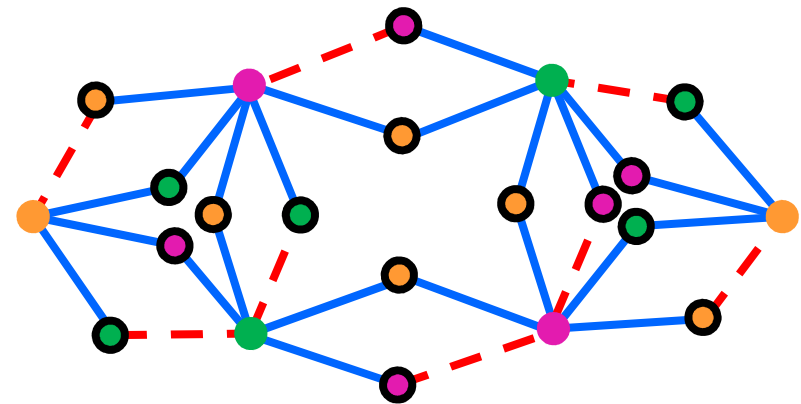


Recent Developments on Homomorphisms and Colourings of Signed Graphs



Éric SOPENA

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Recent Developments on Homomorphisms and Colourings of Signed Graphs



Based on joint work(s) with Julien Bensmail, Clément Charpentier, Reza Naserasr, Théo Pierron, Edita Rollová, Thomas Zaslavsky.

Éric SOPENA

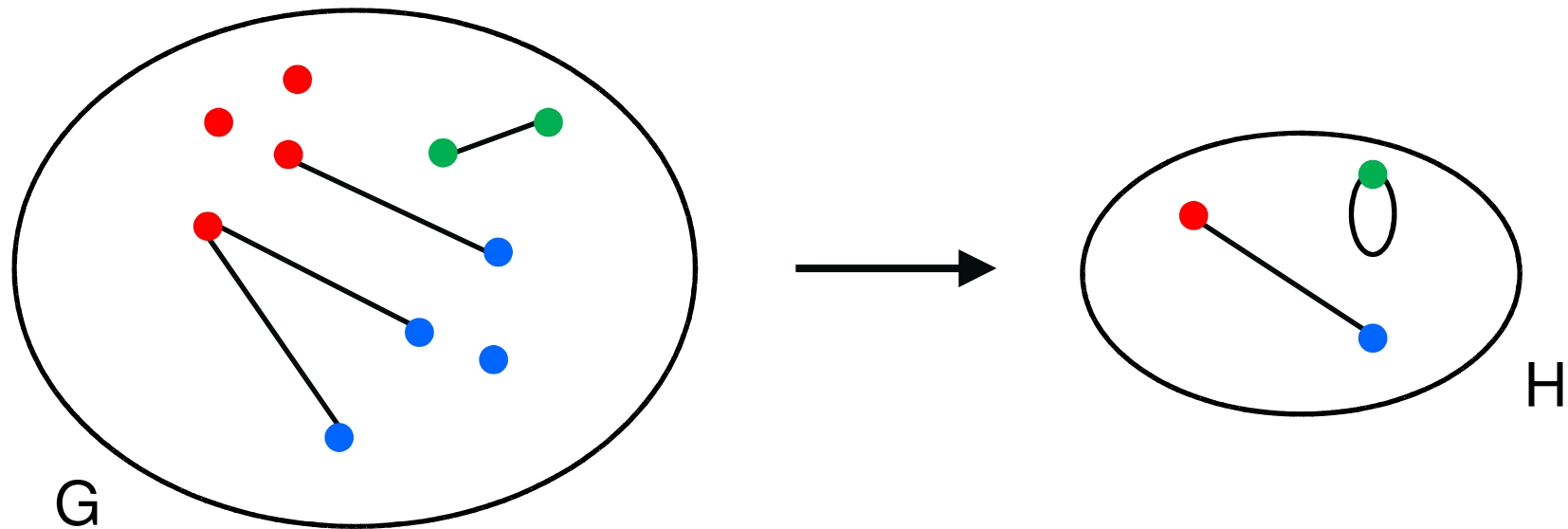
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France

Homomorphisms and colourings of (ordinary) graphs

Homomorphisms of (ordinary, simple) graphs

Homomorphisms



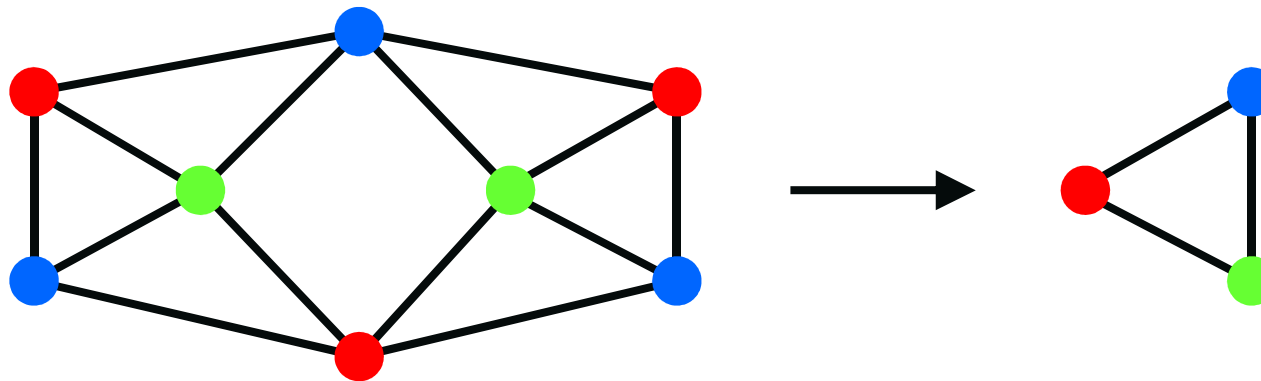
A **homomorphism** from G to H is a mapping $h : V(G) \rightarrow V(H)$ satisfying $xy \in E(G) \Rightarrow h(x)h(y) \in E(H)$.

Note that if H has no loops, then each colour class must be an independent set.

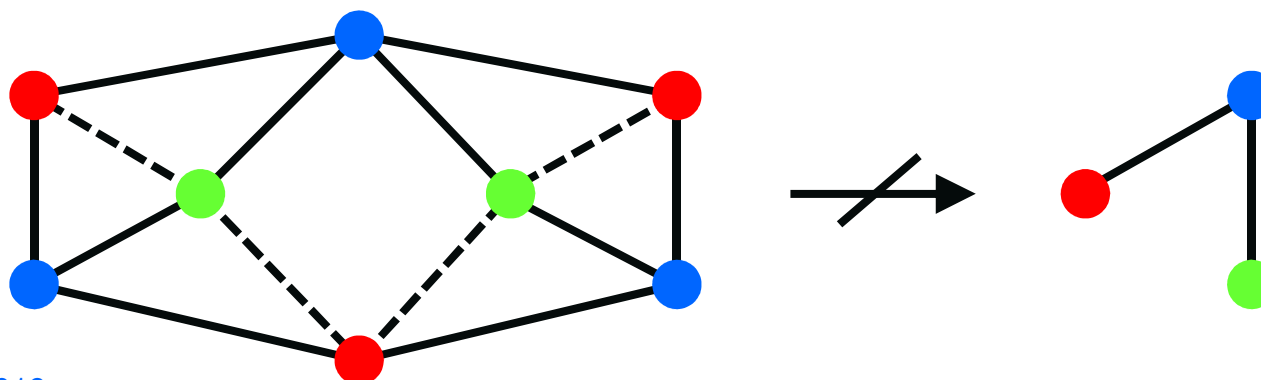
Homomorphisms and (proper vertex) colourings

Homomorphisms vs colourings

A (proper vertex) k -colouring of G can be equivalently defined as a homomorphism from G to the complete graph K_k .



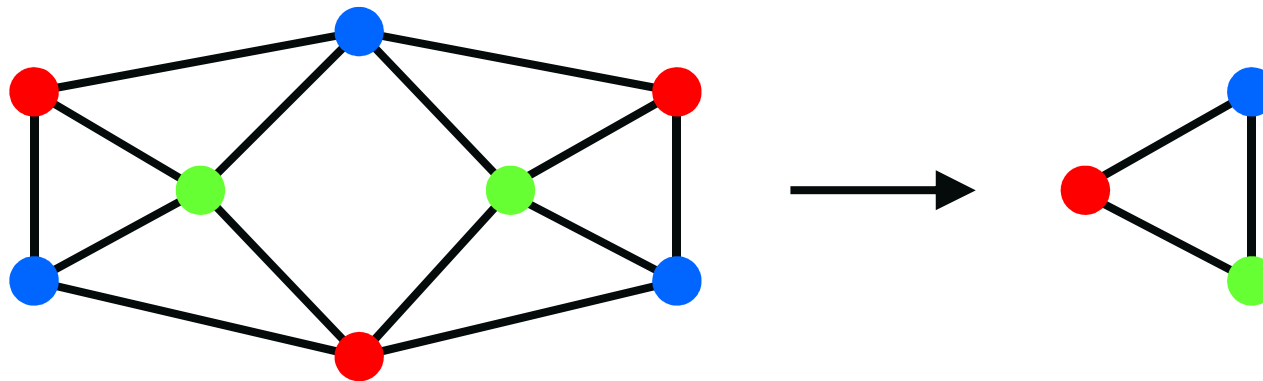
If the target graph H is not complete, then **non-edges** matter!



Homomorphisms and (proper vertex) colourings

Homomorphisms vs colourings

A (proper vertex) k -colouring of G can be equivalently defined as a homomorphism from G to the complete graph K_k .



The target graph H can be obtained from G by a sequence of **identifications** of **non-adjacent** vertices.

