

# Homomorphisms of signed bipartite graphs

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## Abstract

A signature on a graph  $G$  is an assignment of negative or positive sign to the edges. Resigning at a vertex  $v$  is to change the sign of all edges incident to  $v$ . Two signatures are equivalent if one can be obtained from the other by a sequence of resigning. The set of negative edges is normally denoted by  $\Sigma$ . A signed graph, denoted  $(G, \Sigma)$  is a graph  $G$  together with the set of signatures all equivalent to  $\Sigma$ .

A signed minor of  $(G, \Sigma)$  is a signed graph obtained from  $(G, \Sigma)$  by a sequence of deleting vertices or edges, contracting positive edges and resigning. Given two signed graphs  $(G, \Sigma)$  and  $(H, \Sigma_1)$  we say there is a homomorphism of  $(G, \Sigma)$  to  $(H, \Sigma_1)$  if there is a signature  $\Sigma' \equiv \Sigma$  and a mapping  $\phi : V(G) \rightarrow V(H)$  such that  $\phi$  preserves both adjacency and the sign of an edge with respect to  $\Sigma'$ .

Our aim is to study the homomorphism relation between signed graphs where the underlying graph  $G$  is bipartite. We show that this notion captures the notions of chromatic number and graph homomorphisms. In particular we will study Hadwiger's conjecture in this setting. We show that for small values of the chromatic number there are natural strengthening of this conjecture but such extensions will not work for larger chromatic numbers.

**Keywords.** Homomorphism, signed graphs, signed minor and Hadwiger's conjecture.

## 1 Homomorphisms

If there is a homomorphism of  $(G, \Sigma)$  to  $(H, \Sigma_1)$  we write  $(G, \Sigma) \rightarrow (H, \Sigma_1)$ . This relation is a quasi order on the class of all signed graphs. Thus we may use terms such as *bound* and *maximum*.

A cycle with only one, equivalently odd number of, negative edges is called an *unbalanced cycle* and is denoted by  $UC_k$  if the length of the cycle is  $k$ . The following is one of the first theorems in the theory of signed graph homomorphisms.

**Lemma 1.1** *There is a homomorphism of  $UC_k$  to  $UC_\ell$  if and only if  $k \geq \ell$  and  $k = \ell \pmod{2}$ .*

Let  $G$  be a graph; the signed graph  $S(G) = (G^*, \Sigma)$  is obtained by replacing each edge  $uv$  of  $G$  by an unbalanced 4-cycle on four vertices  $ux_{uv}vy_{uv}$ , where  $x_{uv}$  and  $y_{uv}$  are new and distinct vertices. See Figure 1 for an example.

Let  $(K_{k,k}, M)$  be the signed graph where edges of a perfect matching  $M$  are negative. The following theorem shows how to define  $\chi(G)$  using only the notion of homomorphism between signed bipartite graphs.

**Theorem 1.2** *For every  $k \geq 3$  and every graph  $G$ ,  $\chi(G) \leq k$  if and only if  $S(G) \rightarrow (K_{k,k}, M)$ .*

