

# Homomorphisms and colourings of oriented graphs

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SEVENTH CRACOW CONFERENCE ON GRAPH THEORY "RYTRO '14"

September 14-19, 2014

Rytró, Poland

# Preliminary (basic) notions

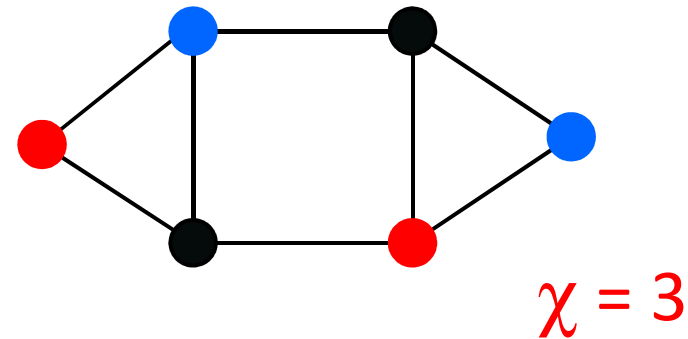
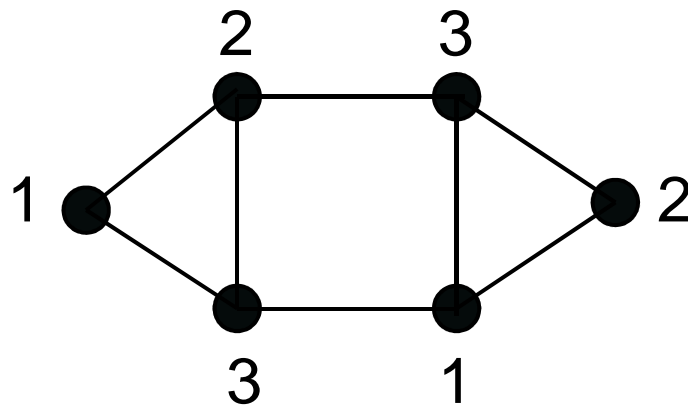
(1)

## Colourings of undirected graphs

A (proper) *k*-colouring of a graph  $G$  is a mapping

$$c : V(G) \rightarrow \{1, 2, \dots, k\}$$

such that every two *adjacent vertices* are assigned *distinct colours*.



The *chromatic number*  $\chi(G)$  of  $G$  is the smallest  $k$  for which  $G$  has a  $k$ -colouring.

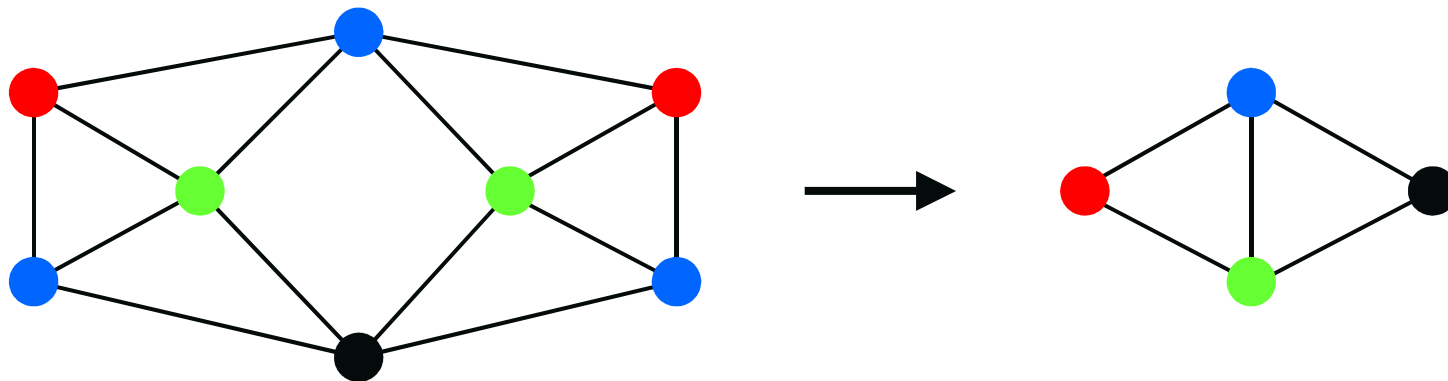
# Preliminary (basic) notions

(2)

## Homomorphisms of undirected graphs

A **homomorphism** from  $G$  to  $H$  is a mapping  $h : V(G) \rightarrow V(H)$  such that :

$$xy \in E(G) \Rightarrow h(x)h(y) \in E(H)$$

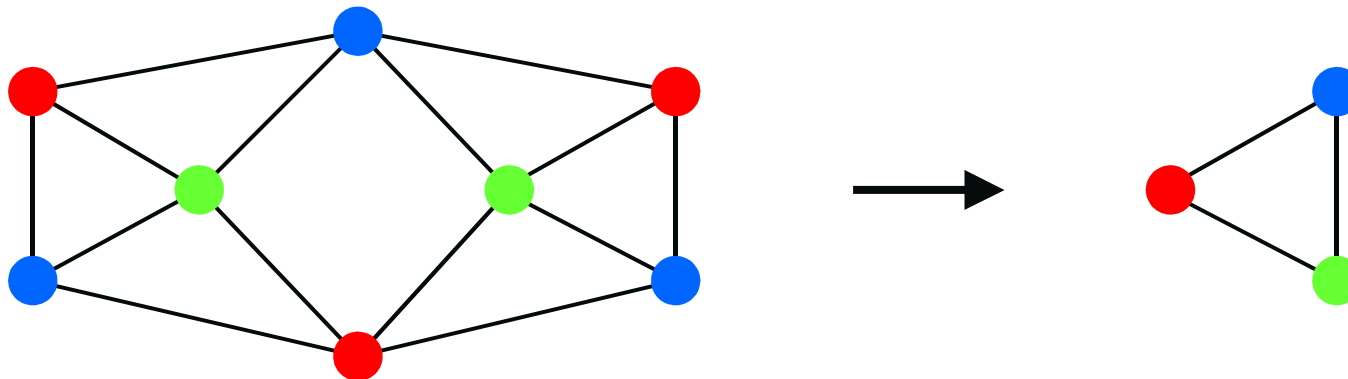


### Notation.

$G \rightarrow H$  : there exists a homomorphism from  $G$  to  $H$

## Colourings vs homomorphisms

A  $k$ -colouring of  $G$  is nothing but a *homomorphism* from  $G$  to  $K_k$ , the *complete graph* on  $k$  vertices.