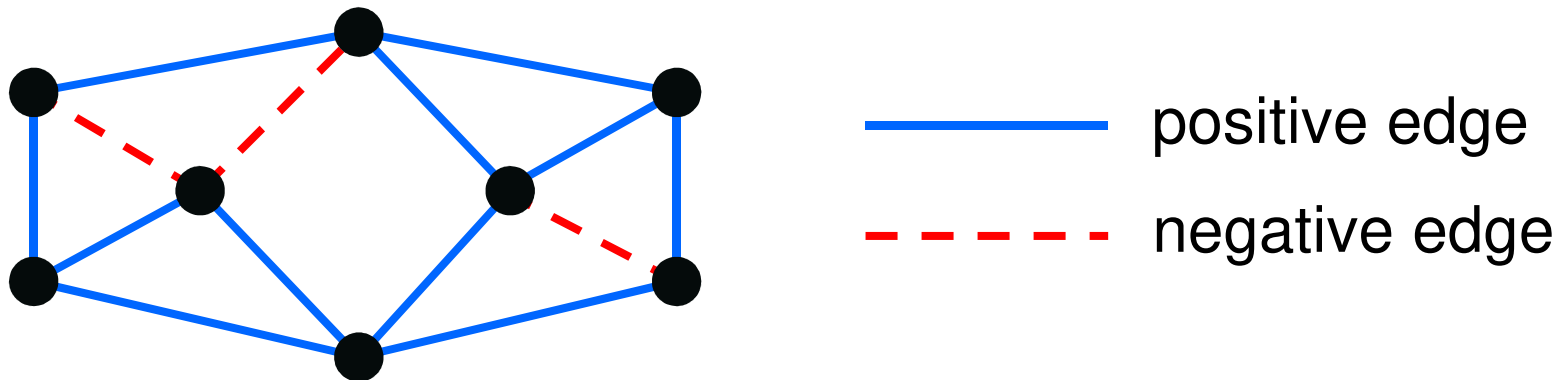
- The **chromatic number** $\chi(G)$ of G is the **smallest order** of a **complete** graph obtained in this way.
- The **achromatic number** $\psi(G)$ of G is the **largest order** of a **complete** graph obtained in this way.

Signed graphs

Signed graphs

Signed graphs

A **signed graph** is a pair (G, σ) , where G is a graph and σ a mapping (the **signature** of (G, σ)) that assigns a **sign** from the set $\{+, -\}$ to each edge of G .

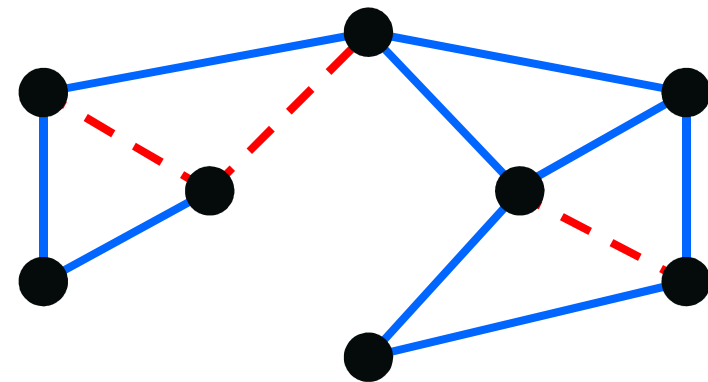
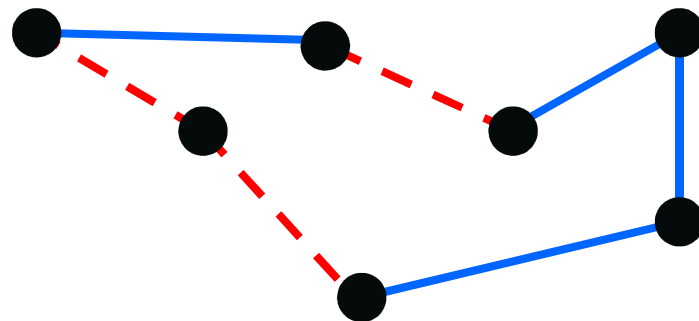


Introduced by [Harary \(1953\)](#), as a model for social networks.

Positive and negative (sub)graphs

Sign of a subgraph

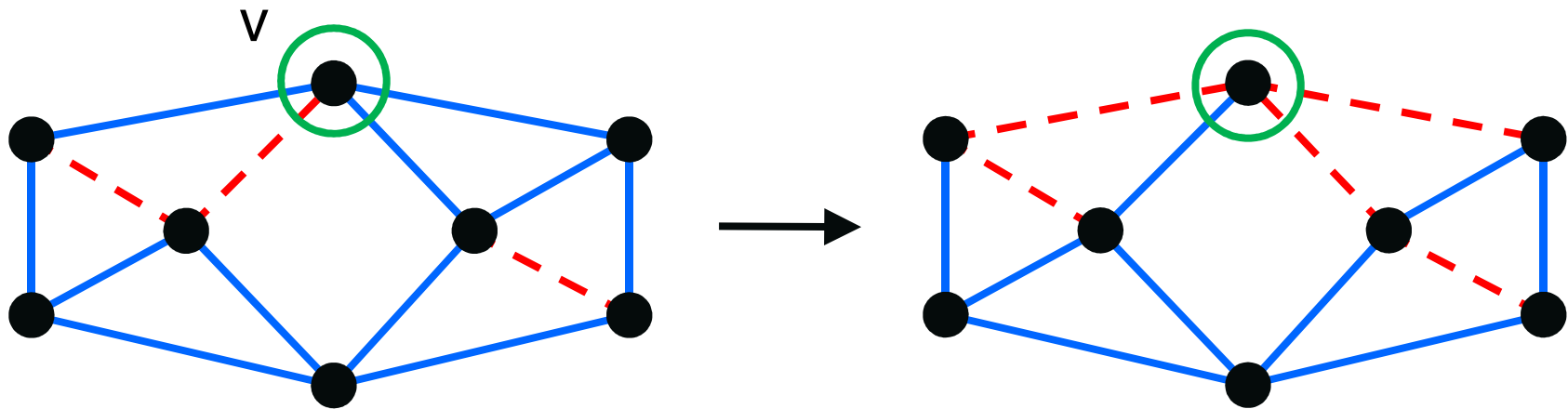
A **subgraph** of a signed graph (G, σ) is **positive** (resp. **negative**) if and only if the **product** of the signs of its edges is **positive** (resp. **negative**).



Vertex switching

Switching at a vertex

Switching at a vertex v in a signed graph (G, σ) consists in changing the sign of all edges incident with v .



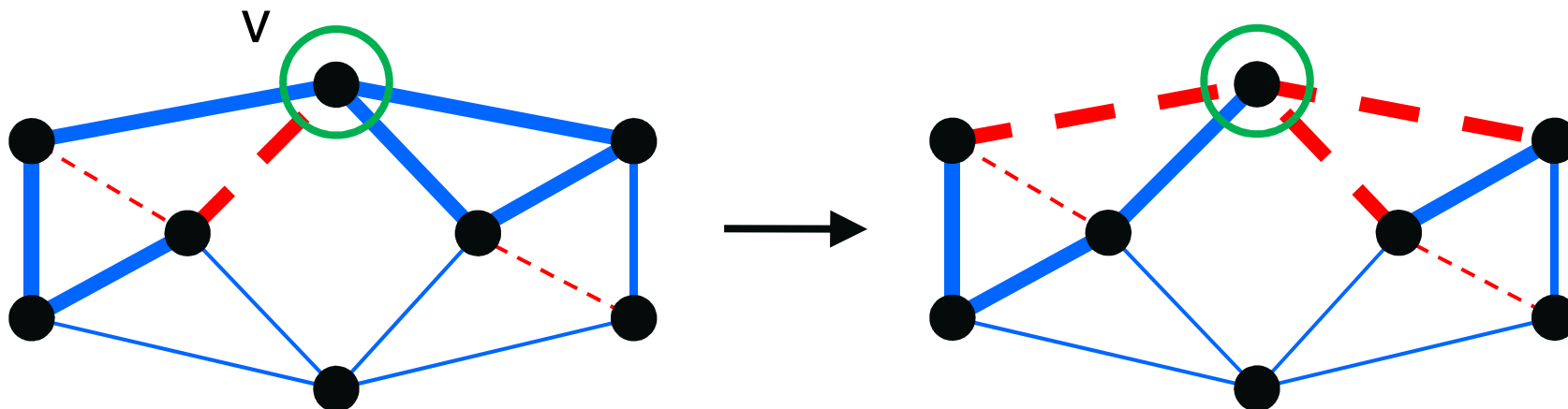
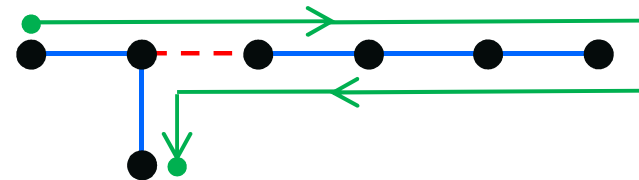
Vertex switching

Vertex switching and closed walks

Switching at a vertex v in a signed graph (G, σ) **does not modify the sign of closed walks** (in particular, of **cycles**) in (G, σ) .

(When computing the sign of a walk, the sign of each edge is counted according to its multiplicity...)

