi_{\Sigma}^{e,1}(G) = |E(G)|$  and agrees with Conjecture 3.1.1 is similar to proving that, in some sense,  $G$  is “locally” antimagic.

### 3. The 1-2-3 Conjecture and the Antimagic Labelling Conjecture

---

Conjecture 3.1.1 can essentially be considered as a combination of the 1-2-3 Conjecture and the Antimagic Labelling Conjecture, as the notions behind it have flavours of both conjectures. As described earlier, proving Conjecture 3.1.1 for some classes of graphs has, to some extent, consequences on the 1-2-3 Conjecture and the Antimagic Labelling Conjecture, or at least on variants of these conjectures.

Our work is focused both on proving Conjecture 3.1.1 for particular classes of nice graphs, and providing upper bounds on  $\chi_{\Sigma}^{e,1}$  for some classes of nice graphs. This chapter is organized as follows. In Section 3.3, we start off by providing support to Conjecture 3.1.1, essentially by showing that the conjecture holds for nice trees. Then we exhibit general upper bounds on  $\chi_{\Sigma}^{e,1}$  in Section 3.4. Finally, we provide, in Section 3.5, refined upper bounds on  $\chi_{\Sigma}^{e,1}$  for nice graphs with maximum average degree at most 3 and nice 2-degenerate graphs.

**Remark.** In 2017, nine months after the submission of our work, a paper published by Haslegrave [Has17] on arXiv proved Conjecture 3.1.1. The proof of Haslegrave is based on probabilistic methods, yielding a non-constructive proof of Conjecture 3.1.1, our work is, however, providing exclusively constructive proofs. Another paper written by Arumugam, Premalatha, Bača and Semaničová-Feňovčíková [APBSF17], independently introduced the same notion of injective nsd edge-weightings, but the authors were interested in minimizing the number of distinct sums induced on the vertices rather than lowering the maximum used weight. However, our Theorem 3.3.3 on trees answers positively to Conjecture 2.3 raised in [APBSF17]. In 2018, a new paper written by Yu, Hu, Yang, Wu and Wang in [YHY<sup>+</sup>18], improved some of the bounds that we have on  $\chi_{\Sigma}^{e,1}$  for general graphs and subcubic graphs, but again using a non-constructive method, the *Combinatorial Nullstellensatz*.

## 3.2 Preliminary results

In this section, we introduce several observations that will be of some use in the next sections.

In the following observation, we start off by pointing out a few situations in which, for a given injective edge-weighting  $\omega$  of a graph  $G$  and a given edge  $uv$  of  $G$ , we necessarily have  $\sigma_{\omega}(u) \neq \sigma_{\omega}(v)$ . We note that the third item is more general, as it implies the other two. Hence we only prove this item.

**Observation 3.2.1.** *Let  $G$  be a graph, and  $\omega$  be an injective edge-weighting of  $G$ . Then, for every edge  $uv$  of  $G$ , we have  $\sigma_{\omega}(u) \neq \sigma_{\omega}(v)$  in any of the following situations:*

1.  $d(u) = 1$  and  $d(v) \geq 2$ ;
2.  $d(u) = d(v) = 2$ ;
3.  $d(u) \geq d(v)$  and  $\min(P_{\omega}(u) \setminus \{\omega(uv)\}) > \max(P_{\omega}(v) \setminus \{\omega(uv)\})$ .

*Proof of item 3.* Let  $m_u := \min(P_\omega(u) \setminus \{\omega(uv)\})$  and  $M_v := \max(P_\omega(v) \setminus \{\omega(uv)\})$ , and suppose that  $d(u) \geq d(v)$  and  $m_u > M_v$ . Hence we have,

$$\sigma_\omega(u) \geq m_u(d(u) - 1) + \omega(uv) > M_v(d(v) - 1) + \omega(uv) \geq \sigma_\omega(v).$$

□

We now observe that to be able to successfully extend a partial nsd edge-weighting to an edge, we need to have sufficiently distinct weights in hand for that purpose.

**Observation 3.2.2.** *Let  $G$  be a graph,  $uv$  be an edge of  $G$ , and  $\omega$  be an nsd edge-weighting of  $G - \{uv\}$  such that  $\sigma_\omega(u) \neq \sigma_\omega(v)$ . Then  $\omega$  can be successfully extended to  $uv$ , using a weight from a set  $W$  of at least  $d(u) + d(v) - 1$  arbitrary distinct positive weights that can be assigned to  $uv$ .*

*Proof.* We note that  $\omega$  currently must satisfy  $\sigma_\omega(u) \neq \sigma_\omega(v)$ , as, otherwise, no matter what weight we assign to  $uv$ , we would have  $\sigma_\omega(u) = \sigma_\omega(v)$ . Under that assumption, the neighbours of  $u$  and  $v$  forbid at most  $d(u) + d(v) - 2$  values of  $\omega(uv)$ , as each value gives a different sum at  $u$  and at  $v$ . Hence if we have  $d(u) + d(v) - 1$  available distinct weights we can extend  $\omega$  to  $uv$  as required. □

Throughout this chapter, several proofs consist of deleting two adjacent edges  $vu_1$  and  $vu_2$  from a nice graph  $G$ , edge-weighting the remaining graph, and correctly extending the weighting to  $vu_1$  and  $vu_2$ . In this regard, we will often refer to the following result, which is about the number of weights that are sufficient to weight  $vu_1$  and  $vu_2$ .

**Observation 3.2.3.** *Let  $G$  be a graph having two adjacent edges  $vu_1$  and  $vu_2$  such that  $G' := G - \{vu_1, vu_2\}$  admits an nsd edge-weighting  $\omega_{G'}$ . Assume further that  $d_G(u_1) \geq d_G(u_2)$ , and set*

$$\mu := (d_G(u_1) + 1) + \max\{0, d_G(v) + d_G(u_2) - d_G(u_1) - 1\}.$$

*Then, assuming we have a set  $W$  of at least  $\mu$  distinct positive weights, we can extend  $\omega_{G'}$  to an nsd edge-weighting of  $G$  by assigning two distinct weights of  $W$  to  $vu_1$  and  $vu_2$ .*

*Proof.* We extend  $\omega_{G'}$  to an nsd edge-weighting  $\omega_G$  of  $G$  by first assigning a weight of  $W$  to  $vu_1$ , and then assigning a distinct weight to  $vu_2$ . We determine, in this proof, the smallest number  $\mu$  of weights that  $W$  should contain so that this strategy has sufficiently many weights to be successfully applied.

We note that extending  $\omega_{G'}$  to  $vu_1$  completely determines the value of  $\sigma_{\omega_G}(u_1)$ , while the value of  $\sigma_{\omega_G}(v)$  is not determined until  $vu_2$  is also weighted. Hence, when first weighting  $vu_1$ , we mainly have to make sure that  $\sigma_{\omega_G}(u_1)$  does not get equal to the sum of weights incident to a neighbour of  $u_1$  different from  $v$ . Also, we should make sure that  $\sigma_{\omega_{G'}}(v) + \omega_G(vu_1)$  does not get equal to  $\sigma_{\omega_{G'}}(u_2)$ , as otherwise we would necessarily get  $\sigma_{\omega_G}(v) = \sigma_{\omega_G}(u_2)$  no matter how we weight  $vu_2$ . There are hence at most  $d_G(u_1)$  conflicts to take into account when weighting  $vu_1$ . Provided  $W$  includes at least  $d_G(u_1) + 1$  distinct

### 3. The 1-2-3 Conjecture and the Antimagic Labelling Conjecture

---

weights, we can hence weight  $vu_1$  correctly, i.e. so that we avoid all conflicts mentioned above, with one weight from  $W$ , since assigning different weights to  $vu_1$  alters  $\sigma_{\omega_G}(u_1)$  in distinct ways.

Now assume  $vu_1$  has been weighted with the additional property that  $\sigma_{\omega_{G'}}(v) + \omega_G(vu_1) \neq \sigma_{\omega_{G'}}(u_2)$ . Since that property holds, Observation 3.2.2 tells us that we can correctly extend  $\omega_{G'}$  to  $vu_2$  provided  $W - \{\omega_G(vu_1)\}$  includes at least  $d_G(v) + d_G(u_2) - 1$  distinct weights. We hence need  $W - \{\omega_G(vu_1)\}$  to include that many distinct weights.

As explained above,  $W$  necessarily includes at least  $d_G(u_1)$  weights that were not assigned to  $vu_1$ . Hence, to make sure, after weighting  $vu_1$ , that  $W$  still includes at least  $d_G(v) + d_G(u_2) - 1$  distinct weights, we need  $W$  to include at least

$$(d_G(v) + d_G(u_2) - 1) - d_G(u_1)$$

other weights. This quantity can be negative, as, notably, the degree of  $u_1$  can be arbitrarily large. Hence if

$$\mu = (d_G(u_1) + 1) + \max\{0, d_G(v) + d_G(u_2) - d_G(u_1) - 1\},$$

as claimed, and  $W$  has size  $\mu$ , then we can achieve the extension of  $\omega_{G'}$  to  $G$  as described earlier.  $\square$

### 3.3 Classes of locally antimagic graphs

As mentioned in Section 3.1, since antimagic graphs verify Conjecture 3.1.1, we directly benefit, in the context of Conjecture 3.1.1, from the investigations on antimagic labellings. Following the survey by J.A. Gallian [Gal17], the following classes of nice graphs hence agree with Conjecture 3.1.1.

**Theorem 3.3.1.** *The classes of known antimagic graphs notably include:*

- *nice paths (Hartsfield, Ringel [HR90]),*
- *wheels (Hartsfield, Ringel [HR90]),*
- *nice regular graphs (Bérci, Bernáth, Vizer [BBV15]),*
- *nice complete partite graphs (Alon, Kaplan, Lev, Roditty, Yuster [AKL<sup>+</sup>04]).*

Consequently, every such graph  $G$  verifies  $\chi_{\Sigma}^{e,1}(G) = |E(G)|$ .

Moreover, for nice graphs with maximum degree 2, we can easily see, as we are assigning positive weights only, that any injective edge-weighting is neighbour-sum-distinguishing. Disjoint unions of nice paths and cycles hence agree with Conjecture 3.1.1.

**Observation 3.3.2.** *Let  $G$  be a nice graph with  $\Delta(G) = 2$ . Then any injective edge-weighting of  $G$  is neighbour-sum-distinguishing.*

### 3.3.1 Forests

One of the main lines of research concerning antimagic labellings is to determine whether nice trees are all antimagic. In the following result, we prove that this question can be answered positively when relaxed to injective nsd edge-weightings. We actually prove a stronger statement that will be useful in the next sections.

**Theorem 3.3.3.** *Let  $F$  be a nice forest. Then, for every set  $W$  of  $|E(F)|$  distinct positive weights, there exists an injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $F$ . In particular, we have  $\chi_{\Sigma}^{e,1}(F) = |E(F)|$ .*

*Proof.* If  $\Delta(F) = 2$ , then the result follows from Observation 3.3.2. So the claim holds whenever  $F$  has size 2. Assume now that the claim is false, and let  $F$  be a counterexample that is minimal in terms of  $n_F + m_F$ , where  $n_F := |V(F)|$  and  $m_F := |E(F)|$ . By the remark above, we have  $m_F \geq 3$ . Let  $W := \{\alpha_1, \dots, \alpha_{m_F}\}$  be a set of distinct positive integers such that  $F$  does not admit an injective nsd  $W$ -edge-weighting. Free to relabel the weights in  $W$ , we may suppose that  $\alpha_1 < \dots < \alpha_{m_F}$ . Due to the minimality of  $F$ , we may assume that  $F$  is a tree (as otherwise we could invoke the induction hypothesis). Furthermore, we may assume that  $F$  has maximum degree at least 3 (otherwise Observation 3.3.2 would apply).

We now successively show that  $F$ , because it is a counterexample to the claim, cannot contain certain structures, until we reach the point where  $F$  is shown to not exist at all, a contradiction. In particular, we focus on the length of the pendent paths of  $F$ , where a *pendent path* of  $F$  is a maximal path  $v_k \dots v_1$ , where  $k \geq 2$ , such that  $d(v_k) \geq 3$ ,  $d(v_{k-1}) = \dots = d(v_2) = 2$ , and  $d(v_1) = 1$ . In the case where  $k = 2$ , we note that the pendent path is a pendent edge, in which case  $v_k = v_2$  and we have  $d(v_2) \geq 3$ . Since  $\Delta(F) \geq 3$ , there are at least three pendent paths in  $F$ .

We start off by showing that the pendent paths of  $F$  all have length at most 2.

**Claim 3.3.4.** *Every pendent path of  $F$  has length at most 2.*

*Proof.* Assume  $F$  has a pendent path  $P := v_k \dots v_1$  with  $k \geq 4$ , where  $d(v_k) \geq 3$ . In this case, let  $F' := F - \{v_{k-1}v_{k-2}, \dots, v_2v_1\}$  be the tree obtained by removing, from  $F$ , all edges of  $P$  but the one incident to  $v_k$ . Clearly,  $F'$  is nice and, due to the minimality of  $F$ , there exists an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_{F'}}\}$ -edge-weighting  $\omega_{F'}$  of  $F'$ , where  $m_{F'} := |E(F')|$ . To prove that the claim holds, we have to prove that we can extend  $\omega_{F'}$  to the edges  $v_{k-1}v_{k-2}, \dots, v_2v_1$ , hence to  $F$ , using weights  $\alpha_{m_{F'}+1}, \dots, \alpha_{m_F}$ , so that we get an injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $F$ , a contradiction.

Due to the length of  $P$ , we have  $|\{\alpha_{m_{F'}+1}, \dots, \alpha_{m_F}\}| \geq 2$ . When weighting the edges  $v_{k-1}v_{k-2}, \dots, v_2v_1$ , we note that we cannot create any sum conflict involving any two consecutive vertices in  $\{v_1, \dots, v_{k-1}\}$ . That is, the incident sums of any two of these vertices can never get equal. This is according to Observation 3.2.1 since we are assigning weights injectively. Hence, when extending  $\omega_{F'}$ , we just have to make sure that  $\sigma(v_{k-1})$  gets different from  $\sigma(v_k)$ , which is possible as we have at least two distinct edge weights to work with. So we can assign a weight to  $v_{k-1}v_{k-2}$  which avoids that conflict, and then

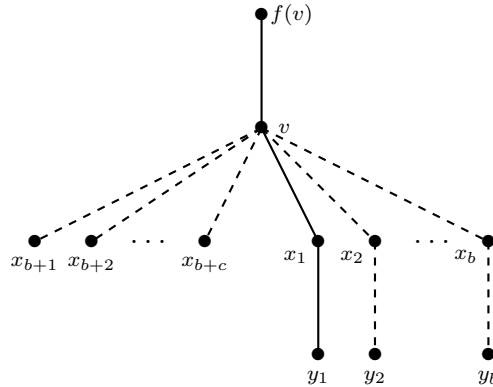


Figure 3.1: Illustrations to Claim 3.3.6.

arbitrarily extend the weighting to the edges  $v_{k-2}v_{k-3}, \dots, v_2v_1$ . This yields an injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $F$ .  $\square$

Now designate a vertex  $r$  with degree at least 3 of  $F$  as being the *root* of  $F$ . For every vertex  $v \neq r$  in  $F$ , we denote by  $f(v)$ , the father of  $v$  in  $F$ . As mentioned in Subsection 2.2.4, a *multifather*  $v$  of  $F$  is a vertex with degree at least 3, *i.e.* having at least two children. If all descendants of  $v$  have degree at most 2, then  $v$  is a *last multifather* of  $F$ . Since  $\Delta(F) \geq 3$ , there are last multifathers in  $F$ .

To further study the structure of  $F$ , we now prove properties of its last multifathers, still under the assumption that  $F$  is rooted at a vertex  $r$  with degree at least 3.

**Claim 3.3.5.** *Vertex  $r$  is not a last multifather.*

*Proof.* Assume that  $r$  is a last multifather. Then  $r$  is the only vertex with degree at least 3 of  $F$ . In other words,  $F$  is a subdivided star. Then by assigning the weights  $\alpha_{m_F}, \alpha_{m_F-1}, \dots, \alpha_1$ , following this order, to the edges of  $F$  as they are encountered during a breadth-first search algorithm performed from  $r$ , we obtain an injective neighbour-sum-distinguishing edge-weighting of  $F$ . To be convinced of this statement, one can refer to Observation 3.2.1.  $\square$

Due to Claim 3.3.5, we may assume that the root  $r$  of  $F$  is not a last multifather. Then all last multifathers of  $F$  (there are some) are different from  $r$ , and hence have a father. We now refine Claim 3.3.4 to the following.

**Claim 3.3.6.** *Every pendent path attached to a last multifather of  $F$  has length 1.*

*Proof.* Let  $v \neq r$  be a last multifather of  $F$ , and assume  $v$  is incident to pendent paths with length 2 (see Figure 3.1). We recall that all pendent paths attached to  $v$  have length at most 2 (Claim 3.3.4), and, since  $v$  is a last multifather, it is incident to at least two pendent paths. Let  $F'$  be the tree obtained from  $F$  by removing all pendent paths attached to  $v$ . Because  $m_{F'} := |E(F')|$  is smaller than  $m_F$ , there exists an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_{F'}}\}$ -edge-weighting  $\omega_{F'}$  of  $F'$ . For contradiction, we

prove below that  $\omega_{F'}$  can be extended correctly to the pendent paths attached to  $v$  using the weights among  $\{\alpha_{m_{F'}+1}, \dots, \alpha_{m_F}\}$  injectively.

Let  $b \geq 1$  be the number of pendent paths of length 2 attached to  $v$  in  $F$ , and let  $vx_1y_1, \dots, vx_by_b$  denote those paths (so that the  $x_i$ 's have degree 2 in  $F$ , while the  $y_i$ 's are leaves). Vertex  $v$  is also adjacent to  $c \geq 0$  leaves  $x_{b+1}, \dots, x_{b+c}$ , which form, with  $v$ ,  $c$  pendent paths of length 1. Since  $v$  is a multifather, we recall that  $b + c = d_F(v) - 1 \geq 2$ .

We extend  $\omega_{F'}$  to the edges of the pendent paths attached to  $v$  in the following way. First, we injectively arbitrarily assign the  $d_F(v) - 2$  weights in  $\{\alpha_{m_F-d_F(v)+3}, \dots, \alpha_{m_F}\}$  to the edges  $vx_2, \dots, vx_{b+c}$ . After that, we assign to the edge  $vx_1$  one of the weights  $\alpha_{m_F-d_F(v)+1}$  or  $\alpha_{m_F-d_F(v)+2}$  chosen so that  $\sigma_{\omega_F}(v)$  is different from  $\sigma_{\omega_{F'}}(f(v))$ . We then assign to  $x_1y_1$  the one weight of  $\alpha_{m_F-d_F(v)+1}$  or  $\alpha_{m_F-d_F(v)+2}$  not assigned to  $vx_1$ . We note that no matter how we complete the extension of  $\omega_{F'}$ ,  $\sigma_{\omega_F}(v)$  will be strictly larger than  $\sigma_{\omega_F}(x_1)$  since  $\sigma_{\omega_F}(v) \geq \alpha_{m_F} + \alpha_{m_F-d_F(v)+1}$ , while  $\sigma_{\omega_F}(x_2) = \alpha_{m_F-d_F(v)+1} + \alpha_{m_F-d_F(v)+2}$ , and  $\alpha_{m_F} > \alpha_{m_F-d_F(v)+2}$  because  $d_F(v) \geq 3$ .

We finish the extension of  $\omega_{F'}$  to  $F$  by arbitrarily injectively assigning the remaining non-used smaller weights to the edges  $x_2y_2, \dots, x_by_b$ . Because all the  $x_i$ 's have degree 2 and the  $y_i$ 's have degree 1, no conflict may arise between those vertices (Observation 3.2.1). Furthermore, since the degree of  $v$  is larger than the degree of the  $x_i$ 's, and the weights assigned to the  $vx_i$ 's are bigger than the weights assigned to the  $x_iy_i$ 's for  $i \geq 2$ , by Observation 3.2.1, no conflict may arise between  $v$  and the  $x_i$ 's for  $i \geq 2$ . Hence we obtained an injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $F$ , a contradiction.  $\square$

We finally study last multifathers of  $F$  being at the maximum distance from  $r$ , we call these vertices the *deepest last multifathers* of  $F$ . From now on, we focus on a fixed deepest last multifather  $v^*$  of  $F$ , which we choose arbitrarily. In the upcoming proof, for any vertex  $v$  of  $F$ , we denote by  $F_v$  the subtree of  $F$  rooted at  $v$  and induced by  $v$  and its descendants in  $F$ . Recall that all children of a last multifather are leaves (Claim 3.3.6).

**Claim 3.3.7.** *Every last multifather  $v$  of  $F_{f(v^*)}$  is a child of  $f(v^*)$ . In particular,  $v$  is a deepest last multifather of  $F$ .*

*Proof.* The claim follows from the fact that if there exists a descendant  $v \neq v^*$  of  $f(v^*)$  being at distance at least 2 from  $f(v^*)$ , then  $v$  would, in  $F$ , be at greater distance from  $r$  than  $v^*$  is. This would contradict the fact that  $v^*$  is a deepest last multifather.  $\square$

Recall that  $f(v^*)$  cannot be incident, in  $F$ , to a pendent path with length at least 3 (Claim 3.3.4). Hence, every child of  $f(v^*)$  is either a leaf (type-1), a degree-2 vertex adjacent to a leaf (type-2, i.e. the inner vertex of a pendent path with length 2), or a deepest last multifather (type-3). See Figure 3.2 for an illustration. Furthermore, we know that  $f(v^*)$  is adjacent to at least one type-3 vertex, which is  $v^*$ . In the following proof, we show that  $v^*$  is actually the only child of  $f(v^*)$  in  $F$ .

**Claim 3.3.8.** *Vertex  $v^*$  is the only child of  $f(v^*)$  in  $F$ .*



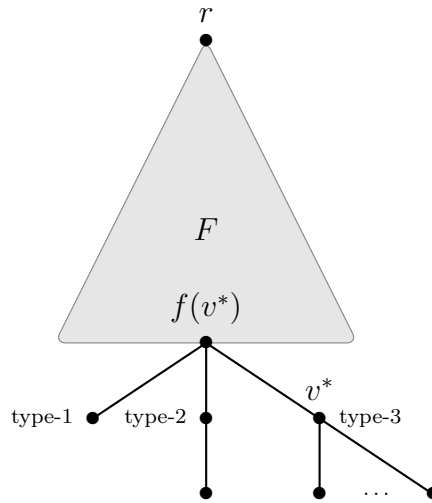


Figure 3.2: Illustration of the three children types mentioned in the proof of Theorem 3.3.3.

*Proof.* Suppose the claim is false, and let  $v \neq v^*$  be another child of  $f(v^*)$ . Let  $x_1$  and  $x_2$  be two leaves adjacent to  $v^*$ , which exist since  $v^*$  is a last multifather, and all pendent paths attached to  $v^*$  have length 1 (Claim 3.3.6).

Assume first that  $v$  is type-2 or type-3, or, in other words, that  $d_F(v) \geq 2$ . In that case,  $v$  is adjacent to at least one leaf, say  $y$ . We here consider  $F' := F - \{vy, v^*x_1, v^*x_2\}$ . Note that  $F'$  remains nice and has fewer edges than  $F$ . Due to the minimality of  $F$ , there exists an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_{F'}}\}$ -edge-weighting  $\omega_{F'}$  of  $F'$ , where  $m_{F'} := |E(F')|$ . We show below that  $\omega_{F'}$  can be extended to the three removed edges with injectively using the three edge weights  $\alpha_{m_F-2}, \alpha_{m_F-1}, \alpha_{m_F}$ , yielding an injective neighbour-sum-distinguishing  $W$ -edge-weighting  $\omega_F$  of  $F$ , a contradiction.

We first assign a weight to  $v^*x_1$  based on the conflicts that may happen when weighting  $vy$ . When assigning any of the three weights to  $vy$ , the only problem which may occur, recall Observation 3.2.1, is that  $\sigma_{\omega_F}(v)$  gets equal to  $\sigma_{\omega_{F'}}(f(v^*))$ . If assigning one of the three weights  $\alpha_{m_F-2}, \alpha_{m_F-1}, \alpha_{m_F}$  to  $vy$  indeed results in that conflict, we assign that weight to  $v^*x_1$ . Otherwise, we assign any of the three weights to  $v^*x_1$ . In any case, no conflict may arise as  $\sigma_{\omega_F}(v^*)$  is still not determined.

We are now left with two weights, which we must assign to  $v^*x_2$  and  $vy$ . Due to the choice of the weight assigned to  $v^*x_1$ , we note that no problem may occur when weighting  $vy$ . Hence, we just have to weight  $v^*x_2$  correctly and assign the remaining weight to  $vy$ . When weighting  $v^*x_2$ , the only problem which may occur, according to Observation 3.2.1, is that  $\sigma_{\omega_F}(v^*)$  gets equal to  $\sigma_{\omega_{F'}}(f(v^*))$ . But, since we have two distinct weights to work with, one of them can be assigned to  $v^*x_2$  so that this conflict is avoided. Thus we can weight  $v^*x_2$  correctly and eventually weight  $vy$  with the remaining weight, resulting in the claimed edge-weighting  $\omega$ .

We may now assume that all children, including  $v$ , of  $f(v^*)$  different from  $v^*$  are

type-1, i.e. leaves. The contradiction can then be obtained quite similarly as in the previous case but with setting  $F' := F - \{f(v^*)v, v^*x_1, v^*x_2\}$ . When weighting  $f(v^*)v$ , we have to make sure, if  $f(v^*) \neq r$ , that  $\sigma_{\omega_F}(f(v^*))$  does not get equal to  $\sigma_{\omega_{F'}}(f(f(v^*)))$ . Note that if  $f(v^*) = r$ , then the situation is actually easier as there is one less conflict to consider. If one of the three available weights  $\alpha_{m_F-2}, \alpha_{m_F-1}, \alpha_{m_F}$ , when assigned to  $f(v^*)v$ , yields a conflict involving  $f(v^*)$  and  $f(f(v^*))$ , then we assign that weight to  $v^*x_1$ . Otherwise, we assign any weight to  $v^*x_1$ . This ensures that, when assigning any of the two remaining weights to  $f(v^*)v$ , no conflict may involve  $f(v^*)$  and  $f(f(v^*))$ . We finally arbitrarily assign the two remaining weights to  $v^*x_2$  and  $f(v^*)v$ . If this results in a neighbour-sum-distinguishing edge-weighting  $\omega_F$  of  $F$ , then we are done. Otherwise, it means that  $\sigma_{\omega_F}(v^*) = \sigma_{\omega_F}(f(v^*))$ . In that case, note that, because all assigned edge weights are distinct, when swapping the values assigned to  $v^*x_2$  and  $f(v^*)v$  by  $\omega_F$  that conflict cannot remain. Furthermore, according to the remarks above, we still do not create any sum conflict involving  $f(v^*)$  and  $f(f(v^*))$ . After the swapping operation  $\omega_F$  hence gets neighbour-sum-distinguishing, a contradiction.  $\square$

We are now ready to finish the proof by showing that, under all information we have obtained,  $F$  actually admits an injective neighbour-sum-distinguishing  $W$ -edge-weighting, a contradiction.

From Claim 3.3.8, we get that  $d_F(f(v^*)) = 2$ , as  $v^*$  is not the root of  $F$ , so  $f(f(v^*))$  exists. Let  $x_1, \dots, x_k$  be the  $k \geq 2$  leaves attached to  $v^*$  in  $F$ , which exist since  $v^*$  is a type-3 vertex. Now consider the tree  $F' := F - \{v^*x_1, \dots, v^*x_k\}$  with size  $m_{F'} := |E(F')|$ . Due to the minimality of  $F$ , there exists an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_{F'}}\}$ -edge-weighting  $\omega_{F'}$  of  $F'$ . We extend  $\omega_{F'}$  to the  $k$  removed edges so that an injective neighbour-sum-distinguishing  $W$ -edge-weighting  $\omega_F$  of  $F$  is obtained, a contradiction. To that aim, we arbitrarily injectively assign the weights  $\alpha_{m_F-k+1}, \dots, \alpha_{m_F}$  to the pendent edges  $v^*v_1, \dots, v^*v_k$  attached to  $v^*$ . Recall that we cannot get sum conflicts involving  $v^*$  and the  $v_i$ 's according to Observation 3.2.1. Furthermore, we have  $d_F(v^*) \geq 3$  while  $d_F(f(v^*)) = 2$  (Claim 3.3.8), and we have used the  $k$  biggest weights of  $W$  to weight the edges  $v^*x_i$ . From this and Observation 3.2.1, we get that, necessarily,  $\sigma_{\omega_F}(v^*) > \sigma_{\omega_{F'}}(f(v^*))$ . Hence,  $\omega_F$  is neighbour-sum-distinguishing, a contradiction.  $\square$

## 3.4 General upper bounds

Towards Conjecture 3.1.1, we exhibit, in this section, for any nice graph  $G$ , an upper bound on  $\chi_{\Sigma}^{e,1}(G)$  of the form  $k \cdot |E(G)|$ , where  $k$  is a fixed constant.

It turns out, first, that some results towards the 1-2-3 Conjecture can be extended to the injective context, hence yielding bounds to our context. This is in particular the case of the weighting algorithm by [KKP10], which was designed to prove that  $\chi_{\Sigma}^e(G) \leq 5$  holds for every nice graph  $G$ . In very brief words, this algorithm initially assigns the list of weights  $\{1, 2, 3, 4, 5\}$  to every edge of  $G$ , which contains the possible weights that any edge can be assigned at any moment of the algorithm. The algorithm then linearly processes the vertices of  $G$  with possibly adjusting some incident edges weights (but staying in the

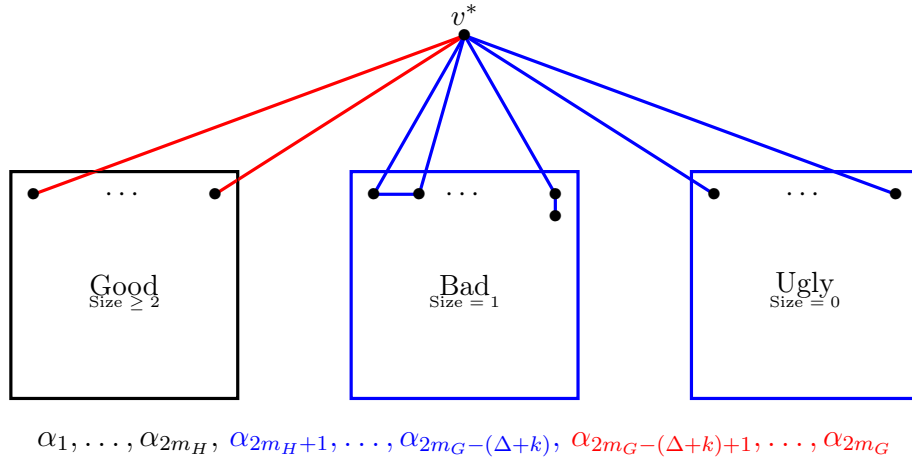


Figure 3.3: Illustration of the three types of components in the proof of Theorem 3.4.1, each edge or component on the figure is weighted using the weights of the same colour.

list  $\{1, 2, 3, 4, 5\}$ ) so that sum conflicts are avoided around any vertex considered during the course.

It is easy to check that this algorithm also works under the assumption that every edge of  $G$  is assigned a (possibly unique) list of five allowed consecutive weights  $\{\alpha - 2, \alpha - 1, \alpha, \alpha + 1, \alpha + 2\}$ . In particular, when applied with non-intersecting such lists assigned to the edges, the algorithm yields an injective nsd edge-weighting, as every edge weight can be assigned to at most one edge. So, applying the algorithm on a nice graph  $G$  with edges  $e_0, \dots, e_{m-1}$  where each edge  $e_i$  is assigned the list  $\{5i + 1, 5i + 2, 5i + 3, 5i + 4, 5i + 5\}$  results in an injective nsd  $(5 \cdot |E(G)|)$ -edge-weighting of  $G$ . From this, we get that  $\chi_{\Sigma}^{e,1}(G) \leq 5 \cdot |E(G)|$  holds for every nice graph  $G$ .

The  $5 \cdot |E(G)|$  bound on  $\chi_{\Sigma}^{e,1}(G)$  above can actually be improved down to  $2 \cdot |E(G)|$  by means of a careful inductive proof scheme, which we describe in the following proof. We actually prove (here and further) a stronger statement to get rid of the non-connected cases.

**Theorem 3.4.1.** *Let  $G$  be a nice graph. Then, for every set  $W$  of  $2 \cdot |E(G)|$  distinct positive weights, there exists an injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $G$ . As a consequence, we have  $\chi_{\Sigma}^{e,1}(G) \leq 2 \cdot |E(G)|$ .*

*Proof.* The proof is by induction on  $n_G + m_G$ , where  $n_G := |V(G)|$  and  $m_G := |E(G)|$ . As it can easily be checked that the claim is true for small values of  $n_G + m_G$ , we proceed to the induction step. Consider hence a value of  $n_G + m_G$  such that the claim is true for all smaller values of this sum.

We may assume that  $G$  is connected, as otherwise induction could be invoked on the different components of  $G$ . Set  $\Delta := \Delta(G)$ . Since we may assume that  $m_G \geq 4$  and  $G$  is nice, we clearly have  $\Delta \geq 2$ . We may even assume that  $\Delta \geq 3$ , as otherwise  $G$

would admit an injective neighbour-sum-distinguishing  $W$ -edge-weighting according to Observation 3.3.2. Consider any vertex  $v^*$  of  $G$  verifying  $d_G(v^*) = \Delta$  and denote by  $u_1, \dots, u_\Delta$  the neighbours of  $v^*$  in  $G$ .

Set  $G' := G - v^*$ . Note that  $G'$  may include components isomorphic to  $K_2$ , and thus be not nice. In this context, we say that a component of  $G'$  is *ugly* if it has no edge, *bad* if it is isomorphic to  $K_2$ , and *good* otherwise. Basically, a bad component of  $G'$  is an edge to which  $v^*$  is joined in  $G$ : either  $v^*$  is adjacent to the two ends of that edge, or  $v^*$  is adjacent to only one of the two ends.

If  $G'$  does not have good components, then  $G$  is a connected graph whose only vertex with degree at least 3 is  $v^*$  such that  $G'$  consists of isolated vertices and isolated edges only. In particular, all vertices of  $G$  but  $v^*$  have degree at most 2, and every degree-2 vertex  $u_i$  adjacent to  $v^*$  is either adjacent to another degree-2 vertex  $u_j$  adjacent to  $v^*$ , or adjacent to a degree-1 vertex. In such a situation, assuming  $W := \{\alpha_1, \dots, \alpha_{2m_G}\}$  where  $\alpha_1 < \dots < \alpha_{2m_G}$ , it can easily be seen that assigning decreasing weights  $\alpha_{2m_G}, \dots, \alpha_1$ , following this order, to the edges of  $G$  as they are encountered while performing a breadth-first search algorithm from  $v^*$ , results in an injective nsd edge-weighting of  $G$ . This notably follows as a consequence of Observation 3.2.1.

Hence we may assume that  $G'$  has good components  $C_1, C_2, \dots, C_k$ . Let  $H$  denote the union of the  $C_i$ 's, and set  $m_H := |E(H)|$ . Since the  $C_i$ 's are nice by definition, so is  $H$ . Furthermore, we have that  $m_H < m_G$ . According to the induction hypothesis, there hence exists an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{2m_H}\}$ -edge-weighting  $w_H$  of  $H$ . In order to get an injective neighbour-sum-distinguishing  $W$ -edge-weighting  $w_G$  of  $G$ , we eventually need to extend  $w_H$  to the remaining edges of  $G$ , *i.e.* to the  $v^*u_i$ 's and the edges of the bad components of  $G'$ .

To that aim, we use only weights among  $\{\alpha_{2m_H+1}, \dots, \alpha_{2m_G}\}$  injectively, *i.e.* we do not use non-used weights among  $\{\alpha_1, \dots, \alpha_{2m_H}\}$  (see Figure 3.3). Let  $u_1, \dots, u_k$  denote the neighbours of  $v^*$  belonging to good components of  $G$ . We start by injectively assigning weights to the edges  $v^*u_1, \dots, v^*u_k$  using  $\Delta+k$  of the weights in  $\{\alpha_{2m_G-(\Delta+k)+1}, \dots, \alpha_{2m_G}\}$ , without raising any sum conflict. This is possible for every considered edge  $v^*u_i$ , since each  $u_i$  has degree at most  $\Delta - 1$  in  $H$  and we have at least  $\Delta + k - (i - 1) \geq \Delta + 1$  different available weights.

We are now left with weighting the edges of  $G$  belonging to the bad components, or being incident with  $v^*$  and a bad component. Assume there are  $m'$  of them. Then we have  $m_G = m_H + k + m'$ , and, since  $k + m' \geq \Delta$ , we have

$$2m_G - (\Delta + k) - 2m_H = k + 2m' - \Delta \geq m'.$$

The set  $\{\alpha_{2m_H+1}, \dots, \alpha_{2m_G-(\Delta+k)}\}$  hence contains sufficiently many weights for weighting all of the  $m'$  remaining edges. To that aim, we assign the weights  $\alpha_{2m_G-(\Delta+k)}, \dots, \alpha_{2m_H+1}$ , following this order (*i.e.* in decreasing order of magnitude), to these  $m'$  remaining edges as they are encountered during a breadth-first search algorithm performed from  $v^*$ .

It can easily be checked that, by the weighting scheme described above, for every neighbour  $v$  of  $v^*$ , the weights in  $P_\omega(v^*) - \omega(vv^*)$  are greater than all the weights in

### 3. The 1-2-3 Conjecture and the Antimagic Labelling Conjecture

---

$P_\omega(v) - \omega(vv^*)$ . Hence, by Observation 3.2.1, vertex  $v^*$  is distinguished from all its neighbours. By the same observation, it can be checked that no sum conflict can involve vertices of  $G - H$ , thus that the resulting injective edge-weighting is neighbour-sum-distinguishing.  $\square$

We now provide a second upper bound on  $\chi_\Sigma^{e,1}(G)$  of the form  $|E(G)| + k$  for every nice graph  $G$ . Here, our  $k$  is a small linear function of  $\Delta(G)$ , making the bound mostly interesting in the context of nice graphs with bounded maximum degree, and generally better than the bound in Theorem 3.4.1 (except in some cases to be discussed later). The proof scheme we employ here is different from the one used to prove Theorem 3.4.1.

**Theorem 3.4.2.** *Let  $G$  be a nice graph. Then, for every set  $W$  of  $|E(G)| + 2\Delta(G)$  distinct positive weights, there exists an injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $G$ . As a consequence, we have  $\chi_\Sigma^{e,1}(G) \leq |E(G)| + 2\Delta(G)$ .*

*Proof.* We may assume that  $G$  is connected. Set  $\Delta := \Delta(G)$ , and let  $n := |V(G)|$  and  $m := |E(G)|$  denote the order and size, respectively, of  $G$ . Also, set  $W := \{\alpha_1, \dots, \alpha_{m+2\Delta}\}$  with  $0 < \alpha_1 < \dots < \alpha_{m+2\Delta}$ . First choose a vertex  $v^*$  with degree  $\Delta$  in  $G$ , and let  $T$  be a spanning tree of  $G$  including all edges incident to  $v^*$ . From  $T$ , we deduce a partition  $V_0 \cup \dots \cup V_k$  of  $V(G)$ , where each part  $V_d$  includes the vertices of  $G$  being at distance  $d$  from  $v^*$  in  $T$ . In particular,  $V_0 = \{v^*\}$ , and, for every vertex  $u$  in a part  $V_i$  with  $i \neq 0$ , there is exactly one edge from  $u$  to  $V_{i-1}$  in  $T$ . We call this edge the *private edge* of  $u$ .

We now describe how to obtain an injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $G$ . We start by assigning the edge weights  $\alpha_1, \dots, \alpha_{m-(n-1)}$  to the edges of  $E(G) \setminus E(T)$  in an arbitrary way. This leaves us with all edges of  $T$  to be weighted, which includes at least one incident (private) edge for every vertex different from  $v^*$ , and all edges incident to  $v^*$ . To weight these edges without creating any conflict, we will first consider all vertices of  $V_k$  and weight their private edges carefully, then do the same for all vertices of  $V_{k-1}$ , and so on layer by layer until all edges of  $T$  are weighted. Fixing any ordering over the vertices of each  $V_i$  for  $i \in \{1, \dots, k\}$ , this weighting scheme yields an ordering  $u_1, \dots, u_{n-1}$  in which the vertices are considered (*i.e.* the  $|V_k|$  first  $u_i$ 's belong to  $V_k$ , the  $|V_{k-1}|$  next  $u_i$ 's belong to  $V_{k-1}$ , and so on; the  $|V_1|$  last  $u_i$ 's belong to  $V_1$ ). We note that the private edges of the  $|V_1|$  last  $u_i$ 's go to  $v^*$ .