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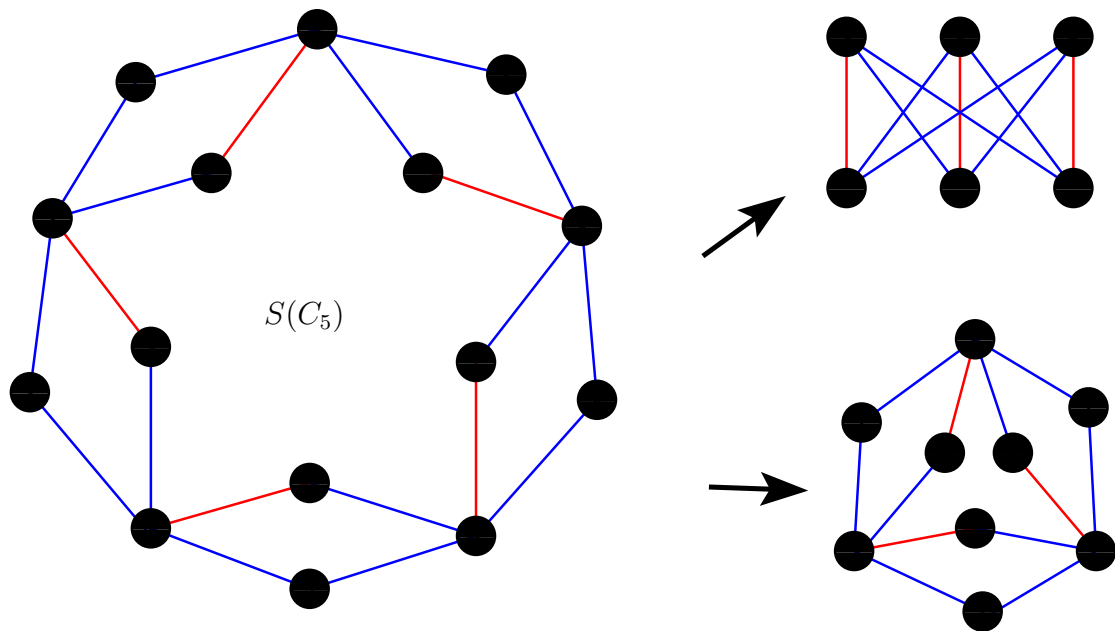


Figure 1:  $S(C_5) \rightarrow (K_{3,3}, M)$  and  $S(C_5) \rightarrow S(C_3)$

**Proof.** It would be enough to prove the theorem for connected graphs. Let  $\varphi : G \rightarrow K_k$  be a  $k$ -coloring of  $G$ . Label vertices in one part of  $K_{k,k}$  with vertices of  $K_k$  (or equivalently with  $k$  colors). We can then regard  $\varphi$  as a partial mapping of  $S(G)$  to  $(K_{k,k}, M)$ . We extend this mapping to the remaining vertices of  $S(G)$  as follows: for each pair  $u$  and  $v$  of adjacent vertices of  $G$ ,  $\varphi$  is extended to  $x_{uv}$  and  $y_{uv}$  in such a way that the image of the unbalanced cycle  $ux_{uv}vy_{uv}$  is an unbalanced 4-cycle in  $(K_{k,k}, M)$ . This is possible simply because  $k \geq 3$ . It is then straightforward to check that this extension is a homomorphism of  $S(G)$  to  $(K_{k,k}, M)$ .

For the converse, assume there is a homomorphism  $\phi$  of  $S(G) = (G^*, \Sigma)$  to  $(K_{k,k}, M)$ . Then  $\phi$  is, in particular, a homomorphism of the bipartite graph  $G^*$  to the complete bipartite graph  $K_{k,k}$ . In the bipartition of  $G^*$ , one part is formed by  $V(G)$  and the other part is the set of new vertices. Thus, the restriction of  $\phi$  on  $V(G)$  is a mapping of  $V(G)$  to  $k$  vertices of one side of  $K_{k,k}$ . Furthermore, if  $uv$  is an edge of  $G$ , then  $u$  and  $v$  must be mapped to distinct vertices because of the unbalanced 4-cycle  $ux_{uv}vy_{uv}$ . Hence this restriction of  $\phi$  is a  $k$ -coloring of  $G$ .  $\square$

In a similar way we show below that the problem of the existence of a homomorphism of a graph  $G$  into a graph  $H$  is captured by the notion of homomorphism between signed bipartite graphs. For a comprehensive study of graph homomorphisms we refer to [3].

**Theorem 1.3** *For every two graphs  $G$  and  $H$ ,  $G \rightarrow H$  if and only if  $S(G) \rightarrow S(H)$ .*

**Proof.** Any homomorphism of  $G$  to  $H$  can easily be extended to a homomorphism of  $S(G)$  to  $S(H)$ .

For the converse, suppose that  $\phi$  is a homomorphism of  $S(G)$  to  $S(H)$ . If  $G$  has no edge, then there is nothing to prove. If  $G$  is bipartite with at least one edge, then  $H$  must also have at least one edge for  $\phi$  to exist and, therefore,  $G$  maps to  $H$ . Thus we may assume  $G$  has at least one odd cycle. Furthermore, we may assume that both  $G$  and  $H$  are connected as we can easily compose homomorphisms on connected components.