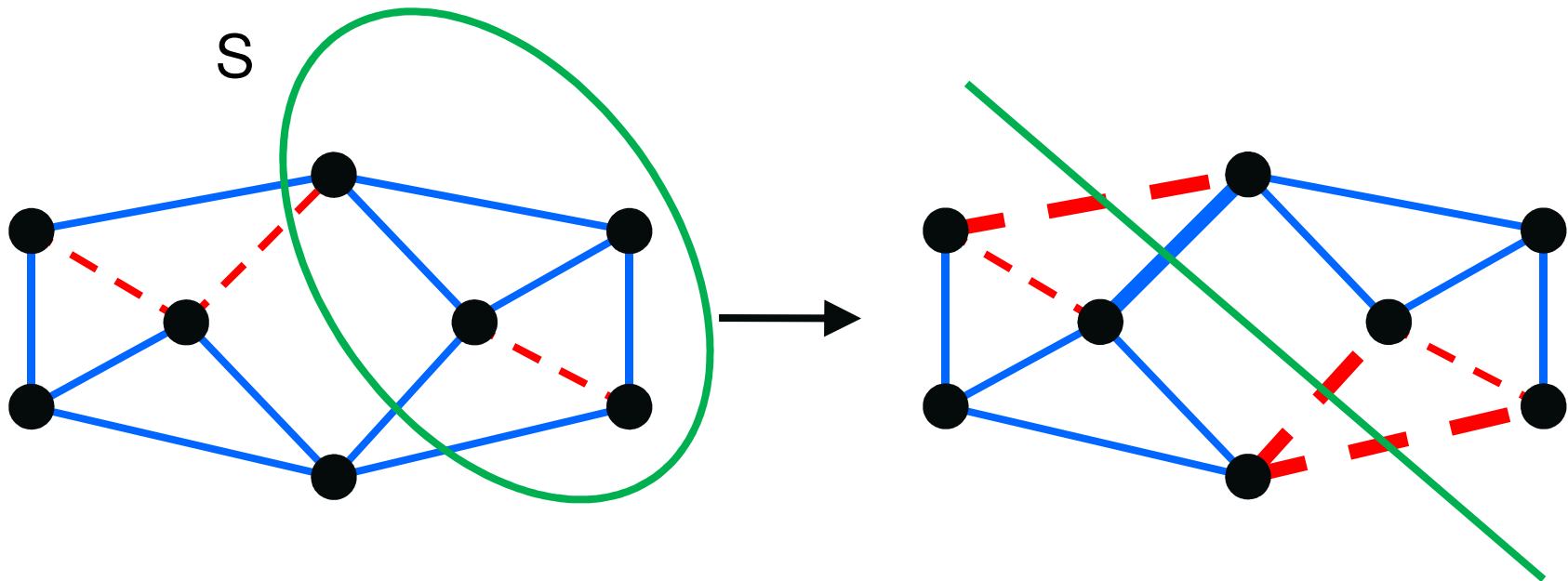
A **positive** walk in a **negative** subgraph:



Vertex switching

Sequence of vertex switchings

Switching a set of vertices $S = \{v_1, \dots, v_k\}$ (in any order) amounts to change the sign of all edges of the **edge-cut** separating S from $V(G) \setminus S$.

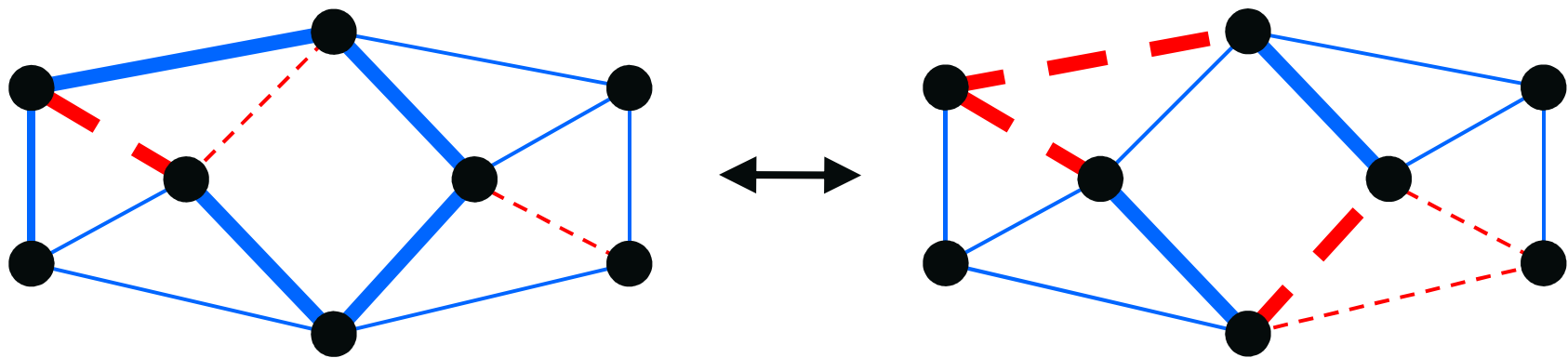


Equivalent signatures

Equivalent signatures

Two signatures σ and σ' of a graph G are **equivalent**, if and only if one can obtain (G, σ') from (G, σ) by a **sequence of vertex switchings**.

Theorem (Zaslavsky, 1982). *Two signatures σ and σ' of a graph G are **equivalent** if and only if (G, σ) and (G, σ') have **the same set of negative cycles**.*



Note. *Equivalence can be decided in time $O(n^2)$.*

Equivalent signatures

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- When two signatures σ and σ' are **equivalent**, the signed graphs (G, σ) and (G, σ') are often considered as “**the same signed graph**”.

(A signed graph is then viewed as an “**equivalence class** of signed graphs”.)

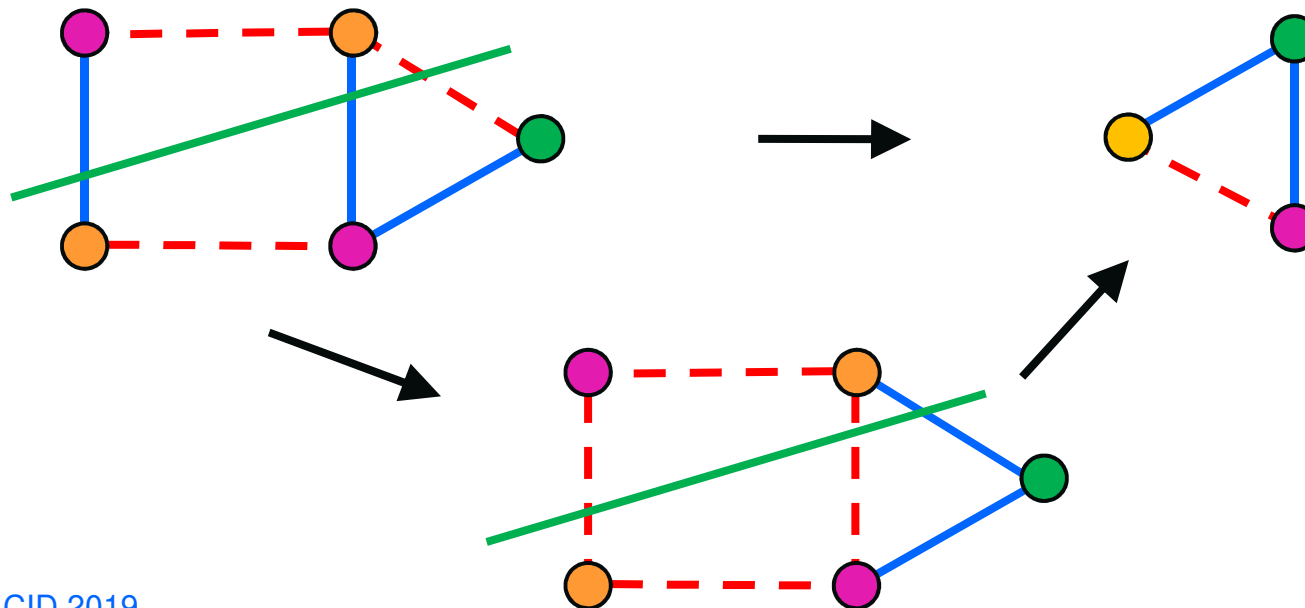
Homomorphisms of signed graphs

Homomorphisms of signed graphs

Homomorphisms of signed graphs

A **homomorphism** from (G, σ) to (H, π) is a homomorphism from G to H that **preserves the sign of closed walks**.

Equivalently (and more intuitively), a **homomorphism** from (G, σ) to (H, π) is a **sign-preserving homomorphism** from (G, σ') to (H, π) , such that σ' is a signature of G **equivalent to σ** .



Four types of walks (closed walks, cycles)

Sign and parity

Each **walk** has a **sign** (the product of the signs of its edges, with multiplicity) and a **parity** (the parity of the number of its edges).

This gives us **four types** of walks (or closed walks, or cycles):

1. positive and even (type **+e**),
2. positive and odd (type **+o**),
3. negative and even (type **-e**),
4. negative and odd (type **-o**).

Note that if **uv** is an edge (of any sign), then the closed walk **uvu** is always a **positive even closed walk**...

Four types of girth

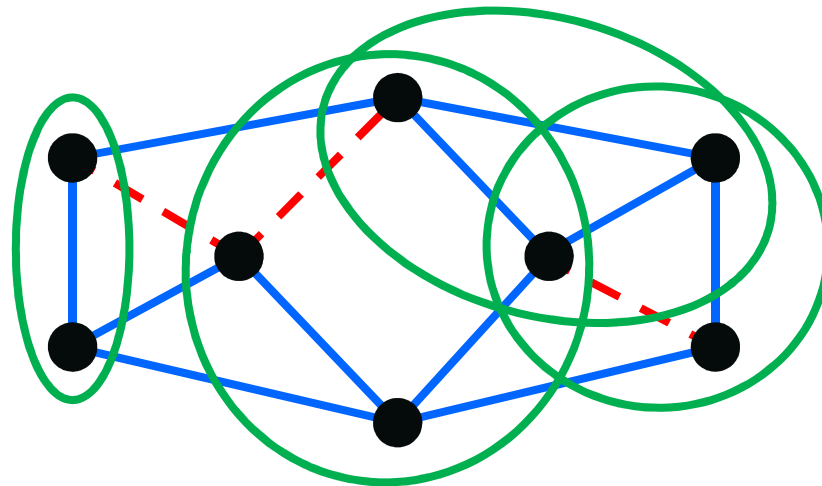
Girth

Girth $g(G)$ of G : length of a shortest cycle in G

These four types of walks allow us to define **four types of girth**:

- The **girth of type $\alpha\beta$** of (G, σ) , $\alpha \in \{+, -\}$ and $\beta \in \{e, o\}$, denoted $g_{\alpha\beta}(G, \sigma)$, is the length of a **shortest closed walk of type $\alpha\beta$** in (G, σ) , or ∞ if no such walk exists.

(Note that the girth of type **$+e$** is always **2**, or ∞ if G has no edges.)



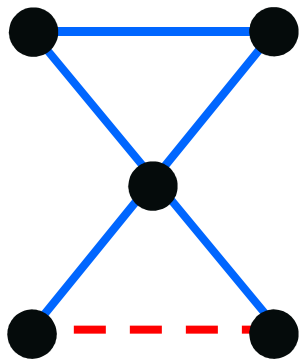
$$\begin{aligned}g_{+e} &= 2, \\g_{+o} &= 3, \\g_{-e} &= 4, \\g_{-o} &= 3.\end{aligned}$$

Note: Computing $g_{\alpha\beta}(G, \sigma)$ can be done in **polynomial time** for every α and β .