To extend the injective neighbour-sum-distinguishing edge-weighting to the edges of  $T$  correctly, we consider the  $u_i$ 's in order, and for each of these vertices, we weight its private edge in such a way that no sum conflict arises. Assume we are currently dealing with vertex  $u_i$ , meaning that all previous  $u_i$ 's have been correctly treated. If  $u_i \notin V_1$ , then we assign to the private edge of  $u_i$  a non-used weight among  $\{\alpha_{m-(n-1)+1}, \dots, \alpha_m\}$  in such a way that  $\sigma(u_i)$  is different from the sums each of the at most  $\Delta - 1$  already processed neighbours of  $u_i$ . Note that, even for the last  $u_i$  not in  $V_1$  to be considered, the number of remaining non-used weights in  $\{\alpha_{m-(n-1)+1}, \dots, \alpha_m\}$  is at least  $\Delta + 1$ , so this weighting extension can be applied to every vertex.

Now, if  $u_i \in V_1$ , then we apply the same strategy but with the weights among  $\{\alpha_{m+1}, \dots, \alpha_{m+2\Delta}\}$ . Again, even for  $u_{n-1}$ , note that this set includes at least  $\Delta + 1$

non-used weights, so we can correctly choose a weight for  $u_{n-1}v^*$  so that  $\sigma(u_{n-1})$  gets different from the sums of the previously-treated vertices. To finish off the proof, we note that, by that strategy, all edges incident to  $v^*$  have been weighted with weights among  $\{\alpha_{m+1}, \dots, \alpha_{m+2\Delta}\}$ . Since  $d(v^*) = \Delta$ , by Observation 3.2.1 we get that  $v^*$  is sum-distinguished from all its neighbours.  $\square$

Note that the  $2 \cdot |E(G)|$  bound from Theorem 3.4.1, can, in several situations, be better than the  $|E(G)| + 2\Delta(G)$  bound from Theorem 3.4.2. To be convinced of that statement, consider the class of graphs obtained by starting from any star with  $\Delta$  leaves  $u_1, \dots, u_\Delta$  and adding no more than  $\Delta - 1$  edges joining pairs of vertices among  $\{u_1, \dots, u_\Delta\}$ .

## 3.5 Refined upper bounds for particular classes of sparse graphs

We now improve the bounds in Section 3.4 to bounds of the form  $|E(G)| + k$ , where  $k$  is a small constant, for several classes of nice graphs  $G$ . Our weighting strategy here relies on removing some edges from  $G$ , then deducing a correct edge-weighting of the remaining graph, and extending that weighting to  $G$ . So that this weighting strategy applies, we focus on rather sparse graph classes with particular properties inherited by their subgraphs. In that respect, we give a special focus to nice 2-degenerate graphs, and nice graphs with maximum average degree at most 3. It is worth recalling that these graphs may have arbitrarily large maximum degree, so Theorem 3.4.2 does not provide the kind of bound we are here interested in.

Throughout this section, when speaking of a  $k$ -vertex, we mean a degree- $k$  vertex. By a  $k^-$ -vertex (resp.  $k^+$ -vertex), we refer to a vertex with degree at most (resp. at least)  $k$ .

### 3.5.1 2-degenerate graphs

In this subsection, we focus on nice 2-degenerate graphs, and exhibit an upper bound on their value of  $\chi_\Sigma^{e,1}$ .

**Theorem 3.5.1.** *Let  $G$  be a nice 2-degenerate graph. Then, for every set  $W$  of  $|E(G)| + 4$  distinct positive weights, there exists an injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $G$ . As a consequence, we have  $\chi_\Sigma^{e,1}(G) \leq |E(G)| + 4$ .*

*Proof.* Assume the claim is false, and let  $G$  be a counterexample that is minimal in terms of  $n_G + m_G$ , where  $n_G := |V(G)|$  and  $m_G := |E(G)|$ . Set  $W := \{\alpha_1, \dots, \alpha_{m_G+4}\}$ . We show below that  $G$  cannot be a counterexample, and thereby get a contradiction. This is done by showing that we can always remove some edges from  $G$  while keeping the graph nice, then deduce an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_{G'}+4}\}$ -edge-weighting  $\omega_{G'}$  of the remaining graph  $G'$ , where  $m_{G'} := |E(G')|$ , and finally extend  $\omega_{G'}$  to get an injective neighbour-sum-distinguishing  $W$ -edge-weighting  $\omega_G$  of  $G$ .

### 3. The 1-2-3 Conjecture and the Antimagic Labelling Conjecture

---

We start by pointing out properties of  $G$  we may assume. Clearly, we may suppose that  $G$  is connected. According to Observation 3.3.2, we may also assume that  $\Delta(G) \geq 3$ , and, therefore, that  $m_G \geq 4$ , as otherwise  $G$  would be a tree, in which case a weighting exists according to Theorem 3.3.3. We note as well that the 1-vertices of  $G$  must be adjacent to vertices with sufficiently large degree.

**Claim 3.5.2.** *Every 1-vertex of  $G$  is adjacent to a  $6^+$ -vertex.*

*Proof.* Assume for contradiction that  $G$  has a 1-vertex  $u$  adjacent to a  $5^-$ -vertex  $v$ . Let  $G' := G - \{uv\}$ . Then  $G'$  is 2-degenerate, and nice as otherwise  $G$  would be a path of length 2 (in which case Theorem 3.3.3 applies). Thus  $G'$  admits an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_{G'}+4}\}$ -edge-weighting  $\omega_{G'}$ , where  $m_{G'} := m_G - 1$ . According to Observation 3.2.2, we can correctly extend  $\omega_{G'}$  to  $uv$ , hence to  $G$ , since we have at least five distinct weights available for that. A contradiction.  $\square$

From Claim 3.5.2, we also deduce the following:

**Claim 3.5.3.**  *$G - u$  is nice for every vertex  $u$ , and therefore  $G - uv$  is nice for every edge  $uv$ .*

*Proof.* Let  $u$  be a vertex of  $G$ , and  $G' := G - u$ . Now assume that  $G'$  is not nice. Since  $G$  is nice, one neighbour of  $u$  has degree 2 in  $G$  and is adjacent to a vertex  $v'$  of degree 1 in  $G$ , with  $v' \neq u$ , a contradiction with Claim 3.5.2. Hence  $G - u$  is nice for every vertex  $u$ . Moreover, it is clear that this implies that  $G - uv$  is also nice for every edge  $uv$ .  $\square$

As a consequence of Claim 3.5.3 and Observation 3.2.2, we immediately get the following:

**Claim 3.5.4.**  *$G$  has no edge  $uv$  with  $d_G(u) + d_G(v) \leq 6$ .*

We are now ready to finish the proof of Theorem 3.5.1. Let  $S_1$  denote the set of  $2^-$ -vertices of  $G$ , and set  $G_1 := G - S_1$ . Since  $\Delta(G) \geq 3$ , graph  $G_1$  has vertices. In particular, since  $G_1$  is 2-degenerate, it has a  $2^-$ -vertex  $v$ . Let us denote as  $d^{2^-}(v)$  the number of neighbours, in  $G$ , of  $v$  belonging to  $S_1$ . Then  $d_G(v) = d^{2^-}(v) + d_{G_1}(v)$ . Moreover, since  $v \notin S_1$ ,  $d_G(v) \geq 3$ . Hence,  $d^{2^-}(v) \geq 1$ . Therefore, by Claim 3.5.4,  $d_G(v) \geq 5$ , and since  $d_{G_1}(v) \leq 2$ ,  $d^{2^-}(v) \geq 3$ .

Now, let  $v_1, v_2, v_3$  be three neighbours of  $v$  belonging to  $S_1$ . We here consider  $G' := G - \{vv_1, vv_2, vv_3\}$ . Note that  $G'$  has to be nice, as otherwise  $G$  would have an edge violating Claim 3.5.4. Due to the minimality of  $G$ , and because  $G'$  is a nice 2-degenerate graph, there exists an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_{G'}+4}\}$ -edge-weighting  $\omega_{G'}$  of  $G'$ . We will extend  $\omega_{G'}$  to  $vv_1, vv_2, vv_3$ , thus to  $G$ , assigning weights among a set of seven weights including those among  $\{\alpha_{m_G+2}, \alpha_{m_G+3}, \alpha_{m_G+4}\}$  in the following way.

We first assign a weight  $\beta_1$  from  $\{\alpha_{m_G+3}, \alpha_{m_G+4}\}$  to the edge  $vv_1$  so that we do not create a sum conflict involving  $v_1$  and its neighbour different from  $v$  (if any), which is clearly possible with two distinct weights. Similarly, we then assign a weight  $\beta_2$  from

$\{\alpha_{m_G+2}, \alpha_{m_G+3}, \alpha_{m_G+4}\} \setminus \{\beta_1\}$  to  $vv_2$  so that we do not create a sum conflict involving  $v_2$  and its neighbour different from  $v$  (if any). Note that due to the choice of  $\beta_1$  and  $\beta_2$ , which are strictly bigger than the weights among  $\{\alpha_1, \dots, \alpha_{m_{G'}+4}\}$ , no matter how we extend the weighting to  $vv_3$  it cannot occur that  $\sigma_{\omega_G}(v)$  gets equal to the sum of weights incident to a neighbour of  $v$  in  $G$  belonging to  $S_1$ . Hence, when extending  $\omega_{G'}$  to  $vv_3$ , we just have to make sure that  $\sigma_{\omega_G}(v_3)$  does not get equal to the sum of weights incident to the neighbour of  $v_3$  different from  $v$  (if any), and that  $\sigma_{\omega_G}(v)$  does not get equal to the sums of weights incident to its at most two neighbours in  $G_1$ . So there are at most three conflicts to take into account while we have five weights in hand to weight  $vv_3$ . Clearly, this is sufficient to extend the weighting.  $\square$

### 3.5.2 Graphs with maximum average degree at most 3

In this subsection, we prove an upper bound on  $\chi_{\Sigma}^{e,1}$  for every nice graph with maximum average degree at most 3.

**Theorem 3.5.5.** *Let  $G$  be a nice graph with  $\text{mad}(G) \leq 3$ . Then, for every set  $W$  of  $|E(G)| + 6$  distinct positive weights, there exists an injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $G$ . In particular, we have  $\chi_{\Sigma}^{e,1}(G) \leq |E(G)| + 6$ .*

*Proof.* Assume there exists a counterexample to the claim, that is, there exists a nice graph  $G$  for which we have  $\text{mad}(G) \leq 3$  but, for a particular set  $W$  including  $|E(G)| + 6$  weights, there is no injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $G$ . We consider  $G$  minimal in terms of  $n_G + m_G$ , where  $n_G := |V(G)|$  and  $m_G := |E(G)|$ . Set  $W := \{\alpha_1, \dots, \alpha_{m_G+6}\}$ , where  $\alpha_1 < \dots < \alpha_{m_G+6}$ . Our ultimate goal in this proof is to show that  $G$  cannot exist. The strategy we employ to this end is essentially to show that  $G$  has a nice subgraph  $H$ , with order  $n_H$  and size  $m_H$ , such that  $H$  has an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_H+6}\}$ -edge-weighting  $\omega_H$  that can be extended to an injective neighbour-sum-distinguishing  $W$ -edge-weighting  $\omega_G$  of  $G$ , contradicting the fact that  $G$  is a counterexample. The main tool we want to use, in order to show that  $H$  has such an edge-weighting, is Theorem 3.4.2. Since  $G$  is a counterexample to the claim, note that Theorem 3.4.2 already implies that  $\Delta(G) \geq 4$ . Furthermore, we may assume that  $G$  is connected, and is not a tree as otherwise Theorem 3.3.3 would apply.

The subgraph  $H$  we consider is obtained by removing all 1-vertices from  $G$ . Of course, we have  $\text{mad}(H) \leq 3$  and it may happen that  $G = H$ . We may as well assume that  $H$  remains nice, as, if it is not the case, then  $G$  would be a tree (a bistar, i.e. a tree having exactly two  $2^+$ -nodes, being adjacent), which is not possible as pointed out above.

In the following result, we observe that, by showing that  $H$  verifies  $\Delta(H) \leq 3$ , then we will get our conclusion.

**Proposition 3.5.6.** *If  $\Delta(H) \leq 3$ , then  $G$  is not a counterexample.*

*Proof.* If  $G = H$ , then  $G$  admits an injective neighbour-sum-distinguishing  $W$ -edge-weighting according to Theorem 3.4.2 since we would have  $\Delta(G) \leq 3$ . So assume that  $G$



### 3. The 1-2-3 Conjecture and the Antimagic Labelling Conjecture

---

has 1-vertices. Since we assume that  $\Delta(H) \leq 3$ , there exists an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_H+6}\}$ -edge-weighting  $\omega_H$  of  $H$ , still according to Theorem 3.4.2.

We now extend  $\omega_H$  to the pendent edges of  $G$ . We successively consider every vertex  $v$  of  $H$  incident to a pendent edge. We start by assigning an arbitrary non-used weight to every pendent edge incident to  $v$ , but one, say  $vu$ .

We claim that we can find a correct weight for  $vu$ . First, we note, according to Observation 3.2.1, that only the neighbours of  $v$  in  $H$  can cause sum conflicts. Hence, when extending  $\omega_H$  to  $vu$ , we just have to make sure, since  $vu$  is the last non-weighted pendent edge incident to  $v$ , that  $\sigma(v)$  does not meet any of the determined sums of the vertices adjacent to  $v$  in  $H$ . By our assumption on  $\Delta(H)$ , there are at most three such vertices, while we have at least seven ways to weight  $vu$  (among  $\{\alpha_1, \dots, \alpha_{m_G+6}\}$ ), each determining a distinct value for  $\sigma(v)$ . We can hence find a correct non-used weight for  $vu$ .

Since the process above can be applied for all vertices of  $H$  incident to a pendent edge in  $G$ , weighting  $\omega_H$  can hence be extended to all pendent edges of  $G$ . Thus  $\omega_H$  can be extended to an injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $G$ , as claimed.  $\square$

It remains to show that  $\Delta(H) \leq 3$ . This is proved by getting successive information concerning the structure of  $H$  so that classical discharging arguments can eventually be employed.

**Claim 3.5.7.** *If  $v \in V(H)$  is adjacent to 1-vertices in  $G$ , then  $d_H(v) \geq 7$ .*

*Proof.* This follows from Observation 3.2.2, as, when removing a pendent edge from  $G$ , applying induction, and putting the edge back, we have seven distinct weights to achieve the extension to  $G$ .  $\square$

**Claim 3.5.8.** *We have  $\delta(H) \geq 2$ .*

*Proof.* If  $\delta(H) = 0$ , then  $G$  is a star, contradicting one of our initial assumptions. Now, if  $\delta(H) = 1$ , then  $G$  includes a vertex  $v$  such that  $d_H(v) = 1$  and  $v$  is incident to pendent edges in  $G$ . But this is impossible as such a  $v$  would not meet the condition in Claim 3.5.7. So  $\delta(H) \geq 2$ .  $\square$

**Claim 3.5.9.** *Graph  $H$  has no two adjacent 2-vertices.*

*Proof.* Suppose that  $H$  has an edge  $uv$  such that  $d_H(u) = d_H(v) = 2$ . Recall that, according to Claim 3.5.7, we have  $d_G(u) = d_G(v) = 2$ . In this case, we consider the graph  $G' := G - \{uv\}$  with size  $m_{G'} := |E(G')|$ . Clearly  $G'$  remains nice (otherwise Claim 3.5.7 would be violated), has  $\text{mad}(G') \leq 3$ , and, due to the minimality of  $G$ , graph  $G'$  admits an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_{G'}+6}\}$ -edge-weighting  $\omega_{G'}$ .

In  $G'$ , we have  $d_{G'}(u) = d_{G'}(v) = 1$ . Let  $u'$  and  $v'$  be the neighbours of  $u$  and  $v$ , respectively, in  $G'$ . Since  $\omega_{G'}$  is injective, we have  $\omega_{G'}(uu') \neq \omega_{G'}(vv')$ . We now note that, under all those assumptions, weighting  $\omega_{G'}$  can easily be extended to an injective neighbour-sum-distinguishing  $W$ -edge-weighting  $\omega_G$  of  $G$ , i.e. to the edge  $uv$ , a contradiction.

---

### 3.5. Refined upper bounds for particular classes of sparse graphs

---

We note that, because  $\omega_{G'}(uu') \neq \omega_{G'}(vv')$  and  $d_G(u) = d_G(v) = 2$ , we cannot get  $\sigma_{\omega_G}(u) = \sigma_{\omega_G}(v)$  when assigning any weight to  $uv$ , recall Observation 3.2.1. So the only constraints we have are that  $\sigma_{\omega_G}(u)$  has to be different from  $\sigma_{\omega_G}(u')$  (which is exactly  $\sigma_{\omega_{G'}}(u')$ ) and  $\sigma_{\omega_G}(v)$  must be different from  $\sigma_{\omega_G}(v')$  (which is exactly  $\sigma_{\omega_{G'}}(v')$ ). These constraints forbid us from assigning, to  $uv$ , at most two of the seven weights that have not been used yet. So we can extend  $\omega_{G'}$  to  $\omega_G$ .  $\square$

**Claim 3.5.10.** *Graph  $H$  has no 2-vertex adjacent to two 3-vertices.*

*Proof.* Assume  $H$  has such a vertex  $v$  with  $d_H(v) = 2$ , and  $v$  has two neighbours  $u_1$  and  $u_2$  verifying  $d_H(u_1) = d_H(u_2) = 3$ . According to Claim 3.5.7, we have  $d_G(v) = 2$ ,  $d_G(u_1) = 3$  and  $d_G(u_2) = 3$ . Let  $G' := G - \{vu_1, vu_2\}$  and  $m_{G'} := |E(G')|$ . Clearly,  $G'$  remains nice with  $\text{mad}(G') \leq 3$ , and, by the minimality of  $G$ , there exists an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_{G'}+6}\}$ -edge-weighting  $\omega_{G'}$ . According to Observation 3.2.3, weighting  $\omega_{G'}$  can be extended to an injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $G$  provided we have at least five distinct edge weights in hand. Since we here have eight non-used edge weights dedicated to weighting  $vu_1$  and  $vu_2$ , the extension of  $\omega_{G'}$  to  $G$  hence exists.  $\square$

**Claim 3.5.11.** *Graph  $H$  has no 3-vertex adjacent to two  $3^-$ -vertices.*

*Proof.* The proof is similar to that of the previous claim. Assume  $H$  has such a 3-vertex  $v$  being adjacent to at least two  $3^-$ -vertices  $u_1$  and  $u_2$ . Again, we set  $G' := G - \{vu_1, vu_2\}$ , and let  $\omega_{G'}$  be an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_{G'}+6}\}$ -edge-weighting of  $G'$ , where  $m_{G'} := |E(G')|$ . Still according to Observation 3.2.3, we know that an extension exists provided we have at least six weights available. So  $\omega_{G'}$  can correctly be extended to  $vu_1$  and  $vu_2$ , as eight edge weights can be used in the present context.  $\square$

Before getting our conclusion, we prove two last claims which are a bit more general than what we actually need.

**Claim 3.5.12.** *Graph  $H$  has no 6-vertex adjacent to two 2-vertices.*

*Proof.* Assume  $H$  has such a 6-vertex  $v$ , and let  $u_1$  and  $u_2$  denote any two of its 2-neighbours. Recall that  $d_H(v) = d_G(v)$ ,  $d_H(u_1) = d_G(u_1)$  and  $d_H(u_2) = d_G(u_2)$  according to Claim 3.5.7. Let  $G' := G - \{vu_1, vu_2\}$  and set  $n_{G'} := |V(G')|$  and  $m_{G'} := |E(G')|$ . Clearly  $G'$  is nice (Claims 3.5.7 and 3.5.9) with  $\text{mad}(G') \leq 3$ , and, since  $n_{G'} + m_{G'} < n_G + m_G$ , there exists an injective neighbour-sum-distinguishing  $\{\alpha_1, \dots, \alpha_{m_{G'}+6}\}$ -edge-weighting  $\omega_{G'}$  of  $G'$ . Again according to Observation 3.2.3, under these conditions, we know that  $\omega_{G'}$  can be extended to  $vu_1$  and  $vu_2$  provided we have at least eight weights available. Since this is precisely the case, we are done.  $\square$

**Claim 3.5.13.** *Graph  $H$  has no 4- or 5-vertex adjacent to at least two  $3^-$ -vertices.*

*Proof.* The proof is very similar to that of Claim 3.5.12, and can be mimicked by letting  $u_1$  and  $u_2$  be two  $3^-$ -vertices adjacent to  $v$ . We then get the same conclusion from Observation 3.2.3.  $\square$

### 3. The 1-2-3 Conjecture and the Antimagic Labelling Conjecture

---

We are now ready to prove that  $H$  has maximum degree 3.

**Claim 3.5.14.** *We have  $\Delta(H) \leq 3$ .*

*Proof.* Assume the contrary, namely that  $\Delta(H) \geq 4$ . We prove the claim by means of the so-called discharging method, through a discharging procedure, based on the following rules.

To every vertex  $v$  of  $H$ , we assign an initial charge  $\rho(v)$  being  $d_H(v) - 3$ . Since  $\text{mad}(H) \leq 3$ , we have

$$\sum_{v \in V(H)} d_H(v) \leq 3 \cdot n_H,$$

which implies that

$$\sum_{v \in V(H)} \rho(v) \leq 0.$$

Without creating or deleting any amount of charge assigned to the vertices, we now transfer a part of the assigned charges from neighbours to neighbours, through three discharging rules applied in two successive steps.

In the sequel, by a *weak 3-vertex* of  $H$  we refer to a 3-vertex adjacent to a 2-vertex (recall that a 3-vertex of  $H$  is adjacent to at most one 2-vertex according to Claim 3.5.11). The first discharging step consists in applying the following rule:

(R1) Every  $4^+$ -vertex transfers  $\frac{1}{4}$  to every adjacent weak 3-vertex.

Once the first discharging step has been performed, we then apply the second step, which consists in applying the following two discharging rules:

(R2) Every weak 3-vertex transfers  $\frac{1}{2}$  to its adjacent 2-vertex.

(R3) Every  $4^+$ -vertex transfers  $\frac{1}{2}$  to every adjacent 2-vertex.

We now compute the final charge  $\rho^*(v)$  that every vertex  $v$  of  $H$  gets once the two steps above have been performed. Recall that  $\delta(H) \geq 2$  according to Claim 3.5.8.

1. If  $v$  is a 2-vertex, then  $v$  is adjacent to a  $4^+$ -vertex, and either a weak 3-vertex or a  $4^+$ -vertex according to Claims 3.5.9 and 3.5.10. Through Rules (R2) and (R3), the two neighbours of  $v$  both transfer  $\frac{1}{2}$  to  $v$ . Hence,  $\rho^*(v) = \rho(v) + 2 \times \frac{1}{2} = 0$ .
2. If  $v$  is a 3-vertex, then  $v$  is either weak, or not. If  $v$  is not weak, it is not concerned by any of Rules (R1), (R2) and (R3), so  $\rho^*(v) = \rho(v) = 0$ . Now assume  $v$  is a weak 3-vertex. According to Claim 3.5.11, vertex  $v$  is adjacent to a 2-vertex  $u$ , and two  $4^+$ -vertices  $z_1$  and  $z_2$ . Through Rule (R1), vertex  $v$  receives  $\frac{1}{4}$  from each of  $z_1$  and  $z_2$ , while, through Rule (R2), vertex  $v$  then transfers  $\frac{1}{2}$  to  $u$ . Therefore,  $\rho^*(v) = \rho(v) + 2 \times \frac{1}{4} - \frac{1}{2} = 0$ .

3. If  $v$  is a 4- or 5-vertex, then  $v$  is adjacent to at most one vertex being either a 2-vertex or weak 3-vertex  $u$  according to Claim 3.5.13. The case where  $\rho^*(v)$  is minimum is when  $v$  is a 4-vertex and  $u$  is a 2-vertex, in which case  $v$  transfers  $\frac{1}{2}$  to  $u$ . In that case, through Rule (R3), we get  $\rho^*(v) = \rho(v) - \frac{1}{2} = \frac{1}{2}$ . So, whenever  $v$  is a 4- or 5-vertex, we get  $\rho^*(v) > 0$ .
4. If  $v$  is a 6-vertex, then  $v$  is adjacent to at most one 2-vertex according to Claim 3.5.12. The case where  $\rho^*(v)$  gets minimum is essentially when  $v$  has one 2-neighbour and five weak 3-neighbours. In that case, following Rules (R1) and (R3), we get  $\rho^*(v) = \rho(v) - 5 \times \frac{1}{4} - \frac{1}{2} = \frac{5}{4}$ . Hence, we always get  $\rho^*(v) > 0$  in that case.
5. If  $v$  is a  $7^+$ -vertex, then  $v$  transfers most charge when  $v$  is adjacent to  $d_H(v)$  2-vertices. In that case, following Rule (R3) we deduce that  $\rho^*(v) = \rho(v) - d_H(v) \times \frac{1}{2}$ . Under the assumption that  $d_H(v) \geq 7$ , observe that  $\rho(v) > d_H(v) \times \frac{1}{2}$ . So, again, we always have  $\rho^*(v) > 0$  in this case.

From the analysis above, we get, because  $\Delta(H) \geq 4$ , that

$$\sum_{v \in V(H)} \rho(v) \leq 0 < \sum_{v \in V(H)} \rho^*(v),$$

which is impossible as we did not create any new amount of charge when applying the discharging procedure. Hence, we have  $\Delta(H) \leq 3$ .  $\square$

The result is a consequence of Proposition 3.5.6 and Claim 3.5.14  $\square$

Theorem 3.5.5 applies to all nice graphs with maximum average degree at most 3. Among the classes of such graphs, we would like to highlight the class of nice planar graphs with girth at least 6, where the *girth*  $g(G)$  of a graph  $G$  is the length of a shortest cycle. We refer the reader to e.g. the article by [BKN<sup>+</sup>99], wherein the authors noticed that, for every planar graph  $G$ , we have

$$\text{mad}(G) < \frac{2g(G)}{g(G) - 2}.$$

This gives that every planar graph  $G$  with  $g(G) \geq 6$  has  $\text{mad}(G) \leq 3$ .

**Corollary 3.5.15.** *Let  $G$  be a nice planar graph  $G$  with  $g(G) \geq 6$ . Then, for every set  $W$  of  $|E(G)| + 6$  distinct weights, there exists an injective neighbour-sum-distinguishing  $W$ -edge-weighting of  $G$ . In particular, we have  $\chi_{\Sigma}^{e,1}(G) \leq |E(G)| + 6$ .*

## 3.6 Discussion

In this chapter, we have introduced and studied Conjecture 3.1.1 which stands, in some sense, as a combination of the 1-2-3 Conjecture and the Antimagic Labelling Conjecture.

### 3. The 1-2-3 Conjecture and the Antimagic Labelling Conjecture

---

In particular, as a support to Conjecture 3.1.1, we have pointed out that some families of nice graphs agree with it, or sometimes almost agree with it, i.e. up to an additive constant term. Although these results can be regarded as a first step towards Conjecture 3.1.1, it is worth emphasizing that our work does not bring anything new towards attacking the 1-2-3 Conjecture and the Antimagic Labelling Conjecture but rather concerns some side aspects of these two conjectures.

As mentioned in Section 3.1, Haslegrave proved Conjecture 3.1.1 using the probabilistic method in [Has17]. However, the proof is not constructive. Moreover, the bounds on  $\chi_{\Sigma}^{e,1}$  obtained in Section 3.4 can also be obtained using Combinatorial Nullstellensatz on the polynomial

$$\prod_{uv \in E(G)} \left( \sum_{u' \in N(u)} X_{uu'} - \sum_{v' \in N(v)} X_{vv'} \right) \times \prod_{e, e' \in E(G), e \neq e'} (X_e - X_{e'}),$$

but it would be a non-constructive proof also.

As further work towards constructive proof of Conjecture 3.1.1, it would be interesting to exhibit, for all nice graphs  $G$ , bounds on  $\chi_{\Sigma}^{e,1}(G)$  of the form  $|E(G)| + k$  for a fixed constant  $k$ . One could as well try to get a bound of the form  $k \cdot |E(G)|$  for some  $k$  between 1 and 2. Obtaining one of these two bounds would already improve the ones we have exhibited in Section 3.4. It is worth mentioning that our bounds in that section can slightly be improved by making some choices in a more clever way. But these improvements would allow us to save a small constant number of weights only, which is far from the desired improvement we have mentioned earlier.

As another direction, we would also be interested in constructive proofs of Conjecture 3.1.1 for other classes of nice graphs being not known to be antimagic yet. Among such classes, let us mention the case of nice bipartite graphs  $G$ , for which we did not manage to come up with an  $|E(G)| + k$  bound on  $\chi_{\Sigma}^{e,1}(G)$ , for any constant  $k$ . Another such class that would be interesting to investigate is the class of nice subcubic graphs. Since we already have a constructive proof that every nice subcubic graph verify  $\chi_{\Sigma}^{e,1}(G) \leq |E(G)| + 6$ , recall Theorems 3.4.2 and 3.5.5.

Nice planar graphs would also be interesting candidates to investigate, as we have been mostly successful with sparse classes of nice graphs. Our result in Corollary 3.5.15 may be regarded as a first step towards that direction.

Our results in this chapter may also be subject to further investigations. In particular, there is still a gap for nice 2-degenerate graphs and graphs with maximum average degree at most 3 between our bounds in Section 3.5 and the bound in Conjecture 3.1.1. One could as well wonder how to generalize our results to nice  $k$ -degenerate graphs and graphs with maximum average degree at most  $k$  for larger values of  $k$ . In particular, it could be interesting to exhibit, for such a graph  $G$ , a constructive proof of a general upper bound on  $\chi_{\Sigma}^{e,1}(G)$  of the form  $|E(G)| + \mathcal{O}(k)$  involving a small function of  $k$ .



# Chapter 4

## Doubling the 1-2-3 Conjecture

As mentioned in Chapter 1, the 1-2-3 Conjecture asks whether every graph with no component isomorphic to  $K_2$  can be 3-edge-weighted so that every two adjacent vertices  $u$  and  $v$  can be distinguished via the sum of their incident weights, that is the incident sums of  $u$  and  $v$  differ by at least 1.

In this chapter we investigate the consequences of requiring a stronger distinction condition on neighbour-sum-distinguishing edge-weightings. Namely, we consider two adjacent vertices distinguished when their incident sums differ by at least 2. As a guiding line, we conjecture that every graph with no component isomorphic to  $K_2$  admits a 5-edge-weighting permitting to distinguish the adjacent vertices in this stronger way.

We prove this conjecture for several classes of graphs, including bipartite graphs and cubic graphs. We then consider algorithmic aspects, and show that it is NP-complete to determine the smallest  $k$  such that a given bipartite graph admits such a  $k$ -edge-weighting. In contrast, we show that the same problem can be solved in polynomial time for a given tree.

The results presented in this chapter were published in [BBSS18].

### 4.1 Introduction

When designing neighbour-sum-distinguishing edge-weightings, the goal is to make adjacent vertices distinguishable via their incident sums. In ordinary neighbour-sum-distinguishing edge-weightings, adjacent vertices are considered distinguished as soon as their incident sums are distinct. We here investigate edge-weightings that permit to distinguish the adjacent vertices in a stronger way. Namely, we require adjacent vertices to have incident sums differing by at least 2. An edge-weighting with this stronger requirement is said to be neighbour-sum-2-distinguishing (*ns2d* for short) throughout. As observed in upcoming Observation 4.2.1, a neighbour-sum-distinguishing  $k$ -edge-weighting can easily be turned into a neighbour-sum-2-distinguishing  $2k$ -edge-weighting. Moreover, since  $K_2$  does clearly not admit any neighbour-sum-2-distinguishing edge-weighting, the notion of nice graphs for neighbour-sum-distinguishing edge-weightings and for neighbour-sum-2-distinguishing edge-weightings coincide. Again, we can thus wonder about the smallest  $k$

such that a given nice graph  $G$  admits a neighbour-sum-2-distinguishing  $k$ -edge-weighting, which we denote by  $\chi_{\Sigma>1}(G)$ .

Our main goal in this chapter is to study how  $\chi_{\Sigma>1}$  behaves in general, especially for graphs for which the parameter  $\chi_{\Sigma}$  is well understood. As noted in upcoming Observation 4.2.1, the 1-2-3 Conjecture, if true, would imply that  $\chi_{\Sigma>1}(G) \leq 6$  holds for every nice graph  $G$ . One could thus naturally wonder about a 1-2-3-4-5-6 Conjecture for neighbour-sum-2-distinguishing edge-weightings. It actually turns out that we did not manage to exhibit nice graphs  $G$  with  $\chi_{\Sigma>1}(G) = 6$ . On the other hand, we prove, throughout this chapter, that for several common classes of nice graphs  $G$  we have  $\chi_{\Sigma>1}(G) \leq 5$ . We are thus tempted to propose the following.

**Conjecture 4.1.1.** *For every nice graph  $G$ , we have  $\chi_{\Sigma>1}(G) \leq 5$ .*

We here give first evidence towards this conjecture. We start in Section 4.2 by raising connections between neighbour-sum-distinguishing edge-weightings and neighbour-sum-2-distinguishing edge-weightings, from which we deduce first bounds on  $\chi_{\Sigma>1}$ . In Section 4.3, we then prove the conjecture for nice bipartite graphs, and investigate further some particular classes of bipartite graphs, from which we exhibit interesting aspects of the problem. The algorithmic aspects are considered in Section 4.4, where we first prove that it is NP-complete to determine the exact value of  $\chi_{\Sigma>1}$ . This statement is showed to remain true even for bipartite graphs, which contrasts with the complexity of determining the exact value of  $\chi_{\Sigma}$  for these graphs. We then show that determining the exact value of  $\chi_{\Sigma>1}$  can be done in polynomial time for trees. Perspectives for future work are gathered in Section 4.5.

## 4.2 Preliminaries

In this section, we point out general properties that actually apply to neighbour-sum- $d$ -distinguishing edge-weightings (for any  $d \geq 1$ ), *i.e.* edge-weightings where the adjacent sums differ by at least  $d$ . For a given graph  $G$ , we denote by  $\chi_{\Sigma>d-1}(G)$  the least integer  $k$  such that  $G$  admits a neighbour-sum- $d$ -distinguishing  $k$ -edge-weighting. Note that the parameter  $\chi_{\Sigma>0}$  actually corresponds to  $\chi_{\Sigma}$ .

Obviously, by multiplying all weights assigned by a neighbour-sum-distinguishing  $k$ -edge-weighting by  $d$ , we obtain a neighbour-sum- $d$ -distinguishing  $dk$ -edge-weighting.

**Observation 4.2.1.** *For every nice graph  $G$  and integer  $d \geq 1$ , we have  $\chi_{\Sigma>d-1}(G) \leq d\chi_{\Sigma}(G)$ . In particular,  $\chi_{\Sigma>1}(G) \leq 2\chi_{\Sigma}(G)$ .*

Observation 4.2.1 already has several implications towards Conjecture 4.1.1. First, the 1-2-3 Conjecture, if true, would imply that  $\chi_{\Sigma>1}(G) \leq 6$  holds for every nice graph  $G$ . Although we still do not know whether the 1-2-3 Conjecture is true, every partial result towards that conjecture can be adapted to Conjecture 4.1.1. In that line, perhaps the most interesting result to consider is the one due to Kalkowski, Karoński and Pfender [KKP10],



who proved that  $\chi_\Sigma(G) \leq 5$  holds for every nice graph  $G$ . In our context, this and Observation 4.2.1 yield the following, which shows that, when requiring adjacent sums to differ by at least  $d$ , the number of needed weights grows linearly with  $d$ .

**Corollary 4.2.2.** *For every nice graph  $G$  and integer  $d \geq 1$ ,  $\chi_{\Sigma>d-1}(G) \leq 5d$ . In particular,  $\chi_{\Sigma>1}(G) \leq 10$ .*

The bound in Corollary 4.2.2 is immediately improved for every graph  $G$  for which we know that  $\chi_\Sigma(G) < 5$  holds. In particular, we have  $\chi_{\Sigma>1}(G) \leq 6$  for every nice graph  $G$  verifying the 1-2-3 Conjecture, which is very close to Conjecture 4.1.1. Let us recall, in particular, that the 1-2-3 Conjecture was verified for nice bipartite graphs, 3-chromatic graphs, nice complete graphs, and regular graphs with sufficiently large degree. We here refer the reader to the survey [Sea12] by Seamone, wherein all such results are gathered.

As stated in Observation 4.2.1, by multiplying all weights assigned by a neighbour-sum-distinguishing  $k$ -edge-weighting by a same integer  $d$ , we get a neighbour-sum- $d$ -distinguishing  $dk$ -edge-weighting since each  $\sigma(v)$  is multiplied by  $d$ . This of course does not have to be true if one decreases (or increases) all weights by a same  $d$ , since, here, the effect on each  $\sigma(v)$  depends on  $d(v)$ . There are situations, however, where this can be done safely.

**Observation 4.2.3.** *Let  $\omega$  be a neighbour-sum-distinguishing  $d$ -edge-weighting,  $d \geq 1$ , of a graph  $G$  such that, for every edge  $uv$  (where  $d(u) \geq d(v)$ ), we have  $\sigma(u) < \sigma(v)$  (resp.  $\sigma(u) > \sigma(v)$ ). Then, by decreasing (resp. increasing) all edge weights by a same integer  $x$ , we get another neighbour-sum-distinguishing  $d$ -edge-weighting of  $G$ .*

Due to the fact that, in the context of Conjecture 4.1.1, we focus on edge-weightings assigning strictly positive weights, when decreasing edge weights we should also make sure that none become zero or negative. Observation 4.2.3 can nevertheless be used when the smallest edge weight value assigned by  $\omega$  is known. As an illustration, we improve Observation 4.2.1 for nice regular graphs.

**Corollary 4.2.4.** *For every nice regular graph  $G$  and integer  $d \geq 1$ , we have  $\chi_{\Sigma>d-1}(G) \leq d\chi_\Sigma(G) - (d - 1)$ . In particular,  $\chi_{\Sigma>1}(G) \leq 2\chi_\Sigma(G) - 1$ .*

Corollary 4.2.4 notably implies that Conjecture 4.1.1 holds for nice complete graphs and 3-colourable regular graphs, as the 1-2-3 Conjecture was proved for such graphs. Other bounds also follow for regular graphs with larger degree, see [Sea12].

### 4.3 Conjecture 4.1.1 for bipartite graphs

Since the 1-2-3 Conjecture holds for nice bipartite graphs (see [KLT04]), from Observation 4.2.1 we directly get that  $\chi_{\Sigma>1}(G) \leq 6$  holds for every nice bipartite graph  $G$ . It was actually proved by Thomassen, Wu and Zhang [TWZ16] that a bipartite graph  $G$  satisfies  $\chi_\Sigma(G) = 3$  if and only if  $G$  is an odd multicactus. *Odd multicacti* can be defined as follows. Start from a collection  $C_1, \dots, C_m$  of  $m \geq 1$  cycles whose lengths are at least 6 and

congruent to 2 modulo 4, and colour the edges of the  $C_i$ 's in a proper way using colours red and green. An odd multicactus is then any connected graph obtained by repeatedly applying the following operation: pick two components  $G_1$  and  $G_2$ , and identify a green edge of  $G_1$  with a green edge of  $G_2$ . Note that, in particular, every cycle whose length is congruent to 2 modulo 4 is an odd multicactus.

From this all, we directly get the following.

**Corollary 4.3.1.** *For every nice bipartite graph  $G$ , we have*

$$\chi_{\Sigma>1}(G) \leq \begin{cases} 4 & \text{if } G \text{ is not an odd multicactus,} \\ 6 & \text{otherwise.} \end{cases}$$

In the next result, we improve Corollary 4.3.1 by completely proving Conjecture 4.1.1 for all nice bipartite graphs.

**Theorem 4.3.2.** *For every nice bipartite graph  $G$ , we have  $\chi_{\Sigma>1}(G) \leq 5$ .*

*Proof.* Let  $A$  and  $B$  denote the two partite sets of  $G$ . We prove a stronger statement, namely that  $G$  admits a 5-edge-weighting such that the vertices from one of the partite sets have incident sum congruent to 2 or 3 modulo 5, while the vertices from the other partite set have incident sum congruent to 0 modulo 5. Such an edge-weighting is clearly neighbour-sum-2-distinguishing. In this setting, note that we can equivalently look for a neighbour-sum-2-distinguishing  $\{0, 1, 2, 3, 4\}$ -edge-weighting.

