

Outerplanar and planar oriented cliques

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Abstract

The clique number of an undirected graph G is the maximum order of a complete subgraph of G and is a well-known lower bound for the chromatic number of G . Every proper k -coloring of G may be viewed as a homomorphism (an edge-preserving vertex mapping) of G to the complete graph of order k . By considering homomorphisms of oriented graphs (digraphs without cycles of length at most 2), we get a natural notion of (oriented) colorings and oriented chromatic number of oriented graphs. An oriented clique is then an oriented graph whose number of vertices and oriented chromatic number coincide. However, the structure of oriented cliques is much less understood than in the undirected case.

In this paper, we study the structure of outerplanar and planar oriented cliques. We first provide a list of 11 graphs and prove that an outerplanar graph can be oriented as an oriented clique if and only if it contains one of these graphs as a spanning subgraph. Klostermeyer and MacGillivray conjectured that the order of a planar oriented clique is at most 15, which was later proved by Sen. We show that any planar oriented clique on 15 vertices must contain a particular oriented graph as a spanning subgraph, thus reproving the above conjecture. We also provide tight upper bounds for the order of planar oriented cliques of girth k for all $k \geq 4$.

1 Introduction and statement of results

An *oriented graph* is a digraph with no cycle of length 1 or 2. By replacing each edge of a simple graph G with an arc (ordered pair of vertices) we obtain an oriented graph \vec{G} ; we say that \vec{G} is an *orientation* of G and that G is the *underlying graph* of \vec{G} . We denote the underlying graph of \vec{G} by $UG(\vec{G})$ or simply G when there is no chance of confusion. We denote by $V(\vec{G})$ and $A(\vec{G})$ the set of vertices and arcs of \vec{G} , respectively. An arc (u, v) (where u and v are vertices) is denoted by \vec{uv} . Two vertices u and v of an oriented graph are *adjacent* if they are linked by an arc.

A *directed 2-path*, or a *2-dipath*, is a directed path on three vertices \vec{uvw} . Here u and w are the *terminal* vertices and v is the *internal* vertex of \vec{uvw} . More generally, a 2-dipath with terminal vertices u, w and internal vertex v is denoted by uvw (this denotes either the 2-dipath \vec{uvw} or the 2-dipath \vec{wvu}).

Colorings of oriented graphs first appeared in the work of Courcelle [3] on the monadic second order logic of graphs. Since then, it has been considered by many researchers, following the work of Raspaud and Sopena [9] on oriented colorings of planar graphs.

An *oriented k -coloring* [12] of an oriented graph \vec{G} is a mapping ϕ from $V(\vec{G})$ to the set $\{1, 2, \dots, k\}$ such that:

- (i) $\phi(u) \neq \phi(v)$ whenever u and v are adjacent, and
- (ii) if \vec{uv} and \vec{wx} are two arcs in \vec{G} , then $\phi(u) = \phi(x)$ implies $\phi(v) \neq \phi(w)$.

We say that an oriented graph \vec{G} is *k -colorable* whenever it admits an oriented k -coloring. The *oriented chromatic number* $\chi_o(\vec{G})$ of an oriented graph \vec{G} is the smallest integer k such that \vec{G} is k -colorable.

Alternatively, one can define the oriented chromatic number by means of homomorphisms of oriented graphs. Let \vec{G} and \vec{H} be two oriented graphs. A homomorphism of \vec{G} to \vec{H} is a mapping $\phi : V(\vec{G}) \rightarrow V(\vec{H})$ which preserves the arcs, that is, $uv \in A(\vec{G})$ implies $\phi(u)\phi(v) \in A(\vec{H})$. The *oriented chromatic number* $\chi_o(\vec{G})$ of an oriented graph \vec{G} is then the minimum order (number of vertices) of an oriented graph \vec{H} such that \vec{G} admits a homomorphism to \vec{H} .

Notice that the terminal vertices of a 2-dipath must receive distinct colors in every oriented coloring because of the second condition of the definition. In fact, for providing an oriented coloring of an oriented graph, only the pairs of vertices which are either adjacent or connected by a 2-dipath must receive distinct colors (that is, for every two non-adjacent vertices u and v which are not linked by a 2-dipath, there exists an oriented coloring which assigns the same color to u and v).

Motivated by this observation, Klostermeyer and MacGillivray [6] introduced the following notion. An *absolute oriented clique*, or simply an *oclique*, is an oriented graph \vec{G} for which $\chi_o(\vec{G}) = |V(\vec{G})|$. Note that ocliques can hence be characterized as those oriented graphs whose any two distinct vertices are either adjacent or connected by a 2-dipath in either direction. In that case we say that the vertices are at *weak directed distance* at most 2 from each other. Clearly, an oriented graph with an oclique of order n as an oriented subgraph has oriented chromatic number at least n . The *absolute oriented clique number* $\omega_{ao}(\vec{G})$ of an oriented graph \vec{G} is the maximum order of an oclique contained in \vec{G} as a subgraph.

The oriented chromatic number $\chi_o(G)$ (resp. absolute oriented clique number $\omega_{ao}(G)$) of a simple graph G is the maximum of the oriented chromatic numbers (resp. absolute oriented clique numbers) of all the oriented graphs with underlying graph G . The oriented chromatic number $\chi_o(\mathcal{F})$ (resp. absolute oriented clique number $\omega_{ao}(\mathcal{F})$) of a family \mathcal{F} of graphs is the maximum of the oriented chromatic numbers (resp. absolute oriented clique numbers) of the graphs from the family \mathcal{F} .

It is easy to observe that the oriented graphs of smallest order to which an oriented graph admits a homomorphism to are always ocliques. Therefore, in order to determine the oriented chromatic number of an oriented graph, it is enough to consider homomorphisms to ocliques.

From the definitions above, clearly we have the following:

Lemma 1.1. *For any oriented graph \vec{G} , $\omega_{ao}(\vec{G}) \leq \chi_o(\vec{G})$.*

One of the first major results proved regarding the oriented chromatic number of planar graphs is the following by Raspaud and Sopena [9].

Theorem 1.2 (Raspaud and Sopena, 1994). *Every planar graph has oriented chromatic number at most 80.*

In the same paper, they also proved that every oriented forest is 3-colorable.

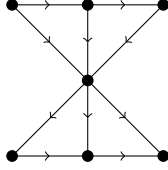


Figure 1: The outerplanar oclique \vec{O} of order 7.

Theorem 1.3 (Raspaud and Sopena, 1994). *Every forest has oriented chromatic number at most 3.*

Later, Sopena [11] proved that every oriented outerplanar graph is 7-colorable and provided an example of an outerplanar oclique of order 7 (Figure 1) to prove the tightness of the result.

Theorem 1.4 (Sopena, 1997). *Every outerplanar graph has oriented chromatic number at most 7.*

The structure of ocliques is much less understood than in the undirected case, where a clique is nothing but a complete graph. For instance, the exact value of the minimum number of arcs in an oclique of order k is not known yet. Füredi, Horak, Pareek and Zhu [4], and Kostochka, Łuczak, Simonyi and Sopena [7], independently proved that this number is $(1 + o(1))k \log_2 k$.

The questions related to the absolute oriented clique number of planar graphs have been first asked by Klostermeyer and MacGillivray [6] in 2002. In their paper they asked: “what is the maximum order of a planar oclique?”, which is equivalent to asking “what is the absolute oriented clique number of planar graphs?”. In order to find the answer to this question, Sopena [13] found a planar oclique of order 15 (Figure 3) while Klostermeyer and MacGillivray [6] showed that there is no planar oclique of order more than 36, improving the upper bound of 80 which can be obtained by using Lemma 1.1 and Theorem 1.2, and conjectured that the maximum order of a planar oclique is 15. Later in 2011, the conjecture was positively settled [10] and we will state it as Theorem 1.8(a) in this article.

Klostermeyer and MacGillivray also showed that any outerplanar oclique of order 7 must contain a particular unique spanning subgraph (oriented).

Theorem 1.5 (Klostermeyer and MacGillivray, 2002). *An oriented outerplanar graph of order at least 7 is an oclique if and only if it contains the outerplanar oclique \vec{O} depicted in Figure 1 as a spanning subgraph.*

Bensmail, Duvignau and Kirgizov [1] showed that given an undirected graph G it is NP-hard to decide if G has an orientation \vec{G} such that \vec{G} is an oclique (the similar problem, but using the directed distance instead of the *weak* directed distance — that is, the length of a shortest directed path in either direction —, was shown to be also NP-complete by Chvátal and Thomassen in [2]). Now it is easy to notice from Theorem 1.5 that an undirected outerplanar graph of order at least 7 can be oriented as an oclique if and only if it contains the graph O (the underlying undirected graph of the oriented graph \vec{O} from Figure 1) as a spanning subgraph. We extend this idea to characterize every outerplanar graph that can be oriented as an oclique in the following result.

Theorem 1.6. *An undirected outerplanar graph can be oriented as an oclique if and only if it contains one of the graphs depicted in Figure 2 as a spanning subgraph.*

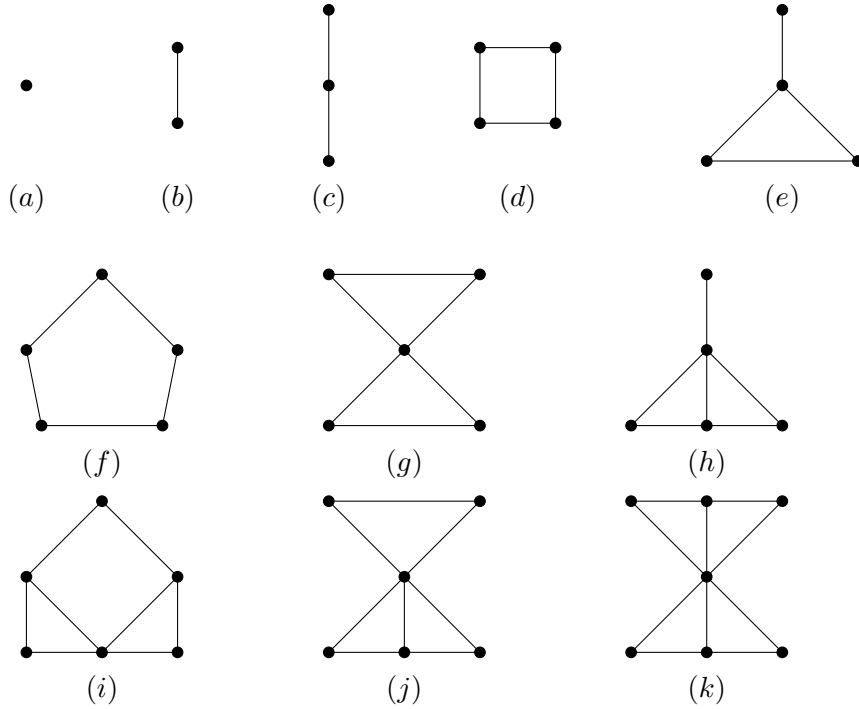


Figure 2: List of edge-minimal oclique spanning subgraphs of all outerplanar ocliques

We also prove a result similar to Theorem 1.5 for planar graphs which implies Theorem 1.8(a) (that is, the absolute oriented clique number of the family of planar graphs is 15).

Theorem 1.7. *A planar oclique has order at most 15 and every planar oclique of order 15 contains the planar oclique \vec{P} depicted in Figure 3 as a spanning subgraph.*

The question regarding the upper bound for the absolute oriented clique number of the families of planar graphs with given girth (length of the smallest cycle in a graph) is also of interest and was asked by Klostermeyer and MacGillivray in [6]. We answer these questions and provide tight bounds.

Let \mathcal{P}_k denote the family of planar graphs with girth at least k . We will prove the following.

Theorem 1.8.

- (a) $\omega_{ao}(\mathcal{P}_3) = 15$.
- (b) $\omega_{ao}(\mathcal{P}_4) = 6$.
- (c) $\omega_{ao}(\mathcal{P}_5) = 5$.
- (d) $\omega_{ao}(\mathcal{P}_k) = 3$ for $k \geq 6$.

In Section 2 we fix the notation to be used in this article and state some useful results. We also define the relative oriented clique number of an oriented graph, which will be used later in a proof. In Section 3, 4 and 5 we prove Theorem 1.6, 1.7 and 1.8, respectively. Finally, we mention in Section 6 some future directions for research on this topic.