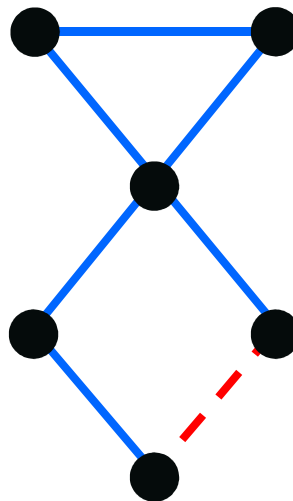
Four types of girth

Closed walks vs cycles

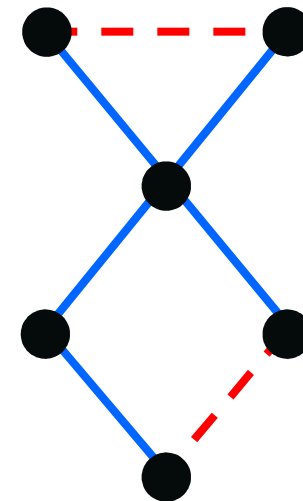
The girth of type $\alpha\beta$ may be realized only by a closed walk which is **not a cycle**:



$$g_{-e} = 6$$



$$g_{-o} = 7$$



$$g_{+o} = 7$$

Three special signed graph classes

Signed graph classes

These four types of girth allow us to define **three types of signed graphs** that are of particular interest:

- The class of **all-positive** signed graphs
 $\{ (G, \sigma) / g_{-e}(G, \sigma) = g_{-o}(G, \sigma) = \infty \}$: **no negative cycle**
(G, σ) can be switched to (G, +)
- The class of **all-negative** signed graphs
 $\{ (G, \sigma) / g_{+e}(G, \sigma) = g_{+o}(G, \sigma) = \infty \}$: **no positive cycle**
(G, σ) can be switched to (G, -)
- The class of signed **bipartite** graphs
 $\{ (G, \sigma) / g_{-o}(G, \sigma) = g_{+o}(G, \sigma) = \infty \}$: **no odd cycle**

Note that signed forests belong to these three classes...

A “no-homomorphism” lemma

Homomorphisms and girth (of ordinary graphs)

There exists a homomorphism from the cycle C_m to the cycle C_n , if and only if (1) m is even, or (2) m and n are odd, and $m \geq n$.

Therefore, $G \rightarrow H$ implies $og(G) \geq og(H)$.

(The odd-girth is the length of a shortest odd cycle.)

Homomorphisms and girth (of signed graphs)

Lemma. *If there exists a homomorphism from (G, σ) to (H, π) , then, for every $\alpha \in \{+, -\}$ and $\beta \in \{e, o\}$, $g_{\alpha\beta}(G, \sigma) \geq g_{\alpha\beta}(H, \pi)$.*

- *This gives four necessary conditions for the existence of a homomorphism.*
- *When are these conditions also sufficient?*

Sparse graphs

Homomorphisms and girth (of signed graphs)

If there exists a homomorphism from (G, σ) to (H, π) , then, for every $\alpha \in \{+, -\}$ and $\beta \in \{e, o\}$, $g_{\alpha\beta}(G, \sigma) \geq g_{\alpha\beta}(H, \pi)$.

Theorem (Charpentier, Naserasr, S., 2019+).

For every signed graph (H, π) , there exists a constant ε such that, if $\text{mad}(G) < 2 + \varepsilon$ and $g_{\alpha\beta}(G, \sigma) \geq g_{\alpha\beta}(H, \pi)$ for every $\alpha \in \{+, -\}$ and $\beta \in \{e, o\}$, then $(G, \sigma) \rightarrow (H, \pi)$.

$\text{mad}(G)$: maximum average degree of G , that is, the maximum of the average degrees of its subgraphs.

Question. *For a given signed graph (H, π) , what is the optimal value of ε ?*

Sparse graphs

Homomorphisms and girth (of signed graphs)

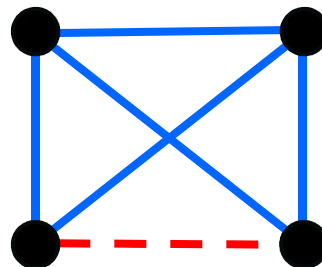
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$\text{mad}(G)$: *maximum average degree* of G , that is, the maximum of the average degrees of its subgraphs.

Example. For (H, π) being



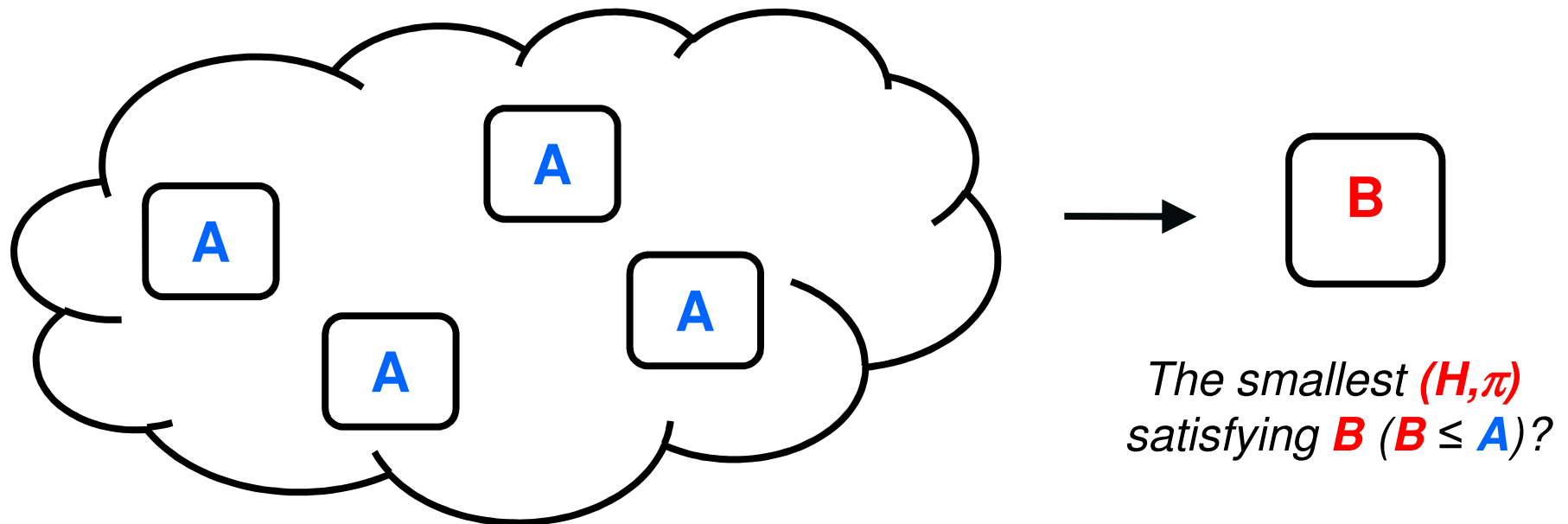
, $\varepsilon = 2 / 3$ (optimal).

Signed planar graphs

A (very) general question about planar graphs

Homomorphisms under girth conditions

Question. Given six integers $a_{\alpha\beta}$ and $b_{\alpha\beta}$, $\alpha\beta \in \{+o, -e, -o\}$, with $a_{\alpha\beta} \geq b_{\alpha\beta}$ for every $\alpha\beta$, what is the smallest signed graph (H, π) with $g_{\alpha\beta}(H, \pi) \geq b_{\alpha, \beta}$ such that every signed planar graph (G, σ) with $g_{\alpha\beta}(G, \sigma) \geq a_{\alpha, \beta}$ admits a homomorphism to (H, π) ?



*The set of signed planar graphs satisfying the girth conditions **A***

A (very) general question about planar graphs

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Example 1.

	(G, σ)	(H, π)
g_{+o}	3	3
g_{-e}	∞	∞
g_{-o}	∞	∞



Answer: $(K_4, +)$

This is 4CT (no negative cycles, no girth restriction)

A (very) general question about planar graphs

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Example 2.

	(G, σ)	(H, π)
g_{+o}	5	3
g_{-e}	∞	∞
g_{-o}	∞	∞



Answer: $(K_3, +)$

This is Grötzsch's Th.
(no negative cycles,
triangle-free planar
graphs)

A (very) general question about planar graphs

Homomorphisms under girth conditions

Question. Given six integers $a_{\alpha\beta}$ and $b_{\alpha\beta}$, $\alpha\beta \in \{+o, -e, -o\}$, with $a_{\alpha\beta} \geq b_{\alpha\beta}$ for every $\alpha\beta$, what is the smallest signed graph (H, π) with $g_{\alpha\beta}(H, \pi) \geq b_{\alpha, \beta}$ such that every signed planar graph (G, σ) with $g_{\alpha\beta}(G, \sigma) \geq a_{\alpha, \beta}$ admits a homomorphism to (H, π) ?

Example 3.

	(G, σ)	(H, π)
g_{+o}	$4k + 1$	$2k + 1$
g_{-e}	∞	∞
g_{-o}	∞	∞



Answer: $(C_{2k+1}, +)$?

This is Jaeger-Zhang's conjecture...

A (very) general question about planar graphs

Homomorphisms under girth conditions

Question. Given six integers $a_{\alpha\beta}$ and $b_{\alpha\beta}$, $\alpha\beta \in \{+o, -e, -o\}$, with $a_{\alpha\beta} \geq b_{\alpha\beta}$ for every $\alpha\beta$, what is the smallest signed graph (H, π) with $g_{\alpha\beta}(H, \pi) \geq b_{\alpha, \beta}$ such that every signed planar graph (G, σ) with $g_{\alpha\beta}(G, \sigma) \geq a_{\alpha, \beta}$ admits a homomorphism to (H, π) ?

Example 4.

	(G, σ)	(H, π)
g_{+o}	3	3
g_{-e}	4	4
g_{-o}	3	3



Answer: ???

“chromatic number” of
signed planar graphs ?
(known: btw. 10 and 40)

A (very) general question about planar graphs

Homomorphisms under girth conditions

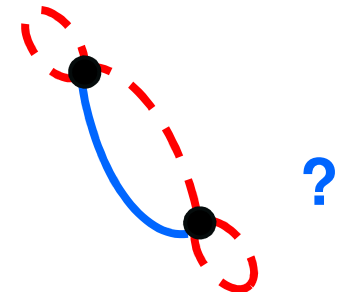
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Example 5.