Assume the edges of  $G$  are weighted in some way. In what follows, we repeatedly apply the following modification procedure for some given  $\alpha \in \{1, 2, 3, 4\}$ . Let  $u$  and  $v$  be two distinct vertices from a same partite set, and let  $P$  be a path (obviously of even length) from  $u$  to  $v$  in  $G$ . The procedure consists in modifying the weights of the edges of  $P$  from one end to the other by alternately subtracting and adding  $\alpha$  (modulo 5), *i.e.*, apply  $-\alpha, +\alpha, -\alpha, \dots, -\alpha, +\alpha$  to the weights along  $P$ . Note that this only alters the incident sums modulo 5 of  $u$  and  $v$ , the two ends of  $P$ , which change by  $-\alpha$  and  $+\alpha$ , respectively. Also, note that the ‘‘orientation’’ of  $P$  is important, as the sum of its first vertex is altered by  $-\alpha$  while the sum of its last vertex is altered by  $+\alpha$ . Hence, in what follows, though  $G$  is not oriented, we consider paths *from* a vertex *to* another one, so that the first and last vertices are clearly identified.

We proceed as follows to obtain the claimed edge-weighting of  $G$ . Assume first that at least one of the two partite sets, say,  $A$ , has even size. Start from all edges being weighted 0. Then repeatedly consider two new vertices  $u$  and  $v$  of  $A$ , and apply the modification procedure above with  $\alpha = 2$  onto a path from  $u$  to  $v$  in  $G$ . As stated, only the incident sums of  $u$  and  $v$  modulo 5 are altered, from 0 to 3 and 2, respectively. Once the process has been applied for all pairs of vertices, we get the desired neighbour-sum-2-distinguishing  $\{0, 1, 2, 3, 4\}$ -edge-weighting of  $G$ .

Assume now that  $A$  and  $B$  both have odd size. Repeating the exact same arguments as in the previous case on pairs of vertices of, say,  $A$ , we can deduce a  $\{0, 1, 2, 3, 4\}$ -edge-weighting of  $G$  where all vertices of  $A$  but one vertex  $u$  have incident sum congruent to 2 or 3 modulo 5, while all vertices of  $B$  have incident sum congruent to 0 modulo 5. In

#### 4. Doubling the 1-2-3 Conjecture

---

particular, also  $u$  has incident sum congruent to 0 modulo 5. If  $A$  has two vertices  $v_1, v_2$  with incident sum congruent to 2 (resp. 3) modulo 5, then we consider one path  $P_1$  from  $u$  to  $v_1$  and one path  $P_2$  from  $u$  to  $v_2$ , and apply the modification procedure for  $\alpha = 1$  (resp.  $\alpha = 4 = -1$  modulo 5) onto  $P_1$  and  $P_2$ . This affects the incident sums of  $v_1$  and  $v_2$  which are now congruent to 3 (resp. 2) modulo 5, while  $u$  now has incident sum congruent to 2 (resp. 3) modulo 5. The edge-weighting is thus as desired.

The only situation where the previous case does not apply is when  $|A| \leq 3$  and  $|B| \leq 3$ . In that case though, it can easily be checked by hand that  $G$  always admits a neighbour-sum-2-distinguishing  $\{0, 1, 2, 3, 4\}$ -edge-weighting.  $\square$

Although Theorem 4.3.2 completely proves Conjecture 4.1.1 for nice bipartite graphs, in the rest of this section we study bipartite graphs further. Namely, we deduce the exact value of  $\chi_{\Sigma>1}$  for nice paths and cycles, and prove a refinement of Conjecture 4.1.1 for odd multicacti. These results support some observations to be raised in concluding Section 4.5.

In all upcoming proofs, we will deal with adjacent degree-2 vertices; for such a configuration, the following observation obviously applies.

**Observation 4.3.3.** *Let  $\omega$  be a neighbour-sum-2-distinguishing edge-weighting of a graph  $G$ , and  $uv$  be an edge such that  $d(u) = d(v) = 2$ . Assuming  $u'$  (resp.  $v'$ ) denotes the neighbour of  $u$  (resp.  $v$ ) different from  $v$  (resp.  $u$ ), the weights  $\omega(u'u)$  and  $\omega(v'v)$  differ by at least 2.*

#### 4.3.1 Paths

We denote by  $P_\ell$  the path of length  $\ell$ . Therefore, the path  $P_1 = K_2$  is not nice. In the next result, we determine the value of  $\chi_{\Sigma>1}(P_\ell)$  for every  $\ell \geq 2$ .

**Theorem 4.3.4.** *For every path  $P_\ell$ ,  $\ell \geq 2$ , we have*

$$\chi_{\Sigma>1}(P_\ell) = \begin{cases} 2 & \text{if } \ell = 2, \\ 3 & \text{if } \ell > 2 \text{ and } \ell \equiv 0, 2, 3 \pmod{4}, \\ 4 & \text{otherwise.} \end{cases}$$

*Proof.* Recall that  $\chi_{\Sigma>1}(P_\ell) \leq 4$  holds for every  $\ell \geq 2$ , by Corollary 4.3.1. Moreover, since  $\chi_{\Sigma>1}(G) = 1$  if and only if  $G$  is a graph such that the degrees of every two adjacent vertices differ by at least 2, we get  $\chi_{\Sigma>1}(P_\ell) \geq 2$  for every  $\ell \geq 2$ .

Let  $v_0, \dots, v_\ell$  denote the vertices of the path  $P_\ell$ , with  $v_i v_{i+1}$  being an edge for every  $i$ ,  $0 \leq i \leq \ell - 1$ . We clearly have  $\chi_{\Sigma>1}(P_2) = 2$  since the weighting  $\omega$  given by  $\omega(v_0 v_1) = \omega(v_1 v_2) = 2$  is neighbour-sum-2-distinguishing. Suppose now that  $\ell \geq 3$ . Then  $P_\ell$  has two adjacent vertices  $v_i$  and  $v_{i+1}$  with degree 2, and Observation 4.3.3 applies. For any 2-edge-weighting  $\omega$  of  $P_\ell$ , we have  $\sigma(v_i) = \omega(v_{i-1} v_i) + \omega(v_i v_{i+1})$  and  $\sigma(v_{i+1}) = \omega(v_i v_{i+1}) + \omega(v_{i+1} v_{i+2})$ . Since  $\omega(v_{i-1} v_i), \omega(v_{i+1} v_{i+2}) \in \{1, 2\}$ , necessarily  $\sigma(v_i)$  and  $\sigma(v_{i+1})$  differ by at most 1, so that  $\omega$  cannot be neighbour-sum-2-distinguishing.

If  $\ell = 3$ , then assigning successive edge weights 1, 3, 3 to the edges of  $P_3$  yields a neighbour-sum-2-distinguishing as it yields successive incident sums 1, 4, 6, 3. So we may suppose from now on that  $\ell \geq 4$ . Under that assumption, note that a neighbour-sum-2-distinguishing 3-edge-weighting  $\omega$  of  $P_\ell$  cannot assign weight 2 since Observation 4.3.3 applies. We thus restrict our attention to  $\{1, 3\}$ -edge-weightings of  $P_\ell$ . Note that the edge  $v_1v_2$  (and similarly  $v_{\ell-1}v_\ell$ ) must be assigned weight 3 so that  $\sigma(v_0)$  and  $\sigma(v_1)$  differ by at least 2. From this, observe that all neighbour-sum-2-distinguishing  $\{1, 3\}$ -edge-weightings of  $P_\ell$  are (up to directional symmetry) 1, 3, 3, 1, 1, 3, 3,  $\dots$ , 3, 3, 1, and 1, 3, 3, 1, 1,  $\dots$ , 3, 3, and 3, 3, 1, 1, 3, 3, 1, 1,  $\dots$ , 3, 3. The claim then follows from the fact that these edge-weightings only apply for particular values of  $\ell$  modulo 4.  $\square$

### 4.3.2 Cycles

For every  $\ell \geq 3$ , we denote by  $C_\ell$  the cycle of length  $\ell$ . In the next result, we determine the value of  $\chi_{\Sigma>1}(C_\ell)$  for every cycle  $C_\ell$ .

**Theorem 4.3.5.** *For every cycle  $C_\ell$ ,  $\ell \geq 3$ , we have*

$$\chi_{\Sigma>1}(C_\ell) = \begin{cases} 3 & \text{if } \ell \equiv 0 \pmod{4}, \\ 5 & \text{otherwise.} \end{cases}$$

*Proof.* Observe first that since cycles satisfy the 1-2-3 Conjecture and are regular, Corollary 4.2.4 implies that  $\chi_{\Sigma>1}(C_\ell) \leq 5$  holds for every cycle  $C_\ell$ .

Let  $v_0, \dots, v_{\ell-1}$  denote the vertices of the cycle  $C_\ell$ , with  $v_i v_{i+1}$  being an edge for every  $i$ ,  $0 \leq i \leq \ell - 1$  (here and in the following, all operations over the subscripts are understood modulo  $\ell$ ). Note first that Observation 4.3.3 implies that no cycle  $C_\ell$  verifies  $\chi_{\Sigma>1}(C_\ell) \leq 2$ . Therefore,  $\chi_{\Sigma>1}(C_\ell) \geq 3$  for every cycle  $C_\ell$ . Still according to Observation 4.3.3, note that no neighbour-sum-2-distinguishing 3-edge-weighting  $\omega$  of  $C_\ell$  can use weight 2. Therefore, a neighbour-sum-2-distinguishing 3-edge-weighting of  $C_\ell$  can only use weights 1 and 3. In such a weighting, the edge weights must follow the pattern 1, 1, 3, 3, 1, 1, 3, 3,  $\dots$  along the cycle, which is possible if and only if  $\ell \equiv 0 \pmod{4}$ .

We now prove that  $\chi_{\Sigma>1}(C_\ell) = 5$  whenever  $\ell \not\equiv 0 \pmod{4}$ . Assume there is a neighbour-sum-2-distinguishing 4-edge-weighting  $\omega$  of some cycle  $C_\ell$ . Note that, by Observation 4.3.3, any two edges  $uv$  and  $vw$  being at distance 2 must be assigned a small (1 or 2) and a big (3 or 4) weight by  $\omega$ . This contradicts the existence of  $\omega$ , and, thus,  $\chi_{\Sigma>1}(C_\ell) = 5$  whenever  $\ell \not\equiv 0 \pmod{4}$ .  $\square$

### 4.3.3 Odd multicacti

Recall that  $\chi_{\Sigma>1}(G) \leq 5$  holds for every odd multicactus  $G$ , according to Theorem 4.3.2. In the next result, we prove that, for odd multicacti, we can even design neighbour-sum-2-distinguishing 5-edge-weightings that do not use weights 2 and 4. This supports Conjecture 4.5.1 we raise in the concluding section.

Observe first that connected multicacti can be defined inductively, as follows. Cycles of length at least 6 and congruent to 2 modulo 4, with edges coloured green and red

alternately, are multicacti. Consider now a multicactus  $G$  whose edges are coloured green and red, and let  $uv$  be a green edge of  $G$ . Then the graph obtained from  $G$  by identifying  $u$  and  $v$  with the end-vertices of a path of length at least 5 and congruent to 1 modulo 4, whose edges are alternately coloured red, green,  $\dots$ , red (from one end to the other), is a multicactus. This operation will be referred to as a *path attachment*. Note that, in any edge-coloured multicactus, the two ends of a green edge have the same degree.

**Theorem 4.3.6.** *Every odd multicactus admits an ns2d  $\{1, 3, 5\}$ -edge-weighting.*

*Proof.* Let  $G$  be an odd multicactus. The proof is by induction on the number of path attachments performed to construct  $G$ . If no such path attachment was made, then  $G$  is a cycle  $C_{4k+2}$ , for some  $k \geq 1$ , and the  $\{1, 3, 5\}$ -edge-weighting obtained by applying the pattern  $1, 1, 3, 3, 5, 5, 1, 1, 3, 3, 5, 5, \dots$  cyclically is clearly a neighbour-sum-2-distinguishing  $\{1, 3, 5\}$ -edge-weighting of  $G$ .

Assume now that  $G$  is not a cycle. Then  $G$  must contain a green edge  $uv$  such that  $u$  and  $v$  are joined by exactly  $x$  paths  $P_1, \dots, P_x$ ,  $x \geq 1$ , with length at least 5 and congruent to 1 modulo 4, and whose internal vertices have degree 2 in  $G$ . In other words, no green edge of the  $P_i$ 's was used to make a path attachment. In the extremal case where the  $P_i$ 's cover all vertices of  $G$  (*i.e.*, all path attachments were performed on only one green edge), we omit one of these paths. Then, when removing all internal vertices of the  $P_i$ 's from  $G$ , we get another connected odd multicactus  $G'$  in which both  $u$  and  $v$  have degree 2. In particular,  $G'$  cannot be reduced to a single edge.

By the induction hypothesis,  $G'$  admits a neighbour-sum-2-distinguishing  $\{1, 3, 5\}$ -edge-weighting  $\omega$ , which we would like to extend to the edges of the  $P_i$ 's, in order to obtain a neighbour-sum-2-distinguishing  $\{1, 3, 5\}$ -edge-weighting of  $G$ . Let us denote by  $u'$  and  $v'$  the neighbours of  $u$  and  $v$ , respectively, different from  $v$  and  $u$ , respectively, in  $G'$ . When extending  $\omega$  to the  $P_i$ 's, we have to make sure that:

1.  $\sigma(u)$  and  $\sigma(u')$  still differ by at least 2;
2.  $\sigma(v)$  and  $\sigma(v')$  still differ by at least 2;
3. both  $\sigma(u)$  and  $\sigma(v)$  differ by at least 2 from the incident sums of their  $x$  neighbours in the  $P_i$ 's;
4.  $\sigma(u)$  and  $\sigma(v)$  still differ by at least 2.

In order to respect the fourth condition above, we will edge-weight the  $P_i$ 's in such a way that  $\sigma(u)$  and  $\sigma(v)$  are altered the same way, *i.e.*, by a same integer  $\alpha$ . To that aim, we will  $\{1, 3, 5\}$ -edge-weight every  $P_i$  in such a way that its two end-edges are assigned the same weight  $\alpha$ . In what follows, the main weighting scheme we use to extend  $\omega$  to the consecutive edges of some  $P_i$  is  $\alpha, \beta, \beta, \gamma, \gamma, \beta, \beta, \dots, \beta, \beta, \gamma, \alpha$  (or reversely), where  $\alpha, \beta, \gamma \in \{1, 3, 5\}$  and  $\alpha \neq \beta$  and  $\beta \neq \gamma$ . Note that, by that edge-weighting, for any such  $\alpha, \beta, \gamma$  no two adjacent inner vertices have their sums differing by strictly less than 2 (recall Observation 4.3.3).

We consider the worst-case scenario where the number  $x$  of  $P_i$ 's is exactly 1. This is indeed the worst case as, as  $x$  gets larger, we get more and more ways to weight the  $P_i$ 's, and thus more and more ways to extend  $\omega$  correctly to  $G$ .

If  $\omega(uv) = 1$ , then we set  $\omega(uv) = 3$ . If no conflict arises, then we proceed with the next step. Otherwise, we proceed as follows. If there exists  $\alpha \in \{1, 3, 5\}$  such that  $\sigma(u) + \alpha$  and  $\sigma(u')$  differ by at least 2, and similarly for  $\sigma(v) + \alpha$  and  $\sigma(v')$ , then we extend  $\omega$  as follows. Note first that, because  $\omega(uv) = 3$ , by the induction hypothesis, we have,  $\sigma(u) \geq 4$  and  $\sigma(v) \geq 6$ , assuming without loss of generality that  $\sigma(u) < \sigma(v)$ . We then extend  $\omega$  to  $P_1$  in the following way; in any case, it can be checked that the resulting edge-weighting of  $G$  is correct:

- If  $\alpha = 1$ , then we assign weights  $1, 3, 3, 1, 1, 3, 3, \dots, 3, 3, 1, 1$  (*i.e.*,  $\alpha = 1$ ,  $\beta = 3$ ,  $\gamma = 1$ ) to the edges of  $P_1$  as going from  $v$  to  $u$ .
- If  $\alpha = 3$ , then we assign weights  $3, 1, 1, 3, 3, 1, 1, \dots, 1, 1, 3, 3$  (*i.e.*,  $\alpha = 3$ ,  $\beta = 1$ ,  $\gamma = 3$ ) as going from  $u$  to  $v$ .
- If  $\alpha = 5$ , then we change  $\omega(uv)$  from 3 to 5 and  $\alpha$  from 5 to 3, so that the previous case now applies.

Now, if no such  $\alpha$  exists, then we set  $\omega(uv) = 5$ . In the present situation,  $\alpha = 5$  now has the desired properties. Similar extension arguments then apply.

So now assume that  $\omega(uv) \in \{3, 5\}$  and there is currently no sum conflict. Again, if there exists  $\alpha \in \{1, 3, 5\}$  with the desired properties, then we are done. Otherwise, note that no conflict may arise when setting  $\omega(uv) = 1$  (as otherwise there would exist an  $\alpha$  when  $\omega(uv) \in \{3, 5\}$ ), and  $\alpha = 1$  is as required. Since  $\omega(uv) = 1$ , we have, say,  $2 \leq \sigma(u) \leq 4$  and  $4 \leq \sigma(v) \leq 6$  ( $\sigma(u) < \sigma(v)$ ). We then extend  $\omega$  to the edges of  $P_1$  as follows:

- If  $\sigma(u) = 2$  and  $\sigma(v) \in \{4, 6\}$ , then we assign weights  $1, 5, 5, 1, 1, 5, 5, \dots, 5, 5, 1, 1$  (*i.e.*,  $\alpha = 1$ ,  $\beta = 5$ ,  $\gamma = 1$ ) to the edges of  $P_1$  as going from  $u$  to  $v$ .
- If  $\sigma(u) = 4$  and  $\sigma(v) = 6$ , then we assign weights  $1, 3, 3, 1, 1, 3, 3, \dots, 3, 3, 1, 1$  (*i.e.*,  $\alpha = 1$ ,  $\beta = 3$ ,  $\gamma = 1$ ) as going from  $v$  to  $u$ .

In all cases, it can be checked that no sum conflict arises in  $G$ , and thus we get a neighbour-sum-2-distinguishing  $\{1, 3, 5\}$ -edge-weighting. This completes the proof.  $\square$

## 4.4 Algorithmic aspects

In this section, we consider the hardness of determining the value of  $\chi_{\Sigma>1}(G)$  for a given graph  $G$ . We first prove, in Subsection 4.4.1, that deciding whether  $\chi_{\Sigma>1}(G) \leq 2$  holds for a given graph  $G$  is NP-complete, even when restricted to 3-degenerate planar bipartite graphs. In Subsection 4.4.2, we prove that, although  $\chi_{\Sigma>1}(T)$  can take any value in

$\{1, 2, 3, 4\}$  for a given tree  $T$  (recall Corollary 4.3.1), deciding the exact value of  $\chi_{\Sigma>1}(T)$  can be done in polynomial time.

It is worth recalling that determining the value of  $\chi_{\Sigma}(G)$  for a given graph  $G$  is NP-complete in general (Dudek and Wajc [DW11]), but can be done in polynomial time when restricted to bipartite graphs (Thomassen, Wu and Yang [TWZ16]). Hence our result in Subsection 4.4.1 shows another difference between the parameters  $\chi_{\Sigma}$  and  $\chi_{\Sigma>1}$ .

#### 4.4.1 General case

Before proceeding with the proof of the main result of this subsection, we first introduce gadgets that we will use to force some weights to be used by any neighbour-sum-2-distinguishing 2-edge-weighting. Each of these gadgets will have a *root vertex* of degree 1 being incident to a *root edge*. We here relax the notion of neighbour-sum-2-distinguishing 2-edge-weighting around the root; that is, we allow a neighbour-sum-2-distinguishing 2-edge-weighting to have adjacent incident sums differing by less than 2, but the incident sum of the root has to be involved in such a conflict. This is because our gadgets will be attached to other graphs via the root, so, in the properties we point out below, the incident sum of the root should not be regarded as fixed.

The gadgets we will construct are called  $(\alpha, S)$ -gadgets, for some given  $\alpha \in \{1, 2\}$  and  $S \subset \mathbb{N}^*$  ( $S \neq \emptyset$ ). Every such gadget  $G$  will satisfy the two following properties:

1. the root edge of  $G$  is necessarily weighted  $\alpha$  by any neighbour-sum-2-distinguishing 2-edge-weighting of  $G$ , and
2.  $S$  is the set of all numbers  $s$  such that there exists a neighbour-sum-2-distinguishing 2-edge-weighting of  $G$  where the (unique) neighbour of the root has incident sum  $s$ .

These gadgets will be used as follows. Let  $H$  be a graph and  $v$  be a vertex of  $H$ . Add to  $H$  an  $(\alpha, S)$ -gadget  $G$  (for some  $\alpha$  and  $S$ ), and identify  $v$  with the root of  $G$ . Then, in any neighbour-sum-2-distinguishing 2-edge-weighting of  $H$ ,  $v$  will necessarily receive weight  $\alpha$  from the root edge of  $G$ , and  $v$  will be adjacent to a vertex whose incident sum belongs to  $S$ . This mechanism can be used both to force particular edge weights to appear around  $v$ , and to ensure that  $\sigma(v)$  does not belong to a particular set (in particular when  $|S| = 1$ ).

We now introduce the gadgets we will use (see Figure 4.1 for an illustration). Consider first a path  $G := u_1u_2u_3$  of length 2. We claim that  $G$  is a  $(2, \{3, 4\})$ -gadget with root  $u_1$ . Indeed, in any neighbour-sum-2-distinguishing 2-edge-weighting  $\omega$  of  $G$  (with the relaxation mentioned above), we have  $\omega(u_1u_2) = 2$ , while  $\omega(u_2u_3)$  can have value either 1 or 2, in which cases we get  $\sigma(u_2) = 3$  and  $\sigma(u_2) = 4$ , respectively.

Now consider a claw  $G$  with vertices  $v, u_1, u_2, u_3$ , where the  $u_i$ 's are the leaves. Add two  $(2, \{3, 4\})$ -gadgets  $G_1$  and  $G_2$  to  $G$ , and identify  $u_3$  and the roots of  $G_1$  and  $G_2$ . We claim that  $G$  is a  $(1, \{3, 4\})$ -gadget with root  $u_1$ . In any neighbour-sum-2-distinguishing 2-edge-weighting  $\omega$  of  $G$ , the vertex  $u_3$  is incident to at least two edges with weight 2 (because of the gadgets  $G_1$  and  $G_2$ ), so that  $\sigma(u_3) = 5$  (if  $\omega(u_3v) = 1$ ) or  $\sigma(u_3) = 6$

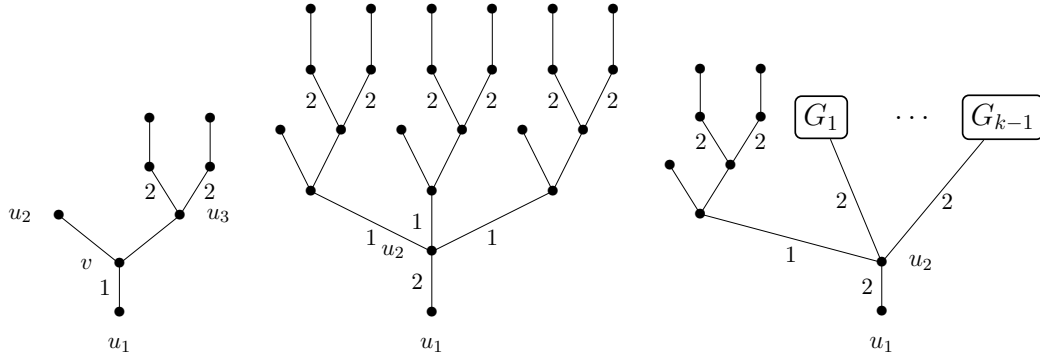


Figure 4.1: The  $(1, \{3, 4\})$ -gadget with root  $u_1$  (left), the  $(2, \{5\})$ -gadget with root  $u_1$  (middle), and the  $(2, \{2k + 1\})$ -gadget with root  $u_1$  (right).

(otherwise). In both cases, we necessarily have  $\omega(vu_2) = \omega(vu_1) = 1$ , so that  $\sigma(v)$  and  $\sigma(u_3)$  differ by at least 2. We thus get  $\sigma(v) = 3$  in the first case, and  $\sigma(v) = 4$  in the second case.

We now describe how to obtain  $(2, S)$ -gadgets with  $S := \{2k + 1\}$  for any  $k \geq 2$ . We first build a  $(2, \{5\})$ -gadget as follows. Start from  $G := u_1 u_2$  being the path of length 1, then add three  $(1, \{3, 4\})$ -gadgets  $G_1, G_2, G_3$  to  $G$ , and identify  $u_2$  and the roots of  $G_1, G_2$  and  $G_3$ . We claim that  $G$  is a  $(2, \{5\})$ -gadget with root  $u_1$ . In any neighbour-sum-2-distinguishing 2-edge-weighting  $\omega$  of  $G$ , the vertex  $u_2$  is incident to at least three edges weighted 1, namely the root edges of the  $G_i$ 's. Now, if  $\omega(u_1 u_2) = 1$ , then  $\sigma(u_2) = 4$ , which creates sum conflicts with vertices from the  $G_i$ 's. So we necessarily have  $\omega(u_1 u_2) = 2$ , in which case  $\sigma(u_2) = 5$ , which is fine since the  $G_i$ 's are  $(1, \{3, 4\})$ -gadgets.

We now turn to the general case. Let  $2k + 1 \geq 7$ , and assume that we have constructed  $(2, S')$ -gadgets with  $S' := \{2k' + 1\}$  for every  $k', 2 \leq k' < k$ . Start from  $G := u_1 u_2$  being the path of length 1. Add  $k - 1$   $(2, \{2k - 1\})$ -gadgets  $G_1, \dots, G_{k-1}$  to  $G$ , as well as one  $(1, \{3, 4\})$ -gadget  $G_0$ , and identify  $u_2$  and the roots of  $G_1, \dots, G_{k-1}$  and  $G_0$ . We claim that  $G$  is a  $(2, \{2k + 1\})$ -gadget with root  $u_1$ . In any neighbour-sum-2-distinguishing 2-edge-weighting  $\omega$  of  $G$ , the  $G_i$ 's force  $\sigma(u_2)$  to have value at least  $2k - 1$ . Depending on whether  $\omega(u_1 u_2) = 1$  or  $\omega(u_1 u_2) = 2$ , we thus have  $\sigma(u_2) = 2k$  or  $\sigma(u_2) = 2k + 1$ , respectively. In the first case, we get sum conflicts between  $u_2$  and its neighbours in  $G_1, \dots, G_{k-1}$  since their incident sums differ by 1. Therefore, we necessarily have  $\omega(u_1 u_2) = 2$ , so that  $\sigma(u_2) = 2k + 1$ , which produces no sum conflict in  $G$ .

Analogous  $(1, S)$ -gadgets with  $S := \{2k + 1\}$ ,  $k \geq 2$ , will also be needed. A  $(1, \{5\})$ -gadget can be obtained as follows. Start from  $G := u_1 u_2$  being the path of length 1, add two  $(2, \{7\})$ -gadgets  $G_1$  and  $G_2$  to  $G$ , and identify  $u_2$  and the roots of  $G_1$  and  $G_2$ . In any neighbour-sum-2-distinguishing 2-edge-weighting  $\omega$  of  $G$ , the  $G_i$ 's force  $u_2$  to be incident to at least two edges with weight 2. So we have  $\sigma(u_2) = 5$  or  $\sigma(u_2) = 6$  depending on whether  $\omega(u_1 u_2) = 1$  or  $\omega(u_1 u_2) = 2$ , respectively. In the second case, however, we get sum conflicts between  $u_2$  and its neighbours in  $G_1$  and  $G_2$ . We thus necessarily have  $\omega(u_1 u_2) = 1$  and  $\sigma(u_2) = 5$ , which produces no sum conflict in  $G$ .



#### 4. Doubling the 1-2-3 Conjecture

---

Now let  $2k + 1 \geq 7$ , and assume that we have constructed  $(1, S')$ -gadgets with  $S' := \{2k' + 1\}$  for every  $k'$ ,  $2 \leq k' < k$ . Start from  $G := u_1u_2$  being the path of length 1, add  $k$   $(2, \{2k + 3\})$ -gadgets  $G_1, \dots, G_k$  to  $G$ , and identify  $u_2$  and the roots of  $G_1, \dots, G_k$ . We claim that  $G$  is a  $(1, \{2k + 1\})$ -gadget. In any neighbour-sum-2-distinguishing 2-edge-weighting  $\omega$  of  $G$ , the  $G_i$ 's force  $\sigma(u_2)$  to be at least  $2k$ . Depending on whether  $\omega(u_1u_2) = 1$  or  $\omega(u_1u_2) = 2$ , we thus have  $\sigma(u_2) = 2k + 1$  or  $\sigma(u_2) = 2k + 2$ , respectively. In the second case, we get sum conflicts between  $u_2$  and its neighbours in  $G_1, \dots, G_k$ , since their incident sums differ by 1. Therefore, we necessarily have  $\omega(u_1u_2) = 1$  and  $\sigma(u_2) = 2k + 1$ , which produces no sum conflict in  $G$ .

Note that all the above-constructed gadgets are trees. With all these gadgets in hand, we now prove the main result of this section.

**Theorem 4.4.1.** *For a given 3-degenerate planar bipartite graph  $G$ , deciding whether  $\chi_{\Sigma > 1}(G) \leq 2$  holds is NP-complete.*

*Proof.* Since the problem is obviously in NP, we proceed with the proof of its NP-hardness. The proof is by reduction from 1-IN-3 SAT which is NP-Complete. The 1-IN-3 SAT problem is a decision problem asking if a given 3CNF formula is a satisfiable 1-in-3 way, *i.e.* if there exists an assigning of truth values to its variables such that each clause contains exactly one literal assigned to the value true. The MONOTONE version of 1-IN-3 SAT adds the restriction that no clause contains a negated variable. This version remains NP-complete (see e.g. [MR08]). From a 3CNF formula  $F$  with no negated variables, we construct a graph  $G$  such that  $F$  is satisfiable in a 1-in-3 way if and only if  $G$  admits a neighbour-sum-2-distinguishing 2-edge-weighting. We may assume that all clauses of  $F$  have three distinct variables, as otherwise  $F$  could be simplified. That is:

- if  $F$  has a clause  $(x_{i_1} \vee x_{i_1} \vee x_{i_1})$ , then  $F$  is not satisfiable in a 1-in-3 way;
- if  $F$  has a clause  $(x_{i_1} \vee x_{i_1} \vee x_{i_2})$ , then  $x_{i_2}$  and  $x_{i_1}$  are forced to *true* and *false*, respectively, by any truth assignment making  $F$  satisfied in a 1-in-3 way.

We denote by  $x_1, \dots, x_n$  the variables of  $F$ , and by  $C_1, \dots, C_m$  its clauses. The construction of  $G$ , which is clearly achieved in polynomial time, is as follows. We start by adding *variable gadgets* in the following way. For each variable  $x_i$  of  $F$ , we add to  $G$  a star  $V_i$  with root  $v_i$  and  $2k_i$  leaves  $u_{i,1}, \dots, u_{i,2k_i}$ , where  $2k_i \geq \max\{10, n_i\}$  is any even integer, and  $n_i$  is the number of clauses of  $F$  that contain  $x_i$ . Next we add  $(1, \{4k_i + 1\})$ -,  $(2, \{4k_i + 3\})$ -,  $(2, \{4k_i + 5\})$ -,  $\dots$ ,  $(2, \{6k_i - 3\})$ - and  $(2, \{6k_i - 3\})$ -gadgets  $G_1, \dots, G_{k_i}$  to  $G$ , and identify  $v_i$  and the roots of  $G_1, \dots, G_{k_i}$ . To avoid any ambiguity, let us emphasize that, for instance, when  $k_i = 5$ , the five added gadgets are one  $(1, \{21\})$ -gadget, one  $(2, \{23\})$ -gadget, one  $(2, \{25\})$ -gadget and two  $(2, \{27\})$ -gadgets.

Because of the  $G_i$ 's, in any neighbour-sum-2-distinguishing 2-edge-weighting  $\omega$  of  $V_i$ , the value of  $\sigma(v_i)$  lies between  $4k_i - 1$  (when all  $v_iu_{i,j}$ 's are assigned weight 1) and  $6k_i - 1$  (when all  $v_iu_{i,j}$ 's are assigned weight 2). Furthermore,  $4k_i - 1$  and  $6k_i - 1$  are both odd. Moreover, we cannot have  $\sigma(v_i) \in \{4k_i, \dots, 6k_i - 2\}$  as otherwise there would be a sum

conflict involving  $v_i$  and one of its neighbours in the  $G_i$ 's. Therefore, either all  $\omega(v_i u_{i,j})$ 's are equal to 1, or all  $\omega(v_i u_{i,j})$ 's are equal to 2. In what follows, we call the vertices  $u_{i,1}, \dots, u_{i,2k_i}$  the *output vertices* of  $V_i$ , and the edges  $v_i u_{i,1}, \dots, v_i u_{i,2k_i}$  the *output edges* of  $V_i$ ,

We now modify  $G$  by considering the clauses of  $F$ . For each clause  $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$  of  $F$ , we add a *clause vertex*  $c_j$  to  $G$ . For each  $V_{i_1}, V_{i_2}, V_{i_3}$ , we then select one output vertex still having degree 1, and identify  $c_j$  and the three selected output vertices. Finally, we add a  $(2, \{7\})$ -gadget  $G_1$  to  $G$ , as well as a  $(2, \{11\})$ -gadget  $G_2$ , and identify  $c_j$  and the roots of  $G_1$  and  $G_2$ . In any neighbour-sum-2-distinguishing 2-edge-weighting  $\omega$  of  $G$ ,  $\sigma(c_j)$  has thus value at least 4 (because of  $G_1$  and  $G_2$ ), and ranges in  $\{7, \dots, 10\}$ . However,  $\sigma(c_j)$  cannot take any value among  $\{7, 8, 10\}$  because of  $G_1$  and  $G_2$ . So we necessarily have  $\sigma(c_j) = 9$ , which occurs only if exactly one of the three output edges originating from  $V_{i_1}, V_{i_2}, V_{i_3}$  is assigned weight 1.

It can be checked that no unexpected sum conflicts (that is, different from those listed above) can arise, in particular thanks to our choice of the  $2k_i$ 's. We now claim that we have the desired equivalence. This directly follows from the following arguments:

- Assigning weight 1 (resp. 2) to an output edge  $v_i c_j$  simulates the fact that variable  $x_i$  brings truth value *true* (resp. *false*) to  $C_j$ .
- Following that equivalence, the fact that, for any  $V_i$ , all output edges of  $V_i$  must be weighted 1 (resp. 2) simulates the fact that setting  $x_i$  to *true* (resp. *false*) by some truth assignment brings the same truth value to every clause containing  $x_i$ .
- The fact that, for every clause vertex  $c_j$ , exactly one incident output edge must be assigned weight 1 simulates the fact that a clause of  $F$  is considered satisfied if and only if it includes exactly one *true* variable.

To complete the proof of the theorem, it suffices to observe the following:

- The PLANAR version of MONOTONE 1-IN-3 SAT remains NP-complete ([MR08]), so we may assume that  $F$  is a planar formula. Since every gadget is a tree, the construction above then yields a planar  $G$ .
- Since every gadget is a tree, the graph  $G$  is 3-degenerate.
- The only cycles in  $G$  are those of the subgraph induced by the  $v_i$ 's and the  $c_j$ 's. Since this subgraph is bipartite, so is  $G$ .

This concludes the proof. □

### 4.4.2 Tree case

In this section, we prove that the counterpart of Theorem 4.4.1 for trees is not true. That is, we prove that determining the value of  $\chi_{\Sigma>1}(T)$  for a given tree  $T$  can be done in polynomial time. Recall that for a tree  $T$ , we always have  $\chi_{\Sigma>1}(T) \leq 4$  (according to Corollary 4.3.1), while  $\chi_{\Sigma>1}(T) = 1$  if and only if, for every two adjacent vertices  $u$  and  $v$  of  $T$ , the values of  $d(u)$  and  $d(v)$  differ by at least 2.

**Theorem 4.4.2.** *For a given tree  $T$ , determining  $\chi_{\Sigma>1}(T)$  can be done in polynomial time.*

*Proof.* For any fixed  $k \in \{1, 2, 3, 4\}$ , we introduce below an algorithm that checks in polynomial time whether  $T$  admits a neighbour-sum-2-distinguishing  $k$ -edge-weighting. So, to determine  $\chi_{\Sigma>1}(T)$ , we can essentially run this algorithm successively with  $k = 1, 2, 3, 4$ . The first value of  $k$  for which the algorithm answers positively is the value of  $\chi_{\Sigma>1}(T)$ .

Designate a node  $r$  of  $T$  as being its *root*. This defines a root-to-leaf orientation of  $T$  in the usual way, where every non-root node  $v$  has a *parent*, and every non-leaf node  $v$  has *children*. By the *descendants* of  $v$ , we refer to the nodes of  $T$  for which we find  $v$  when iterating the parent relationship.

The subtree  $T_v$  of  $T$  rooted at  $v$  is the subtree whose nodes are  $v$  and all its descendants. This subtree  $T_v$  can itself be decomposed into several subtrees, in the following way. Assume that  $v$  has  $d \geq 1$  descendants  $u_1, \dots, u_d$ , ordered following an arbitrary order (supposed to be fixed throughout the proof). Then  $T_v$  can be edge-decomposed into  $d$  subtrees  $T_{v,1}, \dots, T_{v,d}$  being  $T_{u_1} + vu_1, \dots, T_{u_d} + vu_d$ , respectively, whose root,  $v$ , has degree precisely 1. Trees with this property are called *shrubs* throughout. Conversely,  $T_v$  is obtained by identifying the roots of the shrubs  $T_{v,1}, \dots, T_{v,d}$ . For every shrub, we call the edge incident to the root the *root edge*. The non-root end of the root edge is called the *subroot*.

We are now ready to describe our algorithm for deciding whether  $T$  admits a neighbour-sum-2-distinguishing  $k$ -edge-weighting. The rough ideas are the following. The tree  $T$  can be seen as a union of  $d := d(r)$  shrubs  $S_1, \dots, S_d$  whose roots were identified, resulting in  $r$ . A neighbour-sum-2-distinguishing  $k$ -edge-weighting of  $T$  is thus essentially the union of (relaxed, see below) neighbour-sum-2-distinguishing  $k$ -edge-weightings of the  $d$  shrubs attached to  $r$ , with the additional property that the resulting  $\sigma(r)$  does not create any sum conflict. Therefore, in order to construct a neighbour-sum-2-distinguishing  $k$ -edge-weighting of  $T$ , it suffices to find convenient neighbour-sum-2-distinguishing  $k$ -edge-weightings of  $S_1, \dots, S_d$  that can be “glued”. So we need to know, for each shrub  $S_i$  and for every  $\alpha \in \{1, \dots, k\}$ , whether  $S_i$  admits a neighbour-sum-2-distinguishing  $k$ -edge-weighting where the root edge is assigned colour  $\alpha$ , and, for such an edge-weighting of  $S_i$ , which possible incident sums can be obtained for the subroot.

More formally, for a shrub  $S$  with root  $v'$  and subroot  $v$ , we want to compute, for every weight  $\alpha \in \{1, \dots, k\}$ , the set  $X_\alpha(v)$  of possible values of  $\sigma(v)$  by a neighbour-sum-2-distinguishing  $k$ -edge-weighting of  $S$  assigning weight  $\alpha$  to  $v'v$ . Note that a shrub

might be a single edge, and may thus admit no neighbour-sum-2-distinguishing  $k$ -edge-weighting. In that special case, we relax the notion of neighbour-sum-2-distinguishing  $k$ -edge-weighting, and allow the root and the subroot to have the same incident sums.

Assume  $v$  has  $d$  children  $u_1, \dots, u_d$ ,  $d \geq 0$ , and let  $S_1, \dots, S_d$  denote the  $d$  shrubs attached to  $v$  in  $S$ . We claim that each  $X_\alpha(v)$  can be computed in polynomial time from the sets

$$X_1(u_1), \dots, X_k(u_1), X_1(u_2), \dots, X_k(u_2), \dots, X_1(u_d), \dots, X_k(u_d),$$

computed by induction for the shrubs  $S_1, \dots, S_d$ . So, in a way, the sets  $X_1(v), \dots, X_k(v)$  can be computed from smaller shrubs, and deduced successively towards the subroot of  $S$ . We prove this below.

