

# On homomorphisms of planar signed graphs to signed projective cubes

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## Abstract

We conjecture that every planar signed bipartite graph of unbalanced girth  $2g$  admits a homomorphism to the signed projective cube of dimension  $2g - 1$ . Our main result is to show that for a given  $g$ , this conjecture is equivalent to the corresponding case ( $k = 2g$ ) of a conjecture of Seymour claiming that every planar  $k$ -regular multigraph with no odd edge-cut of less than  $k$  edges is  $k$ -edge-colorable.

**Keywords.** Homomorphism, planar signed graph, projective cube, signed graph.

## 1 Introduction

It is a classic result of Tait from 1890 that the Four-Color Theorem (Conjecture at that time) is equivalent to the statement that every cubic bridgeless planar graph is 3-edge-colorable. It is easily observed that if a  $k$ -regular multigraph is  $k$ -edge-colorable, then the number of edges with exactly one end in  $X$ , assuming  $|X|$  is odd, is at least  $k$ . Seymour conjectured in 1975 that for planar graphs the converse is also true:

**Conjecture 1.1 (Seymour [10])** *Every  $k$ -regular planar multigraph with no odd edge-cut of less than  $k$  edges is  $k$ -edge-colorable.*

In this work we focus on a direct extension of the Four-Color Theorem using homomorphisms of signed graphs and its relation with Seymour's conjecture. The theory of homomorphisms of signed graphs includes in particular the theory of graph homomorphisms. A first paper on a comprehensive study of this notion was recently written by the authors of this work. Here we introduce the basic notations but we refer to [8], to [9] and to references mentioned there for more details.

Given a graph  $G$ , a *signature* on  $G$  is a mapping that assigns to each edge of  $G$  either a positive or a negative sign. A signature is normally denoted by the set  $\Sigma$  of negative edges. Given a signature  $\Sigma$  on a graph  $G$ , *resigning* at a vertex  $v$  is to change the sign of each edge incident with  $v$ . Two signatures  $\Sigma_1$  and  $\Sigma_2$  on  $G$  are *equivalent* if one can be obtained from the other by a sequence of resignings or, equivalently, by changing the signs of the edges of an edge-cut. A *signed graph*, denoted  $(G, \Sigma)$ , is a graph together with a class of equivalent signatures where  $\Sigma$  could be any member of the class.

An *unbalanced cycle* in a signed graph  $(G, \Sigma)$  is a cycle having an odd number of negative edges. Note that this is independent of the choice of a representative signature. Furthermore, the notion of unbalanced cycle is, in some sense, an extension of the classic notion of odd cycle, as a cycle of  $(G, E(G))$  is unbalanced if and only if it is an odd cycle. The *unbalanced-girth* of  $(G, \Sigma)$

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is then the shortest length of an unbalanced cycle of  $(G, \Sigma)$ . A cycle that is not unbalanced, i.e., a cycle that has an even number of negative edges (possibly none) is called *balanced*.

Note that if a signed Eulerian graph contains an odd number of negative edges, it must contain an unbalanced cycle. Therefore, if  $W$  is a closed walk in  $G$  with an odd number of negative edges in  $(G, \Sigma)$ , then the subgraph induced by the edges of  $W$  contains an unbalanced cycle.

One of the first theorems in the theory of signed graphs is that the set of unbalanced cycles (equivalently the set of balanced cycles) uniquely determines the associated class of signatures. More precisely:

**Theorem 1.2 (Zaslavsky [11])** *Two signatures  $\Sigma_1$  and  $\Sigma_2$  on a graph  $G$  are equivalent if and only if they induce the same set of unbalanced cycles.*

An important subclass of signed graphs, called *consistent signed graphs*, is the class of signed graphs whose balanced cycles are all of even length and whose unbalanced cycles are all of the same parity. This class itself consists of two parts. When all the unbalanced cycles are odd, then the set of unbalanced cycles of  $(G, \Sigma)$  is exactly the set of odd cycles of  $G$ , thus in this case, by Theorem 1.2,  $E(G)$  is a signature. Such a signed graph will then be called an *odd signed graph*. When all balanced and unbalanced cycles are even, the graph  $G$  must be bipartite, and  $\Sigma$  can be any subset of edges. Such a signed graph will be called an *even signed graph*.