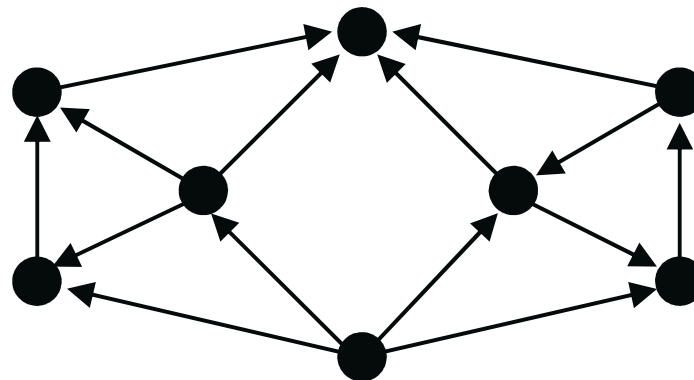


### Remark.

$\chi(G) = k$  if and only if  $G \rightarrow K_k$  and  $G \not\rightarrow K_{k-1}$

## Oriented graphs

An **oriented graph** is an *antisymmetric* digraph (no directed cycle of length 1 or 2).

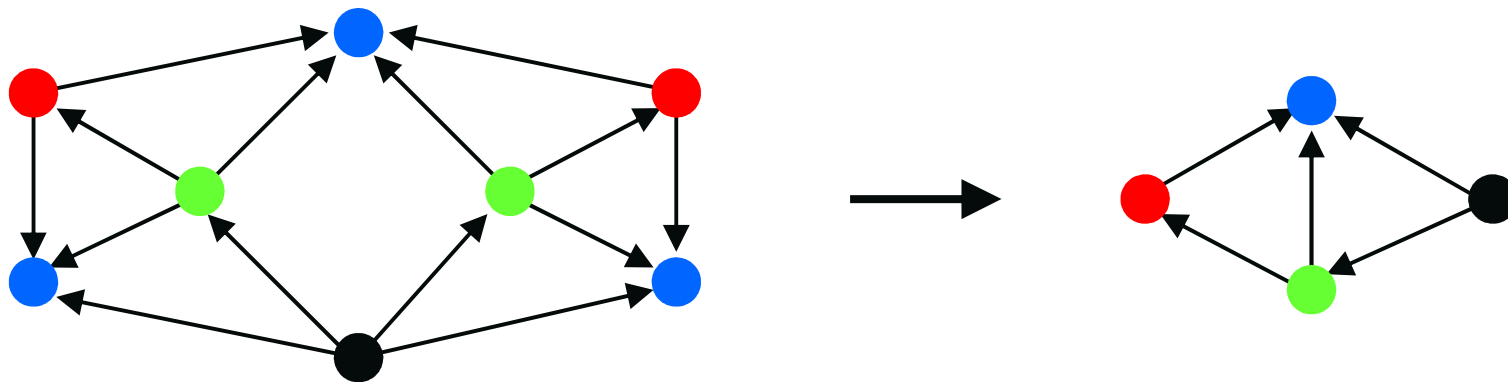


An oriented graph is an *orientation* of its underlying undirected graph, obtained by giving to each edge one of its two possible orientations.

## Homomorphisms of oriented graphs

A **homomorphism** from  $G$  to  $H$  is a mapping  $h : V(G) \rightarrow V(H)$  such that :

$$xy \in E(G) \Rightarrow h(x)h(y) \in E(H)$$



# Oriented colourings

(1)

## Oriented colourings of oriented graphs

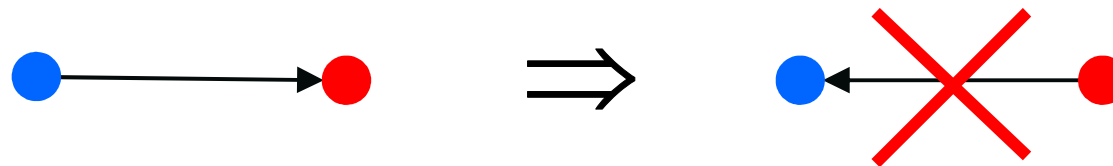
An **oriented  $k$ -colouring** of an oriented graph  $G$  is a mapping

$$c : V(G) \rightarrow \{1, 2, \dots, k\}$$

such that:

$$(1) uv \in E(G) \Rightarrow c(u) \neq c(v)$$

$$(2) uv, xy \in E(G), c(u) = c(y) \Rightarrow c(v) \neq c(x)$$



Hence, all the arcs linking two colour classes (independent sets) **must have the same direction** (non-local condition...).

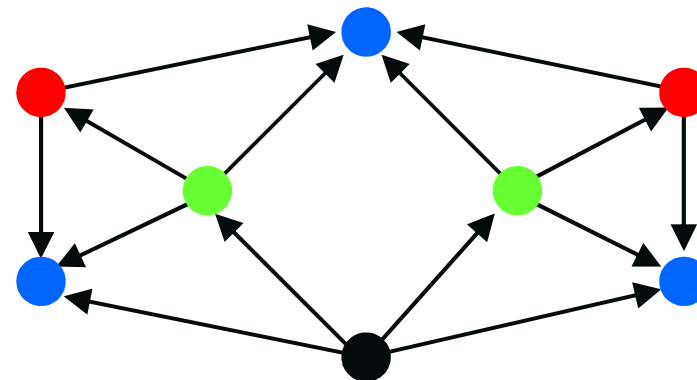
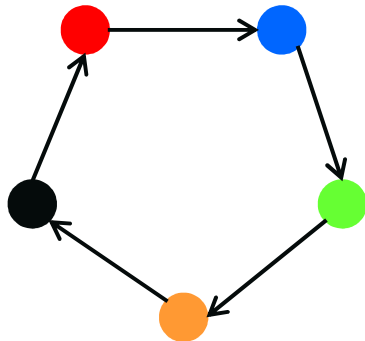
# Oriented colourings

(2)

## Examples.



*Any two vertices linked by a directed path of length 1 or 2 must get distinct colours.*



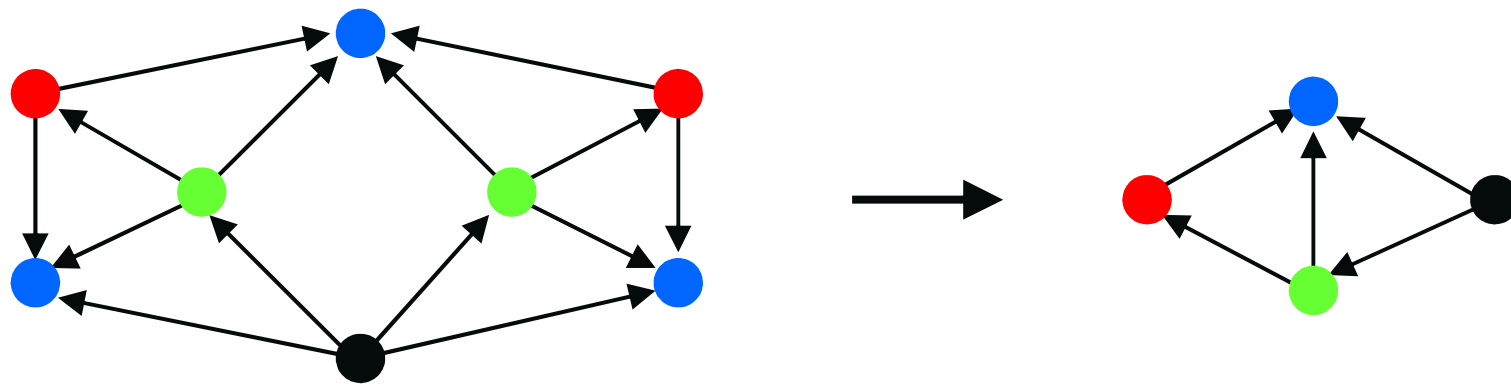


# Oriented colourings

(3)