The base case is when  $S$  is a single edge, that is,  $v$  has no child. If the edge  $v'v$  is assigned any weight  $\alpha \in \{1, \dots, k\}$  by a neighbour-sum-2-distinguishing  $k$ -edge-weighting, then  $\sigma(v) = \alpha$ . So, for such a shrub  $S$ , we have  $X_\alpha(v) = \{\alpha\}$  for every  $\alpha \in \{1, \dots, k\}$ .

Suppose now that  $v$  has  $d \geq 1$  children  $u_1, \dots, u_d$ , and, for each shrub  $S_i$ ,  $1 \leq i \leq d$ , attached to  $v$ , and every  $\alpha \in \{1, \dots, k\}$ , the set  $X_\alpha(u_i)$  has been computed by induction. We now want to compute the sets  $X_1(v), \dots, X_k(v)$ . Since  $d(v) = d + 1$ , by any  $k$ -edge-weighting of  $S$ , the sum  $\sigma(v)$  can take up to  $kd + k - d$  values, namely those among  $\{d + 1, \dots, kd + k\}$ . We repeatedly fix one of those sums  $x$ , and we determine whether  $x$  can be added to some of the sets  $X_1(v), \dots, X_k(v)$ .

Assume we want to determine whether  $x$  has to be added to  $X_\alpha(v)$ , where  $\alpha$  is any value in  $\{1, \dots, k\}$ . Successively consider all partitions  $x_1 + 2x_2 + \dots + kx_k$  of  $x$  into  $x_1 + \dots + x_k = d + 1$  values among  $\{1, \dots, k\}$  only. Recall that we are focusing on computing  $X_\alpha(v)$ , so if  $x_\alpha = 0$ , then we can consider the next partition of  $x$ . Since  $x$  is linear in  $|V(T)|$  and  $k \leq 4$  is fixed, the number of such partitions to consider is polynomial in  $|V(T)|$ . Essentially, we have  $x_1, \dots, x_k \leq |V(T)|$ , meaning that the number of such partitions is roughly  $|V(T)|^{k-1}$ . We now want to know if there is a neighbour-sum-2-distinguishing  $k$ -edge-weighting of  $S$  where  $x_i$  edges incident to  $v$  are assigned weight  $i$ , for every  $i \in \{1, \dots, k\}$ . If such an edge-weighting exists, then for any  $S_i$  to which the weight  $\beta$  is assigned to the root edge,  $X_\beta(u_i)$  contains some value not in  $\{\alpha - 1, \alpha, \alpha + 1\}$ .

Since we are focusing on  $X_\alpha(v)$ , one of the  $x_\alpha$  weights  $\alpha$  around  $v$  will be assigned to  $v'v$ . This leaves us with  $d$  other weights to assign bijectively to the  $vu_i$ 's, with the constraint that if we assign a weight  $\beta$  to  $vu_i$ , then  $X_\beta(u_i)$  should contain a value not among  $\{\alpha - 1, \alpha, \alpha + 1\}$ . If  $\beta$  can indeed be assigned to  $vu_i$  safely, then we call this a *valid assignment*. To find a correct assignment (if any exists), we build a compatibility bipartite graph  $C$  of the valid assignments, as follows. In one side of the bipartition of  $C$ , we put  $d$  vertices corresponding to the  $d$  weights we want to assign. In the other side, we put  $d$  vertices corresponding to the edges  $vu_1, \dots, vu_d$  of  $S$ . We then add an edge joining two vertices of  $C$  if assigning the corresponding weight to the corresponding edge of  $S$  is valid. Now, finding a satisfying assignment of the  $d$  weights to the root edges of the  $S_i$ 's is equivalent to finding a perfect matching in  $C$ , which is known to be doable in polynomial time. If there indeed exists such a perfect matching of  $C$ , then we add  $x$  to  $X_\alpha(v)$ .

We now go back to  $T$ , with the root  $r$  having  $d$  children  $u_1, \dots, u_d$ . For each  $S_i$  of the  $d$  shrubs  $S_1, \dots, S_d$  rooted at  $v$ , we can compute the sets  $X_1(u_i), \dots, X_k(u_i)$  as explained

above. These sets memorize, in a compact way, all possible ways, in terms of incident sums and weights assigned to the root edges, to  $k$ -edge-weight the  $S_i$ 's in a neighbour-sum-2-distinguishing way. Now, again, we can consider every potential incident sum  $x$  as  $\sigma(r)$ , every potential way to partition  $x$  into  $d$  integers among  $\{1, \dots, k\}$ , and, building the compatibility bipartite graph as above, find, if it exists, a valid way to bijectively assign the  $d$  weights to the  $d$  root edges  $vu_1, \dots, vu_d$ . If a valid assignment for a partition of some  $x$  exists, then  $T$  admits a neighbour-sum-2-distinguishing  $k$ -edge-weighting. Otherwise, it does not.

Concerning the complexity aspect, determining  $X_\alpha(v)$  for a shrub of  $T$  with subroot  $v$  can be done in polynomial time. Recall that  $k \leq 4$  is constant. The number of possible sums  $x$  as  $\sigma(v)$  to consider is at most  $k|V(T)|$ . For each of these values of  $x$ , we consider up to  $|V(T)|^{k-1}$  partitions into 1's, 2's,  $\dots$ , and  $k$ 's. Deciding whether there is a valid assignment for one of those partitions can be done in polynomial time, using for instance Edmonds' Blossom Algorithm for computing maximum matchings [Edm65]. The procedure above is almost the same when  $r$  is considered. By all these arguments, the whole procedure can be achieved in polynomial time.  $\square$

## 4.5 Conclusion

In this chapter, we have investigated the consequences on the 1-2-3 Conjecture of requiring adjacent vertices to be distinguishable in a stronger way, namely by asking their incident sums to differ by at least 2. We have addressed Conjecture 4.1.1, to which we did not manage to come up with any counterexample, as an equivalent of the 1-2-3 Conjecture in this context. As a main evidence that our conjecture might be true, we have pointed out some connections between the 1-2-3 Conjecture and Conjecture 4.1.1, and proved the later one for nice bipartite graphs.

Several aspects related to Conjecture 4.1.1 remain unclear to us, and could thus be subject to further work. First, we do not fully understand how the weights 2 and 4 are necessary for our conjecture. In particular, most graphs for which we have proved Conjecture 4.1.1 actually admit neighbour-sum-2-distinguishing  $\{1, 3, 5\}$ -edge-weightings (recall, in particular, Theorem 4.3.6). This supports the following refinement of Conjecture 4.1.1.

**Conjecture 4.5.1.** *Every nice graph admits a neighbour-sum-2-distinguishing  $\{1, 3, 5\}$ -edge-weighting.*

In the context of neighbour-sum-2-distinguishing edge-weightings, Conjecture 4.5.1 might actually be an equivalent to the 1-2-3 Conjecture and more natural than Conjecture 4.1.1. Indeed, in the 1-2-3 Conjecture we aim at getting incident sums differing by at least 1 by using three successive weights  $\alpha - 1, \alpha, \alpha + 1$  differing by 1. In Conjecture 4.5.1, we aim at getting incident sums differing by at least 2 by using three "successive" weights  $\alpha - 2, \alpha, \alpha + 2$  differing by 2. Following this reasoning, perhaps, in general, the following conjecture might be the right direction to consider.

**Conjecture 4.5.2.** *Let  $d \geq 1$  be an integer. Every nice graph admits a neighbour-sum- $d$ -distinguishing  $\{1, d + 1, 2d + 1\}$ -edge-weighting.*

There are intriguing examples, though, such as nice paths  $P_\ell$  of length congruent to 1 modulo 4 (for which  $\chi_{\Sigma>1}(P_\ell) = 4$ , recall Theorem 4.3.4), showing that, for neighbour-sum-2-distinguishing edge-weightings, the weights 2 and 4 are sometimes worth using to get an optimal edge-weighting. More generally, our NP-hardness reduction in the proof of Theorem 4.4.1 shows that there exist complex examples for which an optimal edge-weighting uses weights 1 and 2 only. We believe this could be an interesting aspect to study further.

Although we have proved Conjecture 4.1.1 for nice bipartite graphs (Theorem 4.3.2), we have not proved refined Conjecture 4.5.1 for all such graphs. Another interesting line of research could thus be to generalize Theorem 4.3.6 to all nice bipartite graphs, which would be a first step towards Conjecture 4.5.1.

More directions for future work on neighbour-sum-2-distinguishing edge-weightings are also worth mentioning. Notably, we did not manage to improve the bounds given in Section 4.2 for many classes of graphs. Generally speaking, it does not seem obvious to us how to improve the bound in Corollary 4.2.2, and this would surely require new dedicated tools. Concerning particular classes of graphs, let us mention the case of subcubic graphs. Although we know that cubic graphs comply with Conjecture 4.1.1, and even Conjecture 4.5.1 (recall Corollary 4.2.4), we did not manage to prove that nice subcubic graphs, in general, also do. We believe this would be an appealing first case to consider towards proving Conjecture 4.1.1 for 3-chromatic graphs, for which the 1-2-3 Conjecture holds.

More generally, it would be interesting to consider Conjecture 4.5.2 above. Many of the arguments and techniques used in this work actually generalize to neighbour-sum-distinguishing  $d$ -edge-weightings. For values of  $d$  larger than 2, it is likely that more intriguing phenomenon arise.

# Chapter 5

## The neighbour-sum-distinguishing edge-weighting game

The *neighbour-sum-distinguishing edge-weighting game* on a graph  $G$  is the 2-player game where one player, Alice, is trying to find an nsd edge-weighting of  $G$ , while the other player, Bob, is trying to prevent this.

In this chapter we study the nsd edge-weighting game on various classes of graphs. We present some general results on sufficient conditions for each player to win the game. Then, we discuss the game on some specific classes of graphs. In particular, we prove that Bob wins the game on the complete graph  $K_n$ ,  $n \geq 3$ , whoever starts the game, except when  $n = 4$ . In that case, Bob wins the game on  $K_4$  if and only if he starts the game.

The results presented in this chapter were published in [BPS<sup>+</sup>17].

### 5.1 Introduction

In this chapter we consider a game version of nsd edge-weighting. The *neighbour-sum-distinguishing edge-weighting game* on a graph  $G$  is a 2-player game where the two players, called Alice and Bob, alternately weight an unweighted edge of  $G$ . Alice wins the game if, when all edges are weighted, the so-obtained edge-weighting is an nsd edge-weighting of  $G$ , otherwise Bob wins. Therefore, Bob's objective is to produce an edge-weighting such that two neighbouring vertices get the same sum, while Alice's goal is to prevent him from doing so. The neighbour-sum-distinguishing edge-weighting game on  $G$  with Alice having the first move will be referred to as the *A-game on  $G$* . The nsd edge-weighting game on  $G$  with Bob having the first move will be referred to as the *B-game on  $G$* .

The chapter is organized as follows. In Section 5.2, we provide sufficient conditions on a graph  $G$  ensuring that Alice or Bob wins the *A-game* or the *B-game* on  $G$ . We then consider the nsd edge-weighting game on paths, cycles, stars and double-stars in Section 5.3, on complete graphs in Section 5.4 and on complete bipartite graphs in Section 5.5.

## 5.2 General results

In this section we exhibit sufficient conditions on a graph  $G$  for Alice or Bob to have a winning strategy for the neighbour-sum-distinguishing edge-weighting game on  $G$ . The first two lemmas give sufficient conditions for Bob to win the  $A$ -game or the  $B$ -game. We start with a lemma that give properties of graphs that allow Bob to win the game.

A *balanced edge* in a graph  $G$  is an edge  $uv \in E(G)$  with  $d_G(u) = d_G(v)$ .

**Lemma 5.2.1.** *Let  $G$  be a graph containing a balanced edge.*

1. *If  $|E(G)|$  is even then Bob wins the  $A$ -game on  $G$ .*
2. *If  $|E(G)|$  is odd then Bob wins the  $B$ -game on  $G$ .*

*Proof.* Let  $uv$  be a balanced edge in  $G$  with  $d(u) = d(v) = d$  and  $E_{u,v}$  be the set of edges having exactly one endpoint in  $\{u, v\}$ . We define Bob's strategy as follows:

- If Bob starts the game, he weights any edge from  $E(G) \setminus E_{u,v}$  with any weight.
- If Alice plays in  $E(G) \setminus E_{u,v}$ , then Bob answers in  $E(G) \setminus E_{u,v}$ .
- If Alice assigns the weight  $c$  to an unweighted edge  $uu'$ ,  $u' \neq v$  (resp.  $vv'$ ,  $v' \neq u$ ), then Bob assigns the weight  $c$  to some unweighted edge  $vv'$ ,  $v' \neq u$  (resp.  $uu'$ ,  $u' \neq v$ ).

Since  $|E_{u,v}|$  is even,  $|E(G)|$  and  $|E(G) \setminus E_{u,v}|$  have the same parity. Therefore, Bob can always apply this strategy in both games, so that we eventually get  $\sigma(u) = \sigma(v)$ .  $\square$

A *nice edge* in a graph  $G$  is an edge  $uv$  with  $d_G(v) = 2$  and  $d_G(w) \geq 2$ , where  $w$  is the neighbour of  $v$  distinct from  $u$ . We denote by  $|E(G)|_{\text{nice}}$  the number of nice edges in a graph  $G$ .

**Lemma 5.2.2.** *Let  $G$  be a connected graph.*

1. *If  $|E(G)|$  is odd and  $|E(G)|_{\text{nice}} > \frac{1}{2}|E(G)|$  then Bob wins the  $B$ -game on  $G$ .*
2. *If  $|E(G)|$  is even,  $|E(G)|_{\text{nice}} > \frac{1}{2}|E(G)|$ , and  $G$  contains no  $C_4$  as a subgraph, then Bob wins the  $B$ -game on  $G$ .*

*Proof.* Bob's strategy consists in weighting a non-nice edge whenever possible. Doing so, for the  $B$ -game on  $G$ , Bob will be able to weight a nice edge on his last move whenever  $|E(G)|_{\text{nice}} > \frac{1}{2}|E(G)|$ , regardless of the parity of  $|E(G)|$ .

Suppose first that  $|E(G)|$  is odd, which implies that Bob's last move is the last move of the game. Let  $uv$  be the unweighted nice edge to be weighted by Bob on his last move and let  $w$  be the neighbour of  $v$  distinct from  $u$ . Since  $d(w) \geq 2$ , the sum at  $w$ , say  $M$ , is strictly greater than the weight  $c$  of the edge  $vw$ . Therefore, Bob wins the game by weighting the edge  $uv$  with weight  $M - c$ .



## 5. The neighbour-sum-distinguishing edge-weighting game

---

Suppose now that  $|E(G)|$  is even, which implies that Alice's last move will follow Bob's last move. Observe that the above described strategy is still winning for Bob, unless the vertex  $w$  is incident with another unweighted nice edge, say  $wv'$ . Bob could then apply the above strategy to the edge  $wv'$  unless the neighbour  $w'$  of  $v'$  distinct from  $w$  is incident with a nice edge, that is  $w' = u$ . But in that case  $G$  contains a  $C_4$  linking vertices  $u$ ,  $v$ ,  $w$  and  $v'$ , a contradiction.  $\square$

The last lemma of this section will give sufficient conditions for Alice to win the  $A$ -game or the  $B$ -game. We first recall and introduce some definitions and notation.

A pendent vertex in a graph  $G$  is a vertex with  $d_G(v) = 1$ . For every vertex  $v \in V(G)$ , we denote by  $d_G^p(v)$  the number of pendent neighbours of  $v$ . An internal vertex in a graph  $G$  is a vertex with  $d_G(v) > 1$ . A pendent edge in a graph  $G$  is an edge  $uv$  such that  $u$  or  $v$  is a pendent vertex. An internal edge in a graph  $G$  is an edge  $uv$  such that both vertices  $u$  and  $v$  are internal.

A *partial edge-weighting* of a graph  $G$  is a mapping  $\gamma : E_\gamma \rightarrow \mathbb{N}^*$  where  $E_\gamma \subseteq E(G)$  is the set of weighted edges of  $G$ . The graph  $G$  is said to be *partially weighted* by  $\gamma$  and the corresponding partially edge-weighted graph is denoted by  $(G, \gamma)$ . A partial edge-weighting  $\gamma$  is neighbour-sum-distinguishing if for every edge  $uv$  in  $G$ ,  $\sigma_\gamma(u) \neq \sigma_\gamma(v)$ . A partial edge-weighting  $\gamma$  with  $E_\gamma = E(G)$  is an edge-weighting of  $G$ .

Let  $(G, \gamma)$  be a partially edge-weighted graph. A vertex  $v \in V(G)$  is *fully weighted* if all the edges incident with  $v$  are weighted. An edge  $uv \in E(G)$  is *complete* if both vertices  $u$  and  $v$  are fully weighted, and *safe* if  $\sigma_\gamma(u) \neq \sigma_\gamma(v)$ . Observe that any complete pendent edge is necessarily safe.

**Lemma 5.2.3.** *Let  $G$  be a graph such that  $d_G^p(v) \geq \frac{1}{2} d_G(v) + 1$  for every internal vertex  $v \in V(G)$ .*

- (1) *If  $|E(G)|$  is odd then Alice wins the  $A$ -game on  $G$ .*
- (2) *If  $|E(G)|$  is even then Alice wins the  $B$ -game on  $G$ .*

*Proof.* We first consider the  $A$ -game on  $G$  and let  $|E(G)| = 2k + 1$ . Let us denote by  $\gamma_i^A$  (resp.  $\gamma_i^B$ ) the partial weighting of  $G$  obtained after Alice's  $i$ -th move,  $1 \leq i \leq k + 1$  (resp. after Bob's  $i$ -th move,  $1 \leq i \leq k$ ).

Consider the strategy for Alice given by the following rules:

- (R1) On her first move, Alice weights any internal edge with any weight.
- (R2) If Bob has weighted an internal edge, then Alice weights any internal edge with any weight, if possible.
- (R3) If Bob has weighted the last unweighted internal edge, then Alice chooses a vertex  $w$  incident with at least three unweighted pendent edges and weights any of these unweighted pendent edges with any weight.

- 
- (R4) If Bob has weighted a pendent edge  $uv$  with  $d_G(u) = 1$ , then Alice weights any internal edge incident with  $v$  with any weight, if possible.
  - (R5) If Bob has weighted a pendent edge  $uv$  with  $d_G(u) = 1$ , all internal edges incident with  $v$  are already weighted, and  $v$  is incident with at least two unweighted pendent edges, then Alice weights any internal edge with any weight, if possible.
  - (R6) If Bob has weighted a pendent edge  $uv$  with  $d_G(u) = 1$ , all internal edges are already weighted, and  $v$  is incident with at least two unweighted pendent edges, then Alice chooses a vertex  $w$  incident with at least three unweighted pendent edges and weights any of these unweighted pendent edges with any weight.
  - (R7) If Bob has weighted a pendent edge  $uv$  with  $d_G(u) = 1$ , all internal edges are already weighted, and  $v$  is incident with only one unweighted pendent edge  $u'v$ , then Alice weights the pendent edge  $u'v$  with some weight  $c$  in such a way that every complete edge  $vv'$ , if any, is safe.

We will prove that Alice can always apply this strategy (namely, that Rules (R3) and (R6) can be applied whenever needed), and that this strategy is a winning strategy for Alice. We first claim that after each of Alice's moves, the partially weighted graph  $(G, \gamma_i^A)$  satisfies the following properties:

- (P1) Every complete edge is safe.
- (P2) For every non fully weighted vertex  $v$  incident with at least one unweighted internal edge, the number of weighted pendent edges incident with  $v$  is less than or equal to the number of weighted internal edges incident with  $v$ .
- (P3) Every non fully weighted vertex  $v$  is incident with at least two unweighted pendent edges.

We prove this claim by induction on  $i$ . The three properties clearly hold after Alice's first move. Suppose now that the partially weighted graphs  $(G, \gamma_1^A), \dots, (G, \gamma_p^A)$ ,  $1 \leq p \leq k$ , all satisfy the three properties and that Bob weights the edge  $xy$  on his  $p$ -th move.

Suppose first that  $xy$  is an internal edge. Since  $(G, \gamma_p^A)$  satisfies property (P2),  $xy$  is not a complete edge in  $(G, \gamma_p^B)$ , and thus  $(G, \gamma_p^B)$  satisfies properties (P1) and (P2). Moreover, since  $(G, \gamma_p^A)$  satisfies property (P3), and since  $d_G^p(u) \geq \frac{1}{2}d_G(u) + 1$  and  $d_G^p(v) \geq \frac{1}{2}d_G(v) + 1$ , both vertices  $x$  and  $y$  are incident with at least three unweighted pendent edges, and thus  $(G, \gamma_p^B)$  also satisfies property (P3). According to her strategy, Alice will then apply either Rule (R2) or Rule (R3). If she applies Rule (R2), she weights an unweighted internal edge and, similarly as for Bob's move, we get that  $(G, \gamma_{p+1}^A)$  satisfies the three properties. If all internal edges are weighted, as observed above, both vertices  $x$  and  $y$  are incident with at least three unweighted pendent edges and thus Alice can apply Rule (R3). Again,  $(G, \gamma_{p+1}^A)$  clearly satisfies the three properties.

Suppose now that  $xy$  is a pendent edge, with  $d_G(x) > 1$  and  $d_G(y) = 1$ . Alice will then apply one of the rules (R4), (R5), (R6) or (R7).

## 5. The neighbour-sum-distinguishing edge-weighting game

---

If Alice applies Rule (R4) then, since  $(G, \gamma_p^A)$  satisfies property (P3) and  $x$  was incident with an unweighted internal edge in  $(G, \gamma_p^A)$ ,  $x$  was incident to at least three unweighted pendent edges in  $(G, \gamma_p^A)$  and thus at least two in  $(G, \gamma_p^B)$ . Hence  $(G, \gamma_{p+1}^A)$  satisfies property (P3). Since  $x$  is thus not fully weighted,  $(G, \gamma_{p+1}^A)$  also satisfies property (P1). Moreover, the numbers of pendent and internal weighted edges incident with  $x$  are both increased by one and thus, since  $(G, \gamma_p^A)$  satisfies property (P2),  $(G, \gamma_{p+1}^A)$  also satisfies (P2).

If Alice applies Rule (R5) then, since  $(G, \gamma_p^A)$  satisfies the three properties, both  $(G, \gamma_p^B)$  and  $(G, \gamma_{p+1}^A)$  also satisfy the three properties.

If all internal edges are weighted then, since both  $(G, \gamma_p^A)$  and  $(G, \gamma_p^B)$  satisfy property (P3) and the number of unweighted edges is odd (this follows from the fact that  $|E(G)|$  is odd and the number of weighted edges before Alice's move is even), there exists a vertex  $w$  incident with at least three unweighted pendent edges, so that Alice can apply Rule (R6). Again,  $(G, \gamma_{p+1}^A)$  clearly satisfies the three properties.

Finally, if Alice applies Rule (R7), the choice of the weight  $c$  ensures that  $(G, \gamma_{p+1}^A)$  satisfies the three properties.

We thus get that the final edge-weighting  $\gamma_{p+1}$  is neighbour-sum-distinguishing since the partially weighted graph  $(G, \gamma_{p+1})$  satisfies property (P1).

We now consider the  $B$ -game on  $G$ . We claim that applying the same strategy as before (except Rule (R1) that is no longer valid), Alice wins the  $B$ -game on  $G$ . The proof is similar, up to the applicability of rules (R3) and (R6) which now follows from the fact that  $|E(G)|$  is even and the number of weighted edges before any of Alice's moves is odd.  $\square$

In particular, Lemma 5.2.3 allows us to prove that Alice wins the  $A$ -game or the  $B$ -game on some special trees. A *caterpillar* is a tree  $T$  whose set of internal vertices induces a path, called the *central path* of  $T$ . We then have:

**Corollary 5.2.4** (Special caterpillars). *Let  $T$  be a caterpillar, with central path  $v_1v_2 \dots v_k$ , such that  $d_T(v_1) \geq 4$ ,  $d_T(v_k) \geq 4$  and  $d_T(v_i) \geq 6$  for every  $i$ ,  $2 \leq i \leq k-1$ . We then have:*

1. *If  $|E(G)|$  is odd then Alice wins the  $A$ -game on  $G$ .*
2. *If  $|E(G)|$  is even then Alice wins the  $B$ -game on  $G$ .*

*Proof.* Note that since  $d_T(v_1) \geq 4$ , we have  $d_G^p(v_1) = d_G(v_1) - 1 \geq \frac{1}{2} d_G(v_1) + 1$ . Similarly, we have  $d_G^p(v_k) = d_G(v_k) - 1 \geq \frac{1}{2} d_G(v_k) + 1$  and, since  $d_T(v_i) \geq 6$ ,  $d_G^p(v_i) = d_G(v_i) - 2 \geq \frac{1}{2} d_G(v_i) + 1$  for every  $i$ ,  $2 \leq i \leq k-1$ . Therefore,  $T$  satisfies the conditions of Lemma 5.2.3 and the result follows.  $\square$

## 5.3 Simple graph classes

In this section we study the neighbour-sum-distinguishing edge-weighting game on simple classes of graphs, namely paths, cycles, stars and double-stars.

**Theorem 5.3.1** (Paths and cycles). *Let  $P_{n-1}$  and  $C_n$  respectively denote the path and the cycle on  $n$  vertices,  $n \geq 3$ . We then have:*

1. *Bob wins the  $A$ -game on  $P_n$  if and only if  $n \geq 5$ .*
2. *Bob wins the  $B$ -game on  $P_n$  if and only if  $n \notin \{3, 5\}$ .*
3. *Bob wins the  $A$ -game on  $C_n$  for every  $n$ .*
4. *Bob wins the  $B$ -game on  $C_n$  if and only if  $n \neq 4$ .*

*Proof.* Since every edge-weighting of  $P_3$  is a neighbour-sum-distinguishing edge-weighting, Alice wins both the  $A$ -game and the  $B$ -game on  $P_3$ .

Alice's strategy for the  $A$ -game on  $P_4$  is to first weight the central edge of  $P_4$  and then to weight the last edge with a weight distinct from the weight used by Bob.

Alice's strategy for the  $B$ -game on  $P_5$  is to weight on her first move an edge at distance one from the edge previously weighted by Bob with a weight distinct from the weight used by Bob.

Similarly, Alice's strategy for the  $B$ -game on  $C_4$  is to weight on her first move the edge at distance one from the edge previously weighted by Bob with a weight distinct from the weight used by Bob.

We now prove that Bob wins the game in all the remaining cases. Note that whenever two edges at distance one get the same weight, Bob wins the game. For the  $A$ -game on  $P_n$ ,  $n \geq 5$ , Bob weights on his first move an edge at distance one from the edge previously weighted by Alice, with the same weight. For the  $B$ -game on  $P_4$ , Bob weights first the central edge and then the last edge with the same weight as the one used by Alice. For the  $B$ -game on  $P_n$ ,  $n \geq 6$ , Bob weights first the third edge of  $P_n$  with some weight  $c$ . Alice then cannot prevent Bob from weighting on his second move either the first or the fifth edge with the same weight  $c$ . For the  $A$ -game on  $C_3$ , Bob weights an edge using the same weight as Alice on her first move. For the  $A$ -game on  $C_n$ ,  $n \geq 4$ , Bob weights on his first move an edge at distance one from the edge weighted by Alice, using the same weight. Finally, for the  $B$ -game on  $C_n$ ,  $n \geq 5$ , Bob weights an edge with some weight  $c$  and Alice cannot prevent him from weighting with the same weight  $c$  an edge at distance one from the edge weighted first.  $\square$

Since every edge-weighting of the star graph  $K_{1,n}$ ,  $n \geq 2$ , is a neighbour-sum-distinguishing edge-weighting, we directly get the following:

**Observation 5.3.2.** *For every integer  $n \geq 2$ , Alice wins both the  $A$ -game and the  $B$ -game on  $K_{1,n}$ .*

## 5. The neighbour-sum-distinguishing edge-weighting game

---

The *double-star*  $DS_{m,n}$ ,  $m \geq n \geq 1$ , is obtained from the two stars  $K_{1,m}$  and  $K_{1,n}$  by adding an edge joining their two centers. We prove the following:

**Theorem 5.3.3** (Double-stars).

1. For every integer  $n \geq 1$ , Bob wins the  $B$ -game on  $DS_{n,n}$ .
2. For every integer  $n \geq 1$ , Alice wins the  $A$ -game on  $DS_{n,n}$ .
3. For any integers  $m > n \geq 1$ , Alice wins the  $A$ -game on  $DS_{m,n}$ .
4. For any integers  $m > n \geq 1$ , Alice wins the  $B$ -game on  $DS_{m,n}$ .

*Proof.* Let  $uv$  denote the central edge of the double-star  $DS_{m,n}$ , with  $d(u) = m + 1$  and  $d(v) = n + 1$ . Observe that only the sums at vertices  $u$  and  $v$  may be equal at the end of the game.

Since for every  $n \geq 1$  the central edge of  $DS_{n,n}$  is balanced, Theorem 5.3.3.1 directly follows from Lemma 5.2.1.

We now describe Alice's winning strategy for the  $A$ -game on  $DS_{m,n}$ . On her first move, she weights the central edge with any weight. Then, whenever Bob weights an edge incident with  $u$  (resp.  $v$ ), she weights an edge incident with  $v$  (resp.  $u$ ). On her last move, she weights the last edge with any weight ensuring that the sums at  $u$  and  $v$  are distinct.

Consider now the  $A$ -game on  $DS_{m,n}$ ,  $m > n \geq 1$ . Alice's winning strategy is as follows. On her first move, she weights the central edge with any weight. Then, whenever Bob weights with weight  $c$  an edge incident with  $u$  (resp.  $v$ ), she weights with the same weight an edge incident with  $v$  (resp.  $u$ ), if possible, otherwise she weights any remaining edge with any weight. Since the degrees of vertices  $u$  and  $v$  are distinct, the sums at  $u$  and  $v$  will necessarily be distinct at the end of the game.

Let us finally consider the  $B$ -game on  $DS_{m,n}$ ,  $m > n \geq 1$ . Alice's winning strategy is as follows. If Bob weights the central edge, then Alice weights any edge incident with  $u$  with any weight. If Bob weights with weight  $c$  an edge incident with  $u$  (resp.  $v$ ), she weights with the same weight an edge incident with  $v$  (resp.  $u$ ), if possible, otherwise she weights any remaining edge with any weight. Again, since the degrees of vertices  $u$  and  $v$  are distinct, the sums at  $u$  and  $v$  will necessarily be distinct at the end of the game.

This concludes the proof.  $\square$

## 5.4 Complete graphs

In this section we study the neighbour-sum-distinguishing edge-weighting game on complete graphs. We prove the following:

**Theorem 5.4.1** (Complete graphs).

1. For every integer  $n \geq 3$ , Bob wins the  $A$ -game on  $K_n$ .

2. For every integer  $n \geq 3$ , Bob wins the  $B$ -game on  $K_n$  if and only if  $n \neq 4$ .

*Proof.* Since every edge in  $K_n$  is balanced and the number of edges of  $K_n$  is  $\frac{n(n-1)}{2}$ , we directly get by Lemma 5.2.1 that Bob wins the  $A$ -game on  $K_n$  whenever  $n \equiv 0, 1 \pmod{4}$  and that Bob wins the  $B$ -game on  $K_n$  whenever  $n \equiv 2, 3 \pmod{4}$ .

We now consider the remaining cases. The proof of Theorem 5.4.1 will follow from a series of lemmas.

**Lemma 5.4.2.** *Bob wins the  $A$ -game on  $K_3$ .*

*Proof.* Bob simply weights the second edge with the weight used by Alice on her first move.  $\square$

**Lemma 5.4.3.** *Bob wins the  $A$ -game on  $K_4$ .*

*Proof.* We define Bob's strategy as follows: whenever Alice weights an unweighted edge  $e$  with weight  $c$ , Bob weight the edge  $e'$  parallel with  $uv$  (that is,  $e \cap e' = \emptyset$ ) with the same weight  $c$ . At the end of the game, all vertices clearly get the same sum.  $\square$

**Lemma 5.4.4.** *Alice wins the  $B$ -game on  $K_4$ .*

*Proof.* We define Alice's strategy as follows. On his first move, Bob weights some unweighted edge  $e_1$  with weight  $c_1$ . Alice then weight with the same weight  $c_1$  the edge parallel with  $e_1$ . On his second move, Bob weights some unweighted edge  $e_2$  with weight  $c_2$ . Alice then weight with a weight  $c_3 \neq c_2$  the edge parallel with  $e_2$ . Hence, the two remaining unweighted edges both join vertices with respective distinct sums  $c_1 + c_2$  and  $c_1 + c_3$ , so that Alice wins the game.  $\square$

For larger values of  $n$ , we consider the structure of the graph induced by the unweighted edges before some last rounds. It will be useful to introduce the following notation. We denote by  $R_k$  the graph induced by the  $k$  last remaining (unweighted) edges. The next lemmas concern the graph  $R_4$ . This graph has four edges.

**Remark 5.4.5.** *Let us observe that in the cases we consider, the last even number of moves of the game are played first by Bob and then by Alice, alternately.*

In the following, we will say that we get equality on an edge  $uv$  whenever we get  $\sigma(u) = \sigma(v)$ .

**Lemma 5.4.6.** *If  $R_4$  is a forest then Bob wins.*

*Proof.* Let us consider first the case where  $R_4$  is a path of length four. Denote its vertices by  $x_1, x_2, x_3, x_4, x_5$  and let  $\sigma(x_i)$  be the sums at vertices  $x_i$  at this moment. On his first move, Bob weights the edge  $x_1x_2$  with a huge (much greater than other values of the sums at vertices of  $R_4$ ) weight  $H$ . Alice must weight the edge  $x_4x_5$ , since otherwise Bob in his next move would weight this edge in such a way that he would get the equality on  $x_1x_5$ . But then Bob weights the edge  $x_3x_4$  by  $H'$  such that  $\sigma(x_2) + H = \sigma(x_3) + H'$  and wins the game.

5. The neighbour-sum-distinguishing edge-weighting game

---

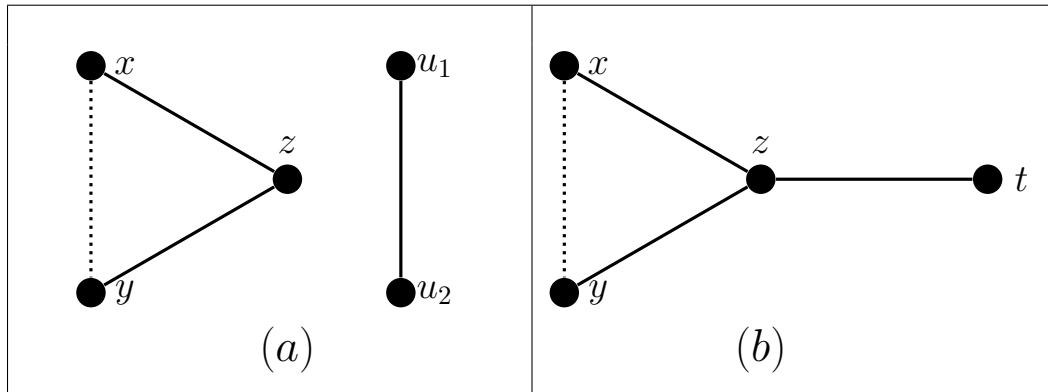


Figure 5.1: Some  $R_4$  graphs: (a)  $K_3 \cup K_2$  and (b)  $T^+$ . The first edge played by Bob is dotted.

Now, let  $R_4$  be a forest different from the path of length four. Then,  $R_4$  has at least three leaves,  $u_1, u_2, u_3$  say, belonging to three distinct edges  $e_1, e_2, e_3$  say, respectively. Bob starts by weighting  $e_1$  with a huge weight  $H$ , and after Alice's move, one of the above mentioned edges, say  $e_3$ , is still not weighted. Bob weights it with a weight  $H'$  in such a way that the equality  $\sigma(u_1) + H = \sigma(u_3) + H'$  holds, where  $\sigma(u_1), \sigma(u_3)$  denote the respective sums just before the four last moves.  $\square$

**Lemma 5.4.7.** *If  $R_4$  is the graph  $K_3 \cup K_2$  then Bob wins.*

*Proof.* Denote the vertices of  $R_4$  as in Figure 5.1(a). On his first move, Bob weights the edge  $xy$  with a huge weight  $H$ .

- If Alice weights the edge  $yz$  (or  $xz$ ), completing the weighting of edges incident to  $y$  (or  $x$ ), then Bob weights  $u_1u_2$  in such a way that he gets the equality on  $u_1y$  (or  $u_1x$ ).

- If Alice weights the edge  $u_1u_2$  then Bob weights  $xz$  in such a way that he gets the equality on  $yz$ .  $\square$

**Lemma 5.4.8.** *If  $R_4 = T^+$  (see Figure 5.1(b)) and  $\sigma(t) > \sigma(z)$ , then Bob wins.*

*Proof.* W.l.o.g. we may assume that  $\sigma(x) \leq \sigma(y)$ . Let  $\sigma(x) = a, \sigma(y) = a + a', \sigma(z) = b, \sigma(t) = b + b'$ , where  $b' > 0$ . Bob weights the edge  $xy$  with a huge weight  $H$ . Alice must weight the edge  $zt$ , since otherwise Bob in his next move weights  $zt$  in such a way that either  $\sigma(x) = \sigma(t)$  and all edges incident with  $x$  or  $t$  are already weighted or  $\sigma(y) = \sigma(t)$  and all edges incident with  $y$  or  $t$  are already weighted, hence Bob would win. Thus, assume that Alice weights  $zt$  with  $M$ . After such moves we have

- $\sigma(x) = a + H,$
- $\sigma(y) = a + a' + H,$
- $\sigma(z) = b + M,$
- $\sigma(t) = b + b' + M.$

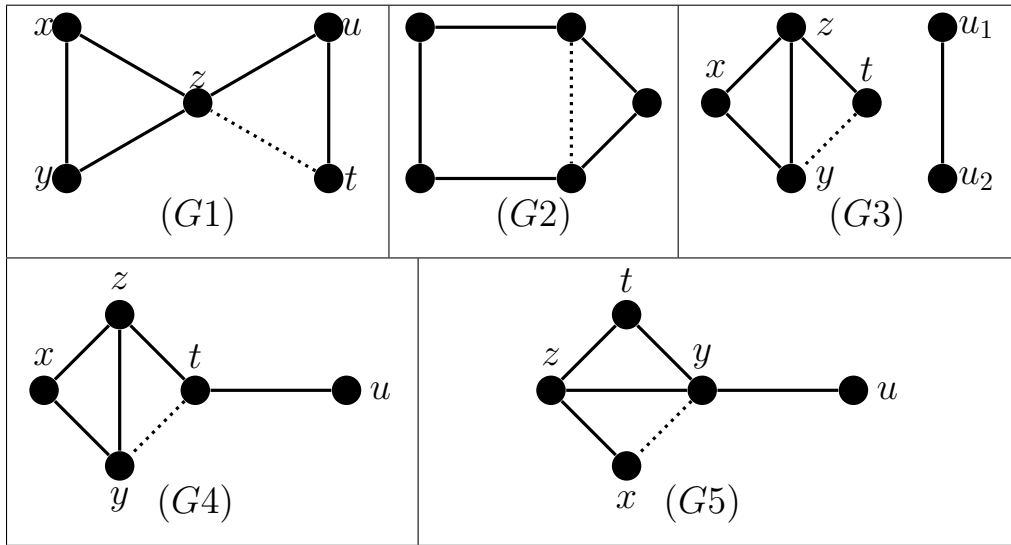


Figure 5.2: Some  $R_6$  graphs. The first edge played by Bob is dotted.