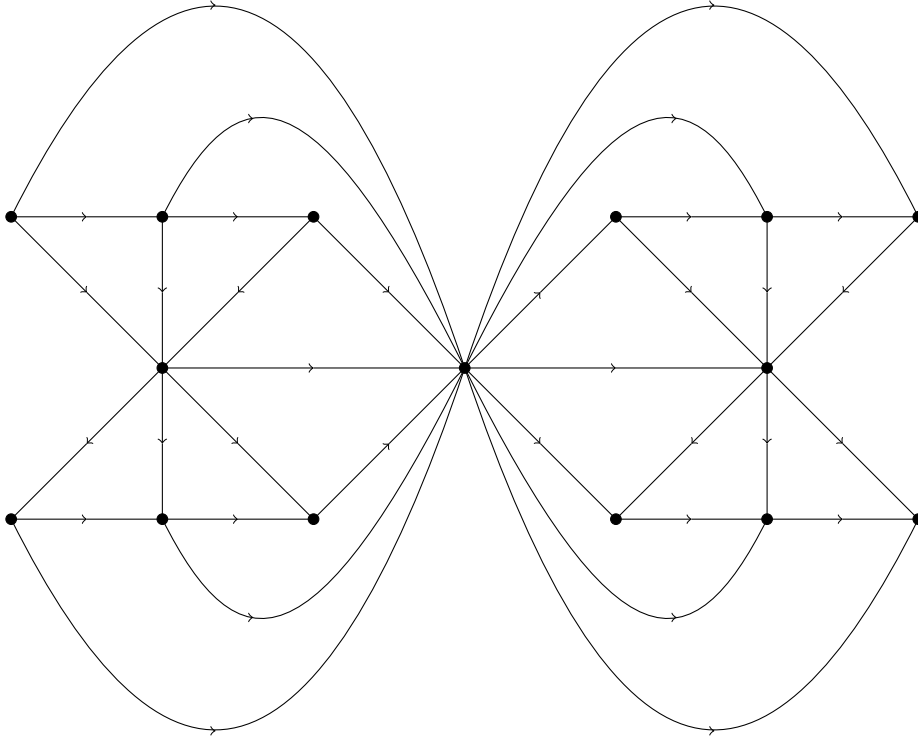


Figure 3: The planar oclique \vec{P} of order 15.

2 Preliminaries

For an oriented graph \vec{G} , every parameter we introduce below is denoted using \vec{G} as a subscript. In order to simplify notation, this subscript will be dropped whenever there is no chance of confusion.

The set of all adjacent vertices of a vertex v in an oriented graph \vec{G} is called its set of *neighbors* and is denoted by $N_{\vec{G}}(v)$. If \vec{uv} is an arc, then u is an *in-neighbor* of v and v is an *out-neighbor* of u . The set of all in-neighbors and the set of all out-neighbors of v are denoted by $N_{\vec{G}}^-(v)$ and $N_{\vec{G}}^+(v)$, respectively. The *degree* of a vertex v in an oriented graph \vec{G} , denoted by $\deg_{\vec{G}}(v)$, is the number of neighbors of v in \vec{G} . Naturally, the *in-degree* (resp. *out-degree*) of a vertex v in an oriented graph \vec{G} , denoted by $\deg_{\vec{G}}^-(v)$ (resp. $\deg_{\vec{G}}^+(v)$), is the number of in-neighbors (resp. out-neighbors) of v in \vec{G} . The *order* $|\vec{G}|$ of an oriented graph \vec{G} is the cardinality of its set of vertices $V(\vec{G})$.

We say that two vertices u and v of an oriented graph *agree* on a third vertex w of that graph if $w \in N^\alpha(u) \cap N^\alpha(v)$ for some $\alpha \in \{+, -\}$ and that they *disagree* on w if $w \in N^\alpha(u) \cap N^\beta(v)$ for $\{\alpha, \beta\} = \{+, -\}$.

A *directed path* of length k , or a *k-dipath*, from v_0 to v_k is an oriented graph with vertices v_0, v_1, \dots, v_k and arcs $\vec{v_0v_1}, \vec{v_1v_2}, \dots, \vec{v_{k-1}v_k}$ where v_0 and v_k are the *terminal* vertices and

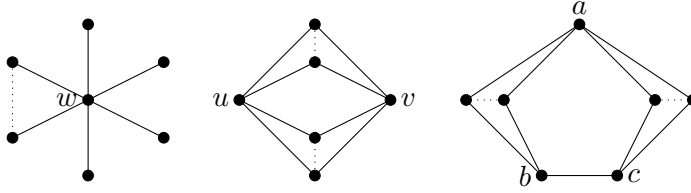


Figure 4: List of all triangle-free planar graphs with diameter 2 (Plesník (1975)).

v_1, \dots, v_{k-1} are *internal* vertices. A *directed cycle* of length k , or a *directed k -cycle*, is an oriented graph with vertices v_1, v_2, \dots, v_k and arcs $\overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, \dots, \overrightarrow{v_{k-1}v_k}$ and $\overrightarrow{v_kv_1}$.

The *directed distance* $\vec{d}_{\vec{G}}(u, v)$ between two vertices u and v of an oriented graph \vec{G} is the smallest length of a directed path from u to v in \vec{G} . We let $\vec{d}_{\vec{G}}(u, v) = \infty$ if no such directed path exists. The *weak directed distance* $\bar{d}_{\vec{G}}(u, v)$ between u and v is then given by $\bar{d}_{\vec{G}}(u, v) = \min\{\vec{d}_{\vec{G}}(u, v), \vec{d}_{\vec{G}}(v, u)\}$.

Let now G be an undirected graph. A *path* of length k , or a *k -path*, from v_0 to v_k is a graph with vertices v_0, v_1, \dots, v_k and edges $v_0v_1, v_1v_2, \dots, v_{k-1}v_k$. The *distance* $d_G(x, y)$ between two vertices x and y of G is the smallest length of a path connecting x and y . The *diameter* $\text{diam}(G)$ of a graph G is the maximum distance between pairs of vertices of the graph.

Triangle-free graphs with diameter 2 have been characterized by Plesník in [8].

Theorem 2.1 (Plesník, 1975). *The triangle-free graphs with diameter 2 are precisely the graphs listed in Figure 4.*

The graphs depicted in Figure 4 are the stars, the complete bipartite graphs $K_{2,n}$ for some positive integer n , and the graph obtained by adding copies of two non-adjacent vertices of the 5-cycle.

A vertex subset D is a *dominating set* of a graph G if every vertex of G is either in D or adjacent to a vertex of D . The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G .

We now define a new parameter for oriented graphs which will be used in our proof. A *relative oriented clique* of an oriented graph \vec{G} is a set $R \subseteq V(\vec{G})$ of vertices such that any two vertices from R are at weak directed distance at most 2 in \vec{G} . The *relative oriented clique number* $\omega_{ro}(\vec{G})$ of an oriented graph \vec{G} is the maximum order of a relative oriented clique of \vec{G} .

The relative oriented clique number $\omega_{ro}(G)$ of a simple graph G is the maximum of the relative oriented clique numbers of all the oriented graphs with underlying graph G . The relative oriented clique number $\omega_{ro}(\mathcal{F})$ of a family \mathcal{F} of graphs is the maximum of the relative oriented clique numbers of the graphs from the family \mathcal{F} .

From the definitions, we clearly have the following extension of Lemma 1.1.

Lemma 2.2. *For any oriented graph \vec{G} , $\omega_{ao}(\vec{G}) \leq \omega_{ro}(\vec{G}) \leq \chi_o(\vec{G})$.*

The relative oriented clique number of outerplanar graphs is at most 7 and this bound is tight.

Theorem 2.3. *Let \mathcal{O} be the family of outerplanar graphs. Then, $\omega_{ro}(\mathcal{O}) = 7$.*

Proof. Note that the oriented outerplanar graph depicted in Figure 1 is an oclique. Hence, by Theorem 1.4 and Lemma 2.2 the result follows. \square

3 Proof of Theorem 1.6

It is easy to verify that each graph in Figure 2 is outerplanar and can be oriented as an oclique. If G is an outerplanar graph which contains any of the graphs from Figure 2 as a spanning subgraph, then we can orient the edges of that spanning subgraph to obtain an oclique and orient all the other edges arbitrarily. Note that with such an orientation G is an outerplanar oclique. This proves the “if” part.

For the “only if” part, first note that a graph cannot be oriented as an oclique if it is disconnected. Now there are only two connected graphs with at most two vertices, namely, the complete graphs K_1 (single vertex) and K_2 (an edge). Both of them are outerplanar and can be oriented as ocliques. Hence, any outerplanar graph on at most two vertices that can be oriented as an oclique must contain one of the graphs depicted in Figure 2(a) and Figure 2(b) as a spanning subgraph.

If G is connected and has 3 vertices it must contain a 2-path as a spanning subgraph. We know that a 2-path is outerplanar and can be oriented as an oclique. Hence, any outerplanar graph on three vertices that can be oriented as an oclique must contain a 2-path (Figure 2(c)) as a spanning subgraph.

If G has at least 4 vertices and is a minimal (with respect to spanning subgraph inclusion) outerplanar graph that can be oriented as an oclique, then $\Delta(G) \geq 2$ as every oriented tree is 3-colorable. Now we will do a case analysis to prove the remaining part.

For the remainder of the proof assume that $V(G) = \{v_1, v_2, v_3, \dots, v_{|G|}\}$ where G is a minimal (with respect to spanning subgraph inclusion) outerplanar graph of order at least 4 that can be oriented as an oclique.

- (i) **For $|G| = 4$ and $\Delta(G) = 2$:** A triangle would force one vertex to have degree zero and hence G can not be oriented as an oclique. So, G must contain a 4-cycle (Figure 2(d)).
- (ii) **For $|G| = 4$ and $\Delta(G) = 3$:** G can not be $K_{1,3}$ since it is 3-colorable (as it is a tree). So we need to add at least one more edge to it. By adding one more edge to it, without loss of generality, we obtain the graph depicted in Figure 2(e).
- (iii) **For $|G| = 5$ and $\Delta(G) = 2$:** G must contain a C_5 (Figure 2(e)) as any other connected graph on five vertices is either a tree or has maximum degree greater than 2.
- (iv) **For $|G| = 5$ and $\Delta(G) = 3$:** Without loss of generality assume that $\deg(v_1) = 3$ and $N(v_1) = \{v_2, v_3, v_4\}$. If $|N(v_1) \cap N(v_5)| < 2$, then there cannot be a 2-dipath between v_5 and a vertex from $N(v_1) \setminus N(v_5)$. If $|N(v_1) \cap N(v_5)| > 2$, that is, $N(v_1) = N(v_5)$, then G will contain $K_{2,3}$ as a subgraph which contradicts the outerplanarity of G . Hence we must have $|N(v_1) \cap N(v_5)| = 2$.

Without loss of generality assume that $N(v_5) = \{v_2, v_3\}$. Since $\Delta(G) = 3$, v_4 must be adjacent to either v_2 or v_3 to allow a 2-dipath between v_4 and v_5 . But then C_5 is a spanning subgraph of G contradicting the minimality of G .

- (v) **For $|G| = 5$ and $\Delta(G) = 4$:** First note that G cannot have four edges as then it will be a tree and hence 3-colorable. Assume first that G has five edges. Without loss of generality assume that $N(v_1) = \{v_2, v_3, v_4, v_5\}$ and that the fifth edge in G is v_2v_3 . Now assume the arc $\overrightarrow{v_2v_1}$ without loss of generality. This implies the arcs $\overrightarrow{v_1v_4}$, $\overrightarrow{v_1v_5}$ to have a 2-dipath between the pair of vertices $\{v_2, v_4\}$ and $\{v_2, v_5\}$ respectively. But then it is not possible

to connect v_4 and v_5 by a 2-dipath. Therefore, G has at least six edges. If G has at least six edges, then G will contain the graph depicted either in Figure 2(g) or in Figure 2(h) as a spanning subgraph.

- (vi) **For $|G| = 6$ and $\Delta(G) = 2$:** The only two connected graphs on six vertices with $\Delta(G) = 2$ are the cycle C_6 on six vertices and the path on six vertices. None of them can be oriented as an oclique as the oriented chromatic number of trees and cycles is bounded above by 3 and 5, respectively.
- (vii) **For $|G| = 6$ and $\Delta(G) = 3$:** Let $N(v_1) = \{v_2, v_3, v_4\}$. Since G is an outerplanar graph we have $|N(v_1) \cap N(v_i)| \leq 2$ for $i \in \{5, 6\}$. Now let $|N(v_1) \cap N(v_5)| = 0$. Then v_2, v_3 and v_4 must be connected by 2-dipaths to v_5 through v_6 contradicting $|N(v_1) \cap N(v_6)| \leq 2$. Hence $|N(v_1) \cap N(v_5)| \in \{1, 2\}$.

First assume that $|N(v_1) \cap N(v_5)| = 1$ and without loss of generality let $N(v_1) \cap N(v_5) = \{v_2\}$. Since $\Delta(G) = 3$, v_2 can be adjacent to at most one of the vertices from $\{v_3, v_4\}$. If v_2 is adjacent to exactly one vertex from $\{v_3, v_4\}$, say v_3 without loss of generality, then v_4 must be connected to v_5 by a 2-dipath through v_6 . Now for having weak directed distance at most 2 between v_3 and v_6 we must either have the edge v_3v_4 or have the edge v_3v_6 (it is not possible to have the edge v_2v_6 as $\Delta(G) = 3$) creating a K_4 -minor or a $K_{2,3}$ respectively, contradicting the outerplanarity of G . Hence, v_2 is non-adjacent to both v_3 and v_4 . In that case, v_5 must be connected to v_3 and v_4 by 2-dipaths through v_6 . This will create a $K_{2,3}$ -minor in G contradicting its outerplanarity.