We claim that in the mapping  $\phi$  from  $V(S(G))$  to  $V(S(H))$  the set  $V(G)$  must be mapped into  $V(H)$ . Since  $V(G)$  is a part in the bipartition of  $S(G)$ , and  $V(H)$  is a part in the bipartition of  $V(S(H))$ , and since  $G$  and  $H$  are both connected,  $\phi$  either maps all vertices in  $V(G)$  to vertices in  $V(H)$  or none of them. Let  $C_{2r+1}$  be an odd cycle of  $G$  and let  $w_1, \dots, w_{2k+1}$  be its vertices, connected in this cyclic order. To complete the proof of our claim we show that vertices of  $C_{2k+1}$

must be mapped into  $V(H)$ . By contradiction suppose that a vertex  $w_i$  of  $C_{2k+1}$  is mapped to a vertex of the form  $x_{uv}$  in  $S(H)$ . Then, because of the unbalanced 4-cycle associated to the edge  $w_i w_{i+1}$  (addition of the index is taken mod  $2k+1$ ) in  $S(G)$ ,  $w_{i+1}$  is mapped to the vertex  $y_{uv}$ . Continuing this process we obtain a 2-coloring of  $C_{2k+1}$  using  $x_{uv}$  and  $y_{uv}$  which is a contradiction.

Thus  $\phi$  maps  $V(G)$  to  $V(H)$ . To show that the restriction of  $\phi$  on these two sets is a homomorphism of  $G$  to  $H$ , let  $uv$  be an edge in  $G$  and let  $UC'$  be the unbalanced 4-cycle associated with this edge in  $S(G)$ . The image of  $UC'$  under  $\phi$  then must be another unbalanced 4-cycle containing  $\phi(u), \phi(v)$  and the other two vertices must be vertices not from  $V(H)$ . This means we have constructed an unbalanced 4-cycle on  $\phi(u)$  and  $\phi(v)$ , but the condition for having such a cycle is to have an edge between  $\phi(u)$  and  $\phi(v)$  in  $H$ . Hence  $\phi$  induces a homomorphism of  $G$  to  $H$ .  $\square$

## 2 Minors

We prove the following minor relation between graphs and their corresponding signed bipartite graphs:

**Theorem 2.1** *For every integer  $n$  and every graph  $G$ ,  $G$  has a  $K_n$ -minor if and only if  $S(G)$  has a  $(K_n, \Sigma)$ -minor for some  $\Sigma$  (equivalently for any  $\Sigma$ ).*

**Proof.** First assume  $(K_n, \Sigma)$  is a signed minor of  $S(G)$  for some  $\Sigma$ . We would like to prove that  $K_n$  is a minor of  $G$ . This is clear for  $n = 1, 2$ . So we assume  $n \geq 3$ . Thus, in producing  $(K_n, \Sigma)$  as a signed minor of  $S(G)$  each vertex of degree 2 in  $S(G)$  is either deleted or identified with one of its neighbours as a result of contracting an incident edge. We define a minor of  $G$  as follows: For each edge  $uv$  of  $G$ , if the corresponding unbalanced 4-cycle is deleted in the process of producing  $(K_n, \Sigma)$  as a signed minor of  $S(G)$ , then delete  $uv$ . If  $u$  and  $v$  are identified through contraction of edges in producing  $(K_n, \Sigma)$  as a signed minor of  $S(G)$ , then contract the edge  $uv$ . Otherwise  $uv$  remains an edge. The resulting minor then must be  $K_n$ .