## Remark.

An oriented  $k$ -colouring of an oriented graph is nothing but a *homomorphism* to a given oriented graph (or *tournament*) with  $k$  vertices.



*The target graph gives the orientation of arcs linking any two colour classes...*

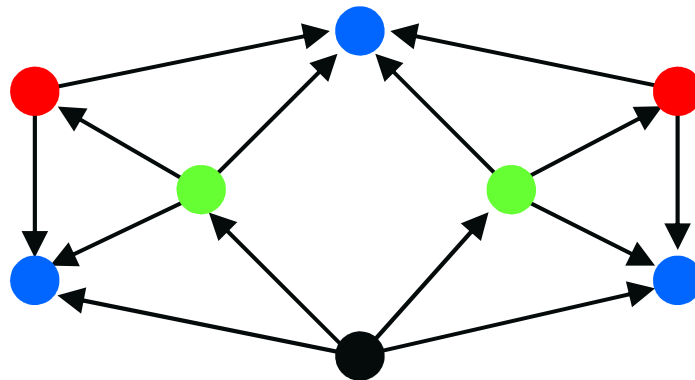
# Oriented chromatic number

(1)

## Oriented chromatic number of oriented graphs

The **oriented chromatic number**  $\chi_o(G)$  of an oriented graph  $G$  is the smallest  $k$  for which  $G$  admits an oriented  $k$ -colouring.

(Or, equivalently, the minimal order of an oriented graph  $H$  such that  $G \rightarrow H$ )

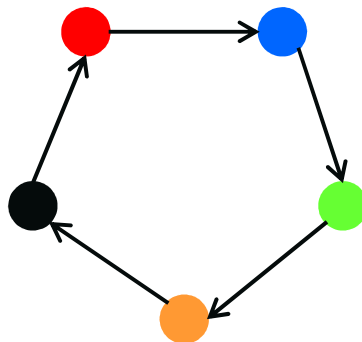
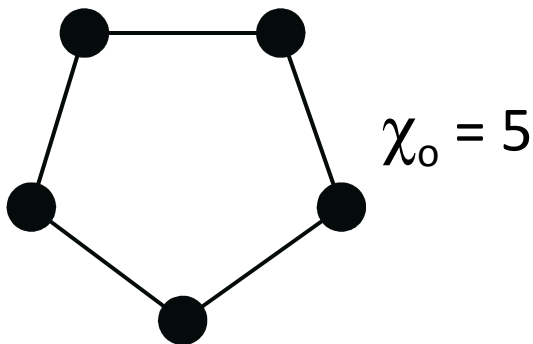


$$\chi_o = 4$$

## Oriented chromatic number of undirected graphs

The **oriented chromatic number**  $\chi_o(U)$  of an *undirected* graph  $U$  is the smallest  $k$  for which *every orientation* of  $U$  admits an oriented  $k$ -colouring:

$$\chi_o(U) = \max \{ \chi_o(G) ; G \text{ is an orientation of } U \}$$



**Observation.**

$$\chi(U) = \min \{ \dots \}$$

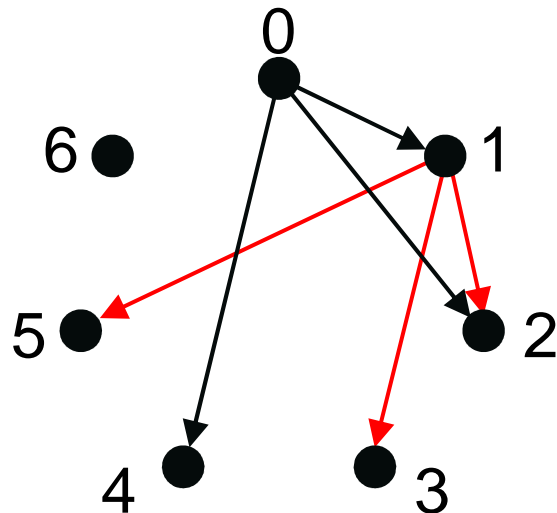
# $\chi_o$ of some graph classes

(1)

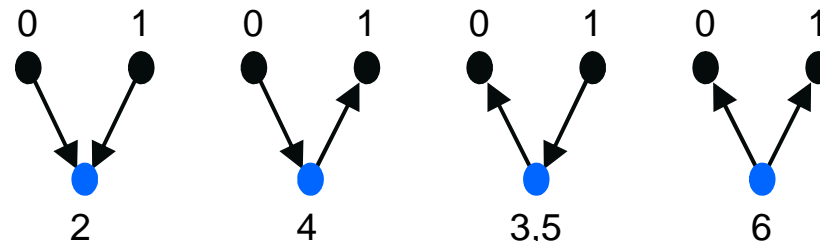
- If  $G$  is a forest, then  $\chi_o(G) \leq 3$  (easy)
- **Theorem.** If  $G$  is an outerplanar graph, then  $\chi_o(G) \leq 7$  (and this bound is tight)

The target graph is the tournament  $QR_7$ , defined as follows:

- $V(QR_7) = \{0, 1, \dots, 6\}$
- $uv \in E(QR_7)$  iff  $v - u \pmod{7} = 1, 2$  or  $4$  (non-zero quadratic residues of 7)



**Property.** For every arc  $uv \in E(QR_7)$ , there exists a vertex  $w$  for every possible orientation of the edges  $uw$  and  $vw$ :