Now assume that $|N(v_1) \cap N(v_5)| = 2$ and let $N(v_1) \cap N(v_5) = \{v_2, v_3\}$. Since G is outerplanar, v_4 cannot be connected to v_5 by a 2-dipath through v_6 as that will create a $K_{2,3}$ -minor. So v_4 must be connected to v_5 by a 2-dipath through either v_2 or v_3 . Note that v_4 cannot be adjacent to both v_2 and v_3 as it will create a K_4 -minor contradicting the outerplanarity of G . Without loss of generality assume that v_4 is connected to v_5 by a 2-dipath through v_3 . Now v_6 cannot be adjacent to v_3 as $\Delta(G) = 3$. To have weak directed distance from v_6 to v_2 and v_4 at most 2 we must have the edges v_4v_6 and v_5v_6 . This will create a $K_{2,3}$ -minor contradicting the outerplanarity of G . Hence we are done with this case as well.

- (viii) **For $|G| = 6$ and $\Delta(G) = 4$:** Let $N(v_1) = \{v_2, v_3, v_4, v_5\}$. Then we have $|N(v_1) \cap N(v_6)| \leq 2$ since G is outerplanar. Let $N(v_1) \cap N(v_6) = N(v_6) = \{v_2\}$. Then v_2 can be a neighbor of at most two vertices from $\{v_3, v_4, v_5\}$ in order to preserve outerplanarity of G and hence the remaining vertex will not have any 2-dipath connecting it to v_6 . So we must have $|N(v_6)| = 2$. Without loss of generality assume that $N(v_6) = \{v_3, v_4\}$. Now v_6 must be connected by 2-dipaths to v_2 and v_5 . These two 2-dipaths must go through v_3 or v_4 . Both 2-dipaths cannot go through the same vertex v_3 (or v_4) as it will create a $K_{2,3}$ -minor contradicting the outerplanarity of G . Therefore, one of the two 2-dipaths must go through v_3 while the other must go through v_4 . This will force G to be the graph depicted in Figure 2(i).
- (ix) **For $|G| = 6$ and $\Delta(G) = 5$:** Assume that $N(v_1) = \{v_2, v_3, v_4, v_5, v_6\}$ and let the vertices v_2, v_3, v_4, v_5 and v_6 be arranged in clockwise order around v_1 in a fixed planar embedding of G . Since G is outerplanar, the induced subgraph $G[v_2, v_3, v_4, v_5, v_6]$ cannot have a cycle as it will create a K_4 -minor, contradicting the outerplanarity of G .

Now without loss of generality assume that $|N^+(v_1)| > |N^-(v_1)|$. Note that if $|N^+(v_1)| \geq 4$ then we cannot have weak directed distance at most 2 between all the vertices of $N^+(v_1)$

keeping the graph outerplanar. Hence we must have $|N^+(v_1)| = 3$ and $|N^-(v_1)| = 2$. Now to have weak directed distance at most 2 between all the vertices of $N^+(v_1)$ and between all the vertices of $N^-(v_1)$, keeping the graph outerplanar, we must have the graph depicted in Figure 2(j) as a spanning subgraph of G .

- (x) **For $|G| = 7$:** It has been proved by Klostermeyer and MacGillivray in [6] that G must contain the graph depicted in Figure 2(k) as a spanning subgraph.

This concludes the proof. □

4 Proof of Theorem 1.7

Goddard and Henning [5] proved that every planar graph of diameter 2 has domination number at most 2 except for a particular graph on nine vertices.

Let \vec{B} be a planar oclique dominated by the vertex v . Note that if we delete the vertex v from \vec{B} the graph we get is outerplanar as all its vertices must be in the same face for being adjacent to v in a planar embedding of \vec{B} . Sopena [12] showed that any oriented outerplanar graph has an oriented 7-coloring (see Theorem 1.4). Hence let c be an oriented 7-coloring of the oriented outerplanar graph obtained from \vec{B} by deleting the vertex v . For $u \in N^\alpha(v)$ let us assign the color $(c(u), \alpha)$ to u for $\alpha \in \{+, -\}$ and the color 0 to v . It is easy to check that this is an oriented 15-coloring of \vec{B} . Hence any planar oclique dominated by one vertex has order at most 15.

Lemma 4.1. *Let \vec{H} be a planar oclique of order 15 dominated by one vertex. Then \vec{H} contains the planar oclique depicted in Figure 3 as a spanning subgraph.*

Proof. Suppose \vec{H} is a triangulated planar oclique of order 15 dominated by one vertex v . Note that $N^\alpha(v)$ is a relative oriented clique in $\vec{H}[N(v)]$ (that is, the oriented subgraph of \vec{H} induced by the neighbors of v , which is actually the oriented graph obtained by deleting the vertex v from \vec{H}) for any $\alpha \in \{+, -\}$. Also note that $\vec{H}[N(v)]$ is an outerplanar graph. Hence, by Theorem 2.3, we have $|N^\alpha(v)| \leq 7$ for any $\alpha \in \{+, -\}$. But we also have

$$|N^+(v)| + |N^-(v)| = 14.$$

Hence we get

$$|N^+(v)| = |N^-(v)| = 7.$$

Now assume $N(v) = \{x_1, x_2, \dots, x_{14}\}$. Moreover, fix a planar embedding of \vec{H} and, without loss of generality, assume that the vertices x_1, x_2, \dots, x_{14} are arranged in a clockwise order around v . The triangulation of \vec{H} forces the edges $x_1x_2, x_2x_3, \dots, x_{13}x_{14}$ and $x_{14}x_1$. We know from the above discussion that there should be two disjoint relative oriented cliques $N^+(v)$ and $N^-(v)$, each of order 7, in the outerplanar graph $\vec{H}[N(v)]$. We already have the cycle $x_1x_2 \dots x_{14}$ forced in the outerplanar graph $\vec{H}[N(v)]$. We will now prove some more structural properties of $\vec{H}[N(v)]$.

As $\vec{H}[N(v)]$ is an outerplanar graph, it must have at least two vertices of degree at most 2. As every vertex of the graph is part of a cycle, there is no vertex of degree at most 1. Hence there are at least two vertices of degree exactly 2 in $\vec{H}[N(v)]$.

Without loss of generality, assume that $\deg_{\vec{H}[N(v)]}(x_2) = 2$ and $x_2 \in N^\alpha(v)$ for some fixed $\alpha \in \{+, -\}$ and let $\{\alpha, \bar{\alpha}\} = \{+, -\}$. Since \vec{H} is triangulated, we must have the edge x_1x_3 . The vertices of $N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ must then be connected to x_2 by 2-dipaths with internal vertex either x_1 or x_3 .

Let four vertices of $N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ be connected to x_2 by 2-dipaths with internal vertex x_1 . Then there will be two vertices, among the above mentioned four vertices, at weak directed distance at least 3 which is a contradiction. So at most three vertices of $N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ can be connected to x_2 by 2-dipaths with internal vertex x_1 . Similarly we can show that at most three vertices of $N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ can be connected to x_2 by 2-dipaths with internal vertex x_3 .

Now suppose there are at least two vertices $x_i, x_j \in N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ that are connected to x_2 by 2-dipaths with internal vertex x_1 and there are at least two vertices $x_k, x_l \in N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ that are connected to x_2 by 2-dipaths with internal vertex x_3 .

Notice that, as the graph \vec{H} is planar, with the given planar embedding of \vec{H} we must have $i, j > k, l$. Now, without loss of generality, we can assume that $i > j$ and $k > l$. But it will be impossible to have weak directed distance at most 2 between x_i and x_l keeping the graph \vec{H} planar. So, at least one of the vertices x_1 or x_3 must be the internal vertex of at most one 2-dipath connecting a vertex of $N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ to x_2 .

If at least one of the vertices x_1 or x_3 belongs to $N^{\bar{\alpha}}(v)$, then we have

$$|N^\alpha(v) \setminus \{x_1, x_2, x_3\}| \geq 5.$$

But then, by the above discussion, we will have a contradiction (there will be at least two vertices of $N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ connected by 2-dipaths with internal vertex x_1 and at least two vertices of $N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ connected by 2-dipaths with internal vertex x_3).

Hence we must have $x_1, x_3 \in N^\alpha(v)$. Without loss of generality, we have three vertices $x_i, x_j, x_k \in N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ connected by 2-dipaths with internal vertex x_1 and one vertex $x_l \in N^\alpha(v) \setminus \{x_1, x_2, x_3\}$ connected by a 2-dipath with internal vertex x_3 . Without loss of generality, we can assume $i > j > k > l$.

Now, to have $\bar{d}(x_i, x_s) \leq 2$ for $s \in \{2, 3, l\}$, the vertices x_2, x_3, x_l must disagree with x_i on x_1 . Also, to have $\bar{d}(x_i, x_k) \leq 2$ and $\bar{d}(x_2, x_l) \leq 2$, we must have the 2-dipaths $x_i x_j x_k$ and $x_2 x_3 x_l$. But then the induced oriented graph $\vec{H}[N^\alpha(v)]$ contains the oriented graph induced by $N^\alpha(a_0)$ of the planar oclique depicted in Figure 3.

Further, notice that no vertex of $N^\alpha(v)$, other than x_2 , has degree 2 in $\vec{H}[N(v)]$. Hence we can infer that a vertex of $N^{\bar{\alpha}}(v)$ has degree 2 in $\vec{H}[N(v)]$. That will imply that the induced oriented graph $\vec{H}[N^{\bar{\alpha}}(v)]$ contains the oriented graph induced by $N^{\bar{\alpha}}(a_0)$ of the planar oclique depicted in Figure 3.

Hence the planar oclique depicted in Figure 3 is a subgraph of \vec{H} . It is easy to check that, regardless of the choice of \vec{H} (it is a triangulation of the planar oclique depicted in Figure 3), if we delete one arc of the oriented subgraph, isomorphic to the planar oclique depicted in Figure 3, of \vec{H} , the oriented graph \vec{H} does no longer remain an oclique. \square

Now, to prove Theorem 1.7, it will be enough to prove that every planar oclique of order at least 15 must have domination number 1. In other words, it will be enough to prove that any planar oclique with domination number 2 must have order at most 14. More precisely, we need to prove the following lemma.

Lemma 4.2. *Let \vec{H} be a planar oclique with domination number 2. Then $|\vec{H}| \leq 14$.*

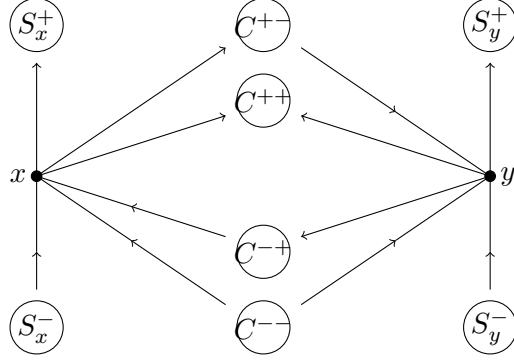


Figure 5: Structure of \vec{G} (not a planar embedding)

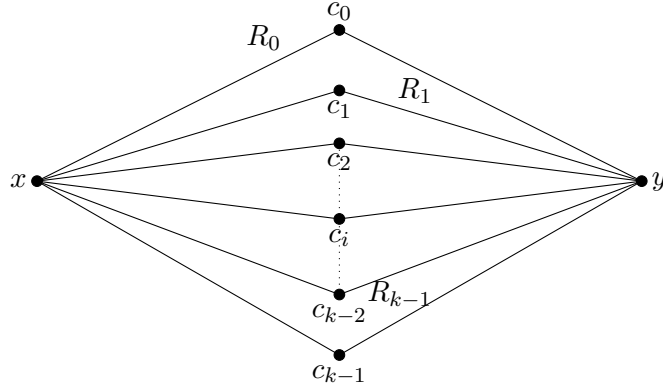


Figure 6: A planar embedding of $UG(\vec{H})$

Let \vec{G} be a planar oclique with $|\vec{G}| > 14$. Assume that \vec{G} is triangulated and has domination number 2.

We define a partial order \prec on the set of all dominating sets of order 2 of \vec{G} as follows: for any two dominating sets $D = \{x, y\}$ and $D' = \{x', y'\}$ of order 2 of \vec{G} , $D' \prec D$ if and only if $|N(x') \cap N(y')| < |N(x) \cap N(y)|$.

Let $D = \{x, y\}$ be a maximal dominating set of order 2 of \vec{G} with respect to \prec . Also, for the remainder of this section, let $t, t', \alpha, \bar{\alpha}, \beta, \bar{\beta}$ be variables satisfying $\{t, t'\} = \{x, y\}$ and $\{\alpha, \bar{\alpha}\} = \{\beta, \bar{\beta}\} = \{+, -\}$.