For the opposite direction, suppose  $K_n$  is a minor of  $G$ . Let  $uv$  be an edge of  $G$ . If the edge  $uv$  is deleted in producing  $K_n$ -minor from  $G$ , then delete all the four edges of corresponding unbalanced 4-cycle. If  $uv$  is contracted, then contract two positive edges of the corresponding unbalanced 4-cycle in  $S(G)$  in such a way that  $u$  and  $v$  are identified after these contractions and delete the other two edges of the unbalanced 4-cycle. Otherwise contract two positive edges of the corresponding unbalanced 4-cycle in such a way that there are two new parallel edges between  $u$  and  $v$ , one positive and one negative. Finally delete all isolated vertices. By allowing multiple edges at the end of this process we get a signed minor of  $S(G)$  which has  $n$  vertices and for each pair  $x$  and  $y$  of vertices two  $xy$  edges, one positive and one negative. For each such pair we delete the negative edge unless  $xy \in \Sigma$  in which case we delete the positive edge. The result is  $(K_n, \Sigma)$  obtained as a signed minor of  $G$ .  $\square$

## 3 Hadwiger's conjecture for signed bipartite graphs

**Conjecture 3.1 (Hadwiger, [2])** *If a graph  $G$  has no  $K_n$ -minor, then it is  $(n-1)$ -colorable.*

By Theorem 1.2, Hadwiger's conjecture can be restated as follows:

**Conjecture 3.2** *Given  $n \geq 4$ , the class  $\mathcal{C} = \{S(G) \mid G \text{ is } K_n\text{-minor-free}\}$  of signed bipartite graphs is bounded by  $(K_{n-1, n-1}, M)$  in the signed graph homomorphism order.*

If the conjecture holds, then the next question would be: what is a natural superclass of  $\mathcal{C}$  which is still bounded by  $(K_{n-1, n-1}, M)$ ?

Hadwiger's conjecture is known to be true for  $n \leq 6$ , thus Conjecture 3.2 is also true for  $n \leq 6$ . For  $n = 4$  we have the following generalization.

**Theorem 3.3** *If  $G$  is a bipartite graph with no  $K_4$ -minor and  $\Sigma$  is any signature on  $G$ , then  $(G, \Sigma) \rightarrow (K_{3,3}, M)$ .*

**Proof.** By adding more edges, if needed, we may assume that  $G$  is edge maximal with respect to being bipartite and having no  $K_4$ -minor. Obviously it is enough to prove the theorem for such edge maximal graphs.

As mentioned before, a classical decomposition theorem for edge-maximal  $K_4$ -minor-free graphs states that every such graph is built from a sequence of triangles starting by one triangle and pasting each new triangle to the graph previously built along an edge. To use the decomposition theorem we add new edges to  $G$ , of green color, until we reach a maximal  $K_4$ -minor-free graph  $G'$ , which obviously is not bipartite anymore. Let  $G''$  be the edge-colored graph obtained from  $G'$  by coloring original positive edges of  $(G, \Sigma)$  in blue, original negative edges of  $(G, \Sigma)$  in red and keeping the green color for edges not in  $G$ .

We claim that there is no triangle in  $G''$  with exactly two green edges. To see this, suppose that  $v_1v_2$  and  $v_1v_3$  are both green and that  $v_2v_3$  is an edge of  $G$ . Since  $G$  is bipartite  $v_2$  and  $v_3$  are in two different parts and thus  $v_1$  is in a different part with respect to one of them. Without loss of generality assume  $v_1$  and  $v_2$  are in different parts. Consider the graph  $G + \{v_1v_2\}$ . By the choice of  $v_2$  this graph is bipartite and since it is a subgraph of  $G'$ , it has also no  $K_4$ -minor but this contradicts the edge maximality of  $G$ .

We now build a new edge-colored graph  $F$  from  $(K_{3,3}, M)$ . The blue and red edges of  $F$  are defined as before and we add green edges between every pair of vertices non adjacent in  $(K_{3,3}, M)$ . The edge-colored graph  $F$  has three types of triangles: (i) triangles with three green edges, (ii) triangles with one green edge and two blue edges, and (iii) triangles with no two edges of the same color. Furthermore it is not hard to verify that each red edge only belongs to triangles of type (iii), each blue edge belongs to triangles of type (ii) or (iii) and each green edge is contained in triangles of each of the three types.