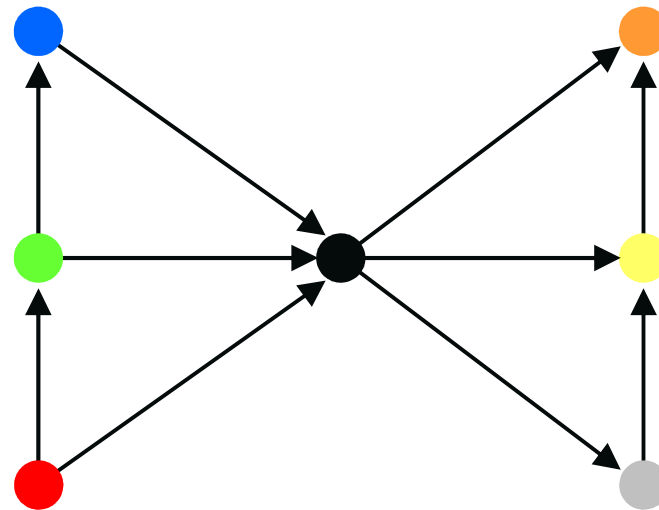


# $\chi_o$ of some graph classes

(1'')

- If  $G$  is a **forest**, then  $\chi_o(G) \leq 3$  (easy)
- **Theorem.** If  $G$  is an **outerplanar graph**, then  $\chi_o(G) \leq 7$  (and this bound is tight)

An outerplanar graph with oriented chromatic number 7:



## Graphs with bounded acyclic chromatic number

A  $k$ -colouring of an undirected graph  $U$  is **acyclic** if every cycle in  $U$  uses *at least 3 colours*.

(In other words, any two colours induce a forest.)

- **Theorem.** If  $G$  has acyclic chromatic number at most  $a$ , then  
 $\chi_o(G) \leq a \cdot 2^{a-1}$  (Raspaud, S., 1994)

*(and this bound is tight)*

(Ochem, 2005)

- **Theorem.** Every planar graph admits an acyclic 5-coloring (*and this bound is tight*) (Borodin, 1979)

- **Corollary.** If  $G$  is planar, then  $\chi_o(G) \leq 80$

*Best known lower bound : 18*

(Marshall, 2012)

# $\chi_0$ of some graph classes

(3)

## Planar graphs

The **girth**  $g(G)$  of  $G$  is the size of a *shortest* cycle in  $G$ .

➤ The best known results are as follows:

girth	lower bound	upper bound	
$\geq 3$	18	80	Marshall 2012 – Raspaud, S., 1994
$\geq 4$	11	40	Ochem, 2004 – Ochem, Pinlou, 2011
$\geq 5$	7	16	Marshall, 2012 – Pinlou, 2009
$\geq 6$	7	11	id. – Borodin, Kostochka, Nešetřil, Raspaud, S., 1999
$\geq 7$	6	7	Nešetřil, Raspaud, S., 1997 – Borodin, Ivanova 2005
$\geq 8$	5	7	Nešetřil, Raspaud, S., 1997 – id.
$\geq 11$	5	6	id. – Ochem, Pinlou, 2008
$\geq 12$	5	5	id. – Borodin, Ivanova, Kostochka, 2007

## Graphs with bounded degree

- Every graph  $G$  with **maximum degree 2**, except the directed cycle on 5 vertices, satisfies  $\chi_o(G) \leq 4$  (easy)



# Oriented cliques (o-cliques)

(1)

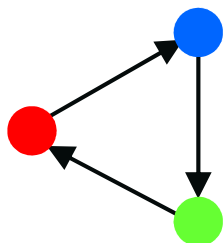
A well-known fact is that the (ordinary) chromatic number  $\chi(G)$  of an undirected graph  $G$  is bounded from below by the clique number  $\omega(G)$  of  $G$  (maximum order of a clique in  $G$ ):  $\chi(G) \geq \omega(G)$ .

Of course, a similar relation holds for oriented graphs...

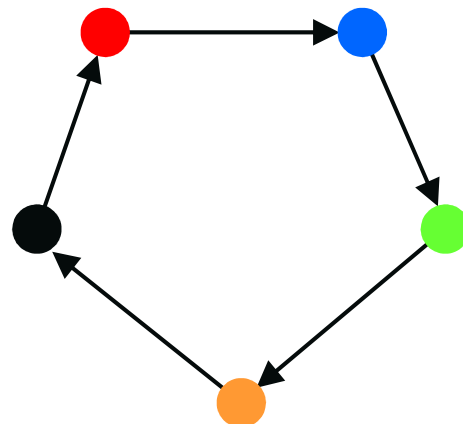
## Oriented cliques

An **oriented clique**  $C$  is an oriented graph satisfying  $\chi_o(C) = |V(C)|$ .

### Examples.



(all tournaments)