Let us fix the following notations (see Figure 5):

$$C = N(x) \cap N(y), C^{\alpha\beta} = N^\alpha(x) \cap N^\beta(y), C_t^\alpha = N^\alpha(t) \cap C, \\ S_t = N(t) \setminus C, S_t^\alpha = S_t \cap N^\alpha(t), S = S_x \cup S_y.$$

Hence we have

$$15 \leq |\vec{G}| = |D| + |C| + |S|. \quad (1)$$

Let \vec{H} be the oriented graph obtained from the induced subgraph $\vec{G}[D \cup C]$ of \vec{G} by deleting all the arcs between the vertices of D and all the arcs between the vertices of C . Note that it is possible to extend the planar embedding of $UG(\vec{H})$ given in Figure 6 to a planar embedding of $UG(\vec{G})$ for some particular ordering of the elements of $C = \{c_0, c_1, \dots, c_{k-1}\}$.

Notice that $UG(\vec{H})$ has k faces, namely the unbounded face F_0 and the faces F_i bounded by edges xc_{i-1} , $c_{i-1}y$, yc_i and c_ix for $i \in \{1, \dots, k-1\}$. Geometrically, $UG(\vec{H})$ divides the plane into

k connected components. The region R_i of \vec{G} is the i^{th} connected component (corresponding to the face F_i) of the plane. The boundary points of a region R_i are c_{i-1} and c_i for $i \in \{1, \dots, k-1\}$, and c_0 and c_{k-1} for $i = 0$. Two regions are adjacent if they have at least one common boundary point (hence, a region is adjacent to itself).

Now, for the different possible values of $|C|$, we want to show that $UG(\vec{H})$ cannot be extended to a planar oclique of order at least 15. Note that, for extending $UG(\vec{H})$ to \vec{G} , we can add new vertices only from S . Any vertex $v \in S$ will be inside one of the regions R_i . If there is at least one vertex of S in a region R_i , then R_i is non-empty and empty otherwise. In fact, when there is no chance of confusion, R_i might represent the set of vertices of S contained in the region R_i .

As any two distinct non-adjacent vertices of \vec{G} must be connected by a 2-dipath, we have the following three lemmas:

- Lemma 4.3.** (a) If $(u, v) \in S_x \times S_y$ or $(u, v) \in S_t^\alpha \times S_t^\alpha$, then u and v are in adjacent regions.
(b) If $(u, c) \in S_t^\alpha \times C_t^\alpha$, then c is a boundary point of a region adjacent to the region containing u .

Proof. (a) If $(u, v) \in S_x \times S_y$ then u and v should be in adjacent regions as otherwise they can neither be adjacent nor have a common neighbor. Thus, they cannot be connected by a 2-dipath.

If $(u, v) \in S_t^\alpha \times S_t^\alpha$ then u and v should be in adjacent regions as otherwise they can neither be adjacent nor have a common neighbor other than t . As u, v agree on t , they cannot be connected by a 2-dipath.

(b) If $(u, c) \in S_t^\alpha \times C_t^\alpha$, then c is a boundary point of a region adjacent to the region containing u as otherwise they can neither be adjacent nor have a common neighbor. Thus, they cannot be connected by a 2-dipath. \square

Lemma 4.4. Let R, R^1 and R^2 be three distinct regions such that R is adjacent to R^i with common boundary point c^i while the other boundary point of R^i is \bar{c}^i , for all $i \in \{1, 2\}$. If $v \in S_t^\alpha \cap R$ and $u^i \in ((S_t^\alpha \cup S_\nu) \cap R^i) \cup (\{\bar{c}^i\} \cap C_t^\alpha)$, then v disagrees with u^i on c^i , for all $i \in \{1, 2\}$. Moreover, if both u^1 and u^2 exist, then $|S_t^\alpha \cap R| \leq 1$.

Proof. The first part of the proof is straightforward. For the moreover part, assume that u^1 and u^2 both exist and $|S_t^\alpha \cap R| \geq 2$. Suppose that $a, b \in S_t^\alpha \cap R$. Note that both a and b are adjacent to t, c^1, c^2 and are enclosed by the cycle induced by x, y, c^1 and c^2 . Now note that a divides the region into three parts. Thus b should lie in one of them. It is thus not possible to connect b with each of t, c^1, c^2 keeping the graph planar. \square

Lemma 4.5. For any arc \vec{uv} in \vec{G} , $|N^\alpha(u) \cap N^\beta(v)| \leq 3$.

Now we ask the question ‘‘How small $|C|$ can be?’’ and try to prove possible lower bounds of $|C|$. The first result regarding the lower bound of $|C|$ is proved below.

Lemma 4.6. $|C| \geq 2$.

Proof. We know that x and y are either connected by a 2-dipath or by an arc. If x and y are adjacent then, as \vec{G} is triangulated, we have $|C| \geq 2$. If x and y are non-adjacent, then $|C| \geq 1$. Hence it is enough to show that we cannot have $|C| = 1$ while x and y are non-adjacent.

If $|C| = 1$ and x and y are non-adjacent, then the triangulation of \vec{G} will force the configuration depicted in Figure 7 as a subgraph of $UG(\vec{G})$, where $C = \{c_o\}$, $S_x = \{x_1, \dots, x_{n_x}\}$

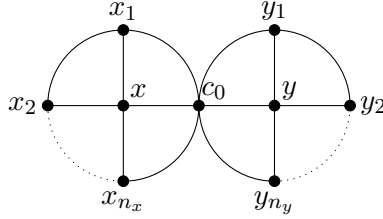


Figure 7: For $|C| = 1$ while x and y are non-adjacent

and $S_y = \{y_1, \dots, y_{n_y}\}$. Without loss of generality, we may assume $|S_y| \geq |S_x|$. Then, by equation (1), we know that $15 \geq |D| + |C| + |S|$, and thus

$$n_y = |S_y| \geq \lceil (15 - 2 - 1)/2 \rceil = 6.$$

Clearly $n_x = |S_x| \geq 3$, as otherwise $\{c_0, y\}$ would be a dominating set with at least two common neighbors $\{y_1, y_{n_y}\}$, which contradicts the maximality of D .

For $n_x = 3$, we know that c_0 is not adjacent to x_2 as otherwise $\{c_0, y\}$ would be a dominating set with at least two common neighbors $\{y_1, y_{n_y}\}$, contradicting the maximality of D . But then x_2 should be adjacent to y_i for some $i \in \{1, \dots, n_y\}$, as otherwise $d(x_2, y) > 2$. Now the triangulation of \vec{G} forces x_2 and y_i to have at least two common neighbors. Also, x_2 cannot be adjacent to y_j for any $j \neq i$, as it would create a dominating set $\{x_2, y\}$ with at least two common neighbors $\{y_i, y_j\}$, contradicting the maximality of D . Hence, x_2 and y_i are adjacent to both x_1 and x_3 . Note that t_{ℓ_t} and t_{ℓ_t+k} are adjacent if and only if $k = 1$, as otherwise $d(t_{\ell_t+1}, t') > 2$ for $1 \leq \ell_t < \ell_t + k \leq n_t$. In this case, again by equation (1), we have

$$n_y = |S_y| \geq 15 - 2 - 1 - 3 = 9.$$

Assume $i \geq 5$. Hence, c_0 is adjacent to y_j for all $j = 1, 2, 3$, as otherwise $d(y_j, x_3) > 2$. This implies $d(y_2, x_2) > 2$, a contradiction. Similarly $i < 5$ would also force a contradiction. Hence $n_x \geq 4$.

For $n_x = 4$, c_0 cannot be adjacent to both x_3 and $x_{n_x-2} = x_2$ as it would create a dominating set $\{c_0, y\}$ with at least two common neighbors $\{y_1, y_{n_y}\}$, contradicting the maximality of D . For $n_x \geq 5$, c_0 is adjacent to x_3 implies, either for all $i \geq 3$ or for all $i \leq 3$, x_i is adjacent to c_0 , as otherwise $d(x_i, y) > 2$. Either of these cases would force c_0 to become adjacent to y_j , as otherwise we would have either $d(x_1, y_j) > 2$ or $d(x_{n_x}, y_j) > 2$ for all $j \in \{1, 2, \dots, n_y\}$. But then we would have a dominating set $\{c_0, x\}$ with at least two common vertices, contradicting the maximality of D . Hence, for $n_x \geq 5$, c_0 is not adjacent to x_3 . Similarly we can show, for $n_x \geq 5$, that c_0 is neither adjacent to x_3 nor to x_{n_x-2} .

So, for $n_x \geq 4$, we can assume without loss of generality that c_0 is not adjacent to x_3 . We know that $d(y_1, x_3) \leq 2$. We have already noted that t_{ℓ_t} and t_{ℓ_t+k} are adjacent if and only if $k = 1$ for any $0 \leq \ell_t < \ell_t + k \leq n_t$. Hence, to have $d(y_1, x_3) \leq 2$, we must have one of the following edges: y_1x_2, y_1x_3, y_1x_4 or y_2x_3 .

The first edge would imply the edges x_2y_j as otherwise $d(x_1, y_j) > 2$ for all $j = 3, 4, 5$. These three edges would then imply $d(x_4, y_3) > 2$. Hence we do not have the edge y_1x_2 .

The other three edges, assuming we cannot have the edge y_1x_2 , would force the edges x_2c_0 and x_1c_0 for having $d(x_2, y) \leq 2$ and $d(x_1, y) \leq 2$. This would imply $d(x_1, y_4) > 2$, a contradiction. Therefore, we cannot have the other three edges too.

Hence we are done. \square

We now prove that, for $2 \leq |C| \leq 5$, at most one region of \vec{G} can be non-empty. Later, using this result, we will improve the lower bound of $|C|$.

Lemma 4.7. *If $2 \leq |C| \leq 5$, then at most one region of \vec{G} is non-empty.*

Proof. Here the undirected underlying graph of \vec{G} is dominated by $D = \{x, y\}$ and the set of common neighbors of x and y is $C = \{c_1, \dots, c_k\}$, for some $k \in \{2, \dots, 5\}$. If we consider only the edges between D and C then a planar embedding of $UG(\vec{G})$ will induce a picture similar to the one depicted in Figure 6.

If $|C| = 2$ and x and y are adjacent, then the region that contains the edge xy is empty, as otherwise the triangulation of \vec{G} would force x and y to have a common neighbor other than c_0 and c_1 . So, for the rest of the proof, we can assume x and y are non-adjacent if $|C| = 2$.

Step 0. We first show that it is not possible to have either $S_x = \emptyset$ or $S_y = \emptyset$ and have at least two non-empty regions. Without loss of generality, assume that $S_x = \emptyset$. Then x and y are non-adjacent, as otherwise y would be a dominating vertex which is not possible.

For $|C| = 2$, if both $S_y \cap R_0$ and $S_y \cap R_1$ are non-empty, then the triangulation of \vec{G} forces either two parallel edges c_0c_1 (one in each region) or a common neighbor of x and y other than c_0, c_1 , a contradiction.

For $|C| = 3, 4$ and 5 , the triangulation of \vec{G} implies the edges $c_0c_1, \dots, c_{k-2}c_{k-1}$ and $c_{k-1}c_0$. Hence every $v \in S_y$ must be connected to x by a 2-dipath through c_i for some $i \in \{1, 2, \dots, k-1\}$. Now assume $|S_y^\alpha| \geq |S_y^{\bar{\alpha}}|$ for some $\alpha \in \{+, -\}$. Then, by equation (1), we have

$$|S_y^\alpha| \geq \lceil (15 - 2 - 5)/2 \rceil = 4.$$

By Lemma 4.3, we know that the vertices of S_y^α will be contained in two adjacent regions for $|C| = 4, 5$. For $|C| = 3$, $S_y^\alpha \cap R_i \neq \emptyset$ for all $i \in \{0, 1, 3\}$ implies $|S_y^\alpha| \leq 3$ by Lemma 4.4. Hence, without loss of generality, we may assume $S_y^\alpha \subseteq R_1 \cup R_2$. If both $S_y^\alpha \cap R_1$ and $S_y^\alpha \cap R_2$ are non-empty then, by Lemma 4.4, each vertex of $S_y^\alpha \cap R_1$ disagrees with each vertex of $S_y^\alpha \cap R_2$ on c_1 . But then, $\{c_1, y\}$ becomes a dominating set with at least six common neighbors (namely c_0, c_2 , and four vertices from S_y^α), which contradicts the maximality of D .

Hence, all the vertices of S_y^α must be contained in one region, say R_1 . Each of them should then be connected to x by a 2-dipath with internal vertex either c_0 or c_1 . However, the vertices that are connected to x by a 2-dipath with internal vertex c_0 should have weak directed distance at most 2 with the vertices connected to x by a 2-dipath with internal vertex c_1 . But it is not possible to connect them unless they are all adjacent to either c_0 or c_1 , in which case it would contradict the maximality of D .

Hence both S_x and S_y are non-empty.