To prove the theorem we now prove the following stronger statement: there exists a suitable “resigning”  $G^*$  of  $G''$  such that  $G^*$  admits a color-preserving homomorphism to  $F$ . By resigning here we mean exchanging the colors red and blue on edges of an edge cut, this can be regarded as a sequence of vertex resigning.

To prove this stronger statement, let  $T_1, \dots, T_k$  be the sequence of triangles obtained from the decomposition of  $G''$  mentioned above. Note that since  $G$  was bipartite, each such triangle contains a green edge. Consider the triangle  $T_1$ . Either it is one of the three types (i), (ii) or (iii), in which case we simply map it to  $F$ , or it has one green and two red edges. Let  $u$  be the common vertex of these two red edges. After resigning at  $u$  we have a triangle of type (ii) and thus we can map it to  $F$ .

By induction, assume now that the graph  $G''_i$ , obtained by pasting the triangles  $T_1, \dots, T_i$ ,  $i < k$ , is mapped to  $F$  and assume that  $T_{i+1}$  is pasted to  $G''_i$  along the edge  $e$ . Let  $v$  be the vertex of  $T_{i+1}$  not incident to  $e$ . If  $T_{i+1}$  is a triangle of one the three types, because of the above mentioned property of  $F$ , we can extend the mapping of  $G''_i$  to  $G''_{i+1}$ , where the colors of the two edges of  $T_{i+1}$  incident with  $v$  are preserved. Otherwise  $T_{i+1}$  has exactly two red edges and one green edge. By resigning at  $v$  we get a triangle that has either one or no red edge, thus obtaining a triangle of type (ii) or (iii). We now extend the homomorphism thanks to the properties of  $F$ . In this process, resigning a vertex would be done at most once, when it is added to the already built part of the graph, so our process is well-defined and the stronger claim is proved.  $\square$

We note that our proof has an algorithmic feature. Given a signed bipartite graph  $(G, \Sigma)$ , where  $G$  is a  $K_4$ -minor-free graph, we can find, in polynomial time, a homomorphism of  $(G, \Sigma)$  to  $(K_{3,3}, M)$ .

Furthermore, we believe that the following stronger statement should also be true:

**Conjecture 3.4** *If  $G$  is bipartite and  $(G, \Sigma)$  has no  $(K_4, E(K_4))$  as a signed minor, then  $(G, \Sigma) \rightarrow (K_{3,3}, M)$ .*

For  $n = 5$  it is shown in [4], using the four-color theorem and a result of [1], that the following holds.

**Theorem 3.5** *If  $G$  is a bipartite planar graph and  $\Sigma$  is any signature on  $G$ , then  $(G, \Sigma) \rightarrow (K_{4,4}, M)$ .*

For large values of  $n$  ( $n \geq 7$ ) Conjecture 3.2 does not extend so nicely. To show this we use the following signed bipartite graph, *Fano*. That is signed graph on  $K_{7,7}$  where vertices on one side are labeled with points of Fano plane and on the other side with lines of the Fano plane. An edge is negative if it connects a line to a point of a line.

**Theorem 3.6** *There exists no value of  $n$  for which *Fano* admits a homomorphism to  $(K_{n,n}, M)$ .*

A proof can be obtained mainly by counting number of distinct copies of  $UC_4$  containing a given edge. For more details we refer to [4]. The following then is an immediate corollary.

**Corollary 3.7** *The class  $\mathcal{C} = \{(G, \Sigma) \mid G \text{ is bipartite and has no } H\text{-minor}\}$  is not bounded by  $(K_{n,n}, M)$  (for no values of  $n$ ) if  $H$  is a graph on at least 15 vertices.*

This shows that for  $n \geq 15$  the reformulation of Hadwiger's conjecture given in Conjecture 3.2 cannot be extended to a general minor closed class of signed bipartite graphs. Even though such an extension was possible for small values of  $n$ .

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