If  $a + H < b + b' + M$ , then Bob weights  $xz$  with  $b + b' + M - (a + H)$ . After his moves  $\sigma(x) = \sigma(t)$  and all edges incident with  $x$  or  $t$  have been weighted and hence Bob wins. Thus, suppose that  $a + H \geq b + b' + M$ . Since  $b' > 0$ , the inequality  $a + H \geq b + b' + M$  implies that  $a + a' + H > b + M$ . In such a case Bob weights  $xz$  with  $a + a' + H - (b + M)$ . After such a move,  $\sigma(y) = \sigma(z)$  and  $yz$  is the only unweighted edge incident with  $y$  or with  $z$ . Thus, no matter how Alice weights the last edge  $yz$ , after her move we still have  $\sigma(y) = \sigma(z)$ .  $\square$

**Observation 5.4.9.** *It is easy to see that if  $R_4 = C_4$  then Bob has no good strategy. This is why he has to avoid  $R_6 = K_{2,3}$  and  $R_6 = K_4$ . How to avoid the second situation is described in Lemma 5.4.11 where we consider the graphs  $R_8$ . In order to avoid  $R_6 = K_{2,3}$  we have to avoid  $R_8 = K_{2,4}$ ,  $R_{10} = K_{2,5}$  and so on up to  $R_{2n} = K_{2,n}$  (see Lemma 5.4.12). The way to avoid  $R_{2n} = K_{2,n}$  is described below (at the beginning of the proof of Theorem 5.4.1).*

Next lemma concerns the graphs  $R_6$  containing the graph  $T^+$ , defined in Figure 5.1(b), as a subgraph. These graphs have six edges.

**Lemma 5.4.10.** *If  $R_6$  contains the graph  $T^+$  as a subgraph and  $R_6 \neq K_4$  then Bob wins.*

*Proof.* If  $R_6$  contains only one cycle then Bob in his first move destroys this cycle and, regardless of Alice's move, the graph  $R_4$  is a forest. Thus, we can apply Lemma 5.4.6.

If  $R_6$  contains more than one cycle, then it is isomorphic to one of the five graphs drawn in Figure 5.2. We consider five cases.

- **Case 1.**  $R_6 = G_1$ . Then Bob plays a huge  $H$  on  $zu$ . If Alice weights  $tu$ , in order to get  $R_4 = T^+$ , Bob weights  $zt$  in such a way that he gets the equality on  $tu$ . If Alice



## 5. The neighbour-sum-distinguishing edge-weighting game

---

weights an edge different from  $tu$ , the graph  $R_4$  is either the forest or the graph  $K_3 \cup K_2$  and we can apply either Lemma 5.4.6 or Lemma 5.4.7.

- **Case 2.**  $R_6 = G_2$ . Then Bob weights the chord of the cycle  $C_5$ . So, after Alice's move, we get the path of length four and we apply again Lemma 5.4.6.
- **Case 3.**  $R_6 = G_3$ . Then Bob plays a huge  $H$  on  $ty$ . If Alice weights  $u_1u_2$ , she gets  $R_4 = T^+$ , but with  $\sigma(t)$  much greater than  $\sigma(z)$  and we can apply Lemma 5.4.8. If Alice weights an edge different from  $u_1u_2$ , the graph  $R_4$  is either a forest or the graph  $K_3 \cup K_2$  and we can apply either Lemma 5.4.6 or Lemma 5.4.7.
- **Case 4.**  $R_6 = G_4$ . Then Bob plays a huge  $H$  on  $ty$ . In order to get  $R_4 = T^+$ , Alice has to weight  $tu$ . Then, she gets  $R_4 = T^+$ , but with  $\sigma(t)$  much greater than  $\sigma(z)$  and we can apply Lemma 5.4.8. If Alice weights an edge different from  $tu$ , the graph  $R_4$  is either the forest or the graph  $K_3 \cup K_2$  and we can apply either Lemma 5.4.6 or Lemma 5.4.7.
- **Case 5.**  $R_6 = G_5$ . Let  $\sigma(v)$  denote the sum at a vertex  $v$  at this moment. Then Bob plays a huge  $H$  on  $ty$ . In order to get  $R_4 = T^+$ , Alice can weight either  $yu$  or  $zt$ .
  - If Alice weights  $zt$ , say by  $M$ , Bob plays on  $yu$  in such a way that the equality on  $tu$  holds. It is possible since the present value at  $t$ , equal to  $\sigma(t) + H + M$ , is much bigger than  $\sigma(u)$ .
  - If Alice weights  $yu$ , say by  $M$ , she gets  $R_4 = T^+$ , but with  $\sigma(t) + H$  at the vertex  $t$  much bigger than  $\sigma(z)$  and we can apply Lemma 5.4.8.

□

The next lemma concerns the graphs  $R_8$  containing the graph  $K_4$  as a subgraph.

**Lemma 5.4.11.** *If  $R_8$  contains the graph  $K_4$  as a subgraph then Bob wins.*

*Proof.* Denote by  $\{u_1, u_2, u_3, u_4\}$  the vertices of the graph  $K_4$ . Consider first the case when  $R_8$  contains one more vertex, say  $u_5$  joined with  $u_2$  and  $u_3$ . Bob starts by weighting the edge  $u_1u_3$ .

If Alice plays on one of the following edges:  $u_1u_2$ ,  $u_1u_4$ ,  $u_5u_2$ ,  $u_5u_3$ , then  $R_6$  is either  $G_4$  or  $G_5$  from Lemma 5.4.10 and Bob wins.

If Alice plays on one of the two edges  $u_2u_3$  or  $u_2u_4$ , then Bob plays on the remaining one and  $R_5$  is the cycle  $C_5$ . In the next move Alice has to create the path  $P_5$ .

If Alice plays on  $u_3u_4$  then we get  $G_1$ , from Lemma 5.4.10 and Bob wins.

Consider now the case when none of the two edges of  $R_8$  which are not edges of  $K_4$  is on a cycle. Denote these edges by  $e_1, e_2$ . Bob weights one of the edges of  $K_4$ .

If Alice weights another edge of  $K_4$  then Bob is able to destroy all cycles in  $R_5$  in his next move. So, Alice has to weight either  $e_1$  or  $e_2$ , but then we get one of the graphs from Lemma 5.4.10 and Bob wins. □

The proof of the following lemma is obvious.

**Lemma 5.4.12.** *If  $R_{2k+2} \neq K_{2,k+1}$  then Bob is able to get  $R_{2k} \neq K_{2,k}$ .*

We are now able to complete the proof of Theorem 5.4.1.

First, we shall show that Bob is able to avoid the graph  $R_{2(n-2)} = K_{2,n-2}$ . Consider the  $A$ -game on  $K_n$  with  $n \equiv 2, 3 \pmod{4}$ . Then  $\frac{n(n-1)}{2}$  is odd. The number of moves before the *critical situation* (where only  $2(n-2)$  unweighted edges remain) is equal to

$$\frac{n(n-1)}{2} - 2(n-2) = \frac{(n-2)(n-3)}{2} + 1$$

which is also odd. If  $R_{2(n-2)} = K_{2,n-2}$  then the weighted edges should form the graph  $K_{n-2} \cup K_2$ . The longest path in this graph is of order  $n-2$ . Therefore, the aim of Bob is to weight a path of order at least  $n-1$  before the critical situation is attained.

Since Alice starts, Bob has  $\frac{(n-2)(n-3)}{4}$  moves. Observe that the first edge weighted by Alice can be used by Bob to build the path. Therefore, he needs only  $n-3$  moves.

We have

$$\frac{(n-2)(n-3)}{4} \geq n-3$$

for  $n \geq 6$ .

In the case of the  $B$ -game, with  $n \equiv 0, 1 \pmod{4}$ , the number  $\frac{n(n-1)}{2}$  is even. The number of moves until the critical situation has been reached is equal to  $\frac{n(n-1)}{2} - 2(n-2) = \frac{(n-2)(n-3)}{2} + 1$ , which is also even. Since Bob starts, he has  $\frac{n^2-5n+8}{4}$  moves. Observe that also in this case the first edge weighted by Alice can be used by Bob to build the path. Therefore, he needs only  $n-3$  moves.

We have

$$\frac{n^2-5n+8}{4} \geq n-3$$

for  $n \geq 5$ .

We define Bob's strategy as follows:

- Until a situation where only  $2(n-2)$  unweighted edges remain is reached, Bob builds a path as long as possible. As we showed above, this strategy allows to avoid the graph  $R_{2(n-2)} = K_{2,n-2}$ .
- Next, Bob continues avoiding  $R_{2k} = K_{2,k}$  (cf. Lemma 5.4.12) until the moment when only eight moves remain.
- If  $R_8$  contains  $K_4$ , then Bob plays as in Lemma 5.4.11. If not, he continues his strategy and avoids  $K_{2,3}$  in the next round.
- Since  $R_6$  is neither  $K_{2,3}$  nor  $K_4$ , Bob continues the game using strategy described in Lemma 5.4.10.

So, Bob wins in every case. □

## 5.5 Complete bipartite graphs

In this section we study the neighbour-sum-distinguishing edge-weighting game on complete bipartite graphs of the form  $K_{2,n}$ ,  $n \geq 2$ . We prove the following:

**Theorem 5.5.1** (Complete bipartite graphs).

1. For every integer  $n \geq 2$ , Bob wins the  $A$ -game on  $K_{2,n}$ .
2. For every integer  $n \geq 2$ , Alice wins the  $B$ -game on  $K_{2,n}$ .

*Proof.* Let us denote by  $\{u, v\} \cup \{x_1, \dots, x_n\}$  the bipartition of  $V(K_{2,n})$ . By Theorem 5.3.1, we know that Bob wins the  $A$ -game on  $K_{2,2} = C_4$  and that Alice wins the  $B$ -game on  $K_{2,2}$ . We can thus assume that  $n \geq 3$ .

We first describe Bob's strategy for the  $A$ -game on  $K_{2,n}$ . Except on his two last moves, whenever Alice weights an edge  $ux_i$  (resp.  $vx_i$ ) with weight  $c_i$ , Bob weights the corresponding "twin" edge  $vx_i$  (resp.  $ux_i$ ) with the same weight  $c_i$ . After the  $(2n - 4)$ -th round, the sums at both vertices  $u$  and  $v$  are thus equal to the same value, say  $\sigma$ , and the unweighted edges are  $ux_1, vx_1, ux_2$  and  $vx_2$ , without loss of generality. Assume, again without loss of generality, that Alice weights the edge  $ux_1$  with weight  $c$ . Bob answers by weighting the edge  $vx_2$  with weight  $\sigma + c$ , so that the sums at vertices  $u$  and  $x_2$  are both equal to  $\sigma + c$ , which implies that Alice loses the game.

We now turn to the  $B$ -game and describe Alice's strategy. Assume that on his  $i$ -th move Bob weights with weight  $c_i$  the edge  $wx_i$ ,  $w \in \{u, v\}$ ,  $1 \leq i \leq n$ . Alice then responds by weighting with the same weight  $c_i$  an edge  $w'x_{i'}$  such that  $i' \neq i$  and  $\{w, w'\} = \{u, v\}$  (note that this is always possible since Bob plays first). At the end of the game, the sum at both vertices  $u$  and  $v$  is thus  $\sum_{i=1}^n c_i$  and the sum at any vertex  $x_i$  is  $c_i + c_{i'}$  with  $i' \neq i$ . Therefore, since  $n \geq 3$ , the sum at vertices  $u$  and  $v$  is strictly greater than the sum at vertex  $x_i$  for every  $i$ ,  $1 \leq i \leq n$ , and thus Alice wins the game.  $\square$

We leave as an open problem the question of determining who wins the  $A$ -game or the  $B$ -game on general complete bipartite graphs  $K_{m,n}$ ,  $3 \leq m \leq n$ .



# Chapter 6

## Locally irregular decompositions of subcubic graphs

A graph  $G$  is locally irregular if every two adjacent vertices of  $G$  have different degrees. A locally irregular decomposition of  $G$  is a partition  $E_1, \dots, E_k$  of  $E(G)$  such that each  $G[E_i]$  is locally irregular. Not all graphs admit locally irregular decompositions, but for those that are decomposable, in that sense, it was conjectured by Baudon, Bensmail, Przybyło and Woźniak [BBPW15] that they decompose into at most three locally irregular graphs.

We here focus on locally irregular decompositions of subcubic graphs, which form an important family of graphs in this context, as all non-decomposable graphs are subcubic. As a main result, we prove that decomposable subcubic graphs decompose into at most five locally irregular graphs, and only at most four when the maximum average degree is less than  $\frac{12}{5}$ . We then consider weaker decompositions, where subgraphs can also include regular components, and prove two relaxations of the conjecture above for subcubic graphs.

The results presented in this chapter were published in [BBH<sup>+</sup>18].

### 6.1 Introduction

Throughout this chapter, we deal with locally irregular decompositions, as defined in Section 1.4.5.

The concept of locally irregular graphs arose in the context of neighbour-sum-distinguishing edge-weightings. The **1-2-3 Conjecture**, raised by Karoński, Łuczak and Thomason [KLT04], and its variants (see the survey [Sea12] by Seamone), are perhaps the most representative examples where locally irregular graphs arise naturally, as the “best graphs” for these problems are precisely the locally irregular ones.

Still in the context of those weighting problems related to locally irregular graphs, there are situations where, though a given graph  $G$  is not locally irregular, knowing that  $G$  decomposes into a certain number of locally irregular graphs may have some consequences. Recall that, by a decomposition of  $G$ , we mean an edge-partition  $E_1, \dots, E_k$  of  $E(G)$ . Note that a decomposition of  $G$  can be equivalently regarded as an edge-colouring

of  $G$ . A decomposition of  $G$  is *locally irregular* when each colour class induces a locally irregular graph. Locally irregular decompositions were formally introduced in [BBPW15] by Baudon, Bensmail, Przybyło and Woźniak, who noted that, in particular contexts, a graph admitting a particular locally irregular decomposition agrees with the 1-2-3 Conjecture, or some of its variants. As a more general perspective, we are interested in determining, given a graph  $G$ , the smallest number of locally irregular subgraphs that  $G$  decomposes into.

Recalling the definition of exceptional graphs in Section 1.4.5, note that all exceptional graphs are subcubic (*i.e.*, have maximum degree at most 3), and are of odd size (number of edges). For these main reasons, we believe that understanding locally irregular decompositions of subcubic graphs is of prime importance.

Concerning decomposable graphs, the main conjecture is that they should admit decompositions into at most three locally irregular graphs.

**Conjecture 6.1.1** ([BBPW15]). *For every decomposable graph  $G$ , we have  $\chi'_{\text{irr}}(G) \leq 3$ .*

Conjecture 6.1.1 was verified for several classes of graphs, including decomposable trees, decomposable complete graphs, and some classes of decomposable bipartite graphs and Cartesian products [BBPW15]. Using probabilistic methods, Conjecture 6.1.1 has also been verified for regular graphs with degree at least  $10^7$  [BBPW15], and for graphs with minimum degree at least  $10^{10}$  [Prz16]. Let us further point out that the bound in Conjecture 6.1.1, if true, would be best possible, since decomposable complete graphs or cycles with length congruent to 2 modulo 4, cannot be decomposed into two locally irregular graphs only. In general, Baudon, Bensmail and Sopena [BBS15] showed that determining the irregular chromatic index of a given graph is an NP-complete problem.

At that moment, though, it was not known whether  $\chi'_{\text{irr}}$  is, in general, bounded above by a constant. This was also not known in the particular case of decomposable bipartite graphs, for which we still do not know whether Conjecture 6.1.1 holds. These two questions were later considered by Bensmail, Merker and Thomassen [BMT17], who proved the following:

**Theorem 6.1.2** ([BMT17]). *For every decomposable graph  $G$ , we have  $\chi'_{\text{irr}}(G) \leq 328$ . Furthermore, if  $G$  is bipartite, then we have  $\chi'_{\text{irr}}(G) \leq 10$ .*

Later, Lužar, Przybyło and Soták improved these bounds in [LPS16] proving the following:

**Theorem 6.1.3** ([LPS16]). *For every decomposable graph  $G$ , we have  $\chi'_{\text{irr}}(G) \leq 220$ . Furthermore, if  $G$  is bipartite, then we have  $\chi'_{\text{irr}}(G) \leq 7$ .*

In this chapter, we consider Conjecture 6.1.1 in the context of bounded-degree graphs, giving a special focus on subcubic graphs. One first point for that is that it is still not known whether decomposable subcubic graphs verify Conjecture 6.1.1. Another important motivation is that subcubic graphs are intimately related to exceptional graphs, as

all exceptional graphs are subcubic. For these two reasons, it is interesting to understand how locally irregular decompositions behave in subcubic graphs.

This chapter is organized as follows. In Section 6.2, we start by recalling some arguments and results from [BMT17] that are used in our proofs, and which we also use to deduce a first upper bound on the irregular chromatic index of decomposable bounded-degree graphs. In the case of decomposable subcubic graphs  $G$ , this yields that  $\chi'_{\text{irr}}(G) \leq 7$  always holds. Through a more involved proof, we decrease, in Section 6.3, this bound down to 5. In Section 6.4, we further decrease this bound down to 4 for decomposable subcubic graphs with maximum average degree less than  $\frac{12}{5}$ . We then consider, in Sections 6.5 and 6.6, two relaxed versions of Conjecture 6.1.1 that were considered by Bensmail and Stevens [BS16], where one allows locally irregular decompositions to also induce subgraphs with regular components. We show that, in this context, the two relaxations of Conjecture 6.1.1 are true for subcubic graphs. We end up this chapter in Section 6.7, where we gather some possible directions for future work.

**Remark:** Right before the submission of the current work, we have been notified of the appearance, on arXiv, of the paper of Lužar, Przybyło and Soták [LPS16], reducing the bounds in Theorem 6.1.2 to 220 and 7, respectively. It was also proved that  $\chi'_{\text{irr}}(G) \leq 4$  holds for every decomposable subcubic graph  $G$ , which improves our main results in Section 6.3, and partially those in Section 6.4. However, the results in the current chapter and [LPS16] were obtained independently, and the proof arguments we use are different from those from [LPS16], and may thus be of interest for future work on locally irregular decompositions. Furthermore, to the best of our knowledge, the questions we consider in Sections 6.5 and 6.6 have not been considered by other authors.

## 6.2 Locally irregular decompositions of bounded-degree graphs

One first ingredient in the proof of Theorem 6.1.2 is a general reduction of Conjecture 6.1.1 to graphs with even size. We generalize it in the following way, where, by a *hereditary family* of graphs, we mean a family of graphs that is closed under taking subgraphs.

**Theorem 6.2.1** ([BMT17]). *Let  $\mathcal{G}$  be a hereditary family of graphs. Then, we have:*

$$\max \{ \chi'_{\text{irr}}(G) : G \in \mathcal{G} \text{ is decomposable} \} \leq \max \{ \chi'_{\text{irr}}(G) : G \in \mathcal{G} \text{ is connected and has even size} \} + 1.$$

Hence, in order to exhibit constant upper bounds on the irregular chromatic index of decomposable graphs among a class  $\mathcal{G}$ , one may focus on the connected even-size graphs of  $\mathcal{G}$  only. One convenient point for focusing on connected even-size graphs is that they are all decomposable. In particular, when considering an even-size subgraph of a graph, we do not have to wonder about whether it is exceptional or not.

The proof of Theorem 6.2.1 relies on the following two lemmas, which we use in the next section.

**Lemma 6.2.2** ([BMT17]). *Let  $G$  be a connected graph with even size. Then, for every vertex  $v$  of  $G$ , there is a path  $P$  of length 2 in  $G$ , such that  $P$  contains  $v$ , and all components of  $G - E(P)$  have even size.*

Recall that, when referring to a *claw*, we mean the star  $K_{1,3}$  on 4 vertices.

**Lemma 6.2.3** ([BMT17]). *Let  $G$  be a decomposable connected graph with odd size. Then, there is, in  $G$ , a claw  $H$  with 0 or 2 of its edges subdivided, such that all components of  $G - E(H)$  have even size.*

Clearly, the graph property “being of maximum degree at most  $k$ ” is a hereditary property. Thus, using Theorem 6.2.1 and Lemma 6.2.2, we can already state a general upper bound on the irregular chromatic index of a decomposable graph with given maximum degree.

**Observation 6.2.4.** *For every decomposable connected graph  $G$  with even size, we have  $\chi'_{\text{irr}}(G) \leq 3\Delta(G) - 3$ .*

*Proof.* We prove the claim by induction on  $|V(G)| + |E(G)|$ . As it can easily be verified whenever  $G$  is small, we proceed with the inductive step. Let  $v$  be a  $\Delta(G)$ -vertex of  $G$ . According to Lemma 6.2.2, we can find, in  $G$ , a path  $P$  of length 2 such that  $P$  contains  $v$ , and all components of  $G' := G - E(P)$  have even size. Since  $G'$  is smaller than  $G$ , all its components have even size, and  $\Delta(G') \leq \Delta(G)$ , there exists a locally irregular  $(3\Delta(G) - 3)$ -edge-colouring of  $G'$ . By that edge-colouring, there is necessarily, in  $G$ , at least one of the  $3\Delta(G) - 3$  colours, say  $\alpha$ , which is not assigned to any edge incident to the vertices of  $P$ . Hence, by assigning colour  $\alpha$  to the edges of  $P$ , we get a locally irregular  $(3\Delta(G) - 3)$ -edge-colouring of  $G$ , since a path of length 2 is locally irregular.  $\square$

**Corollary 6.2.5.** *For every decomposable graph  $G$ , we have  $\chi'_{\text{irr}}(G) \leq 3\Delta(G) - 2$ .*

## 6.3 Locally irregular decompositions of subcubic graphs

Concerning lower bounds on the maximum irregular chromatic index of a decomposable subcubic graph, let us first mention that there are infinitely many subcubic graphs  $G$  verifying  $\chi'_{\text{irr}}(G) = 3$ . This is, in particular, the case for cycles with length congruent to 2 modulo 4 (see [BBPW15]). It is actually NP-complete to decide whether a given cubic graph  $G$  verifies  $\chi'_{\text{irr}}(G) \leq 2$ , implying that much more subcubic graphs, with possibly a more complex structure, can have irregular chromatic index 3. This follows from a result of Dehghan, Sadeghi and Ahadi [DSA13], who proved, in the context of the 1-2-3 Conjecture, that deciding whether a cubic graph has a neighbour-sum-distinguishing 2-edge-weighting is an NP-complete problem. This result implies exactly the claim above, as neighbour-sum-distinguishing 2-edge-weightings and locally irregular 2-edge-colourings are equivalent notions in regular graphs (see [BBPW15]).



## 6. Locally irregular decompositions of subcubic graphs

---

We now turn our attention towards upper bounds on the irregular chromatic index of decomposable subcubic graphs. According to Observation 6.2.4, we know that connected subcubic graphs with even size have irregular chromatic index at most 6. From that, we get, according to Corollary 6.2.5, that decomposable subcubic graphs have irregular chromatic index at most 7. In this section, we decrease these two bounds to 4 and 5, respectively. We actually focus on decomposable connected subcubic graphs with even size that are *strictly subcubic*, meaning that they are not cubic. By proving that they have irregular chromatic index at most 4, we are then able to prove the upper bound 5 on the irregular chromatic index of both cubic graphs with even size, and decomposable (not necessarily strictly) subcubic graphs with odd size.

**Theorem 6.3.1.** *For every decomposable connected strictly subcubic graph  $G$  with even size, we have  $\chi'_{\text{irr}}(G) \leq 4$ .*

*Proof.* Let  $G$  be a counterexample to the claim that is minimal in terms of  $|V(G)| + |E(G)|$ . In other words, we have  $\chi'_{\text{irr}}(G) > 4$ , and every smaller connected strictly subcubic graph with even size has irregular chromatic index at most 4. Our proof consists in showing that  $G$  cannot contain certain configurations, until we get to the point where  $G$  is shown to be cubic, a contradiction.

Recall that a *bridge* of a graph refers to an edge whose deletion disconnects the graph. We start off by showing that  $G$  cannot contain non-pendent bridges, where, by a *pendent bridge*, we mean a bridge which is a pendent edge. In other words, a pendent bridge is a pendent edge, and a non-pendent bridge is a bridge whose deletion results into two components having edges.

**Claim 6.3.2.** *The graph  $G$  has no non-pendent bridge.*

*Proof.* Assume, for a contradiction, that  $G$  has a non-pendent bridge, *i.e.*, an edge  $uv$  such that  $G - uv$  has two components  $G_u$  and  $G_v$  with  $|E(G_u)|, |E(G_v)| > 0$ . Further assume that  $u$  belongs to  $G_u$  while  $v$  belongs to  $G_v$ . Since  $G$  has even size, we have that  $|E(G_u)| + |E(G_v)|$  is odd. We may hence assume that  $G_u$  has even size, while  $G_v$  has odd size. Since  $G_u$  and  $G_v + uv$  are smaller than  $G$ , are strictly subcubic and of even size, we have  $\chi'_{\text{irr}}(G_u), \chi'_{\text{irr}}(G_v + uv) \leq 4$  due to the minimality of  $G$ . Hence, there exist a locally irregular 4-edge-colouring  $\phi_u$  of  $G_u$ , and a locally irregular 4-edge-colouring  $\phi_v$  of  $G_v + uv$ . Since  $d_{G_u}(u) \leq 2$ , and we can freely permute any two colours assigned by  $\phi_u$  to the edges of  $G_u$ , we can make sure that  $\phi_u$  assigns colours among  $\{1, 2\}$  to the edges of  $G_u$  incident to  $u$ . Similarly, since  $d_{G_v+uv}(v) = 1$ , and we can freely permute the colours assigned by  $\phi_v$  to the edges of  $G_v + uv$ , we can make sure that  $\phi_v(uv) = 3$ . Clearly,  $\phi_u$  and  $\phi_v$  give rise to a locally irregular 4-edge-colouring of  $G$ , a contradiction.  $\square$

We now show that  $G$  cannot contain pendent bridges as well. In the upcoming proof, and throughout this chapter, whenever considering a subgraph obtained by removing edges, we also remove its isolated vertices, if any.

**Claim 6.3.3.** *The graph  $G$  has no 1-vertex.*

*Proof.* Assume the contrary, and let  $uv$  be an edge of  $G$  such that  $d(u) = 1$ . We must have  $d(v) = 3$ , as otherwise  $d(v) = 2$  and the other edge incident to  $v$  would be a non-pendent bridge. Let  $w_1$  and  $w_2$  denote the two neighbours of  $v$  different from  $u$ . Consider the graph  $G' := G - uv - vw_1$ . Note that  $G'$  is connected as otherwise  $vw_1$  would be a non-pendent bridge of  $G$  whose existence would contradict Claim 6.3.2. Hence  $G'$  is a strictly subcubic graph with even size, and smaller than  $G$ . There hence exists a locally irregular 4-edge-colouring of  $G'$ . By this edge-colouring, the vertices  $u, v$  and  $w_1$ , because  $d_{G'}(u) = 0$ ,  $d_{G'}(v) = 1$  and  $d_{G'}(w_1) \leq 2$ , are incident to at most three different colours. A non-used colour can hence be assigned to  $uv$  and  $vw_1$ , resulting in a locally irregular 4-edge-colouring of  $G$ , again a contradiction.  $\square$

We gather previous Claims 6.3.2 and 6.3.3 in the following way:

**Claim 6.3.4.** *The graph  $G$  has no bridge.*

Our goal now is to show that  $G$  has no 2-vertex. To that aim, we first show that  $G$  cannot have small cycles, namely triangles ( $C_3$ 's) and squares ( $C_4$ 's).

**Claim 6.3.5.** *The graph  $G$  has no triangle.*

*Proof.* Assume the contrary, and let  $C := uvwu$  be a triangle of  $G$ . If one vertex, say  $u$ , of  $C$  is a 2-vertex, then consider  $G' := G - uv - uw$ . That graph is a strictly subcubic graph, with even size and fewer vertices and edges than  $G$ , which hence admits a locally irregular 4-edge-colouring. Since  $d_{G'}(v), d_{G'}(w) \leq 2$ , at most two different colours are assigned to the edges incident to  $v$  and  $w$  in  $G'$ . This is because a locally irregular graph cannot include a component isomorphic to  $K_2$ . We can thus assign a non-used colour to  $uv$  and  $uw$ , resulting in a locally irregular 4-edge-colouring of  $G$ , a contradiction.

Assume now that  $d(u) = d(v) = d(w) = 3$ . We note that if removing any of the 2-paths  $vwu$ ,  $uvw$  or  $uwv$  from  $G$  results in a connected graph, then we can deduce a locally irregular 4-edge-colouring of the remaining graph, having the additional property that at most three colours are assigned to the at most four remaining edges incident to  $u, v$  and  $w$ . This is again because a locally irregular graph cannot have a component isomorphic to  $K_2$ . Such a colouring can hence be extended to the removed 2-path using one of the non-used colours, hence to  $G$ , a contradiction. Thus, removing any two edges among  $\{vu, uv, vw\}$  disconnects  $G$ . But this contradicts Claim 6.3.4, as this implies that every edge not in  $C$  and incident to  $C$  (there are three of them) is a bridge (either pendent or non-pendent). So  $C$  cannot exist.  $\square$

**Claim 6.3.6.** *The graph  $G$  has no square.*

*Proof.* Assume the contrary, and let  $C := uvwxu$  be a square of  $G$ . First assume that  $C$  has at least one 2-vertex; without loss of generality, we may assume that  $d(u) = 2$ . Consider the graph  $G' := G - ux - uv$ ; this graph is connected, has even size, and is smaller than  $G$ . Therefore, it admits a locally irregular 4-edge-colouring. If one of the four colours is not assigned to one of the at most four edges incident to  $x$  and  $v$  in  $G'$ , then we can obtain a locally irregular 4-edge-colouring of  $G$  by assigning the non-used

colour to  $ux$  and  $uv$ . So we may assume that  $d_G(v) = d_G(x) = 3$ , and that all four edges incident to  $v$  and  $x$  in  $G'$  are assigned different colours. But then, in the 4-edge-colouring, necessarily one of  $wx$  and  $wv$  is isolated in the subgraph induced by its assigned colour, implying that this subgraph is not locally irregular, thus that the 4-edge-colouring is not locally irregular, a contradiction. So, necessarily, one of the four colours does not appear around  $x$  and  $v$  in  $G'$ , and the previous case applies.

Assume now that  $d(u) = d(v) = d(w) = d(x) = 3$ . We denote by  $u', v', w', x'$ , respectively, the neighbour of  $u, v, w, x$ , respectively, which does not belong to  $C$ . Note that  $G' := G - ux - uv$  remains connected as otherwise  $uu'$  would be a bridge in  $G$  (contradicting Claim 6.3.4). Since  $G'$  is of even size and is smaller than  $G$ , it admits a locally irregular 4-edge-colouring  $\phi$ . We show that  $\phi$  can always be extended to a locally irregular 4-edge-colouring of  $G$ , a contradiction.

Similarly as in a previous case, we may assume that  $\phi$  assigns each of the four colours to at least one edge incident to  $u, v$  and  $x$  in  $G'$ . Note that there are exactly five such edges, as  $G$  is simple and does not have triangles by Claim 6.3.5 (in particular,  $v' \neq x$ ). Assume, without loss of generality, that  $\phi(uu') = 1$ . Note first that we cannot have  $\phi(vw) = 1$  or  $\phi(wx) = 1$ . Indeed, in such a situation (say  $\phi(wx) = 1$ ), so that all four colours appear in the neighbourhood of  $u, v, x$ , one would need, without loss of generality,  $\phi(xx') = 2$ ,  $\phi(vw) = 3$  and  $\phi(vv') = 4$ . But then either  $wx$  is an isolated edge in the 1-subgraph<sup>1</sup>, or  $vw$  is an isolated edge in the 3-subgraph, contradicting the fact that  $\phi$  is locally irregular.

So we may assume that  $1 \notin \{\phi(vw), \phi(wx)\}$ . We consider two cases depending on whether  $\phi(vw)$  and  $\phi(wx)$  are equal or not.

- **Case 1:**  $\phi(vw) \neq \phi(wx)$ .

Without loss of generality, assume that  $\phi(vw) = 3$  while  $\phi(wx) = 2$ , and also that  $\phi(xx') = 4$  (since colour 4 appears in the neighbourhood of  $u, v, x$ ). Because the 2-subgraph is locally irregular, we necessarily have  $\phi(wv') = 2$ , which implies, because the 3-subgraph is locally irregular,  $\phi(vv') = 3$ . Therefore, if  $u'$  is a 2-vertex in the 1-subgraph, then we can extend  $\phi$  to  $G$  by setting  $\phi(ux) = \phi(uv) = 1$ . So assume  $u'$  is a 3-vertex in the 1-subgraph. Analogously, if  $v'$  is not a 3-vertex in the 3-subgraph, then we can extend  $\phi$  to  $G$  by setting  $\phi(uv) = 3$  and  $\phi(ux) = 1$ . So assume  $v'$  is a 3-vertex in the 3-subgraph. Now, if  $w'$  is not a 3-vertex in the 2-subgraph, then we can extend  $\phi$  to  $G$  by setting  $\phi(wv) = \phi(wx) = 2$ , and  $\phi(ux) = 1$  and  $\phi(uv) = 3$ . So assume that  $w'$  is a 3-vertex in the 2-subgraph. Again,  $\phi$  can be extended to  $G$  by setting  $\phi(wv) = \phi(wx) = 1$ , and  $\phi(ux) = \phi(uv) = 2$ .

- **Case 2:**  $\phi(vw) = \phi(wx)$ .

We may assume that  $\phi(vw) = \phi(wx) = 2$ , and that  $\phi(vv') = 4$  and  $\phi(xx') = 3$  (because all four colours appear around  $u, v, x$ ). As in the previous case, we may

---

<sup>1</sup>Given any colour  $\alpha$  assigned by an edge-colouring, when mentioning the  $\alpha$ -subgraph, we refer to the subgraph whose edges are the ones assigned colour  $\alpha$ .

assume that  $u'$  is a 3-vertex in the 1-subgraph. If  $w$  is a 3-vertex in the 2-subgraph, then  $\phi$  can be extended to  $G$  by setting  $\phi(ux) = 1$  and  $\phi(uv) = 2$ . So assume that  $w$  is a 2-vertex in the 2-subgraph. Similarly, if  $x'$  is a 3-vertex in the 3-subgraph, then we can extend the colouring by setting  $\phi(xu) = 3$  and  $\phi(uv) = 1$ . So assume that  $x'$  is a 2-vertex in the 3-subgraph. A similar argument shows that we may as well assume that  $v'$  is a 2-vertex in the 4-subgraph. Now consider the value of  $\phi(ww')$ . On the one hand, if  $\phi(ww') = 1$ , then  $\phi$  can be extended to  $G$  by setting  $\phi(xw) = \phi(xu) = 3$ , and  $\phi(vw) = \phi(vu) = 4$ . On the other hand, if  $\phi(ww') \neq 1$ , then  $\phi$  can be extended to  $G$  by setting  $\phi(ux) = \phi(uv) = 2$  and  $\phi(wx) = \phi(wv) = 1$ .

In each case,  $\phi$  can be extended to a locally irregular 4-edge-colouring of  $G$ , a contradiction. So  $G$  cannot contain a square.  $\square$

We now focus on the 2-vertices of  $G$ , which exist, since  $G$  is strictly subcubic and has no 1-vertex (Claim 6.3.3).

**Claim 6.3.7.** *The graph  $G$  has no neighbouring 2-vertices.*

*Proof.* Assume  $G$  has two adjacent 2-vertices  $u$  and  $v$ , and let  $u'uvv'$  be the induced path of length 3 of  $G$  containing  $u$  and  $v$ . Here, we consider the graph  $G' := G - u'u - uv$ . This graph is connected, as otherwise  $u'u$  would be a bridge of  $G$ , contradicting Claim 6.3.4. Furthermore,  $G'$  has even size and is smaller than  $G$ . Hence, there exists a locally irregular 4-edge-colouring of  $G'$ . Since, in  $G'$ , the vertices  $u'$  and  $v$  are a  $2^-$ -vertex and a 1-vertex, respectively, that edge-colouring assigns at most three different colours to edges incident to  $u'$  and  $v$  in  $G'$ . So we can assign a non-used colour to  $u'u$  and  $uv$ , which results in a locally irregular 4-edge-colouring of  $G$ , a contradiction.  $\square$

**Claim 6.3.8.** *The graph  $G$  has no 3-vertex adjacent to two 2-vertices.*

*Proof.* Assume, for contradiction, that  $G$  has a 3-vertex  $v$  adjacent to two 2-vertices  $u_1, u_2$  and another  $2^+$ -vertex  $w$ . Consider the graph  $G' := G - vu_1 - vu_2$ . If  $G'$  is not connected, then necessarily  $w$  belongs to the same component as one of  $u_1$  and  $u_2$  (as, otherwise,  $vw$  would be a bridge in  $G$ , contradicting Claim 6.3.4). Actually,  $w$  belongs to the same component as only one of  $u_1$  and  $u_2$ , as otherwise  $G'$  would be connected. Assume without loss of generality that  $w$  and  $u_2$  belong to the same component of  $G'$ , while  $u_1$  belongs to another component. But then  $vu_1$  is a bridge in  $G$ , which contradicts Claim 6.3.4.

So  $G'$  is necessarily connected. Furthermore, it has even size and is smaller than  $G$ . Hence, there exists a locally irregular 4-edge-colouring of  $G'$ . Since  $d_G(u_1) = d_G(u_2) = 2$ , by that edge-colouring, at most three different colours are assigned to the edges incident to  $v, u_1$  and  $u_2$  in  $G'$ . There is thus a non-used colour that can be assigned to  $vu_1$  and  $vu_2$ , resulting in a locally irregular 4-edge-colouring of  $G$ . This is a contradiction.  $\square$

We are now ready to conclude the proof, by raising a final contradiction. Since  $G$  is strictly subcubic and  $\delta(G) > 1$ , there is a 2-vertex  $v$  in  $G$ . Let  $u_1$  and  $u_2$  be the two neighbours of  $v$  in  $G$ . Because  $G$  has no triangle by Claim 6.3.5, the vertices  $u_1$  and  $u_2$  are not joined by an edge. Furthermore, since  $G$  has no 1-vertex by Claim 6.3.3, nor

## 6. Locally irregular decompositions of subcubic graphs

---

neighbouring 2-vertices by Claim 6.3.7, we have  $d(u_1) = d(u_2) = 3$ . So let  $w_1, w_2$  denote the two neighbours of  $u_1$  different from  $v$ , and  $w_3, w_4$  denote the two neighbours of  $u_2$  different from  $v$ . Since  $G$  has no square by Claim 6.3.6, we have  $N(u_1) \cap N(u_2) = \{v\}$ .

By symmetry, and because  $G$  has no bridge, we may assume that, in  $G - \{v, u_1, u_2\}$ , vertices  $w_1$  and  $w_3$  are in a same component, and  $w_2$  and  $w_4$  are in a same (possibly different) component. Also, the two paths  $P_1 := w_1u_1vu_2w_3$  and  $P_2 := w_1u_1vu_2w_4$  are symmetric, and it is easy to verify that, for some  $i = 1, 2$ , each component of  $G - E(P_i)$  has an even number of edges.

Assume, without loss of generality, that  $P_1$  has that property, and let  $G' := G - E(P_1)$ . Remember that  $G'$  can have up to two components, each of which has even size. Since  $G'$  is strictly subcubic, smaller than  $G$ , and is of even size, there exists a locally irregular 4-edge-colouring  $\phi$  of  $G'$ . We extend  $\phi$  to  $G$ , so that a contradiction is obtained.

Since  $d_{G'}(w_1) \leq 2$  and  $d_{G'}(u_1) = 1$ , the vertices  $w_1$  and  $u_1$  are incident to at most three edge colours by  $\phi$ , namely the colours assigned to  $u_1w_2$  and to the at most two edges incident to  $w_1$  in  $G'$ . So there is a colour  $\alpha_1 \in \{1, 2, 3, 4\}$  such that, when assigning colour  $\alpha_1$  to  $w_1u_1$  and  $u_1v$ , those two edges induce a path of length 2 in the  $\alpha_1$ -subgraph. Analogously, there is a colour  $\alpha_2 \in \{1, 2, 3, 4\}$  such that, when assigning colour  $\alpha_2$  to  $w_4u_2$  and  $u_2v$ , those two edges induce a path of length 2 in the  $\alpha_2$ -subgraph. If  $\alpha_1 \neq \alpha_2$ , then we get a locally irregular 4-edge-colouring of  $G$  by assigning colour  $\alpha_1$  to  $w_1u_1$  and  $u_1v$ , and colour  $\alpha_2$  to  $w_4u_2$  and  $u_2v$ .

