

# A Study of $k$ -dipath Colourings of Oriented Graphs

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## EXTENDED ABSTRACT

Recall that an *oriented graph* is a digraph obtained from a simple, undirected graph by giving each edge one of its two possible orientations. The *directed girth* of an oriented graph  $H$  is defined to be the minimum length of a directed cycle in  $H$ , or infinity if  $H$  has no directed cycle. For a pair of vertices  $u, v \in V(G)$ , the *weak distance* from  $u$  to  $v$ , denoted  $d_{\text{weak}}(u, v)$ , is the least  $k$  such that there is a directed path of length  $k$  from  $u$  to  $v$  or from  $v$  to  $u$ , or infinity if no such path exists.

Recall, also, that if  $G$  and  $H$  are oriented graphs, then a *homomorphism of  $G$  to  $H$*  is a function  $\phi$  from the vertices of  $G$  to the vertices of  $H$  such that  $\phi(x)\phi(y) \in E(H)$  whenever  $xy \in E(G)$ . If  $G$  and  $H$  are oriented graphs such that there is a homomorphism  $\phi$  of  $G$  to  $H$ , then we write  $\phi : G \rightarrow H$ , or  $G \rightarrow H$  if the name of the function  $\phi$  is not important.

Let  $k$  and  $t$  be positive integers, and let  $G$  be an oriented graph. Chen and Wang [2] defined a  *$k$ -dipath  $t$ -colouring* of  $G$  to be an assignment of  $t$  colours to the vertices of  $G$  so that any two vertices joined by a directed path of length at most  $k$  are assigned different colours. A 1-dipath  $t$ -colouring of an oriented graph  $G$  is a colouring of the underlying undirected graph of  $G$ . Figure 1 gives of a 3-dipath 4-colouring of an oriented graph.

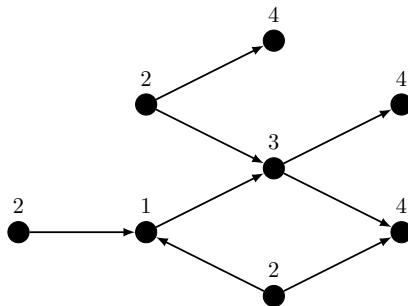


Figure 1: A 3-dipath 4-colouring.

The  *$k$ -dipath chromatic number* of  $G$ , denoted by  $\chi_{kd}(G)$ , is the smallest positive integer  $t$  such that there exists a  $k$ -dipath  $t$ -colouring of  $G$ . Chen and Wang showed that any orientation of a Halin graph has 2-dipath chromatic number at most 7, and there are infinitely many such graphs  $G$  with  $\chi_{2d}(G) = 7$  [2].

For a positive integer  $t$ , an *oriented  $t$ -colouring* of an oriented graph  $G$  is a homomorphism of  $G$  to some oriented graph on  $t$  vertices. Oriented colourings were first introduced in 1994 [3], and have been a topic of considerable interest in the literature since then; see the recent survey by Sopena [8], and also [1], [4] for related topics. Since oriented graphs have no directed cycles of length two, the definition implies that any two vertices of  $G$  joined by a directed path of length at most two are assigned different colours (i.e. they have different images) in an oriented colouring of  $G$ . The 2-dipath chromatic number is of interest, in part, because it

gives a lower bound for the *oriented chromatic number*  $\chi_o(G)$  – the smallest positive integer  $t$  such that  $G$  admits an oriented  $t$ -colouring. It follows from the definition that any oriented colouring of  $G$  is a 2-dipath colouring of  $G$ ; hence  $\chi_{2d}(G) \leq \chi_o(G)$ .

In her Master's thesis [9] (more recently published as [6]), Sherk (née Young) gives a *homomorphism model* for 2-dipath  $t$ -colouring. That is, for each positive integer  $t$ , she describes an oriented graph  $\mathcal{G}_t$  with the property that an oriented graph  $G$  has a 2-dipath  $t$ -colouring if and only if there is a homomorphism of  $G$  to  $\mathcal{G}_t$ . As is common with such theorems, it is possible to use the homomorphism to  $\mathcal{G}_t$  to find a 2-dipath  $t$ -colouring of  $G$ . The existence of this model implies an upper bound for the oriented chromatic number as a function of the 2-dipath chromatic number. It also leads to a proof that deciding whether a given oriented graph has a 2-dipath  $t$ -colouring is Polynomial if the fixed integer  $t \leq 2$ , and NP-complete if  $t \geq 3$ .

A natural question is whether Sherk's results can be generalized to  $k$ -dipath  $t$ -colouring. We seek a model similar to hers, where the homomorphism to the target oriented graph can be used to find a  $k$ -dipath  $t$ -colouring of the given oriented graph. However, we suggest that such model only exists for colouring oriented graphs with no directed cycles of length  $k$  or less. Consider the case of 3-dipath  $t$ -colouring, where  $t \geq 3$ . Suppose there exists a digraph  $H_{3,t}$  with the property that an oriented graph  $G$  has a 3-dipath  $t$ -colouring if and only if there is a homomorphism of  $G$  to  $H_{3,t}$ . The digraph  $H_{3,t}$  has no loops, otherwise all vertices of  $G$  can be assigned the same colour.