Step 1. We now prove that at most four sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, for all $t \in \{x, y\}$ and $i \in \{0, 1, \dots, k-1\}$. It is immediate for $|C| = 2$. For $|C| = 4$ and 5 , the statement follows from Lemma 4.3. For $|C| = 3$, we consider the following two cases:

- (i) Assume $S_t \cap R_i \neq \emptyset$ for all $t \in \{x, y\}$ and for all $i \in \{0, 1, 2\}$. Then, by Lemma 4.4, we have $|S_t \cap R_i| \leq 1$ for all $t \in \{x, y\}$ and for all $i \in \{0, 1, 2\}$. By equation (1), we have

$$15 \leq |\vec{G}| = 2 + 3 + 4 = 9.$$

This is a contradiction.

- (ii) Assume that five out of the six sets $S_t \cap R_i$ are non-empty and that the other one is empty, where $t \in \{x, y\}$ and $i \in \{0, 1, 2\}$. Without loss of generality, we can assume $S_x \cap R_0 = \emptyset$. By Lemma 4.4, we have $|S_t \cap R_i| \leq 1$ for all $(t, i) \in \{(x, 1), (x, 2), (y, 0)\}$. In particular, $|S_x| \leq 2$.

Now, all vertices of $S_t \cap R_i$ are adjacent to c_1 for $i \in \{1, 2\}$, for being at weak directed distance at most 2 from each other, by Lemma 4.4. That means every vertex of S_x is adjacent to c_1 . Hence, there can be at most three vertices in $(S_y \cap R_1) \cup (S_y \cap R_2)$ as otherwise the dominating set $\{c_1, y\}$ would contradict the maximality of D . Hence, $|S_y| \leq 4$.

Therefore, by equation (1) we have

$$15 \leq |\vec{G}| = 2 + 3 + (2 + 4) = 11.$$

This is a contradiction.

Hence, at most four sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, where $t \in \{x, y\}$ and $i \in \{0, 1, \dots, k-1\}$.

Step 2. Assume now that exactly four sets out of the sets $S_t \cap R_i$ are non-empty, for all $t \in \{x, y\}$ and $i \in \{0, \dots, k-1\}$. Without loss of generality, we have the following three cases (by Lemma 4.3):

- (i) Assume the four non-empty sets are $S_x \cap R_1, S_y \cap R_0, S_y \cap R_1$ and $S_y \cap R_2$ (only possible for $|C| \geq 3$). The triangulation of \vec{G} then forces the edges c_0c_{k-1} and c_1c_2 . Lemma 4.4 implies that $S_x \cap R_1 = \{x_1\}$ and that the vertices of $S_y \cap R_0$ and the vertices of $S_y \cap R_2$ disagree with x_1 on c_0 and c_1 , respectively.

For $|C| = 3$, if every vertex from $S_y \cap R_1$ is adjacent to either c_0 or c_1 , then $\{c_0, c_1\}$ will be a dominating set with at least four common neighbors $\{x, y, x_1, c_2\}$, contradicting the maximality of D . If not, then the triangulation of \vec{G} will force x_1 to be adjacent to at least two vertices, from S_y , say y_1 and y_2 . But then, $\{x_1, y\}$ would be a dominating set with at least four common neighbors $\{y_1, y_2, c_0, c_1\}$, contradicting the maximality of D .

For $|C| = 4$ and 5, Lemma 4.3 implies that vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_2$ disagree with each other on y . Now, by Lemma 4.4, any vertex of $S_y \cap R_1$ is adjacent either to c_0 (if it agrees with the vertices of $S_y \cap R_0$ on y) or to c_1 (if it agrees with the vertices of $S_y \cap R_2$ on y). Also, the vertices of $S_y \cap R_0$ and $S_y \cap R_2$ are connected to x_1 by a 2-dipath through c_0 and c_1 respectively. Hence, by Lemma 4.5, we have $|S_y \cap R_0|, |S_y \cap R_2| \leq 3$.

Now, by equation (1), we have

$$|S_y| \geq (15 - 2 - 5 - 1) = 7.$$

Hence, without loss of generality, at least four vertices y_1, y_2, y_3, y_4 of S_y are adjacent to c_0 . But, in that case, $\{c_0, y\}$ is a dominating set with at least five common neighbors $\{y_1, y_2, y_3, y_4, c_{k-1}\}$, contradicting the maximality of D for $|C| = 4$.

For $|C| = 5$, each vertex of $S_y \cap R_1$ disagrees with c_3 on y by Lemma 4.3 and therefore, without loss of generality, all of them are adjacent to c_0 . Then, vertices of $S_y \cap R_1$ disagrees with vertices of $S_y \cap R_2$ on y as well. This implies vertices of $S_y \cap R_2$ agrees with c_3 on y and must be connected to c_3 by 2-dipaths with internal vertex c_2 . Now, by Lemma 4.4, $|S_y \cap R_2| \leq 1$. So, $|S_y| \geq 7$ implies $|S_y \cap (R_0 \cup R_1)| \geq 6$. But every vertex of $S_y \cap (R_0 \cup R_1)$ are adjacent to c_0 . In that case, $\{c_0, y\}$ is a dominating set with at least six common neighbors, contradicting the maximality of D for $|C| = 5$.

- (ii) Assume the four non-empty sets are $S_x \cap R_0, S_x \cap R_1, S_y \cap R_0$ and $S_y \cap R_1$. For $|C| = 2$, every vertex in S is adjacent either to c_0 or to c_1 (by Lemma 4.4). So, $\{c_0, c_1\}$ is a dominating set. Hence, no vertex $w \in S$ can be adjacent to both c_0 and c_1 since otherwise $\{c_0, c_1\}$ would be a dominating set with at least three common neighbors $\{x, y, w\}$, contradicting the maximality of D . By equation (1), we have

$$|S| \geq 15 - 2 - 2 = 11.$$

Hence, without loss of generality, we may assume $|S_x \cap R_0| \geq 3$. Suppose $\{x_1, x_2, x_3\} \subseteq S_x \cap R_0$. In that case, all vertices of $S_x \cap R_0$ must be adjacent to c_0 (or to c_1), as otherwise it would force all vertices of $S_y \cap R_1$ to be adjacent to both c_0 and c_1 (by Lemma 4.4). Without loss of generality, assume that all vertices of $S_x \cap R_0$ are adjacent to c_0 . Then, all vertices $w \in S_y$ will be adjacent to c_0 , as otherwise $d(w, x_i) > 2$, for some $i \in \{1, 2, 3\}$. But then $\{c_0, x\}$ would be a dominating set with at least three common vertices $\{x_1, x_2, x_3\}$, contradicting the maximality of D .

For $|C| = 3, 4$, every vertex of S will be adjacent to c_0 (by Lemma 4.4). By equation (1), we have

$$|S| \geq (15 - 2 - 4) = 9.$$

Hence, without loss of generality, $|S_x| \geq 5$. In that case, $\{c_0, x\}$ is a dominating set with at least five common neighbors $S_x \cup \{y\}$, contradicting the maximality of D for $|C| = 3, 4$.

For $|C| = 5$, every vertex of $S_t \cap R_i$ disagrees with c_{i+2} on t and, therefore, $|S_t \cap R_i| \leq 3$ for $i \in \{0, 1\}$ by Lemma 4.3. Assume $|S_x \cap R_0| = 3$ and $S_x \cap R_0 = \{x_1, x_2, x_3\}$. Moreover, assume without loss of generality that $c_2 \in N^\alpha(x)$. In that case, we must have $\{x_1, x_2, x_3\} \subseteq N^{\bar{\alpha}}(x)$.

Note that x_1, x_2 and x_3 must agree on c_0 in order to be at weak directed distance at most 2 with the vertices of $S_y \cap R_1$. Further, assume that $\{x_1, x_2, x_3\} \subseteq N^\beta(c_0)$. But then, as all the three vertices $\{x_1, x_2, x_3\}$ are adjacent to both x and c_0 , the only way each of them can be at weak directed distance 2 from c_3 is through a 2-dipath with internal vertex x . Hence, we have $c_3 \in N^\alpha(x)$. This implies $x_4 \in N^{\bar{\alpha}}(x)$ for any vertex $x_4 \in S_x \cap R_1$. But then, the vertices of $S_x \cap R_1$ must disagree with vertices of $S_x \cap R_0$ on c_0 , making it impossible for the vertices of $S_y \cap R_0$ to be at weak directed distance at most 2 from x_1, x_2, x_3 and from the vertices of $S_x \cap R_1$. Therefore, we must have $|S_x \cap R_0| \leq 2$.

Similarly, we can prove $|S_t \cap R_i| \leq 2$ for $i \in \{0, 1\}$.

We now show that it is not possible to have $|S_t \cap R_i| = 2$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$. Suppose on the contrary that this is the case. Then, clearly, the vertices of $S_t \cap R_i$ disagree with c_{i+2} and c_{i+3} on t . Hence, the vertices of $S_t \cap R_0$ agree with the vertices of $S_t \cap R_1$ on t . Therefore, the vertices of $S_t \cap R_0$ must disagree with the vertices of $S_t \cap R_1$ on c_0 .

Then it will not be possible to have both the vertices of $S_x \cap R_0$ at weak directed distance at most 2 from all the four vertices of S_y .

Therefore, we have $|S| \leq 7$. Hence, by equation (1), we have

$$15 \leq |\vec{G}| \leq 2 + 5 + 7 = 14.$$

This is a contradiction and we are done.

- (iii) Assume the four non-empty sets are $S_x \cap R_1, S_x \cap R_2, S_y \cap R_0$ and $S_y \cap R_1$ (only possible for $|C| = 3$). In that case, Lemma 4.4 implies that every vertex of $(S_x \cap R_1) \cup (S_y \cap R_0)$ is adjacent to c_0 and that every vertex of $(S_x \cap R_2) \cup (S_y \cap R_1)$ is adjacent to c_1 .

Moreover, the triangulation of \vec{G} forces the edges c_0c_2 and c_1c_2 . It also forces some vertex $v_1 \in S_y \cap R_1$ to be adjacent to c_0 . But this would create the dominating set $\{c_0, c_1\}$ with at least four common neighbors $\{x, y, v_1, c_2\}$ contradicting the maximality of D .

Hence at most three sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, where $t \in \{x, y\}$ and $i \in \{0, 1, \dots, k-1\}$.

Step 3. Now assume that exactly three sets out of the sets $S_t \cap R_i$ are non-empty, where $t \in \{x, y\}$ and $i \in \{0, \dots, k-1\}$. Without loss of generality we have the following two cases (by Lemma 4.3):

- (i) Assume the three non-empty sets are $S_x \cap R_0, S_y \cap R_0$ and $S_y \cap R_1$. The triangulation of \vec{G} implies that the edge c_0c_1 lies inside the region R_1 .

For $|C| = 2$, there exists $u \in S_y \cup R_1$ such that u is adjacent to both c_0 and c_1 , by the triangulation of \vec{G} . Now, if $|S_y \cup R_1| \geq 2$, then some other vertex $v \in S_y \cup R_1$ must be adjacent to either c_0 or c_1 . Without loss of generality, we may assume that v is adjacent to c_0 . Then, every vertex $w \in S_x \cap R_0$ will be adjacent to c_0 , in order to have $d(v, w) \leq 2$. But in that case $\{c_0, y\}$ would be a dominating set with at least three common neighbors $\{c_1, u, v\}$, contradicting the maximality of D .

So we must have $|S_y \cup R_1| = 1$. Assume that $S_y \cup R_1 = \{u\}$. Then, any vertex $w \in S_x \cap R_0$ is adjacent to either c_0 or c_1 . If $|S_x| \geq 5$ then, without loss of generality, we can assume that at least three vertices of S_x are adjacent to c_0 . Now, to have weak directed distance at most 2 from all those three vertices, every vertex of S_y must be adjacent to c_0 . This would create the dominating set $\{c_0, x\}$ with at least three common neighbors, contradicting the maximality of D .

Also $|S_x| = 1$ clearly creates the dominating set $\{c_0, y\}$ (as x_1 is adjacent to c_0 by the triangulation of \vec{G}) with at least three common neighbors (a vertex from $S_y \cap R_0$ by the triangulation of \vec{G} , u and c_1), contradicting the maximality of D .

For $2 \leq |S_x| \leq 4$, c_0 (or c_1) can be adjacent to at most two vertices of $S_y \cap R_0$ since otherwise there would be one vertex $v \in S_y \cap R_0$ which would force c_0 (or c_1) to be adjacent to all vertices of $w \in S_x$ (in order to satisfy $d(v, w) \leq 2$) and create a dominating set $\{c_0, y\}$ that contradicts the maximality of D .

Also, not all vertices of S_x can be adjacent to c_0 (or c_1), as otherwise $\{c_0, y\}$ (or $\{c_1, y\}$) would be a dominating set with at least three common neighbors (u, c_1 (or c_0) and a vertex from $S_y \cap R_0$), contradicting the maximality of D .