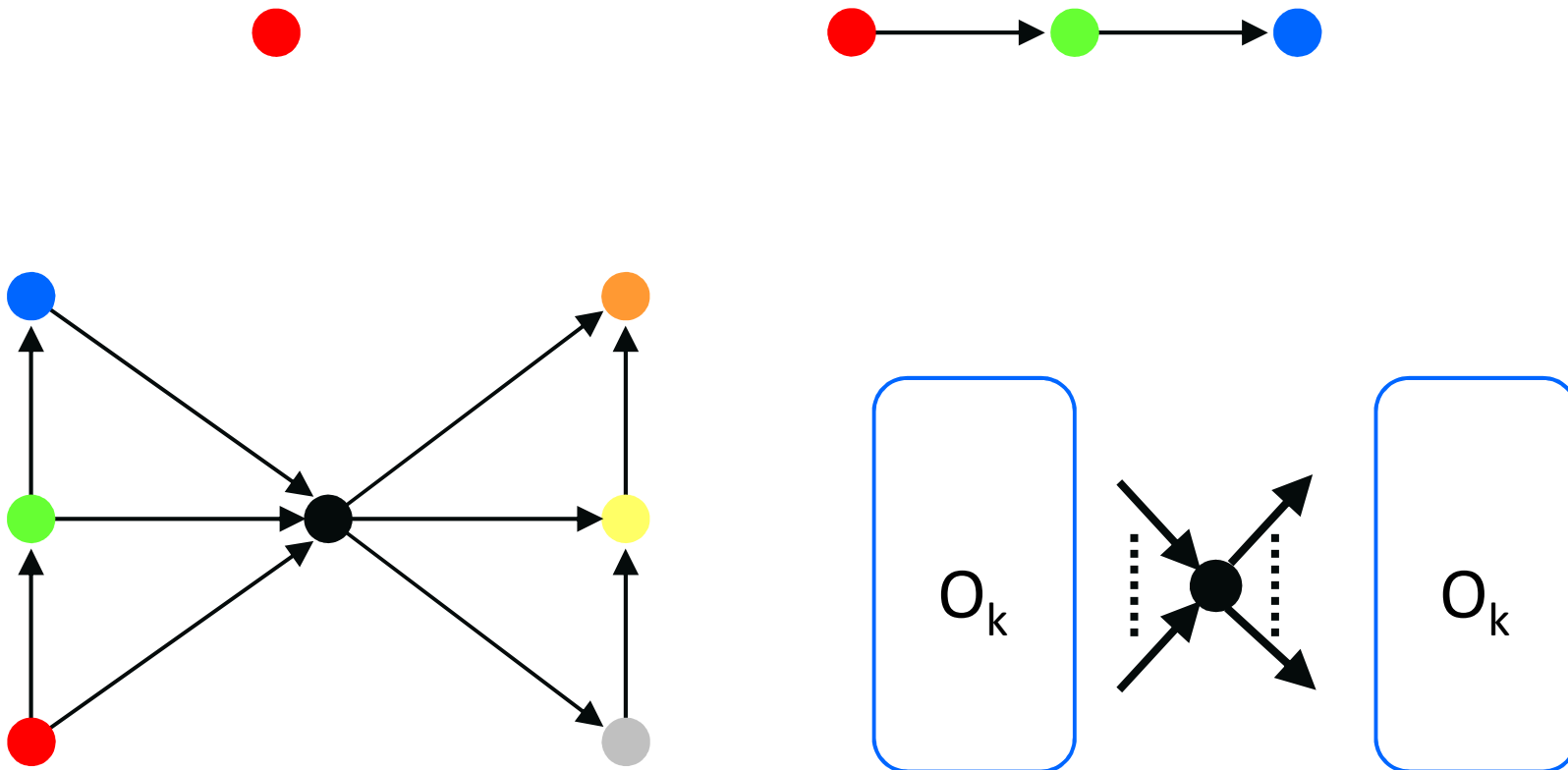


**Remark.** An o-clique is nothing but an oriented graph in which any two vertices are linked by a directed path (in any direction) of length at most 2.

# Oriented cliques (o-cliques)

(2)

## Building oriented o-cliques of order $2^k - 1$



## Structural properties of o-cliques

- **Theorem.** The minimum number of edges in an o-clique of order  $n$  is  $(1 + o(1))n \log_2 n$ .

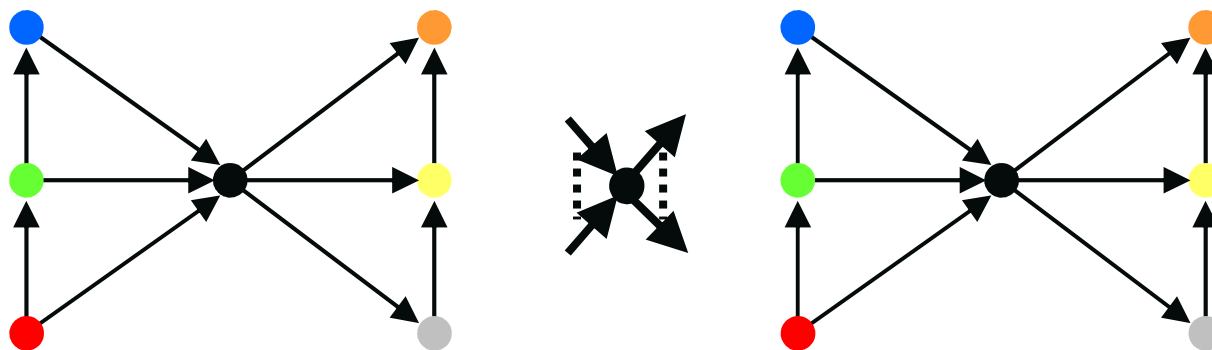
(Füredi, Horak, Parrek, Zhu, 1998 – Kostochka, Łuczak, Simonyi, S., 1999)

- **Theorem.** The order of a *planar* o-clique is at most 36.

(Klostermeyer, MacGillivray, 2002)

- **Theorem.** The maximum order of a *planar* o-clique is 15.

(Sen, 2012)



# Oriented clique numbers...

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(1)

The oriented clique number of an oriented graph may be defined in two different ways...

## Absolute oriented clique number

The **absolute oriented clique number**  $\omega_{ao}(G)$  of an oriented graph  $G$  is the maximum order of an o-clique subgraph of  $G$ .

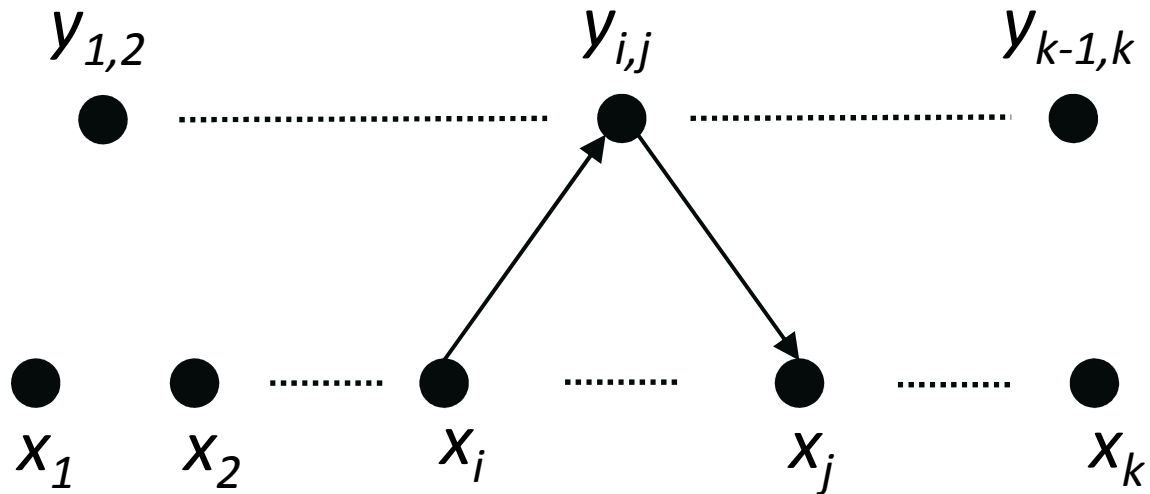
## Relative oriented clique number

The **relative oriented clique number**  $\omega_{ro}(G)$  of an oriented graph  $G$  is the maximum size of a subset  $S$  of  $V(G)$  satisfying: every two vertices in  $S$  are linked (in  $G$ ) by a directed path *of length at most 2*.

# Oriented clique numbers...

(2)

## Example.



$$\omega_{ao} = 3$$
$$\omega_{ro} = k$$

Clearly, for every oriented graph  $G$ , we have:

$$\omega_{ao}(G) \leq \omega_{ro}(G) \leq \chi_o(G)$$

# Oriented clique numbers...

(3)

For planar graphs with given girth, the following is known:

<b>girth</b>	$\omega_{ao}$	$\omega_{ro}$
3	15	$15 \leq \dots \leq 80$
4	6	$10 \leq \dots \leq 26$
5	5	6
6	3	4
$\geq 7$	3	3

(Sen, 2013+)

(Nandy, Sen, S., 2014+)

Consider the following decision problem:

$OCN_k$ : oriented  $k$ -colorability  
INSTANCE: an oriented graph  $G$   
QUESTION: do we have  $\chi_o(G) \leq k$ ?