By definition, the directed 3-cycle has a 3-dipath 3-colouring: assign each vertex a different colour. Thus, there is a homomorphism of the directed 3-cycle to  $H_{3,t}$ . Consequently,  $H_{3,t}$  has a directed 3-cycle. But, now there is a homomorphism of a directed path of length three to  $H_{3,t}$  in which the two end vertices have the same image. Since the ends of a directed path of length three must be assigned different colours in a 3-dipath  $t$ -colouring, this model will not have the desired property. Similar considerations apply to  $k$ -dipath  $t$ -colouring for all pairs of positive integers  $k$  and  $t$  with  $t \geq k$ . Hence, a homomorphism model of the type we seek will not exist if the oriented graphs being coloured can have directed cycles of length  $k$  or less. This is consistent with Sherk's results for 2-dipath colouring of oriented graphs as these graphs have no directed cycles of length two or less.

Let  $G$  be an oriented graph. Define  $G^k$  to be the simple graph formed from  $G$  as follows:

- $V(G^k) = V(G)$ , and
- $E(G^k) = \{uv \mid 0 < d_{\text{weak}}(u, v) \leq k\}$ .

Observe that if  $G$  is an oriented graph with directed girth at least  $k + 1$ , then there is a one-to-one correspondence between  $k$ -dipath colourings of  $G$  and proper colourings of  $G^k$ .

The main result of this work is the construction of a homomorphism model for  $k$ -dipath  $t$ -colouring of oriented graphs with no directed cycles of length  $k$  or less. That is, for all positive integers  $k$  and  $t$  we describe an oriented graph  $\mathcal{G}_{k,t}$  with the property that an oriented graph  $G$ , with no directed cycle of length at most  $k$ , has a  $k$ -dipath  $t$ -colouring if and only if  $G$  admits a homomorphism to  $\mathcal{G}_{k,t}$ . Further, such a homomorphism is sufficient to construct an explicit  $k$ -dipath  $t$ -colouring of  $G$ . The oriented graph  $\mathcal{G}_{k,t}$  is constructed by considering a set of matrices which encode the possible colours of vertices (in a  $k$ -dipath  $t$ -colouring) in an closed  $(k - 1)$ -neighbourhood of a fixed vertex. The set of such matrices is taken as the vertex set of  $\mathcal{G}_{k,t}$ . Adjacency between a pair of matrices encodes that the respective fixed vertices in those matrices have consistent  $(k - 1)$ -neighbourhoods. This construction generalises Sherk's homomorphism model for 2-dipath colouring using vectors to encode the colours of neighbours of a vertex with a fixed colour.

This homomorphism model for  $k$ -dipath  $t$ -colouring captures the following facts.

- If  $t < k$ , then no oriented graph with directed girth at least  $k + 1$  and a  $k$ -dipath  $t$ -colouring can have a directed path of length greater than  $t$ . Any such digraph is  $k$ -dipath  $t$ -colourable.

- If  $t \geq k$ , then any  $k$ -dipath  $t$ -colouring of an oriented graph is also a  $k$ -dipath  $(t+1)$ -colouring, and there exist oriented graphs with  $k$ -dipath chromatic number  $t+1$ .
- Every  $(k+1)$ -dipath  $t$ -colouring of an oriented graph is a  $k$ -dipath  $t$ -colouring

In addition to considering homomorphisms and  $k$ -dipath coloring, we determine the complexity of deciding the existence of a  $k$ -dipath  $t$ -colouring for all pairs of fixed positive integers  $k$  and  $t$ . When instances are restricted to oriented graphs with no directed cycles of length  $k$  or less, it is shown that this problem is NP-complete whenever  $t > k \geq 3$ , and Polynomial if  $t = k$  or  $k \leq 2$ . When there are no restrictions, it is shown that this problem is NP-complete whenever  $k \geq 3$  and  $t \geq 3$ , and Polynomial whenever  $k \leq 2$  or  $t \leq 2$ . These results generalise the result of Sherk for 2-dipath colouring; however, the here reduction is directly from the  $t$ -colouring problem, rather than following from the homomorphism model.

The problem of deciding whether the  $k$ -th power of a graph  $G$  is  $t$ -colourable is known to be NP-complete [7, 5]. Our results imply that if  $t > k$ , the problem of deciding whether the underlying graph of the  $k$ -th power of an oriented graph  $G$  (i.e. two vertices are adjacent if and only if they are at weak distance at most  $k$ ) is  $t$ -colourable is NP-complete, even when restricted to powers of oriented graphs with directed girth at least  $k+1$ , and also that, if  $t = k \geq 3$ , the problem of deciding whether the  $k$ -th power of an oriented graph  $G$  is  $t$ -colourable is NP-complete.

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