	(G, σ)	(H, π)
g_{+o}	3	3
g_{-e}	4	2
g_{-o}	3	1



Answer:



Conjecture of Naserasr and Raspaud (2015), **disproved** by Kardoš and Narboni (2019+).

A special case:
Signed bipartite graphs

Sparse bipartite graphs

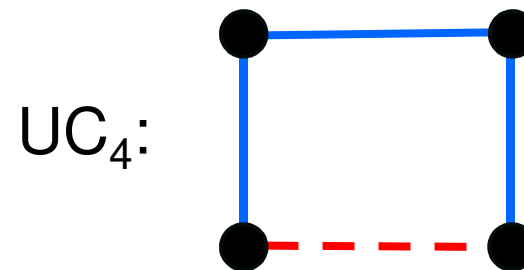
Homomorphisms and girth (of signed graphs)

Question. When are the necessary girth conditions $g_{\alpha\beta}(G, \sigma) \geq g_{\alpha\beta}(H, \pi)$ also sufficient for having $(G, \sigma) \rightarrow (H, \pi)$?

Theorem (Charpentier, Naserasr, S., 2019+).

If G is *bipartite* and $\text{mad}(G) < 2 + 1 / (2g - 1)$, then, for every signature σ on G satisfying $g_{-e}(G, \sigma) \geq 2g$, $(G, \sigma) \rightarrow UC_{2g}$.

UC_{2g} : the *negative (unbalanced) cycle* of length $2g$.



Sparse bipartite graphs

Homomorphisms and girth (of signed graphs)

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For $g = 2$, the value $2 + 1/3 = 7/3$ can be improved as follows:

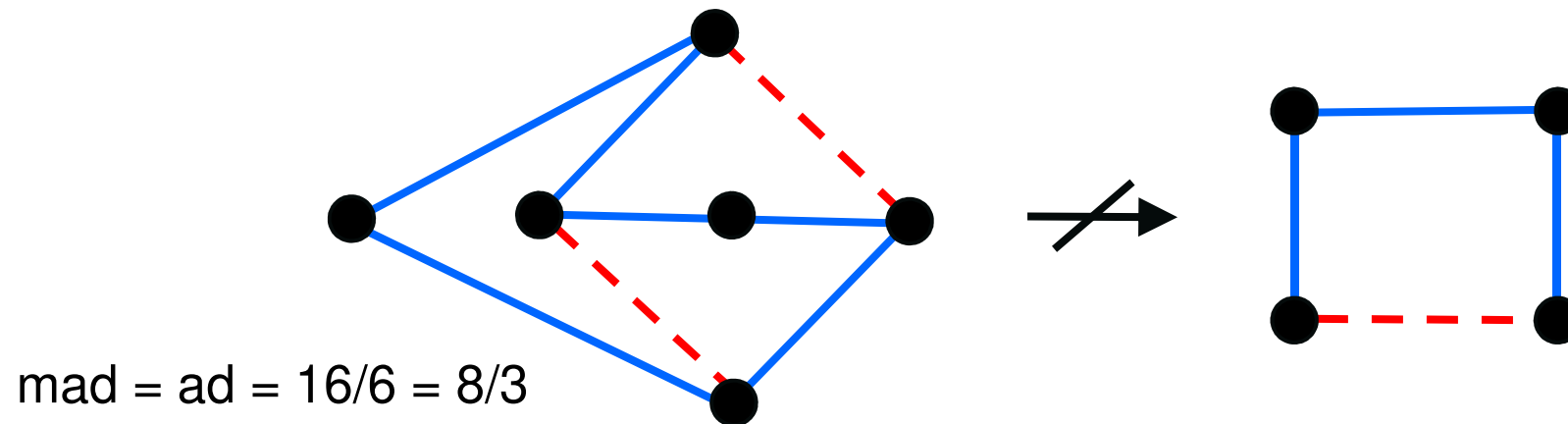
Theorem (Charpentier, Naserasr, S., 2019+).

If G is *bipartite* and $\text{mad}(G) < 8/3$, then $(G, \sigma) \rightarrow UC_4$ for every signature σ on G , and the value $8/3$ is *optimal*.

Sparse bipartite graphs

Homomorphisms and girth (of signed graphs)

Question. When are the necessary girth conditions $g_{\alpha\beta}(G, \sigma) \geq g_{\alpha\beta}(H, \pi)$ also sufficient for having $(G, \sigma) \rightarrow (H, \pi)$?



Theorem (Charpentier, Naserasr, S., 2019+).

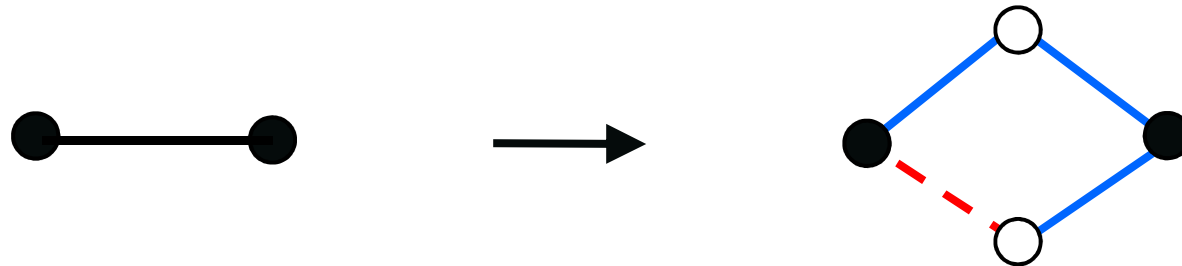
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Signed bipartite graphs

Signed bipartite graphs vs ordinary graphs

The study of homomorphisms of signed bipartite graphs captures the study of homomorphisms of graphs as a special case.

With every graph G , one can associate a **signed bipartite graph** $S(G)$ by replacing each edge of G by a **negative 4-cycle**.

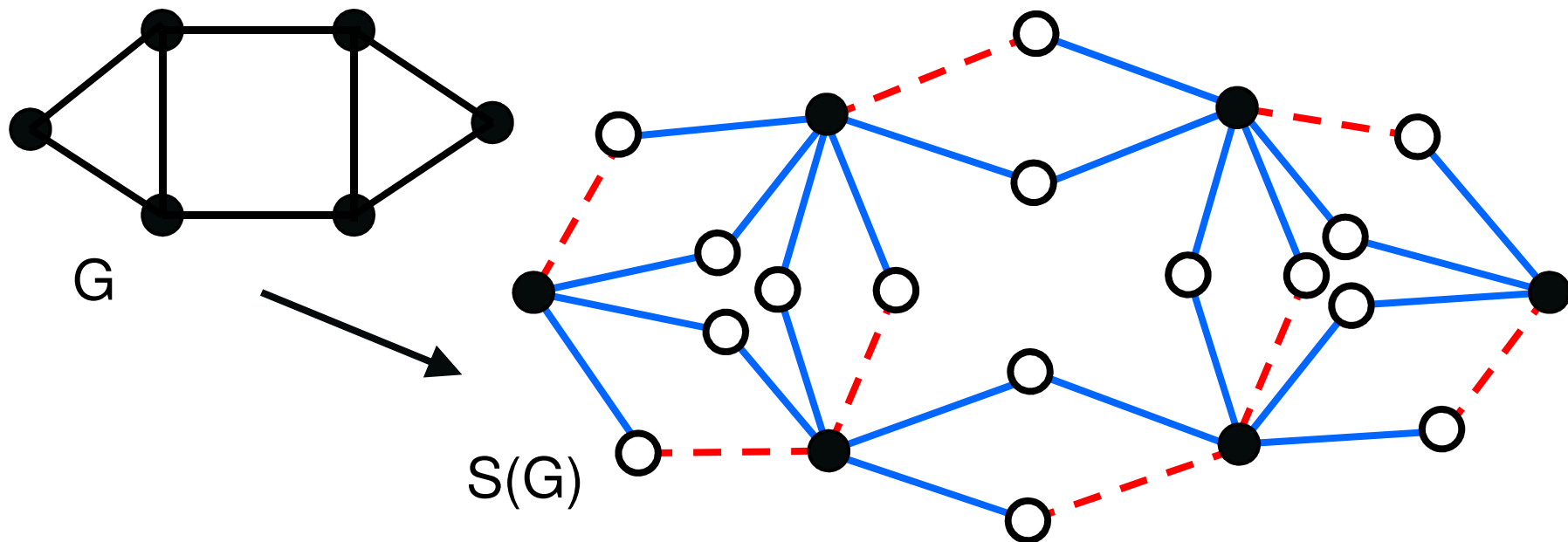


Signed bipartite graphs

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Signed bipartite graphs - homomorphisms

Signed bipartite graphs vs ordinary graphs

The study of homomorphisms of signed bipartite graphs captures the study of homomorphisms of graphs as a special case.

With every graph G , one can associate a signed bipartite graph $S(G)$ by replacing each edge of G by a negative 4-cycle.