Note that, by equation (1), we have

$$|S_y \cap R_0| \geq 10 - |S_x|.$$

Assume $S_x = \{x_1, \dots, x_n\}$, with the triangulation of \vec{G} forcing the edges $c_0x_1, x_1x_2, \dots, x_{n-1}x_n$ and x_nc_1 for $n \in \{2, 3, 4\}$.

For $|S_x| = 2$, at most four vertices of $S_y \cap R_0$ can be adjacent to c_0 or c_1 . Hence, there will be at least four vertices of $S_y \cap R_0$ each connected to x by a 2-dipath through x_1 or

x_2 . Without loss of generality, x_1 will be adjacent to at least 2 vertices of S_y , and hence $\{x_1, y\}$ will be a dominating set contradicting the maximality of D .

For $|S_x| = 3$, without loss of generality, assume that x_2 is adjacent to c_0 . To satisfy $\bar{d}(x_1, v) \leq 2$ for all $v \in S_y \cap R_0$, at least four vertices of S_y will be connected to x_1 by a 2-dipath through x_2 (as, according to previous discussions, at most two vertices of S_y can be adjacent to c_0). This would create the dominating set $\{x_2, y\}$, contradicting the maximality of D .

For $|S_x| = 4$ we have the edges x_2c_0 and x_3c_1 , as otherwise at least three vertices of S_x would be adjacent to either c_0 or c_1 , which is not possible (because it forces all vertices of S_y to be adjacent to c_0 or c_1). Now, each vertex $v \in S_y \cap R_0$ must be adjacent either to c_0 or to x_2 (to satisfy $\bar{d}(v, x_1) \leq 2$) and also either to c_1 or to x_3 (to satisfy $\bar{d}(v, x_4) \leq 2$), which is not possible due to the planarity of \vec{G} .

For $|C| = 3, 4, 5$, by Lemma 4.4, each vertex of S_x disagrees with each vertex of $S_y \cap R_1$ on c_0 . We also have the edge x_1c_2 for some $x_1 \in S_x$ by the triangulation of \vec{G} . By equation (1), we have

$$|S| \geq (15 - 2 - |C|) = 13 - |C|.$$

Hence, $|S_x| \leq 2$ for $|C| = 3, 4$, as otherwise every vertex $u \in S_y$ would be adjacent to c_0 , creating a dominating set $\{c_0, t\}$ with at least $(|C| + 1)$ common neighbors $S_t \cup \{c_1\}$ for some $t \in \{x, y\}$, contradicting the maximality of D . For $|C| = 5$, since all the vertices in $S_x \cap R_0$ agree with each other on x (as they all must disagree with c_2 on x) and on c_0 (as they all disagree with vertices of $S_y \cap R_1$ on c_0), by Lemma 4.5, we have $|S_x \cap R_0| \leq 3$. But if $|S_x \cap R_0| = 3$ then every vertex of S_y will be adjacent to c_0 , creating a dominating set $\{c_0, y\}$ with at least six common neighbors $S_y \cup \{c_1\}$, contradicting the maximality of D .

Hence $|S_x| \leq 2$ for $|C| = 3, 4, 5$.

Now for $|C| = 3$, we can assume that x and y are non-adjacent as otherwise $\{c_0, y\}$ would be a dominating set with at least four common neighbors (x, c_1 and two other vertices each from the sets $S_y \cap R_0, S_y \cap R_1$ by triangulation) contradicting the maximality of D . Hence triangulation will imply the edge c_1c_2 . Now for $|S_x| \leq 2$, either $\{c_0, c_2\}$ is a dominating set with at least four common neighbors $\{x, y, c_1, x_1\}$ contradicting the maximality of D or x_1 is adjacent to at least two vertices $y_1, y_2 \in S_y \cap R_0$ creating a dominating set $\{x_1, y\}$ (the other vertex in S_x must be adjacent to x_1 by triangulation) with at least four common neighbors $\{y_1, y_2, c_0, c_2\}$ contradicting the maximality of D .

For $|C| = 4$ we have $|S_y \cap R_1| \leq 2$ as otherwise we will have the dominating set $\{c_0, y\}$ with at least five common neighbors (c_1 , vertices of $S_y \cap R_1$ and one vertex of $S_y \cap R_0$ by the triangulation of \vec{G}), contradicting the maximality of D . By equation (1), we have

$$\begin{aligned} |S_y \cap R_0| &\geq (15 - |D| - |C| - |S_x| - |S_y \cap R_1|) \\ &\geq (15 - 2 - 4 - 2 - 2) = 5. \end{aligned}$$

Now, at most two vertices of $S_y \cap R_0$ can be adjacent to c_0 as otherwise $\{c_0, y\}$ would be a dominating set with at least five common neighbors (c_1 , vertices of $S_y \cap R_0$ and one vertex of $S_y \cap R_1$ by the triangulation of \vec{G}), contradicting the maximality of D .

Also, by the triangulation of \vec{G} , in R_3 we have either the edge xy or the edge c_2c_3 . But, if we have the edge xy , then $|S_y \cap R_1| = 1$ as otherwise the dominating set $\{c_0, y\}$ would contradict the maximality of D . Hence, by the triangulation of \vec{G} , and in order to have weak directed distance at most 2 from the vertices of $S_x \cup \{x\}$, each vertex of $S_y \cap R_0$ will be adjacent either to c_3 or to x_1 . This will create a dominating set $\{x_1, y\}$ or $\{c_3, y\}$ that contradicts the maximality of D . Hence, we do not have the edge xy (not even in other regions) and we thus have the edge c_2c_3 .

For $|S_x| \leq 2$, the vertices of $S_y \cap R_0$ will be adjacent to either c_3 , c_0 or x_1 in order to have weak directed distance at most 2 from x . But then, the triangulation of \vec{G} will force at least two vertices of $S_y \cap R_0$ to be common neighbors of c_3 and x_1 , or to have the edge c_0c_3 . It is not difficult to check, casewise (drawing a picture for individual cases will help in understanding the scenario), that one of the sets $\{c_0, y\}$, $\{c_3, y\}$ or $\{x_1, y\}$ would then be a dominating set contradicting the maximality of D .

For $|C| = 5$, by Lemma 4.3, each vertex of $S_y \cap R_i$ must disagree with c_{i+2} on y . If the vertices of $S_y \cap R_0$ and the vertices of $S_y \cap R_1$ agree with each other on y , then they must disagree with each other on c_0 , which implies $|S_y \cap R_i| \leq 3$ for all $i \in \{0, 1\}$. If the vertices of $S_y \cap R_0$ and the vertices of $S_y \cap R_1$ disagree with each other on y , then the vertices of $S_y \cap R_i$ must agree with c_{3-i} on y . In that case, by Lemma 4.4, each vertex of $S_y \cap R_i$ must be connected to c_{3-i} by a 2-dipath through c_{4-3i} , which implies $|S_y \cap R_i| \leq 3$ for all $i \in \{0, 1\}$.

Assume $|S_y \cap R_0| = 3$ and $|S_y \cap R_1| = 3$. Then, each vertex of $S_y \cap R_i$ must disagree with both c_{i+2} and c_{i+3} on y . This would imply that the vertices of $S_y \cap R_0$ and the vertices of $S_y \cap R_1$ disagree with each other on c_0 . Now, there would be no way to have weak directed distance at most 2 between a vertex of S_x and all the six vertices of S_y .

Hence we must have $|S_y| \leq 5$. Then, by equation (1), we have

$$15 \leq |\vec{G}| \leq 2 + 5 + (2 + 5) = 14.$$

This is a contradiction, and this concludes this particular subcase.

- (ii) Assume the three non-empty sets are $S_x \cap R_1$, $S_y \cap R_0$ and $S_y \cap R_2$ (only possible for $|C| \geq 3$). By Lemma 4.4, we have $S_x = \{x_1\}$ and the fact that each vertex of $S_y \cap R_i$ disagrees with $c_{i^2/4}$ on x_1 for $i \in \{0, 2\}$. Moreover, the triangulation of \vec{G} implies the edges x_1c_0 , x_1c_1 , $c_{k-1}c_0$, c_0c_1 and c_1c_2 .

For $|C| = 3$, $\{c_0, c_1\}$ is a dominating set with at least four common neighbors $\{x, y, c_2, x_1\}$, contradicting the maximality of D . For $|C| = 4, 5$, every vertex of $S_y \cap R_0$ disagrees with every vertex of $S_y \cap R_2$ on y . Hence, by Lemma 4.5, we have $|S_y \cap R_i| \leq 3$ for all $i \in \{0, 2\}$. By equation (1), we then have

$$\begin{aligned} 15 \leq |\vec{G}| &= |D| + |C| + |S| \\ &\leq [2 + 5 + (1 + 3 + 3)] = 14. \end{aligned}$$

This is a contradiction.

Step 4. Hence, at most two sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, where $t \in \{x, y\}$ and $i \in \{0, 1, \dots, k-1\}$.

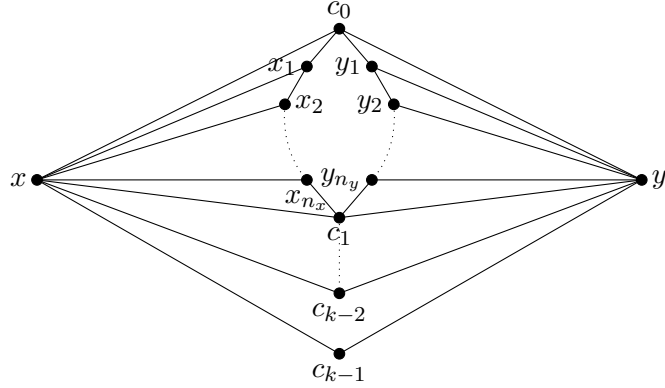


Figure 8: The only non-empty region is R_1

Assume that exactly two sets out of the sets $S_t \cap R_i$ are non-empty, where $t \in \{x, y\}$ and $i \in \{0, \dots, k-1\}$, yet there are two non-empty regions. Without loss of generality, assume that the two non-empty sets are $S_x \cap R_0$ and $S_y \cap R_1$.

The triangulation of \vec{G} would force x and y to have a common neighbor other than c_0 and c_1 for $|C| = 2$ which is a contradiction. For $|C| = 3, 4, 5$ the triangulation of \vec{G} forces the edges $c_{k-1}c_0$ and c_0c_1 . By Lemma 4.4, we know that each vertex of S is adjacent to c_0 . By equation (1), we have

$$|S| \geq (15 - 2 - 5) = 8.$$

Hence, without loss of generality, we may assume $|S_x| \geq 4$. But then $\{c_0, x\}$ would be a dominating set with at least six common neighbors $S_x \cup \{c_{k-1}, c_1\}$, contradicting the maximality of D .

This concludes the proof. \square

The lemma proved above was one of the key steps to prove the theorem. Now we will improve the lower bound on $|C|$.

Lemma 4.8. $|C| \geq 6$.

Proof. For $|C| = 2, 3, 4, 5$, without loss of generality by Lemma 4.7, we may assume R_1 to be the only non-empty region. The triangulation of \vec{G} will then force the configuration depicted in Figure 8 as a subgraph of $UG(\vec{G})$, where $C = \{c_0, \dots, c_{k-1}\}$, $S_x = \{x_1, \dots, x_{n_x}\}$ and $S_y = \{y_1, \dots, y_{n_y}\}$. Without loss of generality, we may assume

$$|S_y| = n_y \geq n_x = |S_x|.$$

Then, by equation (1), we have

$$n_y = |S_y| \geq (15 - 2 - |C| - |S_x|) = 13 - |C| - |S_x|. \quad (2)$$

First of all, assume $n_x = 0$. Then x is not adjacent to y , as otherwise y would dominate the whole graph. So we have the edges $c_0c_1, c_1c_2, \dots, c_{k-1}c_0$ by the triangulation of \vec{G} . Then, by equation (2), we have

$$|S_y| \geq 13 - 5 = 8.$$

Now, to have $\bar{d}(x, y_i) \leq 2$, every y_i must be connected to x by a 2-dipath with internal vertex either c_0 or c_1 . Hence, at least four vertices of S_y must be adjacent to either c_0 or c_1 . Note that c_0 is also adjacent to c_{k-1}, c_1 and that c_1 is also adjacent to c_0, c_2 . But then, the dominating set $\{c_0, y\}$ or $\{c_1, y\}$ will contradict the maximality of D . Hence $n_x \geq 1$.

The proof will now directly follow from the four claims below.

Claim 1: $|C| = 5$ is not possible.

Proof of claim 1: Assume that $|C| = 5$. Then, by equation (2), we have

$$|S_y| \geq 13 - 5 - n_x = 8 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 4$. Now, every vertex of S_y disagrees with c_3 on y . They must also disagree with y on c_2 , as otherwise all of them would be connected to c_2 by 2-dipaths with internal vertex c_1 , which would imply $\bar{d}(y_1, y_4) > 2$. For similar reasons, the vertices of S_y must disagree with c_4 on y .