Given two graphs  $G$  and  $H$ , a *homomorphism* of  $G$  to  $H$  is a mapping  $\phi : V(G) \rightarrow V(H)$  such that if  $xy \in E(G)$  then  $\phi(x)\phi(y) \in E(H)$ . Given two signed graphs  $(G_1, \Sigma_1)$  and  $(G_2, \Sigma_2)$  we say that  $(G_1, \Sigma_1)$  admits a *signed homomorphism*, or a homomorphism for short, to  $(G_2, \Sigma_2)$  if there are signatures  $\Sigma'_1$  and  $\Sigma'_2$  equivalent to  $\Sigma_1$  and  $\Sigma_2$ , respectively, and a homomorphism  $\varphi$  of  $G_1$  to  $G_2$  such that  $\varphi$  also preserves the signs of edges given by  $\Sigma'_1$  and  $\Sigma'_2$ . It is easily observed that the existence of  $\varphi$  is independent of the choice of the signature in the image graph while the choice of the signature of  $(G_1, \Sigma'_1)$  is essential. Thus the binary relation  $(G_1, \Sigma_1) \rightarrow (G_2, \Sigma_2)$ , which denotes the existence of a homomorphism of  $(G_1, \Sigma_1)$  to  $(G_2, \Sigma_2)$ , is an associative one.

One of the first results in the theory of signed graph homomorphisms is the following easy to prove lemma, which establishes conditions for the existence of a homomorphism from an unbalanced cycle to another.

**Lemma 1.3** *There is a homomorphism of  $UC_k$  to  $UC_\ell$  if and only if  $k \geq \ell$  and  $k \equiv \ell \pmod{2}$ .*

Using the definition of *signed projective cube* from the next section, the following conjecture is the main concern of this work:

**Conjecture 1.4** *Every consistent planar signed graph of unbalanced girth  $k$  admits a homomorphism to the signed projective cube of dimension  $k - 1$ .*

In an unpublished manuscript [4], B. Guenin, after introducing the notion of signed graph homomorphisms, proposed that the larger class of consistent signed graphs with an additional property (having no  $(K_5, E(K_5))$ -signed-minor) satisfies the same conclusion. He has shown relations between his conjecture and several other conjectures.

For odd values of  $k$  it turns out that all the edges of both sides are negative, and thus the problem is reduced to a graph homomorphism problem. Therefore, by [7], for each such  $k$  this conjecture is equivalent to Conjecture 1.1 with the corresponding value of  $k$ . In this work we prove the analog for even values of  $k$ . We note that, as it is shown in [8], the restriction of the notion of signed homomorphism to the class of even signed graphs already captures the notion of graph coloring and graph homomorphism through simple and natural graph operations.

## 2 Signed Projective Cubes

The projective cube of dimension  $d$ , denoted  $\mathcal{PC}_d$ , is the Cayley graph  $(\mathbb{Z}_2^d, \{e_1, e_2, \dots, e_d\} \cup \{J\})$  where  $e_i$  is the vector of  $\mathbb{Z}_2^d$  with the  $i$ -th coordinate being 1 and other coordinates being 0 and  $J = (1, 1, 1, \dots, 1)$ . It is easy to check that  $\mathcal{PC}_2$ ,  $\mathcal{PC}_3$  and  $\mathcal{PC}_4$  are isomorphic to  $K_4$ ,  $K_{4,4}$  and the well-known Clebsch graph, respectively.

Using the Cayley definition of  $\mathcal{PC}_d$ , let  $\mathcal{J}$  be the set of edges corresponding to  $J$ . We define the *signed projective cube* of dimension  $d$ , denoted  $\mathcal{SPC}_d$ , to be the signed graph  $(\mathcal{PC}_d, \mathcal{J})$ .