Theorem (Naserasr, Rollová, S., 2015). *For all graphs G and H , $G \rightarrow H$ if and only if $S(G) \rightarrow S(H)$.*

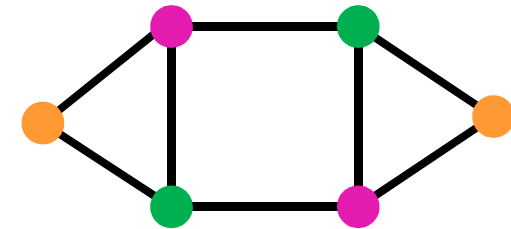
Signed bipartite graphs – chromatic number

Signed bipartite graphs vs ordinary graphs

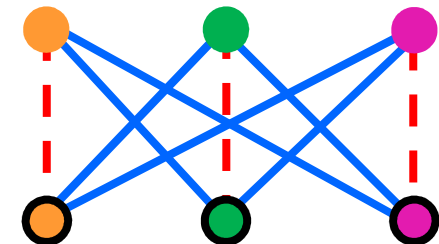
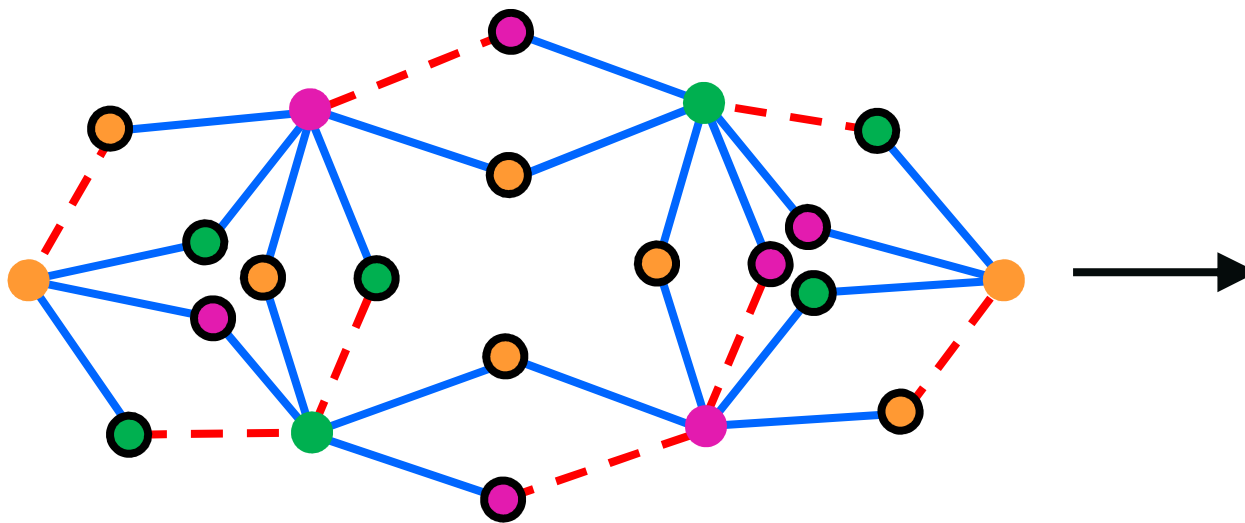
Homomorphisms of signed bipartite graphs also captures the notion of the chromatic number.

Theorem (Naserasr, Rollová, S., 2015).

For every graph G and every $k \geq 3$,
 $\chi(G) \leq k$ if and only if $S(G) \rightarrow (K_{k,k}, \sigma_M)$.



$$\chi(G) = 3$$



Signed bipartite planar graphs

Signed bipartite planar graphs

Recall:

Theorem (Charpentier, Naserasr, S., 2019+).

If G is *bipartite* and $\text{mad}(G) < 8/3$, then $(G, \sigma) \rightarrow UC_4$ for every signature σ on G , and the value $8/3$ is optimal.

Corollary. If G is *bipartite* and *planar*, σ a signature on G such that $g_{-e}(G, \sigma) \geq 8$, then $(G, \sigma) \rightarrow UC_4$.

The conjectured value is **6** instead of **8**...

Conjecture (Naserasr, Rollová, S., 2015). If (G, σ) is a signed *bipartite planar* graph with $g_{-e}(G, \sigma) \geq 4g - 2$, then $(G, \sigma) \rightarrow UC_{2g}$.

A bipartite analogue of Jaeger's conjecture:
if G is a *planar* graph with *girth* at least $4g$, then $G \rightarrow C_{2g+1}$.

Signed bipartite planar graphs

Signed bipartite planar graphs

Similarly:

Theorem (Charpentier, Naserasr, S., 2019+).

If G is *bipartite* and $\text{mad}(G) < 2 + 1 / (2g - 1)$, then, for every signature σ on G satisfying $g_{-e}(G, \sigma) \geq 2g$, $(G, \sigma) \rightarrow UC_{2g}$.

Corollary. If G is *bipartite* and *planar*, σ a signature on G such that $g_{-e}(G, \sigma) \geq 8g - 2$, then $(G, \sigma) \rightarrow UC_{2g}$.

The conjectured value is $4g - 2$ instead of $8g - 2$...

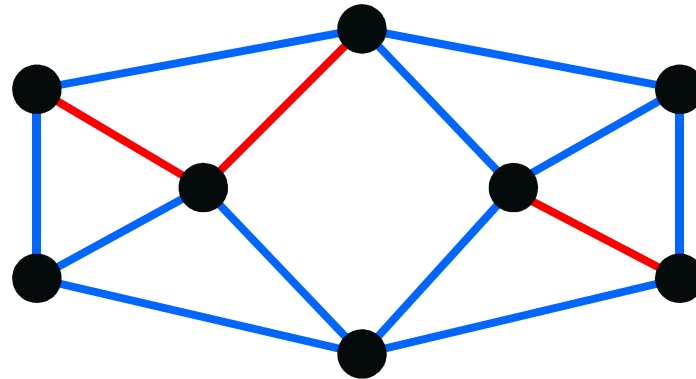
Conjecture (Naserasr, Rollová, S., 2015). If (G, σ) is a signed *bipartite planar* graph with $g_{-e}(G, \sigma) \geq 4g - 2$, then $(G, \sigma) \rightarrow UC_{2g}$.

2-edge-coloured graphs

2-edge-coloured graphs

2-edge-coloured graphs

A 2-edge-coloured graph is a triple (G, B, R) , where G is a graph, and (B, R) is a partition of $E(G)$ into blue and red edges.

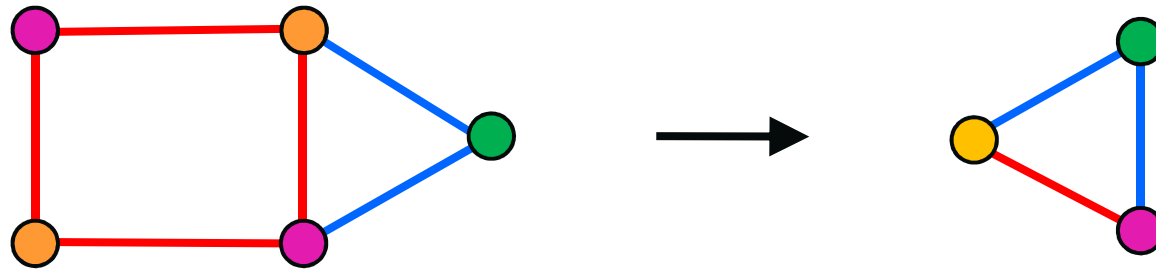


The main difference with signed graphs is that we do not switch vertices...

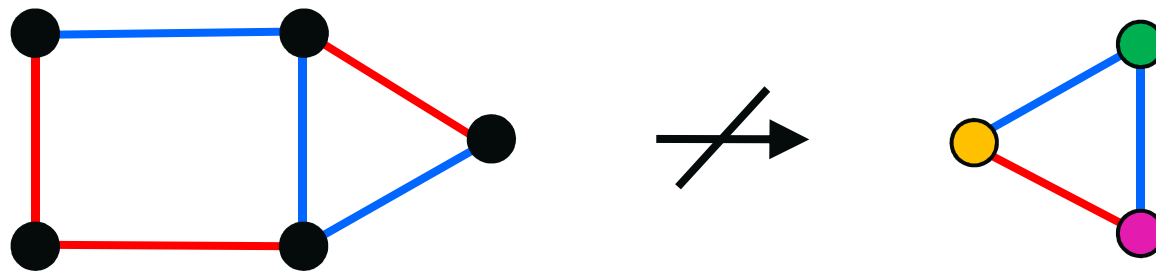
Homomorphisms of 2-edge-coloured graphs

Homomorphisms

A **homomorphism** from (G, B, R) to (G', B', R') is a homomorphism of G to G' that **preserves the edge colours**.



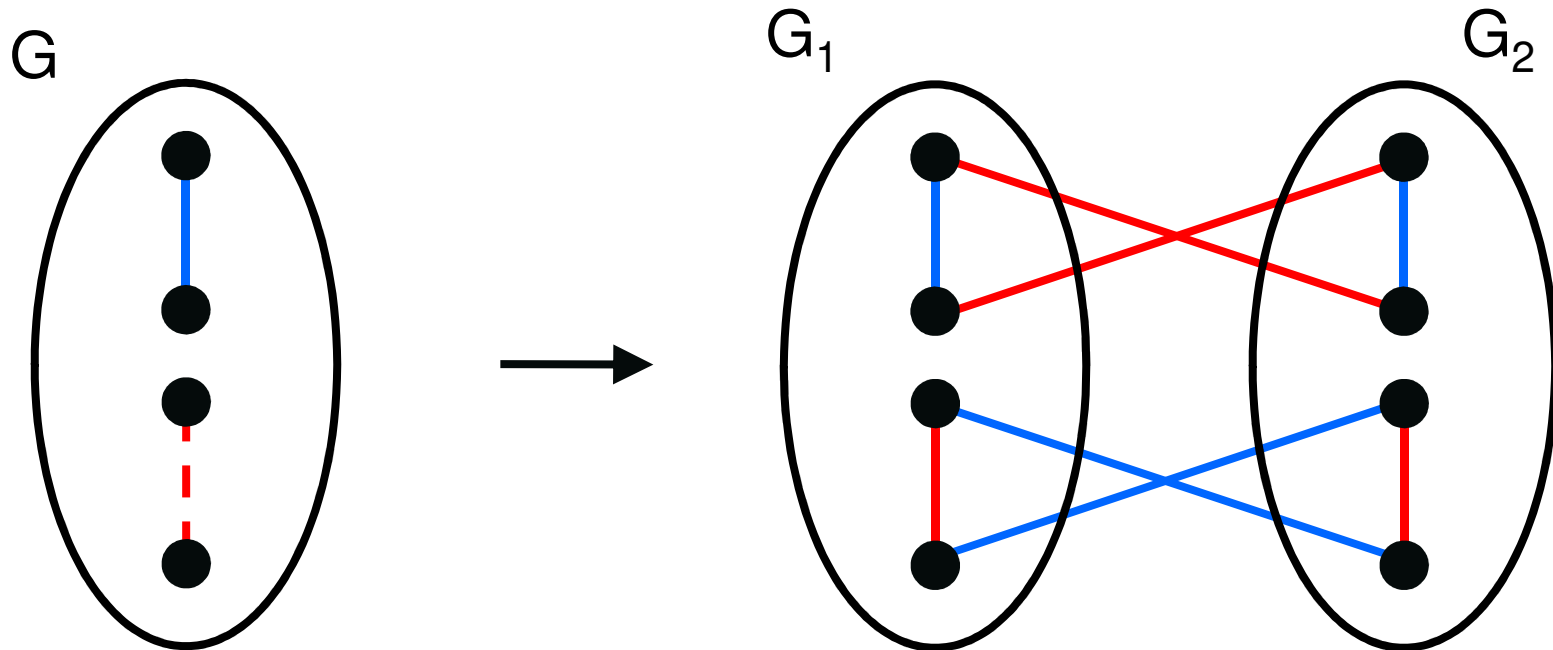
The main difference with signed graphs is that **we do not switch** vertices...