Moreover, the edge c_0c_1 does not exist since it would force each vertex of S_y to be connected to vertices of S_x by 2-dipaths with internal vertex either c_0 or c_1 . In fact, for $n_x \geq 2$, as not all vertices of S_x can be adjacent to both c_0 and c_1 , every vertex of S_y would be connected to the vertices of S_x by 2-dipaths with internal vertex being exactly one of c_0, c_1 , thus implying $\bar{d}(y_1, y_4) > 2$. For $n_x = 1$, as $n_y \geq 7$, at least four vertices of S_y would be connected to the vertices of S_x by 2-dipaths with internal vertex being exactly one of c_0, c_1 , implying $\bar{d}(y_i, y_{i+3}) > 2$ for some $i \in \{1, 2, \dots, n_y\}$. Hence, the edge c_0c_1 does not exist.

Also, if we have the edge y_1y_4 and, without loss of generality, the edge y_1y_3 by the triangulation of \vec{G} , then every vertex of S_x must be connected to y_2 by 2-dipaths with internal vertex y_1 . In this case, $\{y_1, y\}$ is a dominating set with at least n_y common neighbors (c_0 and $n_y - 1$ common neighbors from S_y). Hence, to avoid a contradiction with the maximality of D , we must have $n_y \leq 5$. We must also have $n_x \geq 3$. But then, as every vertex of S_x agree with each other on y_1 and on x (as they all disagree with c_3 on x), they must all disagree with c_1 and c_4 on x to have weak directed distance at most 2 with them. Also the vertices of S_y must disagree with c_1 and c_4 on y to have directed distance at most 2 with them. So, c_1 and c_4 agrees with each other on both x and y , Therefore, to have weak directed distance at most 2 between c_1 and c_4 we must have the 2-dipath connecting c_4 and c_1 with internal vertex c_0 . But this is a contradiction as we can not have the edge c_0c_1 .

Similarly, we cannot have the edge y_1y_3 also. Therefore, y_1 and y_4 must be connected by a 2-dipath with an internal vertex x_j from S_x for some $j \in \{1, 2, \dots, n_x\}$. As we cannot have the edge y_1y_4 , this implies that every vertex of $S \setminus \{x_j\}$ is adjacent to x_j to be at weak directed distance at most 2 from each other. We can then reach a contradiction, exactly as in the case described in the paragraph above.

This proves the claim. ◇

Claim 2: $|C| = 4$ is not possible.

Proof of claim 2: Assume that $|C| = 4$. Then, by equation (2), we have

$$|S_y| \geq 13 - 4 - n_x = 9 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 5$.

We now show that every vertex of S_y disagrees with c_2 and c_3 on y . First note that no vertex can agree with both c_2 and c_3 on y as otherwise it would be adjacent to both c_0 and c_1 , which

is impossible since $n_y \geq 5$. So, if the claim is not true, then some vertices of S_y will agree with c_2 on y and the other vertices of S_y will agree with c_3 on y .

Also at most three vertices of S_y can agree with c_2 (or c_3) on y . So, $n_y \leq 6$. Hence, $n_x \geq 3$.

Now, three vertices of S_y agree on y with, say, c_2 . Then they will all disagree with c_2 on c_1 and every vertex (there are at least three such vertices) of S_x will disagree with those three vertices on c_1 . Then, to have weak directed distance at most 2 between the vertices of S_x , the other vertices (there are at least two such vertices) of S_y should be adjacent to c_1 , which is not possible as they are already connected to c_3 with 2-dipaths with internal vertex c_0 .

The rest of the proof is similar to the proof of Claim 1. Using similar arguments, it is possible to show that the edge c_0c_1 does not exist, that the edge y_1y_4 does not exist and that it is not possible to have a 2-dipath with internal vertex from S_x connecting y_1 and y_4 . \diamond

Claim 3: $|C| = 3$ is not possible.

Proof of claim 3: Assume that $|C| = 3$. Then, by equation (2), we have

$$|S_y| \geq 13 - 3 - n_x = 10 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 5$.

First note that it is not possible to have the edge c_0c_1 , as this will force some three vertices of S_y to be connected to vertices of S_x by 2-dipaths with internal vertex c_0 (or c_1), making $\{c_0, y\}$ (or $\{c_1, y\}$) a dominating set that contradicts the maximality of D .

For $n_y \geq 7$, there are at least four vertices in S_y that agree with each other on y . We need to have weak directed distance at most 2 between them. Let those four vertices be y_i, y_j, y_k, y_l with $i > j > k > l$.

Now, assume that we have the edge y_iy_l . Then, every vertex of S_x will be adjacent to either y_i or y_l . Without loss of generality, assume that every vertex of S_x is adjacent to y_i . But then, $\{y_i, y\}$ would be a dominating set with at least four common neighbors, contradicting the maximality of D . Hence $n_y \leq 6$ and, therefore, we must have $n_x \geq 4$.

For $n_y = 5, 6$, one can show that these cases are not possible without creating a dominating set that contradicts the maximality of D . If one just tries to have weak directed distance at most 2 between the vertices of S , the proof will follow. The proof of this part is also similar to the ones done before and, though a bit tedious, is not difficult to check. \diamond

Claim 4: $|C| = 2$ is not possible.

Proof of claim 4: Assume that $|C| = 2$. Then, by equation (2), we have

$$|S_y| \geq 13 - 2 - n_x = 11 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 6$.

This is actually the easiest of the four claims. The case $n_y \geq 7$ can be argued as in the previous proof. For $n_y = 6$, we must have $n_x \geq 5$. If one just tries to have directed distance at most 2 between the vertices of S , the proof will follow. The proof of this part is also similar to the ones done before and, though a bit tedious, is again not difficult to check. \diamond

This completes the proof of the lemma. \square

Up to now, we have proved that the value of $|C|$ is at least 6. This is an answer to our question ‘‘how small $|C|$ can be?’’. We will now consider the question ‘‘How big $|C|$ can be?’’ and try to provide upper bounds for the value of $|C|$. The following lemma will help us to do so.

Lemma 4.9. *If $|C| \geq 6$, then the following holds:*

(a) $|C^{\alpha\beta}| \leq 3$, $|C_t^\alpha| \leq 6$, $|C| \leq 12$. Moreover, if $|C^{\alpha\beta}| = 3$, then $\vec{G}[C^{\alpha\beta}]$ is a 2-dipath.

(b) $|C_t^\alpha| \geq 5$ (respectively 4, 3, 2, 1, 0) implies $|S_t^\alpha| \leq 0$ (respectively 1, 3, 4, 5, 6).

Proof. (a) If $|C^{\alpha\beta}| \geq 4$, then there will be two vertices $u, v \in C^{\alpha\beta}$ with $d(u, v) > 2$, which is a contradiction. Hence we have the first inequality, which implies the other two.

If $|C^{\alpha\beta}| = 3$, then the only way to connect the two non-adjacent vertices u, v of $C^{\alpha\beta}$ is to connect them with a 2-dipath through the other vertex (other than u, v) of $C^{\alpha\beta}$.

(b) Lemma 4.3(b) implies that if all the elements of C_t^α do not belong to the set of four boundary points of any three consecutive regions (like R, R^1, R^2 in Lemma 4.4), then $|S_t^\alpha| = 0$. Hence, we have $|C_t^\alpha| \geq 5$ implies $|S_t^\alpha| \leq 0$.

By Lemma 4.4, if all the elements of C_t^α belong to the set of four boundary points $c^1, c^2, \overline{c^1}, \overline{c^2}$ of three consecutive regions R, R^1, R^2 (like in Lemma 4.4) and contains both $\overline{c^1}, \overline{c^2}$, then $|S_t^\alpha| \leq 1$. Also $S_t^\alpha \subseteq R$ by Lemma 4.4. Hence we have

$$|C_t^\alpha| \geq 4 \text{ implies } |S_t^\alpha| \leq 1.$$

Assume now that all the elements of C_t^α belong to the set of three boundary points $c^1, c^2, \overline{c^1}$ of two adjacent regions R, R^1 (like in Lemma 4.4) and contain both $\overline{c^1}, c^2$. Then, by Lemma 4.3, $v \in S_t^\alpha$ implies v is in R or R^1 .

Now, if both $S_t^\alpha \cap R$ and $S_t^\alpha \cap R^1$ are non-empty, then each vertex of $(S_t^\alpha \cap R) \cup \{c^2\}$ disagrees with each vertex of $(S_t^\alpha \cap R^1) \cup \{\overline{c^1}\}$ on c^1 (by Lemma 4.4).

Hence, by Lemma 4.5, we have

$$|(S_t^\alpha \cap R) \cup \{\overline{c^1}\}|, |(S_t^\alpha \cap R^1) \cup \{c^2\}| \leq 3.$$

This clearly implies

$$|S_t^\alpha \cap R|, |S_t^\alpha \cap R^1| \leq 2 \text{ and } |S_t^\alpha| \leq 4.$$

Suppose now that we have $|S_t^\alpha| = 4$ and, hence, also $|S_t^\alpha \cap R|, |S_t^\alpha \cap R^1| = 2$. Then, $S_{t'} = \emptyset$ as the only way for a vertex of $S_{t'}$ to be at weak directed distance at most 2 from every vertex of S_t is by being connected by a 2-dipath with internal vertex c_1 , which is impossible as the vertices of $S_t^\alpha \cap R$ disagree with the vertices of $S_t^\alpha \cap R^1$ on c_1 .

In fact, for the same reason, it is impossible to have weak directed distance at most 2 between all the vertices of S_t and t' unless we have the edge tt' (that is the edge xy). But then, the edge tt' makes t a vertex that dominates the whole graph, contradicting the domination number of the graph being 2. Therefore, it is not possible to have $|S_t^\alpha| = 4$. Hence, we have $|S_t^\alpha| \leq 3$ in this case.

Also, if one of $S_t^\alpha \cap R$ and $S_t^\alpha \cap R^1$ is empty then we must have $|S_t^\alpha| \leq 3$ by Lemmas 4.4 and 4.5. Hence, we have

$$|C_t^\alpha| \geq 3 \text{ implies } |S_t^\alpha| \leq 3.$$

Let $R, R^1, R^2, c^1, c^2, \overline{c^1}, \overline{c^2}$ be as in Lemma 4.4 and assume $C_t^\alpha = \{c^1, c^2\}$. By Lemma 4.3, $v \in S_t^\alpha$ implies v is in R, R^1 or R^2 , and also that both $S_t^\alpha \cap R^1$ and $S_t^\alpha \cap R^2$ cannot be non-empty. Hence, without loss of generality, assume $S_t^\alpha \cap R^2 = \emptyset$.

By Lemma 4.4, the vertices of $S_t^\alpha \cap R^1$ disagree with the vertices of $(S_t^\alpha \cap R) \cup \{c^2\}$ on c^1 . Hence, by Lemma 4.5, we have

$$|S_t^\alpha \cap R^1|, |(S_t^\alpha \cap R) \cup \{c^2\}| \leq 3.$$

This implies $|S_t^\alpha| \leq 5$.

Now, if $S_t^\alpha \cap R^1 = \emptyset$ then $S_t^\alpha = S_t^\alpha \cap R$. Let $|S_t^\alpha \cap R| \geq 6$ and consider the induced graph $\vec{\mathcal{O}} = \vec{G}[(S \cap R) \cup \{c^1, c^2\}]$. In this graph, the vertices of $(S_t^\alpha \cap R) \cup \{c^1, c^2\}$ are at weak directed distance at most 2 from each other. Hence, $\chi_o(\vec{\mathcal{O}}) \geq 8$. But this is a contradiction since $\vec{\mathcal{O}}$ is an outerplanar graph and every outerplanar graph has an oriented 7-coloring [12]. Hence,

$$|C_t^\alpha| \geq 2 \text{ implies } |S_t^\alpha| \leq 5.$$

Suppose now that we have $|S_t^\alpha| = 5$. Then we must have $S_{t'} = \emptyset$ as otherwise it is not possible to have weak directed distance at most 2 between the vertices of S .

We also do not have the edge xy as it would contradict the domination number of the graph being 2 (t will dominate the graph). So, by the triangulation of \vec{G} , we have the edges c^1c^2 and c^1c^1 . Hence, each vertex of S_t must be connected to t' with a 2-dipath with internal vertices from $\{c^1, c^2\}$. But then, it will not be possible to have weak directed distance at most 2 between the five vertices of S_t .

Hence,

$$|C_t^\alpha| \geq 2 \text{ implies } |S_t^\alpha| \leq 4.$$

In general, S_t^α is contained in two distinct adjacent regions by Lemma 4.3. Without loss of generality, assume $S_t^\alpha \subseteq R_1 \cup R_2$. If both $S_t^\alpha \cap R_1$ and $S_t^\alpha \cap R_2$ are non-empty then, by Lemma 4.4, we know that the vertices of $S_t^\alpha \cap R_1$ disagree with the vertices of $S_t^\alpha \cap R_2$ on c_1 . Hence, $|S_t^\alpha \cap R_1|, |S_t^\alpha \cap R_2| \leq 3$, which implies $|S_t^\alpha| \leq 6$.