It is not difficult to prove that  $\mathcal{SPC}_d$  is a consistent signed graph and to determine its unbalanced girth. For the proof of the following theorem we refer to [9].

**Theorem 2.1** *All balanced cycles of  $\mathcal{SPC}_d$  are of even length, all unbalanced cycles of  $\mathcal{SPC}_d$  are of the same parity, and the unbalanced girth of  $\mathcal{SPC}_d$  is  $d+1$ . Furthermore, for each unbalanced cycle  $UC$  of  $\mathcal{SPC}_d$  and for each  $x \in \{e_1, e_2, \dots, e_d\} \cup \{J\}$ , there is an odd number of edges of  $UC$  which correspond to  $x$ .*

Using Theorem 2.1 and Theorem 1.2 we get the following corollary.

**Corollary 2.2** *The signed projective cube  $\mathcal{SPC}_{2d}$  is equivalent to  $(\mathcal{PC}_{2d}, E(\mathcal{PC}_{2d}))$ .*

Thus if a signed graph  $(G, \Sigma)$  admits a homomorphism to  $\mathcal{SPC}_{2d}$  then, using the signature  $E(\mathcal{PC}_{2d})$  of  $\mathcal{SPC}_{2d}$ , we conclude that  $(G, \Sigma)$  must be equivalent to  $(G, E(G))$ . On the other hand, since  $\mathcal{SPC}_{2d+1}$  is bipartite, if  $(G, \Sigma)$  maps to  $\mathcal{SPC}_{2d+1}$  then  $G$  must also be bipartite. Thus, in general, consistent signed graphs are the only graphs that can map to signed projective cubes. The following theorem shows that the problem of finding a mapping of a consistent signed graph to a signed projective cube is equivalent to a packing problem.

**Theorem 2.3** *An even (resp. odd) signed graph admits a homomorphism to  $\mathcal{SPC}_{2d-1}$  (resp.  $\mathcal{SPC}_{2d}$ ) if and only if it admits at least  $2d-1$  (resp.  $2d$ ) edge disjoint signatures.*

Theorem 2.3 in this form first appeared in [4]. For even dimensions, i.e., for the case in brackets, since all edges being negative is a signature of  $\mathcal{SPC}_{2d}$ , the problem of finding a signed homomorphism of  $(G, \Sigma)$  to  $\mathcal{SPC}_{2d}$  is reduced to the problem of finding a homomorphism of  $G$  to  $\mathcal{PC}_{2d}$ . For this case, implicitly equivalent formulations can be found in the literature. A proof, for example, can be found in [9].

As a corollary we get that the homomorphism relation between signed projective cubes themselves is very much like that of the homomorphism relation between cycles as given in Lemma 1.3:

**Corollary 2.4** *There is a homomorphism of  $\mathcal{SPC}_d$  to  $\mathcal{SPC}_{d'}$  if and only if  $d \geq d'$  and  $d \equiv d' \pmod{2}$ .*

We can now distinguish two cases of Conjecture 1.4. The first one, for odd values of  $k$ , is a slightly reformulated version of the main conjecture of [7].

**Conjecture 1.4 (odd case)** *Every planar odd signed graph of unbalanced girth at least  $2g+1$  admits a homomorphism to  $\mathcal{SPC}_{2g}$ .*

The case  $g=1$  of this conjecture is indeed the Four-Color Theorem. It is proved in [7] that this is equivalent to Conjecture 1.1 for  $k=2g+1$ . Thus in this paper we examine only the case of Conjecture 1.4 for even values of  $k$ .

**Conjecture 1.4 (even case)** *Every planar even signed graph of unbalanced girth at least  $2g$  admits a homomorphism to  $\mathcal{SPC}_{2g-1}$ .*

### 3 An extension of the Four-Color Theorem

As mentioned before, it has been shown in [8] that the notion of signed homomorphisms on even signed graphs already captures the notion of graph homomorphisms. The operations used to build this connection preserves planarity. Thus any homomorphism theory on planar graphs can be strengthened in the language of signed homomorphisms on planar even signed graphs.

