

## COMPLETE COLORINGS OF SIGNED GRAPHS

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**Abstract.** The aim of this proposal is to study complete colorings of signed graphs, based on the notion of coloring recently introduced by Máčajová, Raspaud and Škovič [3], as an extension of earlier definitions by Zaslavsky [5]. To our knowledge, this notion has not been considered yet, so that one can reasonably expect to obtain interesting results on this topic.

**General context.** A proper  $k$ -coloring of a graph  $G$  is a mapping  $\lambda$  from the set of vertices  $V(G)$  of  $G$  to the set of colors  $\{1, \dots, k\}$  such that  $\lambda(u) \neq \lambda(v)$  for every edge  $uv$  of  $G$ . The chromatic number  $\chi(G)$  of  $G$  is then the smallest integer  $k$  for which  $G$  admits a proper  $k$ -coloring. A proper  $k$ -coloring  $\lambda$  of  $G$  is *complete* if, for every two distinct colors  $i$  and  $j$ ,  $1 \leq i < j \leq k$ , there is an edge  $uv$  in  $G$  with  $\lambda(u) = i$  and  $\lambda(v) = j$ . The *achromatic number*  $\psi(G)$  of  $G$  is then the largest integer  $k$  for which  $G$  admits a complete proper  $k$ -coloring. Complete colorings of graphs have been extensively studied in the literature (see [1] for a comprehensive survey on graph colorings).

A *signed graph* is a pair  $(G, \sigma)$  where  $G$  is a graph and  $\sigma$  a mapping (the *signature* of  $(G, \sigma)$ ) that assigns to each edge of  $G$  a *sign* from the set  $\{+, -\}$ . A proper  $n$ -coloring [3, 5] of a signed graph  $(G, \sigma)$  is a mapping  $\lambda$  from the set of vertices  $V(G)$  of  $G$  to the set of colors  $M_n = \{-k, \dots, k\}$  if  $n = 2k + 1$ , of  $M_n = \{-k, \dots, -1, 1, \dots, k\}$  if  $n = 2k$ , such that  $\lambda(u) \neq \sigma(uv)\lambda(v)$  for every edge  $uv$  of  $G$ . Therefore, the colors assigned to vertices connected by a positive edge must be distinct and can be opposite, while the colors assigned to vertices connected by a negative edge can be identical but cannot be opposite.

**Research proposal.** The goal of this master thesis project is to initiate the study of complete colorings of signed graphs. The definition of such a complete coloring is as follows: A proper  $n$ -coloring  $\lambda$  of a signed graph  $(G, \sigma)$  is *complete* if, for every two distinct integers  $i$  and  $j$ ,  $0 \leq i < j \leq k$  if  $n = 2k + 1$ , or  $1 \leq i < j \leq k$  if  $n = 2k$ , there exists an edge  $uv$  in  $(G, \sigma)$  with  $\lambda(u) = a$ ,  $\lambda(v) = b$ ,  $|a| = i$  and  $|b| = j$ . The achromatic number of a signed graph is then defined in the same way as for ordinary graphs.

It is worth to notice that complete colorings of signed graphs have been studied in [2], but using a rather different notion of signed graph coloring, based on homomorphisms of signed graphs (several types of colorings have been defined for signed graphs, see [4] for a recent survey on this topic).

The work on this project might be organized as follows.

1. Bibliographic review: complete colorings, signed graphs, signed graph colorings...
2. Structural issues: complete colorings and achromatic numbers of special classes of signed graphs.

3. Algorithmic issues: complexity of decision problems related to complete colorings or achromatic numbers.

Depending on the work progress, it may happen that item 2 or item 3 is developed in a deeper way.

## References

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