Signed graphs vs 2-edge-coloured graphs

Universal switching graph

With each signed graph (G, σ) , we associate the 2-edge-coloured graph $USG(G, \sigma)$ obtained as follows:



The signed graph (G, σ)

The 2-edge-coloured graph $USG(G, \sigma)$

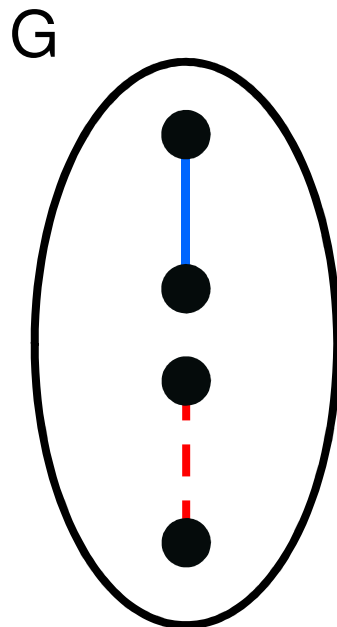
Switching at v in $G \Leftrightarrow$ exchanging v_1 and v_2

Signed graphs vs 2-edge-coloured graphs

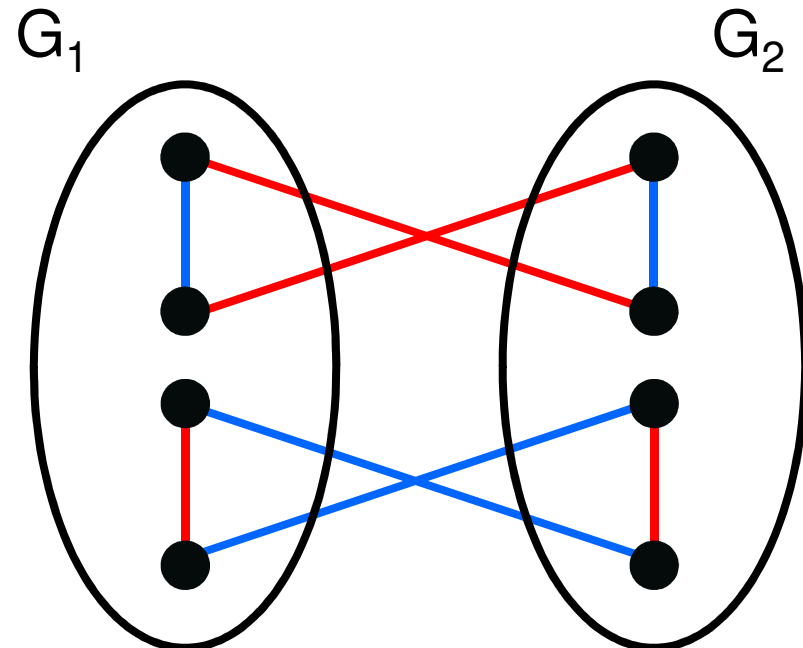
Universal switching graph

Theorem (Brewster and Grave, 2009).

$(G, \sigma) \rightarrow (H, \pi)$ if and only if $USG(G, \sigma) \rightarrow USG(H, \pi)$.



The signed graph (G, σ)



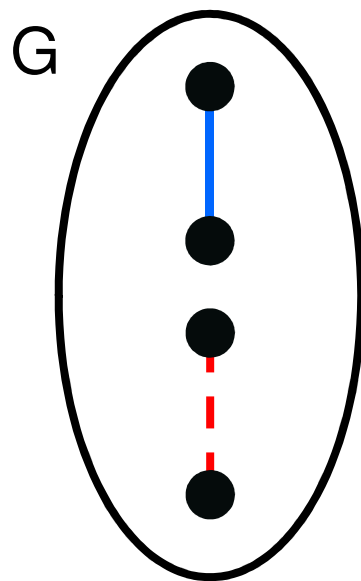
The 2-edge-coloured graph $USG(G, \sigma)$

Signed graphs vs 2-edge-coloured graphs

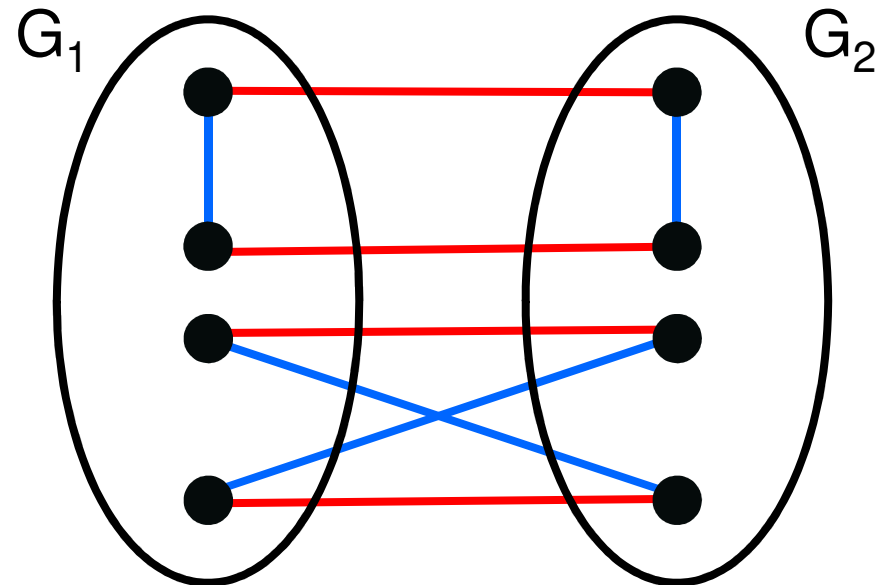
Extended double cover

With each signed graph (G, σ) , we associate the 2-edge-coloured graph $\text{EDC}(G, \sigma)$ obtained as follows:

This extends the [double covering graph](#) introduced by Zaslavsky in 1982.



The signed graph (G, σ)



The 2-edge-coloured graph $\text{EDC}(G, \sigma)$

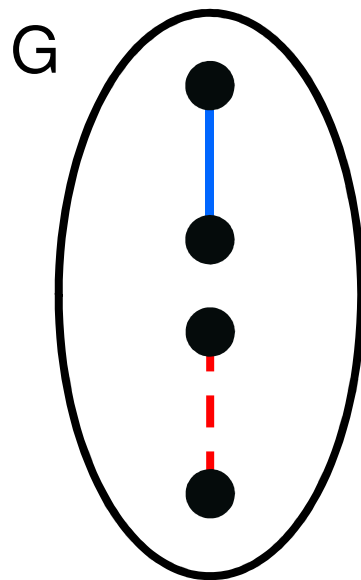
Switching at v in $G \Leftrightarrow$ exchanging v_1 and v_2

Signed graphs vs 2-edge-coloured graphs

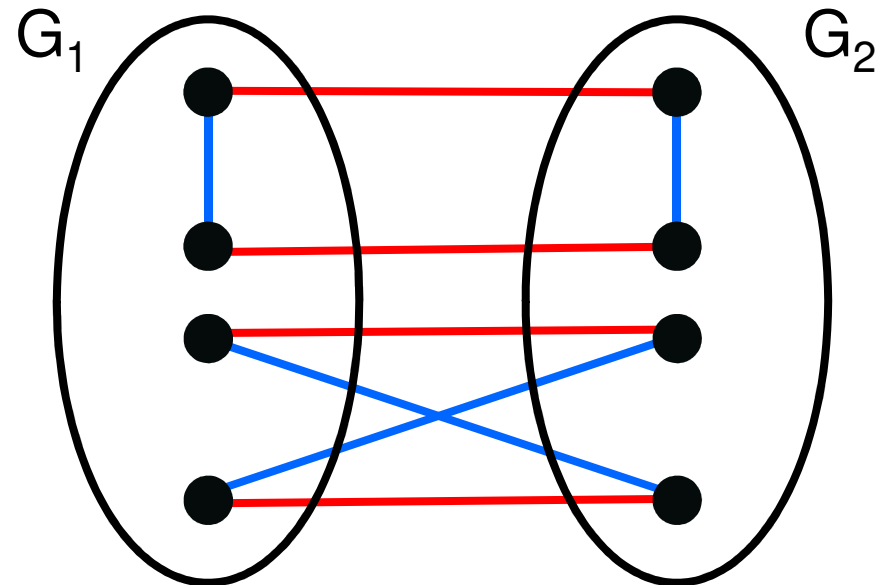
Extended double cover

Theorem (Naserasr, S., Zaslavsky, 2019+).

$(G, \sigma) \rightarrow (H, \pi)$ if and only if $EDC(G, \sigma) \rightarrow EDC(H, \pi)$.



The signed graph (G, σ)



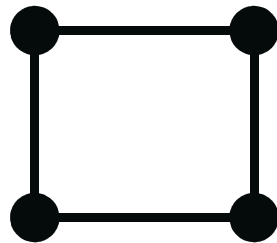
The 2-edge-coloured graph $EDC(G, \sigma)$

Signed projective cubes

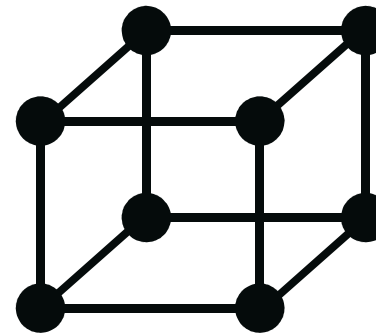
Signed projective cubes

Projective cubes

The n -dimensional projective cube $PC(n)$ is obtained from the n -dimensional hypercube Q_n by adding edges linking antipodal vertices.