We have the following:

➤ **Theorem.**  $OCN_k$  is polynomial if  $k \leq 3$  and NP-complete if  $k \geq 4$ .

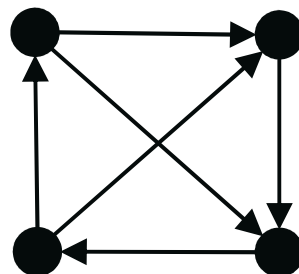
(Klostermeyer, MacGillivray, 2002)

Even even more:

- **Theorem.** For every fixed  $g \geq 3$ ,  $\text{ocN}_4$  is NP-complete for oriented graphs that are planar, with girth at most  $g$ , bipartite, subcubic, with DAG-depth 3, with maximum outdegree 2 and maximum indegree 2, and such that every 3-vertex is adjacent to at most one 3-vertex!

(Guegan, Ochem, 2014+)

- **Remark.** It is *polynomial* to decide whether an oriented graph admits a homomorphism to a tournament  $T$  of order 4, except when  $T$  is the following (contains a directed cycle of length 4):





Another related problem:

$\text{OWD}_k$ : orientation with weak diameter  $k$

INSTANCE: an undirected graph  $U$

QUESTION: does  $U$  admit an orientation with weak diameter  $k$ ?

*weak distance:  $d_w(u,v) = \min \{ d(u,v), d(v,u) \}$*

➤ **Theorem.**  $\text{OWD}_k$  is NP-complete if  $k \geq 2$ .

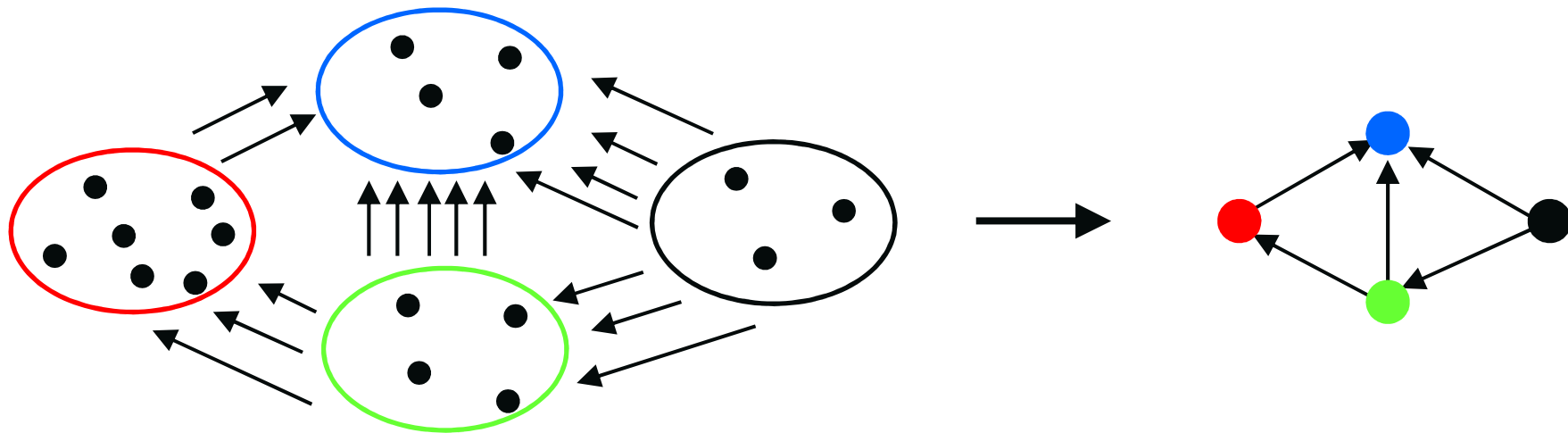
(Bensmail, Duvignau, Kirgizov, 2013+)

➤ **Corollary.** It is NP-complete to decide whether an undirected graph  $U$  admits an orientation which is an o-clique...

# Simple colourings

(1)

(Recall that) An oriented  $k$ -colouring of an oriented  $G$  is a partition of  $V(G)$  into  $k$  independent sets in such a way that all the arcs joining any two such sets have the same direction:



## Simple colourings

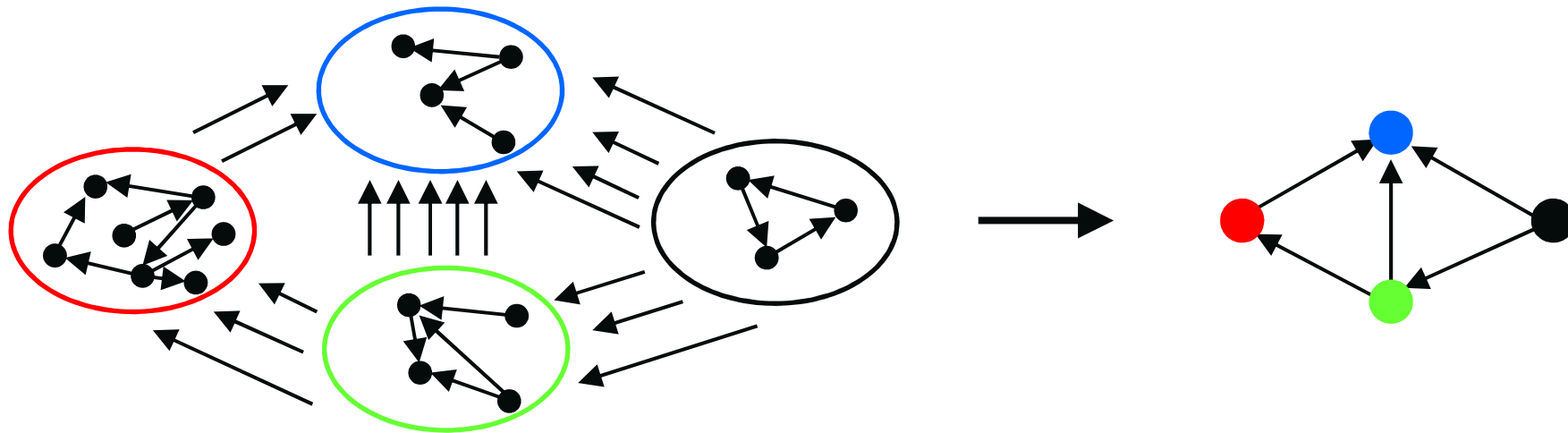
- do not require the parts to be independent sets
- require at least two parts...