Assume now that only one of the two sets $S_t^\alpha \cap R_1$ and $S_t^\alpha \cap R_2$ is non-empty. Without loss of generality, assume $S_t^\alpha \cap R_1 \neq \emptyset$. If $c_0, c_1 \notin C_t^\alpha$ and $|C_t^\alpha| = 1$ then we have $|S_t^\alpha \cap R_1| \leq 3$ by Lemmas 4.4 and 4.5. In the induced outerplanar graph $\vec{\mathcal{O}} = \vec{G}[(S \cap R_1) \cup \{c_1, c_2\}]$, the vertices of $S_t^\alpha \cup (C_t^\alpha \cap \{c_1, c_2\})$ are at weak directed distance at most 2 from each other.

Hence, $7 \geq \chi_o(\vec{\mathcal{O}}) \geq |S_t^\alpha \cup (C_t^\alpha \cap \{c_1, c_2\})|$. Therefore,

$$|C_t^\alpha| \geq 1 \text{ (respectively 0) implies } |S_t^\alpha| \leq 6 \text{ (respectively 7).}$$

Now, when both the equalities hold, we must have $S_{t'} = \emptyset$ as otherwise $C \cup S_t \cup S_{t'}$ would contain an oriented outerplanar graph with oriented chromatic number at least 8, which is not possible, in order to have all the vertices of S at weak directed distance at most 2 from each other.

Now, $S_{t'} = \emptyset$ would imply that the edge xy is not there, as otherwise t would dominate the whole graph. Hence, each vertex of S_t^α must be connected to t' by a 2-dipath with internal vertex c_i for some $i \in \{0, 1, 2\}$. But this would force $|S_t^\alpha \cup C_t^\alpha| \leq 6$ as otherwise the vertices of $S_t^\alpha \cup C_t^\alpha$ would no longer be at weak directed distance at most 2 from each other.

Hence,

$$|C_t^\alpha| \geq 1 \text{ (respectively 0) implies } |S_t^\alpha| \leq 5 \text{ (respectively 6),}$$

and we are done. \square

We now prove that the value of $|C|$ can be at most 5, which contradicts our previously proven lower bound on $|C|$. That actually proves Lemma 4.2.

Lemma 4.10. $|C| \leq 5$.

Proof. Without loss of generality, we can suppose $|C_x^\alpha| \geq |C_y^\beta| \geq |C_y^{\bar{\beta}}| \geq |C_x^{\bar{\alpha}}|$ (the last inequality is forced). We know that $|C| \leq 12$ and that $|C_x^\alpha| \leq 6$ (Lemma 4.9(a)). Therefore, it is enough to show that $|S| \leq 12 - |C|$ for all possible values of $(|C|, |C_x^\alpha|, |C_y^\beta|)$, since it contradicts (1).

For $(|C|, |C_x^\alpha|, |C_y^\beta|) = (12, 6, 6), (11, 6, 6), (10, 6, 6), (10, 6, 5), (10, 5, 5), (9, 5, 5), (8, 4, 4)$ we have $|S| \leq 12 - |C|$, using Lemma 4.9(b).

For $(|C|, |C_x^\alpha|, |C_y^\beta|) = (8, 6, 6), (7, 6, 6), (7, 6, 5), (6, 6, 6), (6, 6, 5), (6, 6, 4), (6, 5, 5)$ we are forced to have

$$|C^{\alpha\beta}| > 3.$$

This is a contradiction by Lemma 4.9(a).

So, $(|C|, |C_x^\alpha|, |C_y^\beta|) \neq (12, 6, 6), (11, 6, 6), (10, 6, 6), (10, 6, 5), (10, 5, 5), (9, 5, 5), (8, 4, 4), (8, 6, 6), (7, 6, 6), (7, 6, 5), (6, 6, 6), (6, 6, 5), (6, 6, 4), (6, 5, 5)$.

We will be done if we prove that $(|C|, |C_x^\alpha|, |C_y^\beta|)$ cannot take the other possible values also. That leaves us checking a lot of cases. We will check just a few cases and observe that the other cases can be checked using similar arguments.

Case 1: Assume $(|C|, |C_x^\alpha|, |C_y^\beta|) = (9, 6, 6)$.

We are then forced to have $|C^{\alpha\beta}| = |C^{\alpha\bar{\beta}}| = |C^{\bar{\alpha}\beta}| = 3$ in order to satisfy the first inequality of Lemma 4.9(a). So, $\vec{G}[C^{\alpha\beta}]$, $\vec{G}[C^{\alpha\bar{\beta}}]$ and $\vec{G}[C^{\bar{\alpha}\beta}]$ are 2-dipaths by Lemma 4.9(a). Without loss of generality, we can assume $C^{\alpha\beta} = \{c_0, c_1, c_2\}$ and $C^{\bar{\alpha}\beta} = \{c_3, c_4, c_5\}$. Hence, by Lemma 4.3, we have $u \in R_1 \cup R_2$ and $v \in R_4 \cup R_5$ for any $(u, v) \in S_y^{\bar{\beta}} \times S_x^{\bar{\alpha}}$. Now, by Lemma 4.3, either $S_y^{\bar{\beta}}$ or $S_x^{\bar{\alpha}}$ is empty. Without loss of generality, assume $S_y^{\bar{\beta}} = \emptyset$. Therefore, we have $|S| = |S_x| = |S_x^{\bar{\alpha}}| \leq 3$ (by Lemma 4.9(b)). So this case is not possible.

Case 2: Assume $(|C|, |C_x^\alpha|, |C_y^\beta|) = (7, 6, 4)$.

So, without loss of generality, we can assume that $\vec{G}[C^{\alpha\beta}]$ and $\vec{G}[C^{\alpha\bar{\beta}}]$ are 2-dipaths, and $C^{\alpha\beta} = \{c_0, c_1, c_2\}$, $C^{\alpha\bar{\beta}} = \{c_3, c_4, c_5\}$ and $C^{\bar{\alpha}\beta} = \{c_6\}$.

By Lemma 4.9, we have $|S_x| \leq 5$ and $|S_y| \leq 3 + 1 = 4$. So we are done if either $S_x = \emptyset$ or $S_y = \emptyset$.

So assume both S_x and S_y are non-empty. First assume that $S_y^{\bar{\beta}} \neq \emptyset$. Then, by Lemma 4.3, we have $S_y^{\bar{\beta}} \subseteq R_5$, $S_x^{\bar{\alpha}} \subseteq R_5 \cup R_6$ and hence $S_y^{\bar{\beta}} = \emptyset$. By Lemma 4.4, the vertices of $S_y^{\bar{\beta}}$ and the vertices of $S_x^{\bar{\alpha}} \cap R_5$ must disagree with c_6 on c_5 while disagreeing with each other on c_5 , which is not possible. Hence, $S_x^{\bar{\alpha}} \cap R_5 = \emptyset$. Also, $|S_x^{\bar{\alpha}} \cap R_6| \leq 3$ as they all disagree on c_5 with the vertices of $S_y^{\bar{\beta}}$. Hence, $|S| \leq 4$ when $S_y^{\bar{\beta}} \neq \emptyset$.

Now assume $S_y^{\bar{\beta}} = \emptyset$ hence $S_y^{\bar{\beta}} \neq \emptyset$. Then, by Lemma 4.3, we have $S_y^{\bar{\beta}} \subseteq R_1 \cup R_2$, $S_x^{\bar{\alpha}} \subseteq R_0 \cup R_1$ and hence $S_y^{\bar{\beta}} = \emptyset$. Assume $S_y^{\bar{\beta}} \cap R_2 = \emptyset$, as otherwise the vertices of $S_x^{\bar{\alpha}}$ would be adjacent to both c_0 and c_1 (to be connected to c_6 and to vertices of $S_y^{\bar{\beta}} \cap R_2$ by a 2-dipath), implying $|S_x^{\bar{\alpha}}| \leq 1$, implying $|S| \leq 5$. If $S_x^{\bar{\alpha}} \cap R_0 \neq \emptyset$ then $|S_y^{\bar{\beta}} \cap R_1| = 1$, $|S_y^{\bar{\beta}} \cap R_1| \leq 1$ and $|S_y^{\bar{\beta}} \cap R_0| \leq 3$, by Lemma 4.4, and hence $|S| \leq 5$. If $S_x^{\bar{\alpha}} \cap R_0 = \emptyset$ then $|S_y^{\bar{\beta}} \cap R_1| \leq 2$, $|S_y^{\bar{\beta}} \cap R_1| \leq 3$ and hence $|S| \leq 5$. So also this case is not possible.

Similarly one can handle the remaining cases. \square

From the above lemmas, we get that every planar oclique of order at least 15 is dominated by a single vertex. Moreover, we also proved that a planar oclique dominated by one vertex

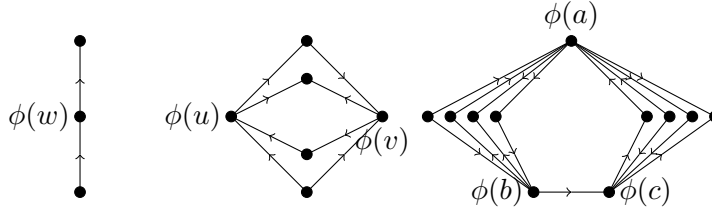


Figure 9: Planar targets with girth at least 4

can have order at most 15. Hence, there is no planar oclique of order more than 15. We also proved that every oclique of order 15 must contain the planar oclique depicted in Figure 3 as a spanning subgraph.

This concludes the proof of Theorem 1.7. \square

5 Proof of Theorem 1.8

(a) The proof directly follows from Theorem 1.7.

(b) In 1975, Plesník [8] characterized and listed all triangle-free planar graphs with diameter 2 (see Theorem 2.1). They are precisely the graphs depicted in Figure 4. Now note that every orientation of those graphs admits a homomorphism to the graphs depicted in Figure 9, respectively (that is, any oriented graph with underlying graph from the first, second and third family of graphs described in Figure 4 admits a homomorphism to the first, second and third oriented graph depicted in Figure 9, respectively).

To prove the homomorphisms, we map the vertices w, u, v and a from Figure 4 to the corresponding vertices $\phi(w), \phi(u), \phi(v)$ and $\phi(a)$ in Figure 9, respectively. The vertices b and c are mapped to the vertices $\phi(b)$ (or $\phi(c)$) and $\phi(c)$ (or $\phi(b)$) depending on the orientation of the edge bc . Without loss of generality, we can assume the edge to be oriented as \overrightarrow{bc} and assume that the vertices b and c map to the vertices $\phi(b)$ and $\phi(c)$, respectively.

Now, to complete the first homomorphism, map the vertices of $N^\alpha(w)$ to the unique vertex in $N^\alpha(\phi(w))$ for $\alpha \in \{+, -\}$.

To complete the second homomorphism, map the vertices of $N^\alpha(u) \cap N^\beta(u)$ to the unique vertex in $N^\alpha(\phi(u)) \cap N^\beta(\phi(v))$ for $\alpha, \beta \in \{+, -\}$.

To complete the third homomorphism, map the vertices of $N^\alpha(a) \cap N^\beta(t)$ to the unique vertex in $N^\alpha(\phi(a)) \cap N^\beta(\phi(t))$ for $\alpha, \beta \in \{+, -\}$ and $t \in \{b, c\}$.

Now, note that the first two oriented graphs depicted in Figure 9 are ocliques of order 3 and 6, respectively, while the third graph is not an oclique but clearly has absolute oriented clique number 5.

Hence, there is no triangle-free planar oclique of order more than 6. Also, the only example of a triangle-free oclique of order 6 is the second graph depicted in Figure 9.

(c) From the proof above, we know that there is no triangle-free planar oclique of order more than 6, and the only example of a triangle-free oclique of order 6 is the second graph depicted in Figure 9, which is a graph with girth 4. Hence, there is no planar oclique with girth at least 5 on more than 5 vertices, while the directed cycle of length 5 is clearly a planar oclique with girth 5.

(d) The 2-dipath is an oclique of order 3. From Plesník’s characterization, the rest of the proof follows easily. \square

6 Conclusions

In this paper we proved three main results regarding the order of planar ocliques. We provided an exhaustive list of spanning subgraphs of outerplanar graphs that admit an orientation with weak directed diameter at most two. Now the question is, can a similar result be proved for planar graphs also?

Question 6.1. *Characterize the set L of graphs such that a planar graph can be oriented as an oclique if and only if it contains one of the graphs from L as a spanning subgraph.*

We partially answer the question by proving that every planar oclique of order 15 must contain a particular oclique as a spanning subgraph. As the order of a planar oclique can at most be 15, a similar study for planar ocliques of order less than 15 will answer the question. We also proved tight upper bounds for the order of planar ocliques of girth at least k for all $k \geq 4$.

We defined the parameter oriented relative oclique number and used it for proving Theorem 1.7. Determining oriented relative clique number for different families of graphs, such as the family of planar graphs, seems to be an interesting direction of research.

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