Assume thus that  $\alpha_1 = \alpha_2$ . Let  $\beta_1 := \phi(u_1w_2)$ . Recall that  $\beta_1 \neq \alpha_1$ . We may assume that  $\beta_1$  is not assigned to any edge incident to  $w_1$  in  $G'$ , as otherwise there would be another colour, different from  $\alpha_2$ , that can be assigned to  $w_1u_1$  and  $u_1v$ , and the previous extension strategy could be applied. We note that if  $w_2$  is a 2-vertex in the  $\beta_1$ -subgraph of  $G'$  induced by  $\phi$ , then a correct extension of  $\phi$  is obtained by assigning colour  $\beta_1$  to  $w_1u_1$  and  $u_1v$ , and colour  $\alpha_2$  to  $w_4u_2$  and  $u_2v$ . Analogously, we can deduce a correct extension when  $w_3$  is a 2-vertex in the  $\beta_2$ -subgraph induced by  $\phi$ , where  $\beta_2 := \phi(u_2w_3)$  (unless  $\beta_2$  appears on an edge incident to  $w_4$ , in which case there would be another colour, different from  $\alpha_1$ , available to colour  $w_4u_2$  and  $u_2v$ ). Therefore, we may assume that  $w_2$  is a 3-vertex in the  $\beta_1$ -subgraph induced by  $\phi$ , and  $w_3$  is a 3-vertex in the  $\beta_2$ -subgraph induced by  $\phi$ . But, then, a locally irregular 4-edge-colouring of  $G$  is obtained by assigning colour  $\beta_1$  to  $u_1w_1$ , colour  $\beta_2$  to  $u_2w_4$ , and colour  $\alpha_1$  to  $vu_1$  and  $vu_2$ .  $\square$

We now use Theorem 6.3.1 to derive corollaries for decomposable subcubic graphs with odd size, and cubic graphs with even size.

**Corollary 6.3.9.** *For every connected decomposable strictly subcubic graph  $G$  with odd size, we have  $\chi'_{\text{irr}}(G) \leq 5$ .*

*Proof.* According to Lemma 6.2.3, one can find, in  $G$ , a claw  $H$  with 0 or 2 of its edges subdivided such that  $G' := G - E(H)$  has components with even size only. All components of  $G'$  are strictly subcubic. So, every component of  $G'$  is a strictly subcubic graph with even size. Hence, there exists a locally irregular 4-edge-colouring of  $G'$  according to Theorem 6.3.1. We can extend it to a locally irregular 5-edge-colouring of  $G$  by assigning colour 5 to all edges of  $H$ , which is locally irregular.  $\square$

**Corollary 6.3.10.** *For every decomposable connected cubic graph  $G$ , we have  $\chi'_{\text{irr}}(G) \leq 5$ .*

*Proof.* If  $G$  has odd size, then the proof can be conducted similarly as the proof of Corollary 6.3.9. So assume  $G$  has even size. Then, according to Lemma 6.2.2, one can find, in  $G$ , a path  $P$  with length 2 such that all components of  $G' := G - E(P)$  have even size (just apply the lemma with any vertex). Again, all components of  $G'$  are strictly subcubic and of even size. So, similarly as in the proof of Corollary 6.3.9, we can deduce a locally irregular 4-edge-colouring of  $G'$  (from Theorem 6.3.1), which we can extend to the edges of  $P$  using colour 5, hence to  $G$ .  $\square$

We summarize Theorem 6.3.1 and Corollaries 6.3.9 and 6.3.10 in the following result, which improves Corollary 6.2.5 for subcubic graphs.

**Theorem 6.3.11.** *For every decomposable subcubic graph  $G$ , we have  $\chi'_{\text{irr}}(G) \leq 5$ .*

## 6.4 Locally irregular decompositions of subcubic graphs with maximum average degree less than $\frac{12}{5}$

In this section, we focus on decomposable graphs with maximum average degree less than  $\frac{12}{5}$ , where the *maximum average degree* of a given graph  $G$  is

$$\text{mad}(G) := \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \text{ is a subgraph of } G \right\}.$$

More precisely, we again focus on connected subcubic graphs with even size, and prove the following, which is our main result in this section.

**Theorem 6.4.1.** *For every connected subcubic graph  $G$  with even size (and thus decomposable) and  $\text{mad}(G) < \frac{12}{5}$ , we have  $\chi'_{\text{irr}}(G) \leq 3$ .*

The upcoming folklore lemma gives a relationship between the maximum average degree and the girth of a planar graph. We provide a short proof for the readers' convenience. Recall that the *girth* of a graph  $G$  is the length of a shortest cycle in  $G$ .

**Lemma 6.4.2.** *For every planar graph  $G$  with girth at least  $g$ , we have  $\text{mad}(G) < \frac{2g}{g-2}$ .*

*Proof.* Let  $G$  be a connected planar graph with girth  $g$ . Assume  $g$  is finite, as, otherwise,  $G$  would be a tree and the result holds. Let  $H$  be a subgraph of  $G$ . Note that  $H$  is planar and has girth at least  $g$ . Hence,  $g|F(H)| \leq 2|E(H)|$ , where  $F(H)$  is the set of faces of  $H$ . From Euler's Formula, we obtain:

$$2g - g|V(H)| + g|E(H)| = g|F(H)| \leq 2|E(H)|.$$

Hence,

$$2g + (g - 2)|E(H)| \leq g|V(H)|,$$

which yields

$$2|E(H)|(2g + (g - 2)|E(H)|) \leq 2|E(H)|g|V(H)|,$$

and eventually that

$$\frac{2|E(H)|}{|V(H)|} \leq \frac{2g|E(H)|}{2g + (g - 2)|E(H)|} < \frac{2g}{g - 2}$$

holds. Since this is true for every subgraph  $H$  of  $G$ , the claim is proved.  $\square$

Hence, from Theorem 6.4.1 and Lemma 6.4.2, we deduce the following corollary.

**Corollary 6.4.3.** *For every connected planar subcubic graph  $G$  with even size and girth  $g(G) \geq 12$ , we have  $\chi'_{\text{irr}}(G) \leq 3$ .*

Since edge removals cannot increase the maximum average degree of a graph, Theorem 6.4.1 can be combined with Theorem 6.2.1, which yields the following (improving Theorem 6.3.11 for some classes of decomposable subcubic graphs):

**Theorem 6.4.4.** *For every decomposable subcubic graph  $G$  with  $\text{mad}(G) < \frac{12}{5}$ , we have  $\chi'_{\text{irr}}(G) \leq 4$ .*

Before proceeding with the proof of Theorem 6.4.1, let us introduce a few definitions and notations that we use throughout. A  $3_k$ -vertex is a 3-vertex adjacent to exactly  $k$  2-vertices. A *bad 2-vertex* is a 2-vertex adjacent to another 2-vertex, while a *good 2-vertex* is a 2-vertex adjacent to two 3-vertices. A *light 3-vertex* is a 3-vertex adjacent to a 1-vertex, while a *heavy 3-vertex* is a 3-vertex adjacent to no 2-vertex. A *bad 3-vertex* is a 3-vertex adjacent to two bad 2-vertices. A vertex is called *deficient* if it is a 2-vertex (bad or good) or a light 3-vertex.

*Proof of Theorem 6.4.1.* The proof is done by induction. Assuming there exists a minimum counterexample  $H$  to the claim, we prove that  $H$  cannot exist. To that aim, we go through two steps. The first step consists in proving the non-existence of some set  $\mathcal{S}$  of subgraphs in  $H$ . Based on the resulting structural properties of  $H$ , we then, through a second step, use the discharging technique in order to obtain a contradiction to the fact that  $H$  has small maximum average degree. More precisely, during this second step, we first define a weight function  $\omega : V(H) \rightarrow \mathbb{R}$  with  $\omega(v) := d(v) - \frac{12}{5}$ . An important observation is that, by our hypothesis on the maximum average degree of  $H$ , the total sum of weights must be strictly negative, since

$$\sum_{v \in V(H)} \omega(v) = \sum_{v \in V(H)} d(v) - \frac{12}{5} \cdot |V(H)|$$

and

$$\sum_{v \in V(H)} d(v) \leq |V(H)| \cdot \text{mad}(H) < \frac{12}{5} \cdot |V(H)|.$$

Next, we define discharging rules to redistribute weights among vertices, resulting, once the discharging process is finished, in a new weight function  $\omega^*$ . During the discharging

process, the total sum of weights is kept fixed. Nevertheless, by the non-existence of  $\mathcal{S}$ , it will follow that  $\omega^*(v) \geq 0$  for all  $v \in V(H)$ . This will lead to the following contradiction

$$0 \leq \sum_{v \in V(H)} \omega^*(v) = \sum_{v \in V(H)} \omega(v) < 0,$$

contradicting the existence of  $H$ .

### Structural properties

Let  $H$  be a counterexample to Theorem 6.4.1 minimizing  $|E(H)| + |V(H)|$ . So, in other words, the graph  $H$  has even size, verifies  $\text{mad}(H) < \frac{12}{5}$  and  $\chi'_{\text{irr}}(H) > 3$ , and every proper subgraph  $H'$  of  $H$  with even size verifies  $\chi'_{\text{irr}}(H') \leq 3$ . In particular, if we consider a subgraph  $H' := H - E$  for some subset  $E \subseteq E(H)$  such that all components of  $H'$  have even size, we get  $\chi'_{\text{irr}}(H') \leq 3$ .

We start off by showing that  $H$ , because it is a minimal counterexample to Theorem 6.4.1, cannot contain certain structures.

**Claim 6.4.5.** *The graph  $H$  satisfies the following:*

1.  $H$  does not contain a non-pendent bridge.
2.  $H$  does not contain a 1-vertex adjacent to a 2-vertex.
3.  $H$  does not contain a 3-vertex adjacent to a 1-vertex and a  $2^-$ -vertex.
4.  $H$  does not contain a path  $uvw$  where  $u, v, w$  are 2-vertices.
5.  $H$  does not contain two adjacent light 3-vertices.
6.  $H$  does not contain a 3-vertex adjacent to three 2-vertices.
7.  $H$  does not contain a 3-vertex adjacent to a bad 2-vertex and to two deficient vertices.
8.  $H$  does not contain two adjacent 3-vertices, such that one of them is adjacent to two bad 2-vertices, while the other one is adjacent to one deficient vertex.

*Proof.* We consider each of these structural properties separately.

1. It can easily be checked that the proof of Claim 6.3.2 can be mimicked in the current context, and still applies, despite we here use three colours only.
2. This just follows from the fact that  $H$  has no non-pendent bridge (Claim 6.4.5.1).
3. Assume  $H$  has a 3-vertex  $v$  adjacent to a 1-vertex  $u_1$  and a  $2^-$ -vertex  $u_2$ . Consider  $H' := H - vu_1 - vu_2$ . We note that  $H'$  remains connected as otherwise  $vu_2$  would be a non-pendent bridge in  $H$ , contradicting Claim 6.4.5.1. So  $H'$  has even size, verifies  $\text{mad}(H') < \frac{12}{5}$ , and is smaller than  $H$ . It hence admits a locally irregular



## 6. Locally irregular decompositions of subcubic graphs

---

3-edge-colouring. Now, because  $d_{H'}(v) = 1$  and  $d_{H'}(u_2) \leq 1$ , there are, by that edge-colouring, at most two different colours assigned to the edges incident to  $v$  and  $u_2$  in  $H'$ . So we can freely extend this locally irregular 3-edge-colouring to  $H$  by assigning to  $vu_1$  and  $vu_2$  one colour non-assigned to any edge incident to  $v$  or  $u_2$  in  $H'$ . This is a contradiction.

4. We consider  $H' := H - uv - vw$ . Note that  $H'$  remains connected as otherwise all four edges incident to  $u, v, w$  would be bridges of  $H$ , contradicting Claim 6.4.5.1 or 6.4.5.2. Now, a locally irregular 3-edge-colouring of  $H'$  can be extended to  $H$  by assigning a same colour to  $uv$  and  $vw$  that does not appear around  $u$  or  $w$  in  $H'$ . This is a contradiction.
5. Assume  $H$  has two adjacent light 3-vertices  $v_1$  and  $v_2$ . Let  $u_1$  and  $u_2$ , respectively, denote the 1-vertex adjacent to  $v_1$  and to  $v_2$ , respectively. Let further  $w$  denote the third neighbour of  $v_1$  different from  $u_1$  and  $v_2$ . By Claim 6.4.5.3, we know that  $d(w) = 3$ . Consider  $H' := H - v_1v_2 - v_2u_2$ . Again,  $H'$  is connected as otherwise  $v_1v_2$  would be a non-pendent bridge in  $H$ , contradicting Claim 6.4.5.1. Thus, there exists a locally irregular 3-edge-colouring of  $H'$ . To see that it can be extended to  $v_1v_2$  and  $v_2u_2$ , hence to  $H$ , we just note that, by that edge-colouring, necessarily  $u_1v_1$  and  $v_1w$  are assigned the same colour. This is because  $d_{H'}(u_1) = 1$  and  $d_{H'}(v_1) = 2$ , and a locally irregular graph cannot include a component isomorphic to  $K_2$ . So, by the edge-colouring of  $H'$ , there are at most two different colours assigned to the edges incident to  $v_1$  and  $v_2$ . Therefore, a non-used colour can freely be assigned to  $v_1v_2$  and  $v_2u_2$ , resulting in a locally irregular 3-edge-colouring of  $H$ , a contradiction.
6. Assume  $H$  has a 3-vertex  $v$  whose three neighbours  $u_1, u_2, u_3$  are 2-vertices. Let further  $w_1, w_2, w_3$ , respectively, denote the neighbour of  $u_1, u_2, u_3$ , respectively, different from  $v$ . Consider  $H' := H - vu_2 - vu_3$ . First, we claim that  $H'$  remains connected. Assume the contrary. Note that the component  $C$  containing  $v$  must also contain one of  $u_2$  and  $u_3$  as otherwise  $vu_1$  would be a non-pendent bridge in  $H$ , contradicting Claim 6.4.5.1. So  $C$  contains  $v$  and, say,  $u_2$ , while it does not contain  $u_3$ . But then  $vu_3$  has to be a non-pendent bridge in  $H$ , contradicting Claim 6.4.5.1. So  $H'$  is indeed connected.

Because  $H'$  has even size, verifies  $\text{mad}(H') < \frac{12}{5}$ , and is smaller than  $H$ , there is a locally irregular 3-edge-colouring  $\phi$  of  $H'$ . We extend  $\phi$  to  $H$ , in the following way. First, if one of the three colours does not appear in the neighbourhood of  $u_2, u_3$  and  $v$ , then we can freely assign that colour to both  $vu_2$  and  $vu_3$ . So, without loss of generality, we may assume  $\phi(u_1v) = 1$ ,  $\phi(u_2w_2) = 2$  and  $\phi(u_3w_3) = 3$ . Because  $\phi$  is locally irregular, necessarily we have  $\phi(u_1w_1) = \phi(u_1v) = 1$ . In particular,  $u_1$  is a 2-vertex in the 1-subgraph induced by  $\phi$ . So we can extend  $\phi$  to  $H$  by just assigning colour 1 to  $vu_2$  and  $vu_3$ . This is correct as  $v$  then becomes a 3-vertex in the 1-subgraph while its neighbours are 2<sup>-</sup>-vertices. Hence, we get a contradiction.

7. The proof of this claim is a bit tedious as it cannot be treated using a common

argument for all cases. So, we basically have to consider all possible combinations of deficient vertices. For the sake of legibility, we describe, for each of these cases, the edges which should be removed from  $H$  (resulting in  $H'$ ), and how to extend a locally irregular 3-edge-colouring  $\phi$  of  $H'$  to  $H$ . In particular, checking whether  $H'$  remains connected can be done similarly as in the previous claim.

Let  $v$  be a 3-vertex of  $H$ , and  $u_1$  be a bad 2-vertex adjacent to  $v$ . We denote by  $u_2$  and  $u_3$  the two deficient neighbours of  $v$  different from  $u_1$ . Recall that  $u_2$  and  $u_3$  cannot both be 2-vertices as otherwise  $v$  would contradict Claim 6.4.5.6. So, there are, essentially, two cases to consider:

- (a) Both  $u_2$  and  $u_3$  are light 3-vertices. Consider  $H' := H - vu_2 - vu_3$ . If a colour of  $\phi$  is not assigned to any of the edges incident to  $u_1, u_2, u_3$  in  $H'$ , then we assign that colour to  $vu_2$  and  $vu_3$ . Note further that, for each of  $u_1, u_2, u_3$ , its two incident edges in  $H'$  are assigned a same colour by  $\phi$  (as otherwise it would not be locally irregular). So we may assume that the two edges incident to  $u_1$  are assigned colour 1, the two edges incident to  $u_2$  are assigned colour 2, and the two edges incident to  $u_3$  are assigned colour 3. Then  $\phi$  can be extended to  $H$  by assigning colour 1 to  $vu_2$  and  $vu_3$ .
- (b) The vertex  $u_2$  is a light 3-vertex while  $u_3$  is a 2-vertex. Consider  $H' := H - vu_1 - vu_2$ . Again, if a colour by  $\phi$  does not appear around  $v, u_1$  and  $u_2$ , then we assign that colour to the two removed edges. Otherwise, we again get the property that, for each of  $u_2, u_3$  and the neighbour  $u'_1$  of  $u_1$  different from  $v$ , the two incident edges in  $H'$  are assigned the same colour. So, without loss of generality, we may assume that the two edges incident to  $u'_1$  in  $H'$  are assigned colour 1, the two edges incident to  $u_2$  are assigned colour 2, and the two edges incident to  $u_3$  are assigned colour 3. Then  $\phi$  can be extended to  $H'$  by assigning colour 3 to  $vu_1$  and  $vu_2$ .

8. In the previous case, we have highlighted the fact that, if  $uv$  is an edge of  $H$  such that  $d(u) > 1$  and  $v$  is deficient, then, in a locally irregular edge-colouring of a subgraph  $H'$  of  $H$  not containing  $uv$ , the at most two edges incident to  $v$  in  $H'$  are assigned the same colour.

Assume  $H$  has two adjacent 3-vertices  $v_1$  and  $v_2$  such that  $v_1$  has a deficient neighbour  $u_1$ , while  $v_2$  is adjacent to two bad 2-vertices  $u_2$  and  $u_3$ . We further denote by  $w$  the neighbour of  $v_1$  different from  $u_1$  and  $v_2$ . Due to the fact that  $u_2$  and  $u_3$  are bad 2-vertices, the only possible triangle in  $H[u_1v_1, v_1v_2, v_2u_2, v_2u_3]$  is formed by  $v_2, u_2, u_3$ . If this triangle exists, then we consider  $H' := H - u_3u_2 - u_2v_2$ , and deduce a locally irregular 3-edge-colouring of  $H'$ , which can easily be extended to  $H$ . So assume that  $H[u_1v_1, v_1v_2, v_2u_2, v_2u_3]$  has not triangle, and consider  $H' := H - u_1v_1 - v_1v_2 - v_2u_2 - v_2u_3$ . First assume that  $H'$  remains connected. Then  $H'$  has even size, satisfies  $\text{mad}(H') < \frac{12}{5}$ , and is smaller than  $H$ . It hence admits a locally irregular 3-edge-colouring, which we extend to  $H$  as follows. The idea is to

## 6. Locally irregular decompositions of subcubic graphs

---

colour, if possible,  $u_1v_1$  and  $v_1v_2$  with a same colour, and  $v_2u_2$  and  $v_2u_3$  with a same colour. Note that  $d_{H'}(u_1) \leq 2$  and  $d_{H'}(v_1) = 1$ ; there is thus a non-used colour  $\alpha$  that can freely be assigned to  $u_1v_1$  and  $v_1v_2$ . Similarly, there is also a non-used colour  $\alpha'$  that can be assigned to  $v_2u_2$  and  $v_2u_3$ . We now note that, even if  $\alpha = \alpha'$ , we get a locally irregular 3-edge-colouring of  $H$  by assigning colour  $\alpha$  to  $v_1u_1$  and  $v_1v_2$ , and colour  $\alpha'$  to  $v_2u_2$  and  $v_2u_3$ .

Lastly, assume that  $H'$  is not connected. The rest of the proof now goes quite similarly as the proof of Theorem 6.3.1. Using similar arguments, it can be checked that  $H'$  has exactly two components  $C_1$  and  $C_2$ . In particular, each of the  $C_i$ 's contains two of  $v_1, u_1, u_2, u_3$  (note that if  $d_H(w) = 1$ , then the configuration can easily be treated by removing the edges  $v_2u_2$  and  $v_2u_3$  of  $H$ ). If  $C_1$  and  $C_2$  both have even size, then induction can be invoked, locally irregular 3-edge-colourings of  $C_1$  and  $C_2$  yield a locally irregular 3-edge-colouring of  $H'$ , which can be extended to  $H$  as previously. So assume that  $C_1$  and  $C_2$  both have odd size. It can be checked that, under all those structural properties,  $H$  can be decomposed into two graphs  $H_1$  and  $H_2$ , such that  $V(H_1) \cap V(H_2) = \{v_1, v_2\}$ , and  $v_1$  and  $v_2$  are 2-vertices in, say,  $H_1$ , and 1-vertices in  $H_2$ . Since  $v_1v_2$  cannot be a non-pendent bridge by Claim 6.4.5.1, the two cases to consider, in order to construct  $H_1$  and  $H_2$ , are the following:

- $C_1$  includes  $u_1$  and  $u_2$  (while  $C_2$  includes  $v_1$  and  $u_3$ ): we add  $u_1v_1$ ,  $v_1v_2$  and  $v_2u_2$  to  $C_1$  to obtain  $H_1$ , and add  $v_2u_3$  to  $C_2$  to obtain  $H_2$ .
- $C_1$  includes  $u_1$  and  $u_3$  (while  $C_2$  includes  $v_1$  and  $u_2$ ): we add  $u_1v_1$ ,  $v_1v_2$  and  $v_2u_3$  to  $C_1$  to obtain  $H_1$ , and add  $v_2u_2$  to  $C_2$  to obtain  $H_2$ .

Then  $H_1$  and  $H_2$ , which have even size, verify  $\text{mad}(H_1), \text{mad}(H_2) < \frac{12}{5}$ , and are smaller than  $H$ , admit locally irregular 3-edge-colourings  $\phi_1$  and  $\phi_2$  (where  $\phi_i$  is that of  $H_i$ ), respectively. Note that, in  $H_1$ , if we have  $\phi_1(v_1v_2) = \alpha_1$ , then  $\alpha_1$  is also assigned to one of the two edges adjacent to  $v_1v_2$  in  $H_1$ . In other words, by  $\phi_1$ , there are only two distinct colours  $\alpha_1, \alpha_2$  assigned to the edges incident to  $v_1$  or  $v_2$ . Furthermore, we have, without loss of generality, that  $v_1$  is only incident to edges assigned colour  $\alpha_1$ , while  $v_2$  is incident to one edge assigned colour  $\alpha_1$ , and one edge assigned colour  $\alpha_2$ .

We would now like to permute some of the colours assigned by  $\phi_2$ , so that  $\phi_1$  and  $\phi_2$  yield a locally irregular 3-edge-colouring of  $H$ . Recall that  $V(H_1) \cap V(H_2) = \{v_1, v_2\}$  and that  $d_{H_2}(v_1) = d_{H_2}(v_2) = 1$ . We start by possibly permuting two colours assigned by  $\phi_2$ , so that the edge incident to  $v_2$  in  $H_2$  is assigned a colour  $\beta$  different from  $\alpha_1$  and  $\alpha_2$ . We then finish the permutation process, by, if needed, permuting the two colours by  $\phi_2$  different from  $\beta$ , so that the edge incident to  $v_1$  in  $H_2$  is assigned a colour different from  $\alpha_1$ . Clearly, three colours are sufficient in order to obtain a correct permutation verifying all these constraints. So we end up with a locally irregular 3-edge-colouring of  $H$ , a contradiction. here

□

To lighten the upcoming discharging process, we will not work directly on  $H$  but rather on a subgraph  $H^-$  of  $H$ . More precisely,  $H^-$  is the graph obtained from  $H$  by removing all 1-vertices of  $H$ , *i.e.*,  $H^- := H - \{v \in V(H), d_H(v) = 1\}$ . Clearly,  $H^-$  is connected and  $\text{mad}(H^-) < \frac{12}{5}$ . Furthermore, from the structural properties of  $H$  exhibited in Claim 6.4.5, one can easily derive the following properties of  $H^-$ .

**Claim 6.4.6.** *According to Claim 6.4.5, the graph  $H^-$  satisfies the following:*

1.  $\delta(H^-) \geq 2$  (Claims 6.4.5.2 and 6.4.5.3).
2.  $H^-$  does not contain a path  $uvw$  where  $u, v, w$  are 2-vertices (Claims 6.4.5.3, 6.4.5.4 and 6.4.5.5).
3.  $H^-$  does not contain a 3<sub>3</sub>-vertex adjacent to at least one bad 2-vertex (Claims 6.4.5.3, 6.4.5.5, 6.4.5.6 and 6.4.5.7).
4. A bad 2-vertex of  $H^-$  is also a bad 2-vertex of  $H$  (Claims 6.4.5.3 and 6.4.5.5).

### Discharging procedure

To each vertex  $v$  of  $H^-$ , we assign an initial charge  $\omega(v) := d_{H^-}(v) - \frac{12}{5}$ . We then carry out the discharging procedure in two steps:

**Step 1.** We here just apply, in  $H^-$ , the following rule:

(R0) Every heavy 3-vertex gives  $\frac{1}{5}$  to each adjacent bad 3-vertex.

Once Step 1 is finished, a new weight function  $\omega'$  results. We proceed then with Step 2:

**Step 2.** We here apply, in  $H^-$ , the following two rules:

(R1) Every 3-vertex gives  $\frac{2}{5}$  to each adjacent bad 2-vertex.

(R2) Every 3-vertex gives  $\frac{1}{5}$  to each adjacent good 2-vertex.

Recall that we denote by  $\omega^*$  the resulting weight function. Let  $v \in V(H^-)$  be a  $k$ -vertex. By Claim 6.4.6.1, we have  $k \geq 2$ . Now, consider the following cases:

- **$k = 2$ .** Observe that  $\omega(v) = -\frac{2}{5}$ . Suppose  $v$  is a bad 2-vertex. By Claim 6.4.6.2, the vertex  $v$  is adjacent to a 3-vertex. Hence, by (R1), we have  $\omega^*(v) = -\frac{2}{5} + \frac{2}{5} = 0$ . If  $v$  is a good 2-vertex, then  $\omega^*(v) = -\frac{2}{5} + 2 \times \frac{1}{5} = 0$  by (R2).
- **$k = 3$ .** Observe that  $\omega(v) = \frac{3}{5}$ . To simplify the analysis, we distinguish two cases:

- Suppose first that  $v$  is adjacent to a bad 2-vertex  $u_1$ . By Claim 6.4.6.3, all neighbours of  $v$  cannot be 2-vertices, so  $v$  is adjacent to at most two 2-vertices (including  $u_1$ ). If  $u_1$  is the only 2-vertex neighbouring  $v$ , then, by (R1), we have  $\omega^*(v) \geq \frac{3}{5} - 1 \times \frac{2}{5} = \frac{1}{5} > 0$ . Now assume  $v$  is adjacent to a second 2-vertex  $u_2$ . If  $u_2$  is a good 2-vertex, then, by (R1) and (R2), we have  $\omega^*(v) \geq \frac{3}{5} - 1 \times \frac{2}{5} - 1 \times \frac{1}{5} = 0$ . Now, if  $u_2$  is a bad 2-vertex, then the third neighbour (different from  $u_1$  and  $u_2$ ) of  $v$  is a heavy 3-vertex, as otherwise  $H$  would contain, according to Claim 6.4.6.4, the configuration described in Claim 6.4.5.8. So, by (R0), we have  $\omega'(v) = \frac{4}{5}$ . Hence, by (R1), we get  $\omega^*(v) = \frac{4}{5} - 2 \times \frac{2}{5} = 0$ .
- Finally, if  $v$  is not adjacent to a bad 2-vertex, then  $\omega^*(v) \geq \frac{3}{5} - 3 \times \frac{1}{5} = 0$  by (R0) and (R2).

Therefore,  $H^-$  cannot exist and consequently  $H$  does not exist either. This completes the proof.  $\square$

## 6.5 $K_2$ -irregular decompositions of subcubic graphs

In this section, and in Section 6.6 as well, we focus on two relaxations of Conjecture 6.1.1 considered by Bensmail and Stevens [BS16]. In particular, we completely verify these two relaxations for subcubic graphs.

The idea is to study how easier it is, for proving Conjecture 6.1.1, to allow any locally irregular decomposition to also include additional regular components. In this section, we focus on  $K_2$ -irregular decompositions (or, analogously,  $K_2$ -irregular edge-colourings), which are decompositions in which every part induces components that are either locally irregular or isomorphic to  $K_2$ . In this definition, it should be understood that, in every subgraph induced by a part of the decomposition, there may be locally irregular components, and some components isomorphic to  $K_2$  as well. For a given graph  $G$ , we denote by  $\chi'_{K_2\text{-irr}}(G)$  the smallest number of colours in a  $K_2$ -irregular edge-colouring of  $G$ . Note that  $\chi'_{K_2\text{-irr}}(G)$  is defined for every graph  $G$  as every proper edge-colouring is  $K_2$ -irregular.

Clearly, we have  $\chi'_{K_2\text{-irr}}(G) \leq \chi'_{\text{irr}}(G)$  for every decomposable graph  $G$ . Hence, Conjecture 6.1.1, if true, would imply that  $\chi'_{K_2\text{-irr}}(G) \leq 3$  holds for every graph  $G$ , unless  $G$  is exceptional. One may thus wonder whether even  $\chi'_{K_2\text{-irr}}(G) \leq 2$  is true for every graph  $G$ . This is actually not the case, as, for example,  $\chi'_{K_2\text{-irr}}(K_4) = 3$ . So, in the context of  $K_2$ -irregular edge-colourings, the conjecture that is analogous to Conjecture 6.1.1 should be the next one, which stands as a relaxation of Conjecture 6.1.1.

**Conjecture 6.5.1.** *For every graph  $G$ , we have  $\chi'_{K_2\text{-irr}}(G) \leq 3$ .*

In the following result, we show that Conjecture 6.5.1 admits an easy proof in the context of subcubic graphs. Recall that this result remains best possible even in this context because of the complete graph  $K_4$ .

**Theorem 6.5.2.** *For every subcubic graph  $G$ , we have  $\chi'_{K_2\text{-irr}}(G) \leq 3$ .*

*Proof.* We prove the claim by induction on  $|V(G)| + |E(G)|$ . As the claim can easily be verified whenever  $G$  is small, we proceed with the general case. Consider any vertex  $v$  of  $G$  and denote by  $u_1, \dots, u_k$  its neighbours, where  $k \leq 3$ . Set  $G' := G - \{vu_1, \dots, vu_k\}$ . Since  $G'$  is smaller than  $G$ , there exists a  $K_2$ -irregular 3-edge-colouring of  $G'$ . Since  $d_{G'}(u_1), \dots, d_{G'}(u_k) \leq 2$ , there are, by the edge-colouring, at most two different colours assigned to the edges incident to each  $u_i$  in  $G'$ . For each  $vu_i$ , let  $\alpha_i$  denote a colour not assigned to an edge incident to  $u_i$  in  $G'$ .

Extending the 3-edge-colouring of  $G'$  to a  $K_2$ -irregular 3-edge-colouring of  $G$  can then be done by assigning, for every  $i \in \{1, \dots, k\}$ , colour  $\alpha_i$  to  $vu_i$ , for the following reasons. First of all, because, for each  $u_i$ , edge  $vu_i$  has been assigned a colour not incident to  $u_i$  in  $G'$ , no conflict involving two vertices of  $G'$  may arise. This is because the degrees of the  $u_i$ 's in the 1-, 2-, and 3-subgraphs of  $G'$  that contain them are not altered by the extension. Then, since each  $u_i$  is a 1-vertex in the  $\alpha_i$ -subgraph induced by the resulting edge-colouring of  $G$ , it cannot be that  $v$  and  $u_i$  are involved in a conflict: the only situation where  $v$  and  $u_i$  have the same degree in the  $\alpha_i$ -subgraph is when this degree is exactly 1, in which case  $v$  and  $u_i$  belong to a component isomorphic to  $K_2$  in the  $\alpha_i$ -subgraph. Thus, we necessarily end up with a  $K_2$ -irregular 3-edge-colouring of  $G$ .  $\square$

## 6.6 Regular-irregular decompositions of subcubic graphs

In this section, we focus on *regular-irregular decompositions* (or, analogously, *regular-irregular edge-colourings*), which are more general than  $K_2$ -irregular decompositions considered in Section 6.5. Here, we allow every subgraph induced by a part of a decomposition to have components being either locally irregular or regular. So,  $K_2$ -irregular decompositions are nothing but regular-irregular decompositions where one requires all induced regular subgraphs to be 1-regular. For a given graph  $G$ , we denote by  $\chi'_{\text{reg-irr}}(G)$  the smallest number of colours in a regular-irregular edge-colouring of  $G$ . Since we have  $\chi'_{\text{reg-irr}}(G) \leq \chi'_{K_2\text{-irr}}(G)$  for every graph  $G$ , again every graph is decomposable in that manner. Note further that if  $G$  is regular, then  $\chi'_{\text{reg-irr}}(G) = 1$ .

Regular-irregular decompositions were considered by Bensmail and Stevens [BS16], who conjectured the following.

**Conjecture 6.6.1** (Bensmail, Stevens [BS16]). *For every graph  $G$ , we have  $\chi'_{\text{reg-irr}}(G) \leq 2$ .*

Conjecture 6.6.1 is known to hold for a few classes of graphs, including trees and some other classes of bipartite graphs [BS16]. We here give further evidence to the conjecture by showing it to hold for subcubic graphs as well.

**Theorem 6.6.2.** *For every subcubic graph  $G$ , we have  $\chi'_{\text{reg-irr}}(G) \leq 2$ .*

*Proof.* The proof consists in edge-colouring with colours red and green two edge-disjoint subgraphs  $\mathcal{C}$  and  $F$  of  $G$ , in the following way:

## 6. Locally irregular decompositions of subcubic graphs

---

1. We consider, as  $\mathcal{C}$ , a collection of vertex-disjoint cycles of  $G$ , and assign colour, say, red, to all edges of  $\mathcal{C}$ .
2. Set  $F := G - E(\mathcal{C})$ . Then, we edge-colour  $F$  in a regular-irregular way with colours red and green, in such a way that all edges in  $F$  being adjacent, in  $G$ , to edges of  $\mathcal{C}$  are assigned colour green.

If  $F$  can be edge-coloured as described, then we note that the components of the red subgraph induced by the edge-colouring of  $\mathcal{C}$  are disjoint, in  $G$ , from the components of the red subgraph induced by the edge-colouring of  $F$ . So the 2-edge-colourings of  $\mathcal{C}$  and  $F$  yield a regular-irregular 2-edge-colouring of  $G$ .

Start from  $\mathcal{C}$  being empty, and, until this procedure cannot be repeated, pick any cycle  $C$  of  $G - E(\mathcal{C})$  and move the edges of  $C$  to  $\mathcal{C}$ . Once this process stops, the following holds, basically because  $G$  is subcubic.

**Claim 6.6.3.** *The subgraph  $F := G - E(\mathcal{C})$  is a forest. Furthermore, for every vertex  $v$  of  $G$  having incident edges in  $\mathcal{C}$  and incident edges in  $F$ , we have  $d_{\mathcal{C}}(v) = 2$  and  $d_F(v) = 1$ .*

Assign colour red to all edges in  $\mathcal{C}$ . When referring to a *leaf edge* of  $F$ , we mean an edge that is incident to a leaf of  $F$ . We note that there are, in  $F$ , some leaves that are special in the sense that they have both incident edges in  $\mathcal{C}$  (two edges) and in  $F$  (one leaf edge). We refer to these leaves as *frontier leaves*, and derive this concept to *frontier leaf edges*, which are leaf edges of  $F$  whose at least one end is a frontier leaf. Note that a component of  $F$  can be isomorphic to  $K_2$ , in which case this component is a frontier leaf edge which potentially joins two frontier leaves.

Following the explanations above, we assign colour green to all frontier leaf edges of  $F$ . Note that  $F$  might have non-frontier leaf edges. We assign colour green to these edges as well. It now remains to show that the non-coloured (*i.e.*, non-leaf) edges of  $F$  can be assigned colours red and green, without modifying the pre-colouring we have described, in a regular-irregular way. In other words, we now want to prove the following:

**Claim 6.6.4.** *Every subcubic tree  $T$  admits a regular-irregular 2-edge-colouring, such that all leaf edges are assigned colour 1.*

*Proof.* All along this proof, we see  $T$  as a tree whose leaf edges have been pre-assigned colour 1, and we extend this pre-colouring until a regular-irregular 2-edge-colouring is attained.

The proof is by induction on the size of  $T$ . As base cases, we note that the claim is true whenever  $|E(T)| \leq 3$ . Indeed, if  $T$  has diameter at most 2, then  $T$  is a star on at most three edges being all assigned colour 1. The 1-subgraph is then exactly  $T$ , which is either regular (one edge) or locally irregular (two or three edges). On the other hand, if  $T$  has diameter 3, then  $T$  is the path of length 3 whose two end-edges are assigned colour 1. We here get a regular-irregular 2-edge-colouring (with the desired additional property) of  $T$  by assigning colour 2 to the middle-edge.

Assume thus that the claim holds whenever  $|E(T)|$  is smaller than some value, and consider the next value of  $|E(T)|$ . To begin with, if  $\Delta(T) \leq 2$ , then  $T$  is a path whose two end-edges are assigned colour 1. If the length of  $T$  is odd, then we obtain the desired regular-irregular 2-edge-colouring of  $T$  by assigning colours 1 and 2 alternately, from one end-edge to the other. When the length of  $T$  is even, the claimed edge-colouring can be obtained by applying this colouring scheme starting from the second edge of  $T$ . In particular, the first two edges of  $T$  get assigned colour 1 and thus induce a path of length 2, which is locally irregular, in the 1-subgraph.

We may thus assume that  $\Delta(T) = 3$  since  $T$  is subcubic. By a *pendent path* of  $T$ , we refer to a maximal path  $u_1u_2 \dots u_k$  of  $T$  such that  $u_1$  is a leaf, all internal vertices  $u_2, \dots, u_{k-1}$  are 2-vertices, and  $u_k$  is a 3-vertex. Since  $T$  has 3-vertices, there are at least three pendent paths in  $T$ . If  $T$  has a pendent path  $P$  with length at least 3, then the desired regular-irregular 2-edge-colouring of  $T$  can be obtained in the following way. Let  $P := u_1 \dots u_k$  where  $d(u_1) = 1$  and  $d(u_k) = 3$ . Due to the length of  $P$ , we have  $k \geq 4$ . We consider  $T' := T - u_1u_2 - u_2u_3$  and assign colour 1 to  $u_3u_4$  in  $T'$ . Since  $T'$  is subcubic, smaller than  $T$ , and has all its leaf edges assigned colour 1, there is, by the induction hypothesis, a regular-irregular 2-edge-colouring of  $T'$  which is as claimed. This edge-colouring can be extended to the claimed regular-irregular 2-edge-colouring of  $T$  by assigning colour 2 to  $u_2u_3$  and colour 1 to  $u_1u_2$ .

We may thus assume that all pendent paths of  $T$  have length 1 or 2. If  $T$  has only one vertex  $v$  with  $d(v) = 3$ , then  $T$  is a subdivided claw all of whose leaf edges are assigned colour 1. We here extend the pre-colouring by just assigning colour 2 to all non-coloured edges of  $T$ . Note that these edges are edges that are incident to  $v$  and belong to pendent paths with length 2. The resulting edge-colouring is clearly regular-irregular since the 1- and 2-subgraphs include stars only.

Now assume that  $T$  has at least two 3-vertices, and let  $r$  denote any of them. We designate  $r$  as the root of  $T$ , which defines, in the usual way, a (virtual) orientation of  $T$  from its root to its leaves. Following that orientation, we say that a vertex  $v \neq r$  of  $T$  is a *multifather* if  $v$  has exactly two children (and is hence a 3-vertex as  $v$  also has a father). A multifather of  $T$  is said *last* if all of its descendants are 2<sup>-</sup>-vertices. In other words, a last multifather is a 3-vertex with two pendent paths attached (which are of length 1 or 2). Furthermore, a last multifather is said *deepest* if it is at maximum distance from  $r$  in  $T$ .