(Nešetřil, Smolíková, 2000)

# Simple colourings

(1')

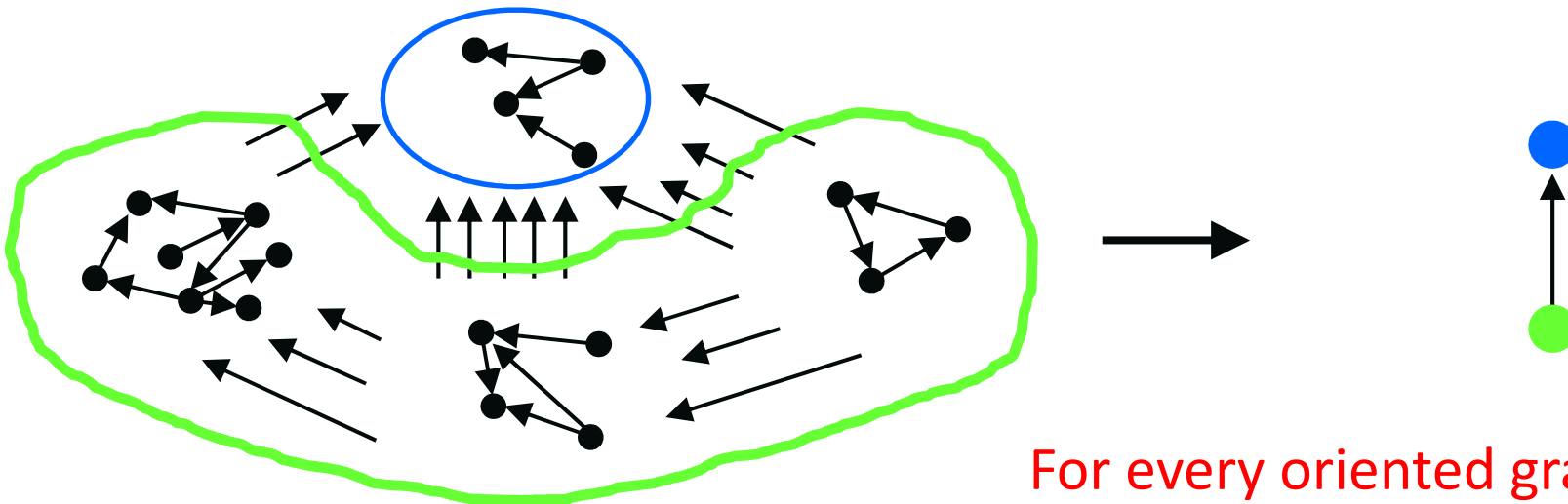
(Recall that) An oriented  $k$ -colouring of an oriented  $G$  is a partition of  $V(G)$  into  $k$  independent sets in such a way that all the arcs joining any two such sets have the same direction:



## Simple colourings

- do not require the parts to be independent sets
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## Simple colourings

- do not require the parts to be independent sets
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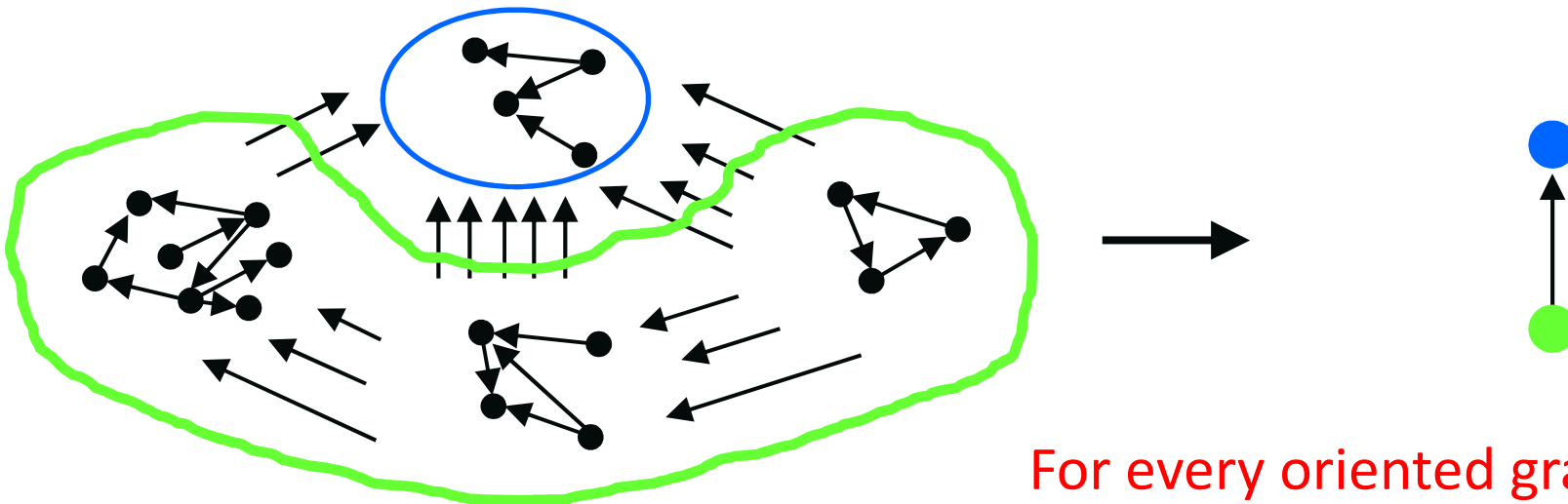
(Nešetřil, Smolíková, 2000)

For every oriented graph  $G$ ,  
 $\chi_s(G) \leq \chi_o(G)$

# Simple colourings

(2)

- **Theorem.** The maximum oriented chromatic number of planar graphs and the maximum simple chromatic number of planar graphs coincide... (Smolíková, 2000)



## Simple colourings

For every oriented graph  $G$ ,  
 $\chi_s(G) \leq \chi_o(G)$

- do not require the parts to be independent sets
- require at least two parts... (Nešetřil, Smolíková, 2000)

# 2-dipath colouring

(1)

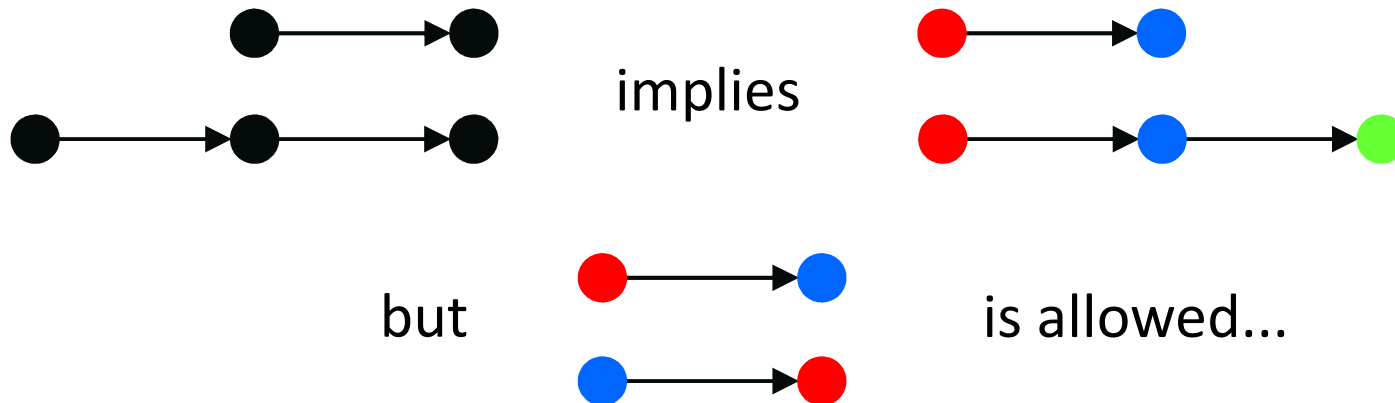
In 2006, Chen and Wang introduced another weaker version of oriented colouring:

## 2-dipath colouring

A **2-dipath  $k$ -colouring** of an oriented graph  $G$  is a mapping

$$c : V(G) \rightarrow \{1, 2, \dots, k\}$$

such that any two vertices linked by a **directed path of length 1 or 2** get distinct colours.



# 2-dipath colouring

(2)

**Remark.** A 2-dipath colouring of an oriented graph  $G$  may also be viewed as a  $L(1,1)$ -labelling of  $G$  (using directed distance).

We have the following results:

- For every oriented graph  $G$ ,  $\chi_{2d}(G) \leq \chi_o(G)$  (definition)
- For every oriented graph  $G$ ,  $\chi_{2d}(G) \geq \omega_{ro}(G) \geq \omega_{ao}(G)$  (definition)

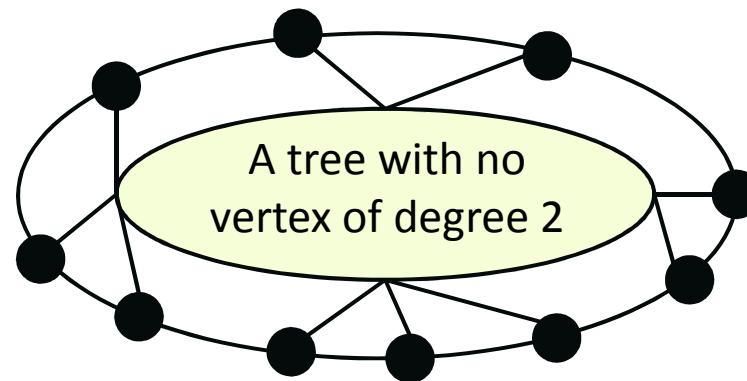
From these observations, we get:

- If  $G$  is an oriented outerplanar graph, then  $\chi_{2d}(G) \leq 7$ , and this bound is tight  
(recall that there exists an outerplanar o-clique of order 7)
- If  $G$  is an oriented planar graph, then  $\chi_{2d}(G) \leq 80$ , and there exist planar graphs with 2-dipath chromatic number 15.

# 2-dipath colouring

(3)

- **Theorem.** If  $G$  is an oriented Halin graph, then  $\chi_{2d}(G) \leq 7$  (and this bound is tight).  
(Chen, Wang, 2006)



- **Theorem.** Determining whether an oriented graph is 2-dipath  $k$ -colourable is polynomial if  $k \leq 2$  and NP-complete if  $k \geq 3$   
(MacGillivray, Sherk, 2014)



# A homomorphism model

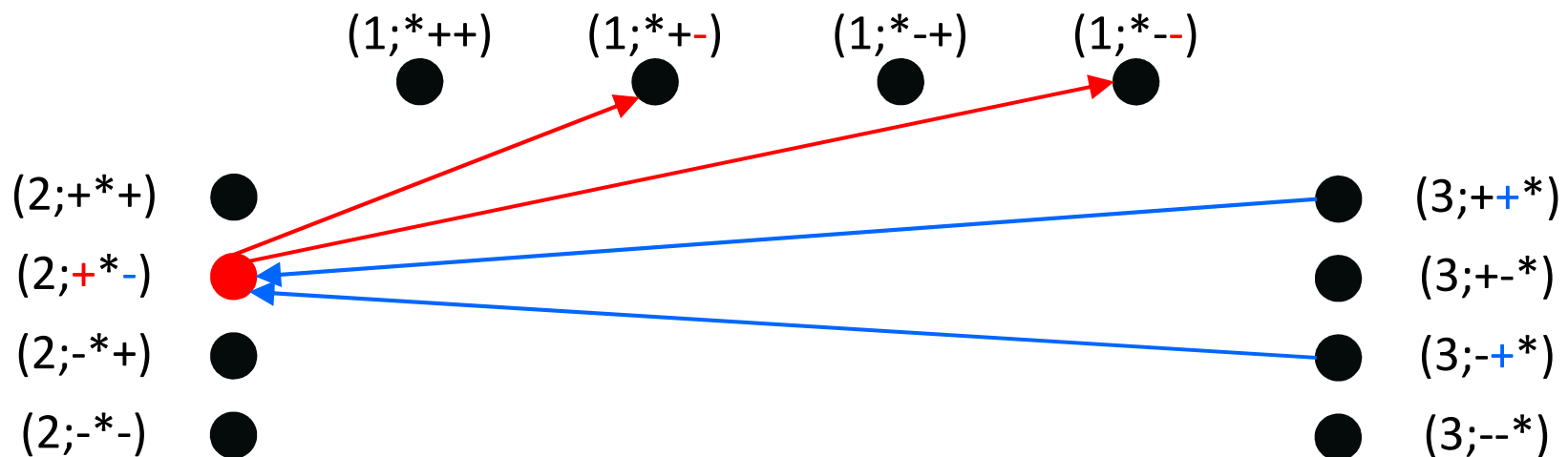
(1)

[MacGillivray, Sherk, 2014]

Let  $G_k$ ,  $k \geq 1$ , be the oriented graph defined as follows:

- $V(G_k) = \{ (u_0 ; u_1, \dots, u_k) : 1 \leq u_0 \leq k, u_i \in \{+, -\} \text{ if } i \neq u_0, u_{u_0} = * \}$
- $E(G_k) = \{ (u_0 ; u_1, \dots, u_k)(v_0 ; v_1, \dots, v_k) : u_{v_0} = +, v_{u_0} = - \}$

**Example.** The oriented graph  $G_3$  ( $3 \cdot 2^2 = 12$  vertices)



# A homomorphism model

(2)

- **Theorem.** For every  $k \geq 1$  and every oriented graph  $G$ ,
- $$\chi_{2d}(G) \leq k \text{ iff } G \rightarrow G_k$$

(MacGillivray, Sherk, 2014)