A key lemma in the study of homomorphism properties of a planar graph is the folding lemma of Klostermeyer and Zhang [6]. This lemma implies that for each planar graph  $G$  of shortest odd cycle length  $2r + 1$ , and for each  $k \leq r$ , there is a planar homomorphic image  $H$  of  $G$  where every face of  $H$  is of length  $2k + 1$  and the shortest odd cycle of  $H$  is also of length  $2k + 1$ . By considering unbalanced cycles instead of odd cycles, we will get the same result for the class of planar even signed graphs. Following notations and ideas of Section 4 in [6] we can prove the following lemma (see [9] for a proof).

**Lemma 3.1 (Folding lemma)** *Let  $G$  be a planar even signed graph of unbalanced girth at least  $g$ . If  $C = v_0 \cdots v_{r-1} v_0$  is a facial cycle of  $G$  with  $r \neq g$ , then there is an integer  $i \in \{0, \dots, r - 1\}$  such that the graph  $G'$  obtained from  $G$  by identifying  $v_{i-1}$  and  $v_{i+1}$  (subscripts are taken modulo  $r$ ) is still of unbalanced girth  $g$ .*

By repeated application of this lemma we conclude that:

**Corollary 3.2** *Given a planar even signed graph  $(G, \Sigma)$  of unbalanced girth at least  $g$ , there is a homomorphic image  $(H, \Sigma_1)$  of  $(G, \Sigma)$  such that: (i)  $H$  is planar, (ii)  $(H, \Sigma_1)$  is an even signed graph, (iii)  $(H, \Sigma_1)$  is of unbalanced girth  $g$  and (iv) every facial cycle of  $(H, \Sigma_1)$  is an unbalanced cycle of length  $g$ .*

We are now ready to prove the main theorem of the paper.

**Theorem 3.3** *The following two statements are equivalent:*

- (i) *Every planar  $2g$ -regular multigraph with no odd edge-cut of less than  $2g$  edges is  $2g$ -edge-colorable.*
- (ii) *Every planar even signed graph of unbalanced girth at least  $2g$  admits a homomorphism to  $\mathcal{SPC}_{2g-1}$ .*

**Proof.** First assume that every planar even signed graph of unbalanced girth at least  $2g$  admits a homomorphism to  $\mathcal{SPC}_{2g-1}$  and let  $G$  be a planar  $2g$ -regular multigraph with no odd edge-cut of less than  $2g$  edges. Using Tutte's matching theorem we can easily verify that  $G$  admits a perfect matching. Let  $M$  be a perfect matching of  $G$ . Let  $G^D$  be the dual of  $G$  with respect to some embedding of  $G$  on the plane. Since  $G$  is  $2g$ -regular,  $G^D$  is clearly bipartite. Let  $M^D$  be the edges in  $G^D$  corresponding to the edges of  $M$ . It is now easy to check that  $(G^D, M^D)$  is a planar even signed graph of unbalanced girth  $2g$ . Therefore, by our main assumption,  $(G^D, M^D)$  admits a homomorphism to  $\mathcal{SPC}_{2g-1}$ . This mapping induces a  $2g$ -edge-coloring on  $G^D$  (not necessarily a proper edge-coloring) using colors  $e_1, \dots, e_{2g-1}, J$ . By Theorem 2.1 every unbalanced cycle has received exactly  $2g$  different colors. In particular each face of  $G^D$ , which is an unbalanced cycle of length  $2g$ , has received all  $2g$  colors. Thus reassigning these colors to their corresponding edges in  $G$  will result in a proper  $2g$ -edge-coloring of  $G$ .