We first claim that if  $T$  has a deepest last multifather  $v$  such that at least one of its two attached pendent paths  $P_1$  and  $P_2$  has length 2, then we can deduce the desired regular-irregular 2-edge-colouring of  $T$ . This follows from the following arguments. First assume that  $P_1 := u_1u_2v$  and  $P_2 := u'_1u'_2v$  have length 2. In that case, we consider  $T' := T - u_1u_2 - u_2v - u'_1u'_2 - u'_2v$ . Assuming  $f(v)$  denotes the father of  $v$  in  $T$ , we assign colour 1 to  $vf(v)$  in  $T'$ . Since  $T'$  is subcubic, smaller than  $T$ , and has all of its leaf edges assigned colour 1, we can deduce a regular-irregular 2-edge-colouring of  $T'$  which is as required. This edge-colouring can be extended to  $T$  by assigning colour 2 to  $vu_2$  and  $vu'_2$  (and still assigning colour 1 to  $u_1u_2$  and  $u'_1u'_2$ ). Now assume that  $P_2 := u'_1v$  has length 1 (while  $P_1$  is as previously). We here consider  $T' := T - u_1u_2 - u_2v - u'_1v$  in which  $vf(v)$



## 6. Locally irregular decompositions of subcubic graphs

---

is assigned colour 1, and a regular-irregular 2-edge-colouring of  $T'$  (with the additional property). We now extend that edge-colouring to  $T$ . If, by assigning colour 2 to  $u_2v$  (and still assigning colour 1 to  $u_1u_2$  and  $u'_1v$ ), we do not get a regular-irregular edge-colouring of  $T$ , that is only because, in the resulting 1-subgraph,  $f(v)$  and  $v$  are 2-vertices. In that situation, the desired regular-irregular 2-edge-colouring of  $T$  is obtained by assigning colour 1 to  $u_2v$ .

Hence, we may assume that  $P_1 := u_1v$  and  $P_2 := u'_1v$  have length 1. Note that if  $f(v) = r$ , then, by definition of a deepest last multifather, every vertex of  $T$  is at distance at most 2 from  $r$ . In that situation, again, by assigning colour 2 to all non-leaf edges of  $T$ , we directly get a regular-irregular 2-edge-colouring which is as desired. So assume  $f(v) \neq r$ , meaning that  $f(v)$  has a father  $f(f(v))$  in  $T$ . In case  $f(v)$  is a 2-vertex, *i.e.*, is not a multifather, we consider  $T' := T - vu_1 - vu'_1 - vf(v)$ , in which the edge  $f(v)f(f(v))$  is assigned colour 1. Here, a regular-irregular 2-edge-colouring of  $T$  is obtained by assigning colour 2 to  $vf(v)$  and colour 1 to  $vu_1$  and  $vu'_1$ .

When  $d(f(v)) = 3$ , there are, according to all assumptions we have made so far, three possibilities concerning the child  $v'$  of  $f(v)$  different from  $v$ : either 1)  $v'$  is a leaf, 2)  $v'$  has one child  $w_1$  being a leaf, or 3)  $v'$  is a deepest last multifather with two children  $w_1$  and  $w'_1$  that are leaves.

In case 1), we consider  $T' := T - vu_1 - vu'_1 - vf(v)$ . According to the induction hypothesis,  $T'$  admits a regular-irregular 2-edge-colouring which is as desired. Recall that  $v'f(v)$  is assigned colour 1 by that colouring. On the one hand, if  $f(v)f(f(v))$  is assigned colour 1, then we can extend the colouring to  $T$  by assigning colour 2 to  $vf(v)$  and colour 1 to  $vu_1$  and  $vu'_1$ . On the second hand, if  $f(v)f(f(v))$  is assigned colour 2, then we get a correct extension by assigning colour 1 to all of  $vf(v)$ ,  $vu_1$  and  $vu'_1$ .

In case 2), we consider  $T' := T - vu_1 - vu'_1$  and assign colour 1 to the leaf edge  $v'f(v)$  of  $T'$ . Again, according to the induction hypothesis, we can find a regular-irregular 2-edge-colouring of  $T'$  which is as desired. Note that if  $f(v)$  is not a 3-vertex in the 1-subgraph induced by that edge-colouring, then we can extend the edge-colouring to  $T$  by assigning colour 1 to both  $vu_1$  and  $vu'_1$ . So we may assume that all three edges incident to  $f(v)$  in  $T'$  are assigned colour 1. In that case, by assigning colour 1 to  $vu_1$  and  $vu'_1$ , and modifying the colour of  $vf(v)$  and  $v'f(v)$  to 2, we get a 2-edge-colouring of  $T$  which is regular-irregular and as desired. In particular, the component of the 1-subgraph that contains  $f(v)$  remains locally irregular, or becomes a  $K_2$ .

Finally, in case 3), we again consider  $T' := T - vu_1 - vu'_1$  in which the leaf edge  $v'f(v)$  is assigned colour 1. Note that  $f(v)$  cannot be a 3-vertex in the 1-subgraph induced by any given regular-irregular 2-edge-colouring of  $T'$  since otherwise  $f(v)$  and  $v'$  would be adjacent 3-vertices in the 1-subgraph. So, necessarily,  $f(v)$  is a  $2^-$ -vertex in the 1-subgraph, and the edge-colouring can be extended to  $T$  by assigning colour 1 to  $vu_1$  and  $vu'_1$ .

Thus, a regular-irregular 2-edge-colouring of  $T$  with the desired additional property always exists. This concludes the proof.  $\square$

Following Claim 6.6.4, there is thus a regular-irregular edge-colouring of  $F$  with colours

red and green, such that all frontier leaf edges are green. Together with the edges of  $\mathcal{C}$  being assigned colour red, this yields the claimed regular-irregular 2-edge-colouring of  $G$ , hence our conclusion.  $\square$

## 6.7 Conclusion

In this work, we have studied locally irregular decompositions in subcubic graphs. Although we did not manage to prove Conjecture 6.1.1 for decomposable subcubic graphs, we have showed that they decompose into at most 5 locally irregular subgraphs, which improves by 2 the upper bound given by Corollary 6.2.5.

One first direction for future work could be to try pushing this bound further down. As pointed out in the introduction, our bound has been recently improved down to 4 by Lužar, Przybyło and Soták [LPS16]. The next step would thus be to completely prove Conjecture 6.1.1 for decomposable subcubic graphs, or at least subclasses of decomposable subcubic graphs. We actually made a first step towards this direction when we considered subcubic graphs with bounded maximum average degree, and proved the conjecture for some of them. As examples, let us mention that the cases of subcubic bipartite graphs and subcubic planar graphs sound quite appealing to us. It might be interesting studying how locally irregular decompositions behave in these graphs.

Another direction for future work could be to consider locally irregular decompositions of graphs with larger, but fixed, maximum degree. Recall that we have provided an upper bound on their irregular chromatic index in Corollary 6.2.5. As a first step, it could be interesting to investigate how lower this bound can be pushed down for decomposable graphs with maximum degree 4. More generally, it could also be interesting to improve the method in the proof of Observation 6.2.4, in order to obtain better bounds on the irregular chromatic index of bounded-degree graphs.

# Chapter 7

## A general decomposition theory for 1-2-3 Conjecture and locally irregular decompositions

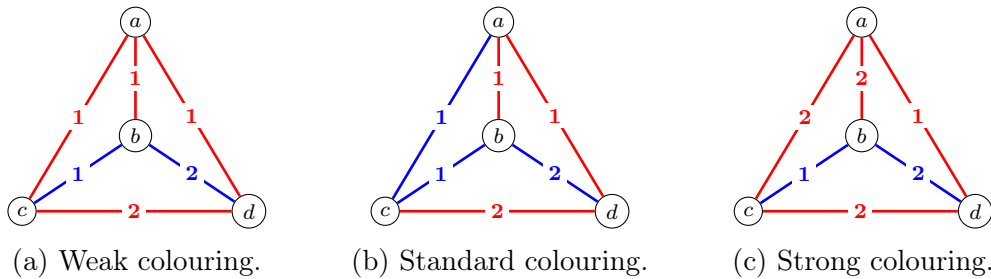
In this chapter, we propose an approach that generalizes the 1-2-3 Conjecture and locally irregular decompositions, involving coloured weights and sums. As a consequence, we get another interpretation of several existing results related to the 1-2-3 Conjecture. We also come up with new related conjectures, to which we give some support.

### 7.1 Introduction

In previous chapters, we explored several variants of nsd edge-weightings and locally irregular decompositions, and presented the links between these two notions. In this chapter, we aim at introducing a general decompositional theory enclosing neighbour-sum-distinguishing edge-weightings and locally irregular decompositions. This theory is based on the following observations. A locally irregular  $\ell$ -edge-colouring of a graph  $G$  is, put differently, a decomposition of  $G$  into graphs  $G_1, \dots, G_\ell$  verifying  $\chi_\Sigma^e(G_1), \dots, \chi_\Sigma^e(G_\ell) = 1$ . The other way around, a neighbour-sum-distinguishing  $k$ -edge-weighting of  $G$  can be seen as a 1-edge-colouring where the only colour class induces a graph, that is precisely  $G$ , whose value of  $\chi_\Sigma^e$  is  $k$ .

These observations led us to combine the notions of neighbour-sum-distinguishing edge-weightings and locally irregular edge-colourings, in the following way. Let  $\ell, k$  be two positive integers, and  $G$  be a graph. To each edge  $e$  of  $G$ , we assign, via a colouring  $\omega$  a pair  $(\alpha, \beta)$ , where  $\alpha \in \{1, \dots, \ell\}$  and  $\beta \in \{1, \dots, k\}$ , which can be regarded as a *coloured weight* (with value  $\beta$  and colour  $\alpha$ ). Now, for every vertex  $v$  of  $G$ , and every colour  $\alpha \in \{1, \dots, \ell\}$ , one can compute the *weighted  $\alpha$ -degree*  $\sigma_\alpha(v)$ , being the sum of weights with colour  $\alpha$  incident to  $v$ . So, with every vertex  $v$  is associated a palette  $(\sigma_1(v), \dots, \sigma_\ell(v))$  of  $\ell$  coloured weighted degrees.

When working on variants of the 1-2-3 Conjecture, the intent is to design edge-

Figure 7.1: Three  $(2, 2)$ -colourings of  $K_4$ .

weightings  $\omega$  that allow to distinguish adjacent vertices, according to some distinction condition. When dealing with the notions introduced in the previous paragraph, there are many ways for asking for distinction, as several coloured sums are available; in this chapter, we will focus on the following three distinction variants, which sound the most natural to us:

- *Weak distinction*: two adjacent vertices  $u$  and  $v$  of  $G$  are considered distinguished if there is an  $\alpha \in \{1, \dots, \ell\}$  such that  $\sigma_\alpha(u) \neq \sigma_\alpha(v)$ .
- *Standard distinction*: two adjacent vertices  $u$  and  $v$  of  $G$  are considered distinguished if, assuming  $\omega(uv) = (\alpha, \beta)$ , we have  $\sigma_\alpha(u) \neq \sigma_\alpha(v)$ .
- *Strong distinction*: two adjacent vertices  $u$  and  $v$  of  $G$  are considered distinguished if, for every  $\alpha \in \{1, \dots, \ell\}$ , we have  $\sigma_\alpha(u) = \sigma_\alpha(v) = 0$ , or  $\sigma_\alpha(u) \neq \sigma_\alpha(v)$ .

Assuming  $\omega$  verifies one of the weak, standard or strong distinction condition for every pair of adjacent vertices, we say that  $\omega$  is a *weak, standard* or *strong*  $(\ell, k)$ -edge-colouring, and that  $G$  is *weakly, standardly* or *strongly*  $(\ell, k)$ -coloured. We also say that  $G$  is *weakly, standardly* or *strongly*  $(\ell, k)$ -colourable, if there are  $\ell', k' \geq 1$  with  $\ell' \leq \ell$  and  $k' \leq k$  such that  $G$  can be weakly, standardly or strongly  $(\ell', k')$ -coloured, respectively.

We provide, in Figure 7.1, an illustration of these concepts on  $K_4$ , the complete graph on four vertices, where the two colours are represented by solid and dashed edges. By the “incident solid sum” of a vertex, we here mean the sum of weights assigned to its incident solid edges. It can be checked that, in Figure 7.1.(a), the depicted  $(2, 2)$ -colouring is a weak colouring. It is however not a standard  $(2, 2)$ -colouring as vertices  $c$  and  $d$  are joined by a solid edge but their incident solid sum equals 3. The colouring in Figure 7.1.(b) is a standard  $(2, 2)$ -colouring which is not a strong colouring, in particular because vertices  $a$  and  $c$  both have incident solid sum 2. The colouring in Figure 7.1.(c) is a strong  $(2, 2)$ -colouring.

This chapter is organized as follows. As already mentioned, the notions of weak, standard and strong  $(\ell, k)$ -colourings can be employed to generalize neighbour-sum-distinguishing edge-weightings and locally irregular edge-colourings. In Section 7.2, we explore these connections. In particular, we recall known results and translate them in our new setting.

Playing with the parameters  $\ell$  and  $k$  and the distinction conditions, we also come up with new problems, some of which we believe are of independent interest. In particular, we wonder whether almost all graphs can be weakly, standardly, or even strongly  $(2, 2)$ -coloured. If true, this would imply side decomposition results related to the 1-2-3 Conjecture. The strong, standard and weak versions of that question are formally introduced in Section 7.3. They are then studied in Sections 7.4, 7.5 and 7.6, respectively.

## 7.2 Previous results and connections to $(\ell, k)$ -colourings

We start, in Section 7.2.1, by making first observations and remarks on weak, standard and strong colourings. We then survey, in Section 7.2.2, some of the results from literature that are directly connected to these notions. More precisely, we explain which notions in the literature are encompassed by weak, standard and strong colourings, and, by rephrasing known results under that new terminology, we exhibit first results.

### 7.2.1 Early observations

First of all, we note that, according to the definitions, every result holding for some version of  $(\ell, k)$ -colourings also holds for the weaker versions. This is why, throughout Sections 7.4 to 7.6, we start by considering strong colourings, then standard colourings, and, finally, weak colourings.

**Observation 7.2.1.** *A strong  $(\ell, k)$ -colouring is also a standard  $(\ell, k)$ -colouring. Analogously, a standard  $(\ell, k)$ -colouring is also a weak  $(\ell, k)$ -colouring.*

In general, though, it can be observed that the converse direction is not true, *i.e.*, that a given  $(\ell, k)$ -colouring does not necessarily fulfil stronger distinction conditions. A good illustration for that is the fact that  $K_3$  can be weakly  $(2, 2)$ -coloured but not standardly  $(2, 2)$ -coloured. There are situations, though, where all distinction conditions behave similarly. We state a few of them below.

First of all, we recall that, for some values of  $\ell$  and  $k$ , some versions of  $(\ell, k)$ -colourings are equivalent to other kinds of distinguishing colourings and weightings. Most of these observations are straightforward, and thus do not need a formal proof. In particular, it can easily be checked that some of these results do not hold for stronger or weaker versions of our colouring variants.

**Observation 7.2.2.** *Weak, standard and strong  $(1, k)$ -colourings and neighbour-sum-distinguishing  $k$ -edge-weightings are equivalent notions.*

**Observation 7.2.3.** *Standard  $(k, 1)$ -colourings and locally irregular  $k$ -edge-colourings are equivalent notions.*

Let  $G$  be a graph, and  $\omega$  be an edge-weighting of  $G$ . For each vertex  $v$  of  $G$ , one can compute its *multiset*  $\mu(v)$  of incident weights induced by  $\omega$ . We say that  $\omega$  is *neighbour-multiset-distinguishing* if no two adjacent vertices of  $G$  get the same multiset of incident

weights. Note that having  $\sigma(u) \neq \sigma(v)$  for an edge  $uv$  of  $G$  implies that  $\mu(u) \neq \mu(v)$  (but the converse is not necessarily true). For this reason, neighbour-multiset-distinguishing edge-weightings have been studied as a weaker form of neighbour-sum-distinguishing edge-weightings.

The point for mentioning neighbour-multiset-distinguishing edge-weightings is that they relate to our notion of weak colourings.

**Observation 7.2.4.** *Weak  $(k, 1)$ -colourings and neighbour-multiset-distinguishing  $k$ -edge-weightings are equivalent notions.*

In Observation 7.2.2, we noticed that, for  $(1, k)$ -colourings, all three distinction conditions are equivalent. In the following result, we point out another context where the three colouring variants coincide.

**Observation 7.2.5.** *In regular graphs, weak, standard and strong  $(2, 1)$ -colourings are equivalent notions.*

## 7.2.2 Previous results

In this section, we restate, in our terminology, several results from the literature on distinguishing weightings and colourings to derive the existence of particular  $(1, k)$ - or  $(\ell, 1)$ -colourings. In other words, we here point out how our colouring concepts encapsulate existing distinguishing weightings and colourings.

This section is not intended to be a full survey on variants of the 1-2-3 Conjecture. Hence, we intentionally focus on those existing results that are closely related to our investigations; for more details, please refer to the survey [Sea12] by Seamone.

### Neighbour-sum-distinguishing edge-weightings

Recall that, according to Observation 7.2.2, being strongly  $(1, k)$ -colourable is equivalent to being neighbour-sum-distinguishing  $k$ -edge-weightable. Thus, all general constant upper bounds on  $\chi_{\Sigma}^e$  yield results on strong colourability (hence on the weaker variants as well, recall Observation 7.2.1).

In the context of neighbour-sum-distinguishing edge-weightings, the leading conjecture is the 1-2-3 Conjecture. If true, that conjecture would imply that every nice graph is strongly  $(1, 3)$ -colourable. Recall that nice graphs are exactly those graphs with no isolated edges.

**Conjecture 7.2.6.** *Every nice graph is strongly  $(1, 3)$ -colourable.*

To date, the best result towards the 1-2-3 Conjecture was given by Kalkowski, Karoński and Pfender [KKP10], who proved that  $\chi_{\Sigma}^e(G) \leq 5$  holds for every nice graph  $G$ . As said above, this result can be stated as follows, using our terminology.

**Theorem 7.2.7.** *Every nice graph is strongly  $(1, 5)$ -colourable.*

The 1-2-3 Conjecture was shown to hold for several common classes of nice graphs, such as complete graphs and 3-colourable graphs. There exist graphs  $G$  verifying  $\chi_{\Sigma}^e(G) = 3$ , such as complete graphs of order at least 3. One natural question is thus whether such graphs are easy to characterize. Dudek and Wajc settled the question in the negative [DW11], by showing that determining the exact value of  $\chi_{\Sigma}^e(G)$  is an NP-complete problem. Later on, Ahadi, Dehghan and Sadeghi [DSA13] proved that this remains true when restricted to regular (cubic) graphs. This result is of prime interest, as all distinguishing weighting and colouring notions considered in this chapter tend to be equivalent when 1) only two weights or colours are considered, and 2) the graph is regular (recall Observation 7.2.5). This result, by itself, directly establishes the general hardness of weak, standard and strong colourings.

It took some time to settle this complexity question for bipartite graphs. The full characterization of connected bipartite graphs  $G$  with  $\chi_{\Sigma}^e(G) = 3$  was given by Thomassen, Wu and Zhang [TWZ16], who proved that they are exactly the *odd multicacti*. These graphs can be constructed as follows. Start from  $m \geq 1$  cycles  $C_1, \dots, C_m$  whose lengths are at least 6 and congruent to 2 modulo 4, and colour the edges of the  $C_i$ 's using colours red and green alternately. Then, an odd multicactus is any connected graph obtained from the  $C_i$ 's via repeated applications of the following operation: pick two components  $G_1$  and  $G_2$ , and identify a green edge of  $G_1$  with a green edge of  $G_2$ . Said differently, an odd multicactus is obtained by identifying edges of particular cycles in a tree-like fashion. In particular, every cycle with length congruent to 2 modulo 4 is an odd multicactus.

**Theorem 7.2.8** (Thomassen, Wu, Zhang [TWZ16]). *A connected bipartite graph  $G$  verifies  $\chi_{\Sigma}^e(G) = 3$  if and only if  $G$  is an odd multicactus.*

### Locally irregular edge-colourings

By Observation 7.2.3, we get that locally irregular  $k$ -edge-colourings are precisely standard  $(k, 1)$ -colourings. We thus survey some of the research on locally irregular edge-colourings, as they transfer to standard  $(k, 1)$ -colourings.

Baudon, Bensmail, Przybyło and Woźniak conjectured that every decomposable graph  $G$  should decompose into at most three locally irregular graphs, *i.e.*,  $\chi'_{\text{irr}}(G) \leq 3$ . Due to Observation 7.2.3, this conjecture can be restated as follows:

**Conjecture 7.2.9.** *Every decomposable graph is standardly  $(3, 1)$ -colourable.*

The best known bound on  $\chi'_{\text{irr}}$  is 220 due to Lužar, Przybyło and Soták [LPS16]. We can thus state the following:

**Theorem 7.2.10.** *Every decomposable graph is standardly  $(220, 1)$ -colourable.*

Baudon, Bensmail, Przybyło and Woźniak verified Conjecture 7.2.9 for several decomposable graph classes [BBPW15], including complete graphs, some bipartite graphs, some Cartesian products, and regular graphs with degree at least  $10^7$ . Later on, Przybyło [Prz15]

verified the conjecture for graphs with minimum degree at least  $10^{10}$ . The complexity aspects were considered by Baudon, Bensmail and Sopena [BBS15], who proved that, for a given graph  $G$ , deciding whether  $\chi'_{\text{irr}}(G) = 2$  is NP-complete in general, while determining  $\chi'_{\text{irr}}(G)$  can be done in polynomial time when  $G$  is a tree.

### Neighbour-multiset-distinguishing edge-weightings

As mentioned in the Chapter 1, all neighbour-sum-distinguishing edge-weightings are neighbour-multiset-distinguishing, but the converse is not always true. The connection between these two notions was first considered by Karoński, Łuczak and Thomason in the paper introducing the 1-2-3 Conjecture [KLT04]. The first formal study of neighbour-multiset-distinguishing edge-weightings may be attributed to Addario-Berry, Aldred, Dalal and Reed, who, later on, gave improved results towards a “multiset version” of the 1-2-3 Conjecture [ABADR05]. In our terminology, this conjecture reads as follows:

**Conjecture 7.2.11.** *Every nice graph is weakly  $(3, 1)$ -colourable.*

So far, the best result towards Conjecture 7.2.11 is hence due to Addario-Berry, Aldred, Dalal and Reed, who proved that all nice graphs admit neighbour-multiset-distinguishing 4-edge-weightings [ABADR05].

**Theorem 7.2.12.** *Every nice graph is weakly  $(4, 1)$ -colourable.*

All graph classes verifying the 1-2-3 Conjecture also verify Conjecture 7.2.11. Additionally, the latter conjecture was also verified for graphs with minimum degree at least 1000, see [ABADR05].

## 7.3 New problems

As seen in Section 7.2, some of the  $(1, k)$ -colouring and  $(\ell, 1)$ -colouring variants correspond to distinguishing weighting and colouring notions already considered in the literature. In particular, for such values of  $\ell$  and  $k$ , there is still some gap between the corresponding conjectures and the best results we know to date. One way to get some sort of side progress, could be to prove the existence of  $(\ell, k)$ -colourings (for some distinction condition) where  $\ell + k$  or  $\max\{\ell, k\}$  is as small as possible.

In particular, the main problem we consider in the rest of this paper, which corresponds to minimizing  $\max\{\ell, k\}$ , and to which we could not find any obvious counterexample, reads as follows. By a *nicer graph*, we mean a graph with no isolated edges and triangles.

**Conjecture 7.3.1.** *Every nicer graph is strongly  $(2, 2)$ -colourable.*

The main reason for suspecting that  $K_2$  and  $K_3$  might be the only connected graphs admitting no strong  $(2, 2)$ -colourings is that they are the only connected exceptional graphs (recall the exact characterization in Subsection 7.2.2) admitting no neighbour-sum-distinguishing 2-edge-weightings.



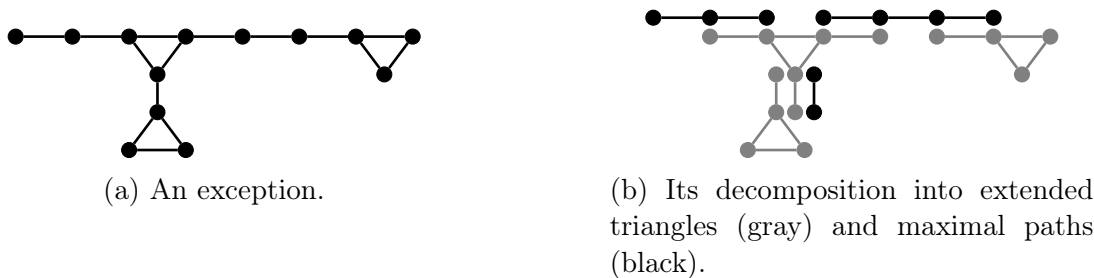


Figure 7.2: Decomposing an exception as described in the proof of Observation 7.3.2.

**Observation 7.3.2.** *Every connected exception different from  $K_2$  and  $K_3$  satisfies Conjecture 7.3.1.*

*Proof.* Let  $G$  be a connected exception different from  $K_2$  and  $K_3$ . We consider several cases corresponding to the three families of connected exceptions given by the definition:

- If  $G$  is an odd-length path, then  $G$  is a connected bipartite graph different from an odd multicactus, thus verifies  $\chi_{\Sigma}^e(G) \leq 2$  according to Theorem 7.2.8, and hence admits strong  $(1, 2)$ -colourings.
- If  $G$  is an odd-length cycle with length at least 5, then  $G$  can be decomposed into two paths  $P_r, P_b$  with length at least 2. In particular, the end-vertices of  $P_r$  (and similarly  $P_b$ ) are not adjacent in  $G$ , and we have  $\chi_{\Sigma}^e(P_r), \chi_{\Sigma}^e(P_b) \leq 2$ . By considering a strong  $(1, 2)$ -colouring of  $P_r$  (with red weights) and a strong  $(1, 2)$ -colouring of  $P_b$  (with blue weights), we eventually get a strong  $(2, 2)$ -colouring of  $G$ .
- Finally assume that  $G \in \mathfrak{T} \setminus \{K_3\}$ . By contracting the triangles (there is at least one such) of  $G$  to vertices, we obtain a tree  $R(G)$  with maximum degree 3, whose some nodes (*triangle nodes*) correspond to triangles of  $G$ , while some nodes (*normal nodes*) correspond to real vertices. Furthermore, by definition, any path of  $R(G)$  joining two triangle nodes has odd length, and any path joining a triangle node and a pendant normal node has even length. We can consider  $G$  as a collection of triangles with at most three pendant edges attached (*extended triangles*), and paths with one or two ends attached to a triangle (*maximal paths*) (see Figure 7.2 for an example). The pendant edges attached to the extended triangles, as well as the end-edges incident to triangles of the maximal paths, are called *attachment edges*. According to these definitions,  $G$  can be constructed from extended triangles and maximal paths by glueing their attachment edges. In particular, every attachment edge belongs to one extended triangle and one maximal path.

Necessarily  $R(G)$  has a degree-1 node  $r$ , being either a triangle node (pendant triangle in  $G$ ) or a normal node (pendant vertex in  $G$ ). Consider the (virtual) orientation of the edges of  $R(G)$  from  $r$  towards the leaves. We construct a strong  $(2, 2)$ -colouring (assigning weights coloured red and blue) iteratively, by extending a colouring along extended triangles and maximal paths following the ordering given

by the orientation of the attachment edges. Since  $R(G)$  is a tree, note that once an attachment edge is coloured, this provides a pre-colouring of the next extended triangle or maximal path to be coloured.

We start constructing the colouring from  $r$ . In  $G$ , node  $r$  corresponds either to an end-vertex of a maximal path  $P$  (normal node), or to a triangle  $T$  (triangle node). In the first case, let  $P := v_1 \dots v_{2k}$ ; then we just assign red weights  $1, 2, 2, 1, 1, \dots$  along  $P$ . In the second case, let  $T := v_1 v_2 v_3 v_1$ , and let  $v'_1$  denote, without loss of generality, the neighbour of  $v_1$  outside  $T$ ; we here assign red weight 1 to  $v_3 v_2$  and red weight 2 to  $v_2 v_1$ , and blue weight 1 to  $v_3 v_1$  and blue weight 2 to  $v_1 v'_1$ . In any case, it can be checked that the colouring is correct so far.

We now proceed to the general case, *i.e.*, we consider a maximal path  $P$  or extended triangle  $T$  whose one attachment edge is coloured, and we extend the colouring to all its other attachment edges in  $G$ . Consider first a maximal path  $P := v_1 \dots v_k$  whose attachment edge  $v_1 v_2$  was assigned, say, a red weight. We here extend the colouring to all edges of  $P$  by assigning red weights (with value 1 or 2) to its edges  $v_2 v_3, \dots, v_{k-1} v_k$  successively. Note that this can be done correctly, as, when a red weight is being assigned to an edge  $v_i v_{i+1}$ , we just have to make sure that the red sum of  $v_i$  avoids the red sum of  $v_{i-1}$ , which is possible since we have two red weights to play with.

We are left with the case where the colouring must be extended to an extended triangle  $T := v_1 v_2 v_3 v_1$  whose one attachment edge, say  $v_1 v'_1$ , was previously assigned, say, a red weight. We here consider cases depending on the number of additional attachment edges:

- If  $v_1 v'_1$  is the only attachment edge of  $T$ , then we assign a red weight to  $v_1 v_2$  so that the red sum of  $v_1$  does not get equal to the red sum of  $v'_1$ . We then assign blue weights  $1, 2$  or  $2, 1$  to  $v_1 v_3$  and  $v_3 v_2$  in such a way that the blue sum of  $v_1$  does not get equal to the blue sum of  $v'_1$ .
- Assume  $v_2 v'_2$  is the only other attachment edge of  $T$ . We here assign a red weight to  $v_1 v_3$  in such a way that the red sum of  $v_1$  does not get equal to the red sum of  $v'_1$ . We then assign blue weights  $1, 2, 1$  or  $2, 1, 1$  to  $v_2 v_1$ ,  $v_2 v_3$  and  $v_2 v'_2$  in such a way that the blue sum of  $v_1$  does not get equal to the blue sum of  $v'_1$ .
- Lastly, assume  $v_2 v'_2$  and  $v_3 v'_3$  are attachment edges. First, we assign blue weight 1 to  $v_1 v_2$  and blue weight 2 to  $v_1 v_3$ . We now assign red weight 1 to  $v'_2 v_2$ , red weight  $\alpha$  to  $v_2 v_3$  and red weight 2 to  $v_3 v'_3$ , where  $\alpha$  is the red weight of  $v'_1 v_1$ .

In any of these cases, it can be checked that the colouring extension is correct. So this covers all cases of the proof.  $\square$

The rest of this chapter is dedicated to providing evidences towards Conjecture 7.3.1. We do it gradually, by first considering, in Section 7.4, Conjecture 7.3.1 in its literal form.

We then consider its standard version (in Section 7.5), before finally considering its weak version (in Section 7.6).

## 7.4 Strong $(\ell, k)$ -colouring

In this section, we consider Conjecture 7.3.1 in its literal form, namely:

**Strong Conjecture.** *Every nicer graph is strongly  $(2, 2)$ -colourable.*

We verify the Strong Conjecture for nice complete graphs and bipartite graphs. Recall that every result on strong  $(2, 2)$ -colourings directly transfers to standard and weak  $(2, 2)$ -colourings.

We start off with complete graphs.

**Theorem 7.4.1.** *For every  $n \geq 4$ , the graph  $K_n$  is strongly  $(2, 2)$ -colourable.*

*Proof.* We prove the claim by induction on  $n$ . To ease the proof, we prove a stronger statement, namely that every complete graph  $K_n$  admits a strong  $(2, 2)$ -colouring with red and blue weights such that either there is no vertex incident to red edges only, or there is no vertex incident to blue edges only.

As a base step, consider  $K = K_4$ . Note that  $K$  can be decomposed into two paths  $P_r$  and  $P_b$  of length 3. To get a strong  $(2, 2)$ -colouring, we proceed as follows. Consider first the edges of  $P_r$  from one end to the other, and assign them red weights 1, 2, 2, respectively. Similarly, then consider the edges of  $P_b$  from one end to the other, and assign them blue weights 1, 2, 2, respectively. Since  $P_r$  and  $P_b$  span all vertices of  $K$ , each vertex gets a non-zero red sum and a non-zero blue sum. This, by itself, guarantees that the additional requirement is fulfilled (*i.e.*, there is no monochromatic vertex). Now, due to how the red weights were assigned, it can easily be seen that the obtained red sums are 1, 2, 3, 4; hence no two vertices get the same red sums. As this is also the case for the blue sums, we have thus constructed a strong  $(2, 2)$ -colouring of  $K$ .

We now prove the general case. Let  $K = K_n$  (where  $n \geq 5$ ), and remove one vertex  $v$  from  $K$ . We end up with a graph isomorphic to  $K_{n-1}$ , which, by the induction hypothesis, admits a strong  $(2, 2)$ -colouring with colours red and blue. Furthermore, we may, without loss of generality, assume that, by this colouring, there is no vertex incident to red edges only. We extend this colouring to  $K$ , *i.e.*, to the edges incident to  $v$ , by assigning red weight 2 to all those edges. As a result, all red sums of the vertices of  $V(K) \setminus \{v\}$  rise by 2, and since every two of them were different, they still are after the extension. Now, note that the red sum of  $v$  is precisely  $2(n-1)$ , which is strictly greater than all the other red sums since all vertices of  $V(K) \setminus \{v\}$  are incident to blue edges. Furthermore, the blue sums of the vertices of  $V(K) \setminus \{v\}$  have not been altered, while  $v$  has blue sum 0 – so no two non-zero blue sums are the same. We thus get a strong  $(2, 2)$ -colouring of  $K$ , and it can be noted that no vertex is incident to blue edges only, as additionally required.  $\square$

We now prove the Strong Conjecture for bipartite graphs. Recall that a connected bipartite graph  $G$  verifies  $\chi_{\Sigma}^e(G) = 3$  if and only if it is an odd multicactus (Theorem 7.2.8).

**Theorem 7.4.2.** *Every nice bipartite graph  $G$  is strongly  $(2, 2)$ -colourable.*

*Proof.* We can assume that  $G$  is connected. If  $G$  is not an odd multicactus, then  $\chi_{\Sigma}^e(G) \leq 2$ , and, equivalently,  $G$  is strongly  $(1, 2)$ -colourable. So let us now assume that  $G$  is an odd multicactus. By construction, note that  $G$  necessarily has a degree-2 vertex  $v$ . Furthermore,  $G$  is 2-connected, so the graph  $G' := G - \{v\}$  is connected. Also,  $G'$  is not an odd multicactus (to be convinced of this, note that it has degree-1 vertices and that one of its partite sets is of even cardinality). So  $G'$  is strongly  $(1, 2)$ -colourable.

Consider thus a strong  $(1, 2)$ -colouring of  $G'$  assigning red weights. We extend this colouring to a strong  $(2, 2)$ -colouring of  $G$ , *i.e.*, to the edges  $u_1v$  and  $vu_2$  incident to  $v$ , by just assigning blue weights 1 and 2 to  $u_1v$  and  $vu_2$ , respectively. As no new edge was assigned a red weight, the adjacent red sums are still different in  $G$ . Furthermore, the only three non-zero blue sums are all different, as they are equal to 1, 2 and 3.  $\square$

In the rest of this section, we confirm that odd multicacti are a peculiar class of nice bipartite graphs for the distinguishing colouring notions we consider, in the following sense.

**Theorem 7.4.3.** *The connected nice bipartite graphs that cannot be strongly  $(1, 1)$ -,  $(1, 2)$ - or  $(2, 1)$ -coloured are exactly the odd multicacti.*

The proof of Theorem 7.4.3 relies on the following result on locally irregular decompositions of odd multicacti, which we believe is of independent interest, as there is still no known characterization of bipartite graphs  $G$  verifying  $\chi'_{\text{irr}}(G) \leq 2$ .

**Lemma 7.4.4.** *For every odd multicactus  $G$ , we have  $\chi'_{\text{irr}}(G) = 3$ .*

*Proof.* Let  $G$  be an odd multicactus. As such (recall the description in Subsection 7.2.2),  $G$  has edges coloured red and green “alternatively”. To avoid any confusion with the colours, in the rest of the proof we refer to the green edges of  $G$  as its *attachment edges*, while we refer to the red edges as its *support edges*.

Since  $G$  is an odd multicactus, by construction there has to be an attachment edge  $uv$  such that  $u$  and  $v$  are joined by several disjoint non-trivial paths  $P_1, \dots, P_k$  of length congruent to 1 modulo 4, whose removal does not disconnect the graph. In some sense, the  $P_i$ 's are leaves in the tree representation of the construction of  $G$ . It is easy to see that, in a locally irregular 2-edge-colouring of  $G$ , necessarily every two subsequent support edges of the  $P_i$ 's must have different colours. Since the  $P_i$ 's have length congruent to 1 modulo 4, this means that, from the point of view of  $uv$ , colouring the  $P_i$ 's is similar to colouring  $k$  parallel edges joining  $uv$ . Said differently, if the multigraph  $G'$ , obtained by replacing the  $P_i$ 's by  $k$  parallel (attachment) edges joining  $u$  and  $v$ , admits no locally irregular 2-edge-colouring, so neither does  $G$ . This operation, consisting in contracting non-trivial paths joining a “leaf” attachment edge, is called a **contraction** below.

By repeatedly applying contractions (note that the argument above works even if the non-trivial paths have parallel attachment edges), we get a series of multigraphs  $G = G_0, G_1, \dots, G_m = G'$  such that 1) if  $G_{i+1}$  admits no locally irregular 2-edge-colourings, then so does not  $G_i$ , and 2)  $G'$  consists of two vertices joined by several

parallel (attachment) edges. Now, it should be clear that  $G'$  admits no locally irregular 2-edge-colourings, which gives our conclusion for  $G$ .  $\square$

We can now prove Theorem 7.4.3:

*Proof of Theorem 7.4.3.* Let  $G$  be a connected nice bipartite graph. If  $G$  is not an odd multicactus, then  $\chi_{\Sigma}^e(G) \leq 2$  (Theorem 7.2.8), and hence  $G$  is strongly  $(1, 2)$ -colourable. So we may assume that  $G$  is an odd multicactus, and thus that  $G$  is not strongly  $(1, 2)$ -colourable. In that case, according to Lemma 7.4.4,  $G$  admits no locally irregular 2-edge-colourings, hence no strong  $(2, 1)$ -colourings.  $\square$

## 7.5 Standard $(\ell, k)$ -colouring

We here consider the standard weakening of Conjecture 7.3.1:

**Standard Conjecture.** *Every nicer graph is standardly  $(2, 2)$ -colourable.*

Note that a standard  $(\ell, k)$ -colouring is nothing but a decomposition into  $\ell$  graphs admitting neighbour-sum-distinguishing  $k$ -edge-weightings. From that perspective, it could be interesting to wonder whether graphs, in general, decompose into a constant number of graphs verifying the 1-2-3 Conjecture. We believe this is an interesting aspect to consider, as not many graphs are known to verify the 1-2-3 Conjecture.

Towards the Standard Conjecture, we thus also raise the following related conjecture, which is, in a sense, a weakening of the 1-2-3 Conjecture:

**Conjecture 7.5.1.** *Every nice graph decomposes into two graphs verifying the 1-2-3 Conjecture.*

In this section, towards the Standard Conjecture, we first improve Theorem 7.2.10 by showing that all nice graphs admit standard  $(40, 3)$ -colourings. We then prove the Standard Conjecture 7.5.1 for nicer 2-degenerate graphs and subcubic graphs, before proving Conjecture 7.5.1 for nice 9-colourable graphs.

### 7.5.1 Standard $(40, 3)$ -colourability

The proof of the following result follows the lines of one in [BMT17], where Bensmail, Merker and Thomassen proved that decomposable graphs can be decomposed into at most 328 locally irregular graphs.

**Theorem 7.5.2.** *Every decomposable graph  $G$  is standardly  $(40, 3)$ -colourable.*

*Proof.* In  $G$ , we can find a locally irregular subgraph  $H_1$  such that  $G - E(H_1)$  has all of its components being of even size ([BMT17], Lemma 2.1). If  $G$  already had even size, then  $H_1$  is empty. Still calling  $G$  the remaining graph, we can decompose  $G$  into a graph  $H_b$  with minimum degree at least  $10^{10}$  and a  $(2 \cdot 10^{10} + 2)$ -degenerate graph

$H_c$  whose all components are of even size ([BMT17], Lemma 4.5). On the one hand, according to a result of Przybyło [Prz15], we can decompose  $H_b$  into three (possibly empty) locally irregular graphs  $H_{b,1}, H_{b,2}, H_{b,3}$ . On the other hand,  $H_c$  can be decomposed into 36 bipartite graphs  $H_{c,1}, \dots, H_{c,36}$  whose all components are of even size ([BMT17], Theorem 4.3).

