

SWITCHING HOMOMORPHISMS OF SIGNED DIGRAPHS

Supervisors: Florent Foucaud, `florent.foucaud@u-bordeaux.fr`

Éric Sopena, `eric.sopena@u-bordeaux.fr`

Research team: Combinatorics and Algorithms (Graphs and Optimization)

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Abstract. The aim of this proposal is to study switching homomorphisms of signed digraphs, based on the notion of switching homomorphisms of undirected signed graphs studied in [4, 5]. To our knowledge, this notion has not been considered yet, so that one can reasonably expect to obtain interesting results on this topic.

General context. Given two directed graphs (digraphs) G and H , a *homomorphism* of G to H is a vertex-mapping f of $V(G)$ to $V(H)$ that preserves adjacencies, that is, if there is an arc from x to y in G , then there is an arc from $f(x)$ to $f(y)$ in H as well. When such a homomorphism exists, we write $G \rightarrow H$. When the considered digraphs are symmetric (that is, for every arc we also have its reverse arc), this is equivalent to classic homomorphisms of undirected graphs. See the book [2] for a general reference on (di)graph homomorphisms.

A related notion is the one of *oriented chromatic number*, see [7] for a survey. For a simple digraph G (a digraph is *simple* if it has no loops or parallel arcs), its oriented chromatic number, denoted $\chi_o(G)$, is the smallest order of a tournament T such that $G \rightarrow T$. (A *tournament* is a simple digraph where every pair of vertices is joined by one arc.)

A *signed digraph* is a pair (G, σ) where G is a digraph and σ a mapping (the *signature* of (G, σ)) that assigns to each arc of G a *sign* from the set $\{+, -\}$. The introduction of signed digraphs almost 50 years ago was motivated by modeling environmental and socio-political problems [6], but unlike their undirected counterparts, they have not been studied much since then (see [1, 3] for some later references). Signed (di)graphs come with an operation called *switching*: to switch at a vertex v in (G, σ) is to change the signs of all arcs incident with v .

We propose to extend the notion of homomorphisms from digraphs to signed digraphs. Similarly to the corresponding definition for undirected signed graphs from [4], a *switching homomorphism* of a signed digraph (G, σ) to a signed digraph (H, τ) consists of:

- (i) a sequence of switchings in (G, σ) resulting in (G, σ') ,
- (ii) followed by a homomorphism f of G to H that also preserves the signs of the arcs. More precisely, if there is an arc from x to y in (G, σ') , we have $\sigma'(xy) = \tau(f(x)f(y))$.

Switching homomorphisms of undirected signed graphs have been extensively studied, see [5] for a recent update on the topic.

Similarly, we may define the *switching oriented chromatic number* of a simple signed digraph, as the smallest order of a signed tournament to which it admits a homomorphism.

Research proposal. The goal of this master thesis is to initiate the study of switching homomorphisms and the switching oriented chromatic number of signed digraphs.

The work on this project might be organized as follows.

1. Bibliographic review: signed graphs, homomorphisms of digraphs, oriented chromatic number...
2. Structural issues: switching oriented chromatic numbers of special classes of signed digraphs.
3. Algorithmic issues: complexity of decision problems related to switching homomorphisms of signed digraphs.

Depending on the work progress, it may happen that item 2 or item 3 is developed in a deeper way.

References

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