Now we assume that every planar  $2g$ -regular multigraph with no odd edge-cut of less than  $2g$  edges is (properly)  $2g$ -edge-colorable. Let  $(G, \Sigma)$  be a plane even signed graph of unbalanced girth  $2g$ . We would like to prove that this signed graph admits a homomorphism to  $\mathcal{SPC}_{2g-1}$ . By Corollary 3.2 we may assume that each face of  $(G, \Sigma)$  is an unbalanced cycle of length exactly  $2g$ . Let  $G^D$  be the dual of  $G$  with respect to its embedding on the plane. Obviously  $G^D$  is a  $2g$ -regular multigraph, furthermore it is easy to check that  $G^D$  has no odd edge-cut of strictly less than  $2g$  edges (this is the dual of having unbalanced girth at least  $2g$ ). Thus, by our main

assumption,  $G^D$  is  $2g$ -edge colorable. Let  $M_i$  be one of the color classes, which, therefore, is a perfect matching. Let  $\Sigma_i$  be the edges of  $G$  corresponding to the edges of  $G^D$  in  $M_i$ . We claim that  $\Sigma_i$  is equivalent to  $\Sigma$ . This is the case because in both  $(G, \Sigma_i)$  and  $(G, \Sigma)$  each face is an unbalanced cycle, and any other cycle is unbalanced if and only if it bounds an odd number of faces. That means that the sets of unbalanced cycles in both signatures are the same and the claim follows by Theorem 1.2. To complete the proof note that we have partitioned edges of  $G$  into  $2g$  sets  $\Sigma_i$  each being a signature of  $(G, \Sigma)$ . Thus, by Theorem 2.3,  $(G, \Sigma)$  admits a homomorphism to  $\mathcal{SPC}_{2g-1}$ .  $\square$

Since Seymour's conjecture is verified up to  $k \leq 8$ , see [5], [2], [3] and [1], we conclude that:

**Corollary 3.4** *Every planar even signed graph of unbalanced girth 4 (6 and 8, respectively) admits a homomorphism to  $\mathcal{SPC}_3$  ( $\mathcal{SPC}_5, \mathcal{SPC}_7$ , respectively).*

Note that if  $G$  is a simple bipartite graph, then the unbalanced girth of  $(G, \Sigma)$  is at least 4. Furthermore, note that  $\mathcal{SPC}_3$  is isomorphic to  $(K_{4,4}, M)$  where  $M$  is a perfect matching of  $K_{4,4}$ . Therefore:

**Corollary 3.5** *Every planar even signed graph admits a homomorphism to  $(K_{4,4}, M)$ .*

Using Theorem 6.2 of [8] it follows that this corollary is stronger than the Four-Color Theorem. This fact, in the edge-coloring formulation, was already proved by P. Seymour [10].

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## References

- [1] M. Chudnovsky, K. Edwards and P. Seymour. Edge-colouring eight-regular planar graphs. Manuscript (2012), available at <http://arxiv.org/abs/1209.1176>.
- [2] Z. Dvořák, K. Kawarabayashi and D. Král'. Packing six T-joins in plane graphs. Manuscript (2010), available at <http://arxiv.org/abs/1009.5912>.
- [3] K. Edwards. Optimization and Packings of T-joins and T-cuts. M.Sc. Thesis, McGill University (2011).
- [4] B. Guenin. Packing odd circuit covers: A conjecture, manuscript.
- [5] B. Guenin. Packing T-joins and edge-colouring in planar graphs. *Mathematics of Operations Research*, to appear.
- [6] W. Klostermeyer, C.Q. Zhang.  $(2 + \epsilon)$ -coloring of planar graphs with large odd girth, *J. Graph Theory* 33 (2) (2000), 109–119.
- [7] R. Naserasr. Homomorphisms and edge-colourings of planar graphs. *J. Combin. Theory Ser. B* 97 (2007), no. 3, 394–400.
- [8] R. Naserasr, E. Rollová and É. Sopena. Homomorphisms of signed graphs, submitted. Available at <http://www.labri.fr/perso/sopena/Publications>.
- [9] R. Naserasr, E. Rollová and É. Sopena. Homomorphisms of planar signed graphs to signed projective cubes, manuscript.
- [10] P. Seymour. *Matroids, Hypergraphs and the Max.-Flow Min.-Cut Theorem*. D. Phil. Thesis, Oxford (1975), page 34.
- [11] T. Zaslavsky. Signed graphs. *Discrete Applied Math.* 4(1) (1982), 47–74.