Recall that every locally irregular graph  $H$  verifies  $\chi_{\Sigma}^e(H) = 1$ . Furthermore, all nice bipartite graphs verify the 1-2-3 Conjecture. From these arguments, using a set of 40 coloured weights 1, 2, 3 to independently weight the edges of each of the  $H_i$ 's and the  $H_{i,j}$ 's,  $i \in \{a, b, c\}$ ,  $j \in \{1, \dots, 36\}$ , we eventually get a standard  $(40, 3)$ -colouring of  $G$ .  $\square$

Since all connected nice exceptional graphs are 3-colourable, they verify the 1-2-3 Conjecture (see [Sea12]), and are thus standardly  $(1, 3)$ -colourable. Together with Theorem 7.5.2, this yields the following:

**Theorem 7.5.3.** *Every nice graph  $G$  is standardly  $(40, 3)$ -colourable.*

## 7.5.2 The Standard Conjecture for 2-degenerate graphs and subcubic graphs

Recall that a graph is *2-degenerate* if every of its subgraphs has a vertex with degree at most 2. A *subcubic graph* is a graph  $G$  with maximum degree at most 3. If all vertices of  $G$  have degree precisely 3, then we call  $G$  *cubic*. Furthermore, if  $G$  is connected and not cubic, *i.e.*,  $G$  has vertices with degree 1 or 2, then we say that  $G$  is *strictly subcubic*.

We first prove the Standard Conjecture for 2-degenerate graphs (with a few exceptions). More precisely, we prove:

**Theorem 7.5.4.** *Every nicer 2-degenerate graph  $G$  is standardly  $(2, 2)$ -colourable.*

Our proof of Theorem 7.5.4 relies on the following lemma, which is proved later in this section.

**Lemma 7.5.5.** *Every nicer 2-degenerate graph  $G$  decomposes into two nice forests.*

*Proof of Theorem 7.5.4.* According to Lemma 7.5.5, we can decompose  $G$  into two forests  $F_r$  and  $F_b$  none of which has an isolated edge. Since every nice tree  $T$  verifies  $\chi_{\Sigma}^e(T) \leq 2$  (*i.e.*, admits standard  $(1, 2)$ -colourings), each of  $F_r$  and  $F_b$ , independently, admits a standard  $(1, 2)$ -colouring; let  $\omega_r$  and  $\omega_b$  be any such standard  $(1, 2)$ -colourings of  $F_r$  and  $F_b$ , respectively. To get a standard  $(2, 2)$ -colouring of  $G$ , we consider all weights assigned by  $\omega_r$  and  $\omega_b$ , and colour red those weights originating from  $\omega_r$ , while we colour blue those weights originating from  $\omega_b$ .  $\square$

We are left with proving Lemma 7.5.5.

*Proof of Lemma 7.5.5.* Throughout the proof, which is by induction on  $|V(G)| + |E(G)|$ , we assume that  $G$  is connected. As a base case, it can be checked that the claim is true whenever  $|V(G)| \leq 4$ . In particular, under all conditions,  $G$  is either 1) a nice tree (in which case the claim holds trivially), 2) a triangle with a pendant vertex attached (which decomposes into two paths of length 2), 3) two triangles glued along an edge (which decomposes into a path of length 2 and a star with three leaves), or 4) a cycle of length 4 (which decomposes into two paths of length 2).

Let us thus proceed to the proof of the general case (in particular,  $|V(G)| \geq 5$ ). First assume that  $G$  has a degree-1 vertex  $v$ . Denote by  $u$  the neighbour of  $v$  in  $G$ , and let  $G' := G - \{v\}$ . Since  $|V(G)| \geq 5$ , note that  $G'$  cannot be  $K_2$  or  $K_3$ . So, by the induction hypothesis,  $G'$  decomposes into a red nice forest and a blue nice forest. Assuming  $u$  belongs to the red forest, we extend that decomposition to  $G$  by adding  $vu$  to the red forest.

Thus, we may assume that  $G$  has a degree-2 vertex  $v$ , with neighbours  $u_1, u_2$ . We distinguish two cases:

- First case:  $v$  is a cut-vertex. Let  $H_1$  and  $H_2$  be the two components of  $G - \{v\}$ , where  $u_i$  belongs to  $H_i$  for  $i = 1, 2$ , and set  $G_1 := H_1 + \{u_1v\}$  and  $G_2 := H_2 + \{u_2v\}$ . Since  $G$  has no degree-1 vertex, note that none of  $G_1$  and  $G_2$  is isomorphic to  $K_2$ . Also,  $v$  has degree 1 in both  $G_1$  and  $G_2$ , so none of  $G_1$  and  $G_2$  is isomorphic to  $K_3$ . By the induction hypothesis,  $G_1$  and  $G_2$  decompose into two nice forests. Note that these two decompositions, when combined in  $G$ , altogether form a decomposition of  $G$  into two nice forests.
- Second case:  $v$  is not a cut-vertex. Thus none of  $vu_1$  and  $vu_2$  is a cut-edge. Thus,  $G' := G - \{vu_1\}$  is not isomorphic to  $K_2$  or  $K_3$ , and, by the induction hypothesis,  $G'$  decomposes into two nice forests, say red and blue. Assume  $vu_2$  belongs to the red forest. If  $u_1$  belongs to the blue forest, then we obtain a decomposition of  $G$  by adding  $vu_1$  to the blue forest. So assume  $u_1$  belongs to the red forest only. If  $u_1$  and  $u_2$  belong to different trees of the red forest, then we can directly add  $vu_1$  to the red forest.

Thus, lastly suppose that  $u_1$  and  $u_2$  belong to the same tree of the red forest. Note that when moving  $vu_2$  from the red to the blue forest, and adding  $vu_1$  to the blue forest, then the obtained blue subgraph remains a forest, and cannot have any tree isomorphic to  $K_2$ . The only problem, here, is that the red forest might now include a tree isomorphic to  $K_2$ . Since  $u_1$  and  $u_2$  belonged to the same tree of the red forest, this means that  $vu_2u_1$ , a path of length 2, was exactly a tree of the red forest. In that situation,  $u_2$  is a cut-vertex of  $G$ , and  $u_1$  also has degree 2 - its neighbours are  $v$  and  $u_2$ . Said differently,  $vu_1u_2v$  is a pendant triangle of  $G$  attached at  $u_2$ .

Now, since  $G$  is not  $K_3$ , then  $u_2$  belongs to the blue forest in the decomposition of  $G'$ . To obtain the desired decomposition of  $G$ , we can here just add  $vu_2$  to the blue forest (which indeed remains a forest), and add  $vu_1$  and  $u_1u_2$  to the red forest (to which we add a path of length 2).

This concludes the proof.  $\square$

We now extend the previous results to nicer subcubic graphs.

**Lemma 7.5.6.** *Every nicer subcubic graph  $G$  decomposes into two nice forests.*

*Proof.* Throughout the proof, which is by induction on  $|V(G)| + |E(G)|$ , it is assumed that  $G$  is connected. As the claim is true whenever  $|V(G)| \leq 4$  ( $G$  is either strictly subcubic and the result follows from Lemma 7.5.5, or isomorphic to  $K_4$ , which decomposes into two paths of length 3), we proceed to the proof of the general case.

We now consider the general case  $|V(G)| \geq 5$ . If  $G$  is strictly subcubic, then  $G$  is 2-degenerate, in which case the result follows from Lemma 7.5.5. So let us assume that  $G$  is cubic. Let  $v$  be a (degree-3) vertex of  $G$ , with neighbours  $u_1, u_2, u_3$ . Note that if all edges among the  $u_i$ 's exist, then  $G$  is  $K_4$  while  $|V(G)| \geq 5$ , a contradiction. Hence, assume without loss of generality that  $u_1u_2$  is not an edge of  $G$ . Consider the graph  $G' := G - \{v\} + \{u_1u_2\}$ . Note that, although  $G'$  might consist of up to two components, none of them is isomorphic to  $K_2$  or  $K_3$  as  $G$  is cubic. So all components are subcubic, and they decompose into two nice forests, say red and blue.

Consider the decomposition of  $G'$ . Suppose that  $u_1u_2$  belongs to the red forest. We consider the same decomposition in  $G$ , except that, since  $G$  does not contain the edge  $u_1u_2$ , we replace it, in the red forest, by the two edges  $u_1v$  and  $vu_2$ . Note that, in  $G$ , the red subgraph remains a nice forest. It thus remains to add  $vu_3$  to either the red or blue forest. If  $u_3$  belongs to the blue forest, then we are done when adding  $vu_3$  to the blue forest. So assume that the two edges, different from  $vu_3$ , incident to  $u_3$  belong to the red forest. If  $v$  and  $u_3$  belong to different trees of the red forest, then we can freely add  $vu_3$  to the red forest. So lastly suppose that we are not in that case.

All of  $u_1, u_2, u_3$  belong to the same tree, say  $T$ , of the red forest. In  $T$ , let us assume that  $u_3$  is closer to  $u_2$  than it is closer to  $u_1$ . In other words, in  $T$ , the only path from  $u_3$  to  $u_1$  passes through  $u_2$ . Let us remove  $vu_2$  from  $T$ . In the red forest,  $T$  is disconnected into two trees  $T'$  and  $T''$ , where  $T'$  contains  $u_2$  and  $u_3$ , while  $T''$  contains  $v$  and  $u_1$ . Note that  $T'$  is not isomorphic to  $K_2$ , since  $u_3$  remains of degree 2 in that tree. If  $T''$  also has this property, then we get a desired decomposition of  $G$  when adding  $vu_2$  and  $vu_3$  to the blue forest (recall that  $u_3$  originally did not belong to the blue forest). So we may assume that  $T''$  is actually isomorphic to  $K_2$ , which means that  $u_1$  had degree 1 in  $T$ . In this situation, we obtain the desired decomposition of  $G$  by adding  $vu_1$  and  $vu_3$  to the blue forest.  $\square$

A similar proof as that used to prove Theorem 7.5.4, but using Lemma 7.5.6 instead of Lemma 7.5.5, now yields the following.

**Theorem 7.5.7.** *Every nicer subcubic graph is standardly  $(2, 2)$ -colourable.*

### 7.5.3 Conjecture 7.5.1 for 9-colourable graphs

To prove Conjecture 7.5.1 for all nicer 9-colourable graphs, we essentially prove that 9-colourable graphs, in general, decompose into two nice 3-colourable graphs. With such



a result in hand, we can then use the fact that nice 3-colourable graphs verify the 1-2-3 Conjecture.

**Lemma 7.5.8.** *Assume that a nicer graph  $G$  can be 2-edge-coloured with red and blue so that the induced red subgraph  $G_R$  and blue subgraph  $G_B$  satisfy  $\chi(G_R) = r$  and  $\chi(G_B) = b$  with  $r, b \geq 2$ . Then  $G$  can be 2-edge-coloured in such a way that  $\chi(G_R) \leq r$ ,  $\chi(G_B) \leq b$ , and  $G_R$  and  $G_B$  are nice.*

*Proof.* The edges of  $G$  will be coloured or recoloured during the proof. Changing the colour of an edge actually means that that edge is added to one of  $G_R$  and  $G_B$ , and, conversely, removed from the second subgraph.

Let us start by raising a few comments on how edge additions and removals affect the parameters and structure of the subgraph we are interested in:

- Adding an edge to a graph can, in general, increase its chromatic number; however, the addition of a pendant edge, or, more generally, of pendant paths does not increase the chromatic number (unless when the graph is edgeless). The addition of an edge such that at least one of its ends was not isolated in the graph does not increase the number of isolated edges.
- Removing edges from a graph can, in general, reduce the chromatic number. It can also produce new isolated edges; but this can only happen when the removed edge lies on a path and is incident with a pendant edge of this path.

The proof is by induction on  $|V(G)|$ . As it can easily be seen that the statement is true when  $|V(G)| \leq 4$ , we may consider the general case  $|V(G)| \geq 5$ . We can assume that  $G$  is connected. To show that  $G$  can be 2-edge-coloured as claimed, we consider three cases:

**Case 1.**  $G$  has a pendant edge, *i.e.*,  $\delta(G) = 1$ .

Let  $uv$  be a pendant edge of  $G$  with  $d(u) = 1$ . Any 2-edge-colouring of  $G$  with  $\chi(G_R) = r \geq 2$  and  $\chi(G_B) = b \geq 2$  induces a 2-edge-colouring of the graph  $G' := G - u$  with  $\chi(G'_R) = r \geq 2$  and  $\chi(G'_B) = b \geq 2$ , or  $G'$  becomes monochromatic. In the first case, since  $G'$  has at least four vertices, we may assume that the graphs  $G'_R$  and  $G'_B$  have no isolated edges. At least one colour, say red, is present at vertex  $v$ . Then we colour the edge  $uv$  red. In the second case, *i.e.*, when  $G'$  is monochromatic, say red, we colour the edge  $uv$  red. In both cases we do not increase the chromatic number of  $G'_R$ .

**Case 2.**  $\delta(G) = 2$ .

Let  $u$  be a vertex with  $d(u) = 2$ . Denote by  $v, w$  its neighbours and let  $G' := G - u$ . As above, any 2-edge-colouring of  $G$  with  $\chi(G_R) = r \geq 2$  and  $\chi(G_B) = b \geq 2$  induces a 2-edge-colouring of the graph  $G'$  with  $\chi(G'_R) = r \geq 2$  and  $\chi(G'_B) = b \geq 2$ , or  $G'$  becomes monochromatic. Suppose first that  $G'$  is connected. Then the lemma holds for  $G'$  and we may suppose that the graphs  $G'_R$  and  $G'_B$  have no isolated edges, or  $G'$  is monochromatic, say red, with  $\chi(G'_R) = r' \geq 2$ . If we are able to find in  $G'$  two edges of different colours incident with  $v$  (say red) and  $w$  (say blue), respectively, then we colour the edge  $uv$  red

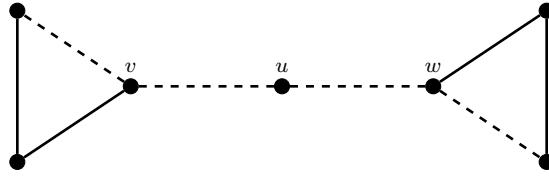


Figure 7.3: A decomposition into two nice bipartite graphs mentioned in the proof of Lemma 7.5.8. Solid edges stand for red edges. Dashed edges stand for blue edges.

and the edge  $uw$  blue. Note that this adds pendant edges to  $G'_R$  and  $G'_B$ , so we do not increase the chromatic number of these graphs and we do not create isolated edges. If the vertices  $v, w$  are incident with edges of one colour only, say red, or  $G'$  is monochromatic, say red, then we colour both edges  $uv, uw$  blue. Again, we do not increase the chromatic number of the graphs  $G'_R$  and  $G'_B$  and we do not create isolated edges.

Consider now the case where  $G - u$  has two components  $G_1$  and  $G_2$ . If neither  $G_1$  nor  $G_2$  is isomorphic to  $K_3$ , then we apply induction hypothesis to  $G_1$  and  $G_2$  and proceed as above, that is:

- if we are able to find in  $G_1$  and  $G_2$  two edges of different colours incident with  $v$  (say red) and  $w$  (say blue), respectively, then we colour the edge  $uv$  red and the edge  $uw$  blue; and
- if the vertices  $v, w$  are incident with edges of one colour only, say red, then we colour both edges  $uv, uw$  blue.

Suppose now that only one of these components, say  $G_1$ , is isomorphic to  $K_3$  and denote its vertices by  $v, v_1, v_2$ . Now, we apply the induction hypothesis to  $G_2$  and if we are able to find an edge coloured, say, blue, incident with  $w$ , then we colour the edge  $uw$  blue. Next, we colour red the edges  $uv$  and  $vv_1$  and we colour blue the edges  $uv_2$  and  $v_2v_1$ .

If both components  $G_1$  and  $G_2$  are isomorphic to  $K_3$ , then one possible 2-edge-colouring without isolated edges is given in Figure 7.3.

**Case 3.**  $\delta(G) \geq 3$ .

We start from a 2-edge-colouring of  $G$  with  $\chi(G_R) = r \geq 2$  and  $\chi(G_B) = b \geq 2$  which minimizes the number of isolated edges in  $G_R$  and  $G_B$ . We will show that if the number of these is still positive, then we can get rid of any given such isolated edge, without creating a new one, and thus get a contradiction.

Let us suppose that  $uv$  is an isolated edge of  $G_B$ . Since  $d(u) \geq 3$  and  $d(v) \geq 3$ , neither  $u$  nor  $v$  is isolated in  $G_R$ . If the vertices  $u, v$  belong to two different components of  $G_R$ , then we recolour the edge  $uv$  red. Such an operation cannot increase the chromatic number of  $G_R$ .

Hence, the vertices  $u, v$  belong to one component of  $G_R$ . Then there is a *red path*  $P$  (containing only red edges) joining  $u$  and  $v$  in  $G_R$ . Denote this path by  $uw_1 \dots w_l v$ , where  $l \geq 1$ . Since the vertices  $u$  and  $v$  are of degree at least 3 in  $G$  and of degree 1 in the blue

graph, they are of degree at least 2 in the red graph. Denote by  $u_1, \dots, u_p$  the neighbours of  $u$  in  $G_R$  and by  $v_1, \dots, v_q$  the neighbours of  $v$  in  $G_R$ , different from  $w_1, w_l$ , respectively. We have  $p, q \geq 1$ .

If  $p \geq 2$ , then we recolour  $uw_1$ , the first edge of the red path  $P$ , blue. From the point of view of the blue graph, we add to a component of  $G_B$  a pendant path  $w_1uv$  of length 2. Since  $b \geq 2$ , this operation does not increase the chromatic number of  $G_B$ . From the point of view of the red graph, we delete an edge  $w_1u$  but we do not create a new isolated edge. Indeed, none of the red edges incident with  $u$  becomes isolated, because there remain  $p \geq 2$  of them. On the other hand, none of the edges incident with  $w_1$  becomes isolated because they are incident with the edge  $w_1w_2$  which lies on the path joining  $w_1$  with  $v$  and is not isolated even in the case  $w_2 = v$  because of the edge  $vv_1$ .

If  $p = 1$ , and by symmetry  $q = 1$ , then we can proceed as above except when the red degree of  $u_1$  is 1 *i.e.*,  $uu_1$  is a pendant edge in the red graph. Then we recolour this edge blue. Since  $b \geq 2$  this operation does not increase the chromatic number of  $G_B$ . Again, from the point of view of the blue graph, we add to a component of  $G_B$  a pendant path  $u_1uv$  of length 2, and from the point of view of the red graph, we delete a pendant edge. Both operations preserve the chromatic numbers of the red and blue graphs.  $\square$

We now prove the second key lemma of this section.

**Lemma 7.5.9.** *Every 9-colourable graph  $G$  decomposes into an  $r$ -colourable graph  $G_R$  and a  $b$ -colourable graph  $G_B$  with  $r, b \leq 3$ .*

*Proof.* Let  $G$  be a graph with  $\chi(G) \leq 9$ . It is easy to see that it is sufficient to consider the case where  $G$  is a complete graph of order  $n \leq 9$ . So let  $G = K_n$  and let  $x_1, \dots, x_n$  denote the vertices of  $G$ . We 2-edge-colour  $G$  with colours red and blue, yielding two subgraphs  $G_R$  and  $G_B$ , respectively, as follows. An edge  $x_i x_j$  is coloured red if and only if  $i = j \pmod 3$ . Otherwise, *i.e.*, when  $i \neq j \pmod 3$ , the edge  $x_i x_j$  is coloured blue.

Clearly,  $\chi(G_R) = 3$  for  $n \geq 3$ . Furthermore, since  $n \leq 9$ , there are at most three numbers congruent to 0 (or to 1, or to 2) modulo 3. Thus,  $G_B$  contains either isolated edges or triangles, so  $\chi(G_B) \leq 3$ .  $\square$

We are now ready to prove that nice 9-colourable graphs verify Conjecture 7.5.1.

**Theorem 7.5.10.** *Every nice 9-colourable graph  $G$  is standardly (2, 3)-colourable.*

*Proof.* If  $G$  is 3-colourable, then we have  $\chi_{\Sigma}^e(G) \leq 3$  (see [KLT04]), or, in other words,  $G$  is standardly (1, 3)-colourable. Now assume that  $G$  is at least 4-chromatic. By Lemma 7.5.9, it can be decomposed into two 3-colourable graphs: an  $r$ -colourable graph  $G_R$  and a  $b$ -colourable graph  $G_B$  with  $r, b \leq 3$ . Since  $G$  is at least 4-chromatic we have also  $r, b \geq 2$ .

We distinguish two cases:

- If  $G$  has no isolated triangles, then, by Lemma 7.5.8, it can be decomposed into two nice graphs  $G_R$  and  $G_B$  with  $\chi(G_R) \leq r$  and  $\chi(G_B) \leq b$  with  $r, b \geq 3$ . So, both of them verify the 1-2-3 Conjecture, and thus admit standardly (1, 3)-colourings. Combining standardly (1, 3)-colourings of  $G_R$  and  $G_B$ , we get a standard (2, 3)-colouring of  $G$ .

- If  $G$  has isolated triangles, then we remove those from  $G$ , apply the previous point to get a  $(2, 3)$ -colouring of what is left, and then give the same color to all the edges of the isolated triangles. Finally, by weighting the edges of each isolated triangle with weights  $1, 2, 3$  (in any colour), we can extend the  $(2, 3)$ -colouring to  $G$ .  $\square$

We note that the approach above can be generalized to show that, in general, any nice graph  $G$  decomposes into a certain number, function of  $\chi(G)$ , of graphs fulfilling the 1-2-3 Conjecture.

## 7.6 Weak $(\ell, k)$ -colouring

We finally consider the weaker form of Conjecture 7.3.1 (which would follow from the 1-2-3 Conjecture, if it turned out to be proved):

**Weak Conjecture.** *Every nice graph is weakly  $(2, 2)$ -colourable.*

Towards the Weak Conjecture, we here first prove that all nice graphs are weakly  $(3, 2)$ - and  $(2, 4)$ -colourable. Both proofs are based on the fact that every nice graph admits a neighbour-sum-distinguishing 5-edge-weighting, as proved by Kalkowski, Karoński and Pfender [KKP10]. We then prove that graphs with minimum degree at least 59 are weakly  $(2, 3)$ -colourable.

### 7.6.1 Weak $(3, 2)$ - and $(2, 4)$ -colourability

We first prove that every nice graph is weakly  $(3, 2)$ -colourable.

**Theorem 7.6.1.** *Every nice graph  $G$  is weakly  $(3, 2)$ -colourable.*

*Proof.* Slight modifications of the proof of Kalkowski, Karoński and Pfender [KKP10] allow to show that every nice graph even admits a neighbour-sum-distinguishing  $\{s - 2, s - 1, s, s + 1, s + 2\}$ -edge-weighting, for any integer  $s$ . Let thus  $\omega$  be a neighbour-sum-distinguishing  $\{-2, -1, 0, 1, 2\}$ -edge-weighting of  $G$ . We deduce a weak  $(3, 2)$ -colouring of  $G$  by modifying and colouring the weights of  $\omega$ , as follows:

- we colour red every edge with value in  $\{1, 2\}$ ;
- we colour blue every edge with value in  $\{-2, -1\}$ , and multiply its value by  $-1$ ;
- we colour green every edge with value  $0$ , and change its value to  $1$ .

The key point is that, through  $\omega$ , every two adjacent vertices  $u$  and  $v$  are only distinguished via their incident edges with weight in  $\{-2, -1, 1, 2\}$ . Said differently the edges with weight  $0$  are useless for distinguishing  $u$  and  $v$ . This implies that, in the obtained  $(3, 2)$ -colouring, it is not possible that both the red and blue sums of  $u$  and  $v$  are equal. From this reasoning, we get that the resulting  $(3, 2)$ -colouring is indeed a weak  $(3, 2)$ -colouring.  $\square$

We now prove that every nice graph can be weakly  $(2, 4)$ -coloured.

**Theorem 7.6.2.** *Every nice graph  $G$  is weakly  $(2, 4)$ -colourable.*

*Proof.* Since  $G$  is nice, it admits a neighbour-sum-distinguishing 5-edge-weighting  $\omega$  according to the result of Kalkowski, Karoński and Pfender [KKP10]. We deduce a weak  $(2, 4)$ -colouring of  $G$  by 2-colouring and (possibly) altering the weights assigned by  $\omega$ , as follows:

- we colour red every edge with value in  $\{1, 2, 3, 4\}$ ;
- we colour blue every edge with value 5, and change its value to 1.

Consider an edge  $uv$  of  $G$ . Note that if the red sums of  $u$  and  $v$  are equal, then their blue sums cannot be equal too: in such a situation, we would get  $\sigma_\omega(u) = \sigma_\omega(v)$ , a contradiction. So we get a weak  $(2, 4)$ -colouring.  $\square$

### 7.6.2 Graphs with $\delta \geq 59$ are weakly $(2, 3)$ -colourable

Before proceeding to the proof of the main result of this section, we first need to introduce two observations.

**Observation 7.6.3.** *Every graph  $G$  decomposes into two subgraphs  $G_1$  and  $G_2$  such that:*

- for every vertex  $v$  of  $G$ , we have  $d_{G_1}(v) \in \left\{ \left\lfloor \frac{d_G(v)}{2} \right\rfloor, \left\lfloor \frac{d_G(v)}{2} \right\rfloor + 1 \right\}$ , and
- for every even-degree vertex  $v$  of  $G$  except possibly one, we have  $d_{G_1}(v) = \frac{d_G(v)}{2}$ .

*Proof.* If the subset  $U \subseteq V$  of the vertices of odd degree in  $G$  is non-empty, add a new vertex  $u$  and join it by a single edge with every vertex in  $U$ ; denote the obtained graph by  $G'$  (if  $U = \emptyset$ , set  $G' = G$ ). As the degrees of all vertices in  $G'$  are even, there exists an Eulerian tour in it. We then traverse all edges of  $G'$  once along this Eulerian tour, starting at  $u$  if it exists, and colour them alternately red and blue. Then the red edges in  $G$  induce its subgraph  $G_1$  consistent with our requirements.  $\square$

**Observation 7.6.4.** *Every graph  $G$  has an orientation  $D$  such that, for every vertex  $v$ , we have  $d_D^+(v) \geq \left\lfloor \frac{d_G(v)}{2} \right\rfloor$ .*

*Proof.* Analogously as in the proof of Observation 7.6.3, if the subset  $U \subseteq V$  of the vertices of odd degree in  $G$  is non-empty, then add a new vertex  $u$  and join it by a single edge with every vertex in  $U$ ; denote the obtained graph by  $G'$  (if  $U = \emptyset$ , set  $G' = G$ ). As the degrees of all vertices in  $G'$  are even, there exists an Eulerian tour in it. By traversing it once we obtain an orientation of  $G'$  with equal in- and out-degrees for all vertices. This yields the desired orientation  $D$  of  $G$ .  $\square$

We now prove the main result:

**Theorem 7.6.5.** *Every graph  $G$  with  $\delta(G) \geq 59$  is weakly  $(2, 3)$ -colourable.*

*Proof.* We may suppose that  $G$  is connected. Let  $G_1 = (V, E_1)$  and  $G_2 = (V, E_2)$  be the subgraphs of  $G$  obtained by applying Observation 7.6.3, where  $x$  is a vertex of even degree in  $G$  for which  $d_{G_1}(x) \neq \frac{d_G(x)}{2}$  if it exists (let  $x$  be any fixed vertex of  $G$  otherwise). We produce a weak  $(2, 3)$ -colouring of  $G$  by colouring and weighting  $G_1$  and  $G_2$  separately.

We colour red all edges of  $G_1$ , and blue all edges of  $G_2$ . Initially we weight all the edges with 2. Denote by  $\omega_1, \omega_2$  the temporary weightings of  $G_1, G_2$ . In what follows,  $\omega_1$  and  $\omega_2$  will be subject to changes, but, for the sake of the proof, we still call them  $\omega_1$  and  $\omega_2$ . At every step of our construction, the colour of any vertex  $v$ , denoted  $c(v)$ , will be understood as the pair  $(\sigma_{\omega_1}(v), \sigma_{\omega_2}(v))$ , where  $\sigma_{\omega_i}(v) := \sum_{u \in N_{G_i}(v)} \omega_i(uv)$  for  $i = 1, 2$ .

Let  $D_1$  and  $D_2$  be auxiliary orientations of  $G_1$  and  $G_2$ , respectively, consistent with Observation 7.6.4. It is straightforward to verify that for every  $v \in V \setminus \{x\}$  we then must have:

$$(d_{D_1}^+(v) + 1)(d_{D_2}^+(v) + 1) > 4d_G(v). \quad (7.1)$$

Let further  $O_i(v) := \{uv \in E_i : u \in N_{D_i}^+(v)\}$  denote the set of edges incident with a given vertex  $v$  which correspond to arcs out-going from  $v$  in  $D_i$ . Hence  $|O_i(v)| = d_{D_i}^+(v)$  for  $i = 1, 2$ .

Let us define the following family of pairwise disjoint four-element sets of pairs of integers

$$\mathfrak{B} = \{\{(2p, 2q), (2p, 2q + 1), (2p + 1, 2q), (2p + 1, 2q + 1)\} : p, q \in \mathbb{Z}\}.$$

Fix an arbitrary ordering  $v_1, \dots, v_n$  over the vertices in  $V$  with  $v_1 = x$ . We will analyse the  $v_i$ 's one after another consistently with this ordering. At each Step  $j$  we will choose some set  $B_j \in \mathfrak{B}$ , different from all such sets already fixed for the neighbours of  $v_j$  in  $G$ , and we will modify weights of the edges in  $O_1(v_j) \cup O_2(v_j)$  so that  $c(v_k) \in B_k$  for every  $k \leq j$ . Note that if we are able to achieve this using only weights 1, 2, 3 on the edges, after Step  $n$  we will then obtain a desired weighting of  $G_1$  and  $G_2$ .

Step 1 is trivial, so assume we analyse Step  $j$  for some  $j \in \{2, \dots, n\}$ , and thus far all our requirements have been fulfilled. Note that we cannot choose at most  $d_G(v_j)$  sets from  $\mathfrak{B}$  for  $v_j$  (these already assigned to neighbours of  $v_j$  in  $G$ ). We however may always modify the weight of every  $v_k v_j \in O_i(v_j)$  by 1 so that  $c(v_k) \in B_k$  (if  $v_k$  has already  $B_k$  assigned),  $i = 1, 2$ . This way we may obtain at least  $|O_1(v_j)| + 1$  distinct values of the first coordinate of  $c(v_j)$  and at least  $|O_2(v_j)| + 1$  distinct values for the second one. This altogether yields a list of available pairs of cardinality at least

$$(|O_1(v_j)| + 1)(|O_2(v_j)| + 1) \geq (d_{D_1}^+(v) + 1)(d_{D_2}^+(v) + 1) > 4d_G(v)$$

for  $c(v_j)$  due to Inequality (7.1) above. At least one of these pairs must thus not belong to any  $B_k$  for  $v_k \in N_G(v_j)$ . We choose any such pair and fix as  $B_j$  the set in  $\mathfrak{B}$  which includes this pair, performing at the same time modifications of weights (to 1 or 3) of (some of) the edges in  $O_1(v_j) \cup O_2(v_j)$  so that  $c(v_j) \in B_j$  afterwards. Note that by our

*7. A general decomposition theory for 1-2-3 Conjecture and locally irregular decompositions*

---

choice of the sets  $O_i(v)$ , the weight of each edge might be modified only once, and hence belongs to  $\{1, 2, 3\}$ .

After Step  $n$ , we thus obtain desired weightings of  $G_1$  and  $G_2$ . □





# Bibliography

- [ABADR05] L. Addario-Berry, R.E.L. Aldred, K. Dalal, and B.A. Reed. Vertex colouring edge partitions. *Journal of Combinatorial Theory, Series B*, 94(2):237–244, 2005.
- [ABDM<sup>+</sup>07] L. Addario-Berry, K. Dalal, C. McDiarmid, B.A. Reed, and A. Thomason. Vertex-colouring edge-weightings. *Combinatorica*, 27:1:1–12, 2007.
- [ABDR08] L. Addario-Berry, K. Dalal, and B.A. Reed. Degree constrained subgraphs. *Discrete Applied Mathematics*, 156(7):1168–1174, 2008. GRACO 2005.
- [AKL<sup>+</sup>04] N. Alon, G. Kaplan, A. Lev, Y. Roditty, and R. Yuster. Dense graphs are antimagic. *Journal of Graph Theory*, 47(4):297–309, 2004.
- [APBSF17] S. Arumugam, K. Premalatha, M. Bača, and A. Semaničová-Feňovčíková. Local antimagic vertex coloring of a graph. *Graphs and Combinatorics*, 33(2):275–285, 2017.
- [BBH<sup>+</sup>18] O. Baudon, J. Bensmail, H. Hocquard, M. Senhaji, and É. Sopena. On locally irregular decompositions of subcubic graphs. *Opuscula Mathematica*, 38(6), 2018.
- [BBPW15] O. Baudon, J. Bensmail, J. Przybyło, and M. Woźniak. On decomposing regular graphs into locally irregular subgraphs. *European Journal of Combinatorics*, 49:90–104, 2015.
- [BBS15] O. Baudon, J. Bensmail, and É. Sopena. On the complexity of determining the irregular chromatic index of a graph. *Journal of Discrete Algorithms*, 30:113–127, 2015.
- [BBSS18] O. Baudon, J. Bensmail, M. Senhaji, and É. Sopena. Neighbour-Sum-2-Distinguishing Edge-Weightings: Doubling the 1-2-3 Conjecture. *Discrete Applied Mathematics*, 2018.
- [BBV15] K. Bérczi, A. Bernáth, and M. Vizer. Regular graphs are antimagic. *The Electronic Journal of Combinatorics*, 22(3), 2015.

- [BKN<sup>+</sup>99] O. V. Borodin, A. V. Kostochka, J. Nešetřil, A. Raspaud, and É. Sopena. On the maximum average degree and the oriented chromatic number of a graph. *Discrete Mathematics*, 206(1-3):77–89, 1999.
- [BM08] A. Bondy and U.S.R. Murty. *Graph Theory*, volume 244 of *Graduate Texts in Mathematics*. Springer-Verlag, London, 2008.
- [BMT17] J. Bensmail, M. Merker, and C. Thomassen. Decomposing graphs into a constant number of locally irregular subgraphs. *European Journal of Combinatorics*, 60:124–134, 2017.
- [BPP<sup>+</sup>17] O. Baudon, M. Piłśniak, J. Przybyło, M. Senhaji, É. Sopena, and M. Woźniak. Equitable neighbour-sum-distinguishing edge and total colourings. *Discrete Applied Mathematics*, 222:40–53, 2017.
- [BPS<sup>+</sup>17] O. Baudon, J. Przybyło, M. Senhaji, E. Sidorowicz, É. Sopena, and M. Woźniak. The neighbour-sum-distinguishing edge-colouring game. *Discrete Mathematics*, 340(7):1564–1572, 2017.
- [Bro41] R. L. Brooks. On colouring the nodes of a network. *Mathematical Proceedings of the Cambridge Philosophical Society*, 37(2):194–197, 1941.
- [BS16] J. Bensmail and B. Stevens. Edge-partitioning graphs into regular and locally irregular components. *Discrete Mathematics and Theoretical Computer Science*, 17(3):43–58, 2016.
- [BSSL17] J. Bensmail, M. Senhaji, and K. Szabo Lyngsie. On a combination of the 1-2-3 Conjecture and the Antimagic Labelling Conjecture. *Discrete Mathematics & Theoretical Computer Science*, Vol. 19 no. 1, August 2017.
- [CLWY11] G. Chang, C. Lu, J. Wu, and Q. Yu. Vertex-coloring edge-weightings of graphs. *Taiwanese Journal of Mathematics*, 15(4):1807–1813, 2011.
- [CLZ15] D. W. Cranston, Y.-C. Liang, and X. Zhu. Regular graphs of odd degree are antimagic. *Journal of Graph Theory*, 80(1):28–33, 2015.
- [Die17] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin Heidelberg, 5<sup>th</sup> edition, 2017.
- [DLY12] Y. Duan, H. Lu, and Q. Yu. L-factors and adjacent vertex-distinguishing edge-weighting. *East Asian Journal on Applied Mathematics*, 2(2):83–93, 2012.
- [DO15] A. Davoodi and B. Omoomi. On the 1-2-3-conjecture. *Discrete Mathematics & Theoretical Computer Science*, 17(1):67–78, 2015.
- [DSA13] A. Dehghan, M.-R. Sadeghi, and A. Ahadi. Algorithmic complexity of proper labeling problems. *Theoretical Computer Science*, 495:25–36, 2013.

- [DW11] A. Dudek and D. Wajc. On the complexity of vertex-coloring edge-weightings. *Discrete Mathematics & Theoretical Computer Science*, 13:45–50, 2011.
- [Edm65] J. Edmonds. Paths, trees, and flowers. *Canadian Journal of Mathematics*, 17:449–467, 1965.
- [Gal17] J. A. Gallian. A dynamic survey of graph labelling. *Electronic Journal of Combinatorics*, #DS6, 20<sup>th</sup> ed., 2017.
- [Has17] J. Haslegrave. Proof of a local antimagic conjecture. arXiv:1705.09957, 2017.
- [HR90] N. Hartsfield and G. Ringel. *Pearls in Graph Theory*. Graduate Texts in Mathematics. Academic Press, San Diego, 1990.
- [Kal10] M. Kalkowski. A note on 1,2-conjecture. In PhD thesis, Adam Mickiewicz University in Poznań, 2010.
- [KKP09] M. Kalkowski, M. Karoński, and F. Pfender. Vertex colouring edge weightings with integer weights at most 6. *Rostocker Mathematisches Kolloquium*, 64:39–43, 2009.
- [KKP10] M. Kalkowski, M. Karoński, and F. Pfender. Vertex-coloring edge-weightings: Towards the 1-2-3-conjecture. *Journal of Combinatorial Theory, Series B*, 100(3):347–349, 2010.
- [KLT04] M. Karoński, T. Łuczak, and A. Thomason. Edge weights and vertex colours. *Journal of Combinatorial Theory, Series B*, 91(1):151–157, 2004.
- [KNN<sup>+</sup>12] M. Khatirinejad, R. Naserasr, M. Newman, B. Seamone, and B. Stevens. Vertex-colouring edge-weightings with two edge weights. *Discrete Mathematics & Theoretical Computer Science*, 14:1–20, 2012.
- [LPS16] B. Lužar, J. Przybyło, and R. Soták. New bounds for locally irregular chromatic index of bipartite and subcubic graphs. Technical report, 2016.
- [LYY09] H. Lu, X. Yang, and Q. Yu. On vertex-coloring edge-weighting of graphs. *Frontiers of Mathematics in China*, 4(2):325–334, 2009.
- [LYZ11] H. Lu, Q. Yu, and C. Zhang. Vertex-coloring 2-edge-weighting of graphs. *European Journal of Combinatorics*, 32:21–27, 2011.
- [Men27] K. Menger. Zur allgemeinen kurventheorie. *Fundamenta Mathematicae*, 10:96–115, 1927.
- [Mer99] M. Meringer. Fast generation of regular graphs and construction of cages. *Journal of Graph Theory*, 30:137–146, 1999.

- [MR08] W. Mulzer and G. Rote. Minimum-weight triangulation is np-hard. *Journal of the ACM*, 55(2):Article no. 11, 2008.
- [Prz15] J. Przybyło. Asymptotically optimal neighbour sum distinguishing colourings of graphs. *Random Structures & Algorithms*, 47(4):776–791, 2015.
- [Prz16] J. Przybyło. On decomposing graphs of large minimum degree into locally irregular subgraphs. *Electronic Journal of Combinatorics*, 23(2):#P.31, 2016.
- [PW10] J. Przybyło and M. Woźniak. On a 1,2 conjecture. *Discrete Mathematics & Theoretical Computer Science*, 12:101–108, 2010.
- [Sea12] B. Seamone. The 1-2-3 conjecture and related problems: a survey. <https://arxiv.org/abs/1211.5122>, 2012.
- [TWZ16] C. Thomassen, Y. Wu, and C. Zhang. The 3-flow conjecture, factors modulo  $k$ , and the 1-2-3-conjecture. *Journal of Combinatorial Theory, Series B*, 121:308–325, 2016.
- [Viz64] V. G. Vizing. On an estimate of the chromatic class of a  $p$ -graph. *Metody Diskretnogo Analiza*, 3:25–30, 1964.
- [WY08] T. Wang and Q. Yu. On vertex-coloring 13-edge-weighting. *Frontiers of Mathematics in China*, 3(4):581–587, 2008.
- [YHY<sup>+</sup>18] X. Yu, J. Hu, D. Yang, J. Wu, and G. Wang. Local antimagic labeling of graphs. *Applied Mathematics and Computation*, 322:30–39, 2018.