Q_2

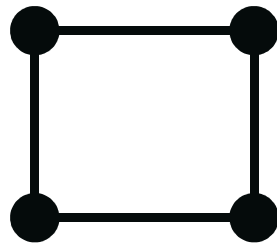


Q_3

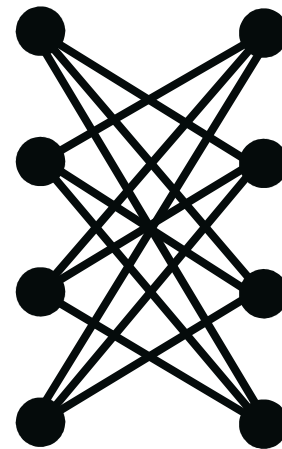
Signed projective cubes

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Q_2

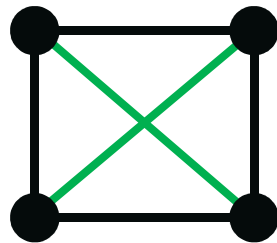


Q_3

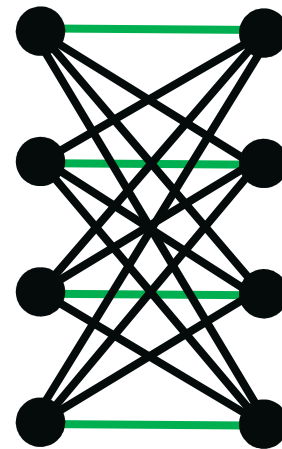
Signed projective cubes

Projective cubes

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PC(2)

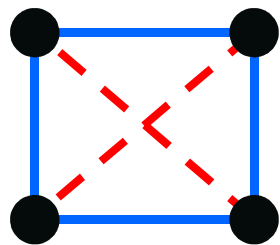


PC(3)

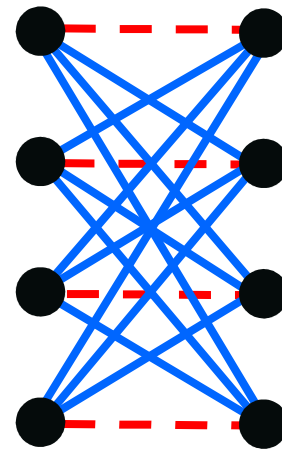
Signed projective cubes

Signed projective cubes

The n -dimensional signed projective cube $\text{SPC}(n)$ is the projective cube $\text{PC}(n)$ where the set of negative edges is the set of edges linking antipodal vertices.



$\text{SPC}(2)$

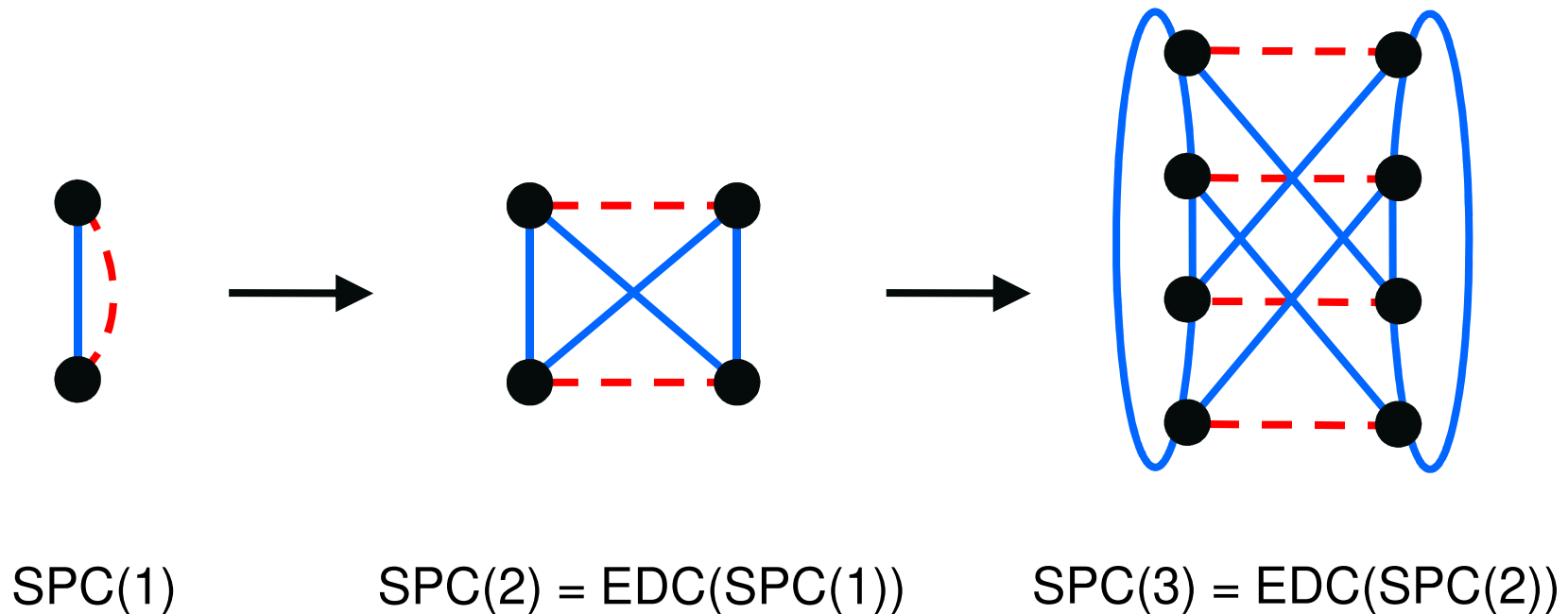


$\text{SPC}(3)$

Signed projective cubes

Signed projective cubes

The n -dimensional signed projective cube $\text{SPC}(n)$ is the extended double cover of the signed projective cube $\text{SPC}(n-1)$.

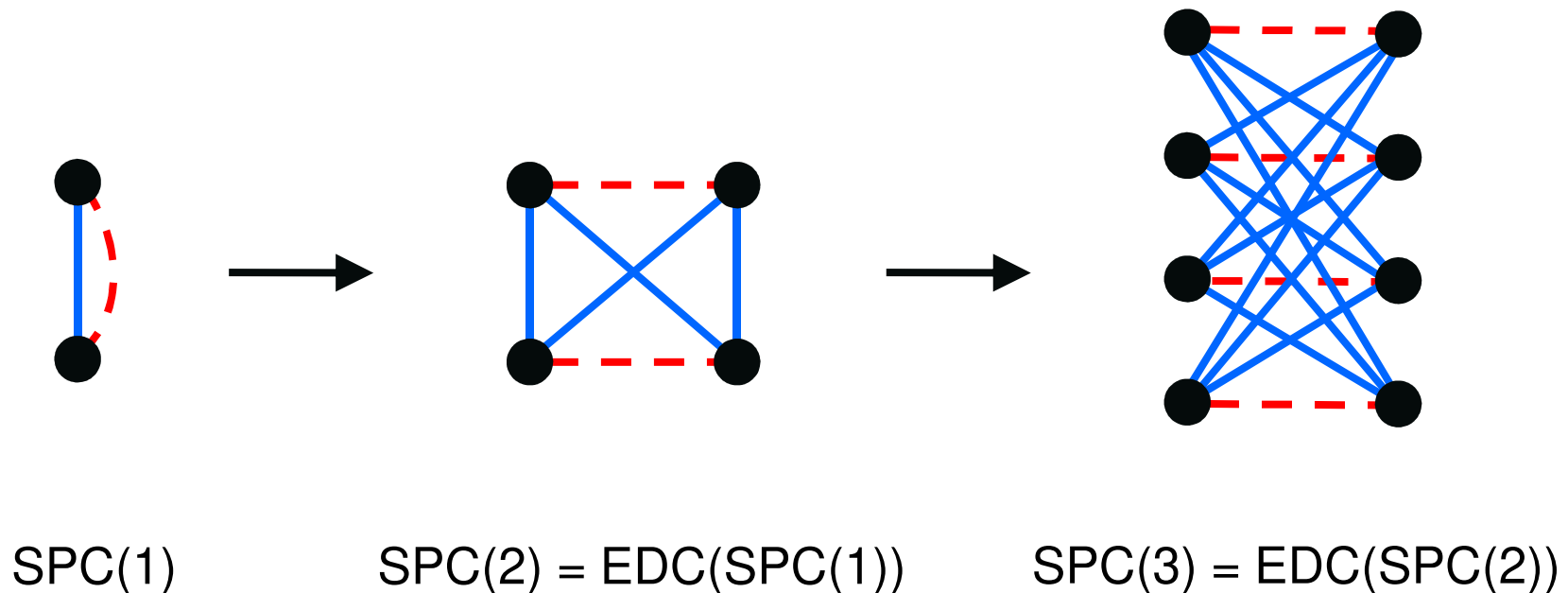


Signed projective cubes

Signed projective cubes

The n -dimensional signed projective cube $\text{SPC}(n)$ is the **extended double cover** of the signed projective cube $\text{SPC}(n-1)$.

Remark. $\text{SPC}(n)$ is *bipartite* if n is odd, and *all-negative* if n is even.



Signed projective cubes

Homomorphisms to signed projective cubes

The n -dimensional signed projective cube $\text{SPC}(n)$ is the **extended double cover** of the signed projective cube $\text{SPC}(n-1)$.

Remark. $\text{SPC}(n)$ is **bipartite** if n is odd, and **all-negative** if n is even.

Conjecture (Guenin, 2005 / Naserasr, 2007).

If (G, σ) is a signed **planar** graph, then $(G, \sigma) \rightarrow \text{SPC}(k)$ iff the “girth conditions” $g_{\alpha\beta}(G, \sigma) \geq g_{\alpha\beta}(\text{SPC}(k))$ are satisfied.

Theorem (Naserasr, Rollová, S., 2013).

This conjecture is equivalent to Seymour’s conjecture: “**Every k -regular planar multigraph with no odd edge-cut of less than k edges is k -edge-colourable**”.

Signed projective cubes

Homomorphisms to signed projective cubes

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This conjecture is equivalent to Seymour’s conjecture: “Every k -regular planar multigraph with no odd edge-cut of less than k edges is k -edge-colourable”.

- $k = 1$: easy,
- $k = 2$: the Four-Colour Theorem,
- $k = 3$: implies the Four-Colour Theorem,
- $k = 3, \dots, 7$: proved, using the Four-Colour Theorem...

Chromatic number(s) of signed graphs

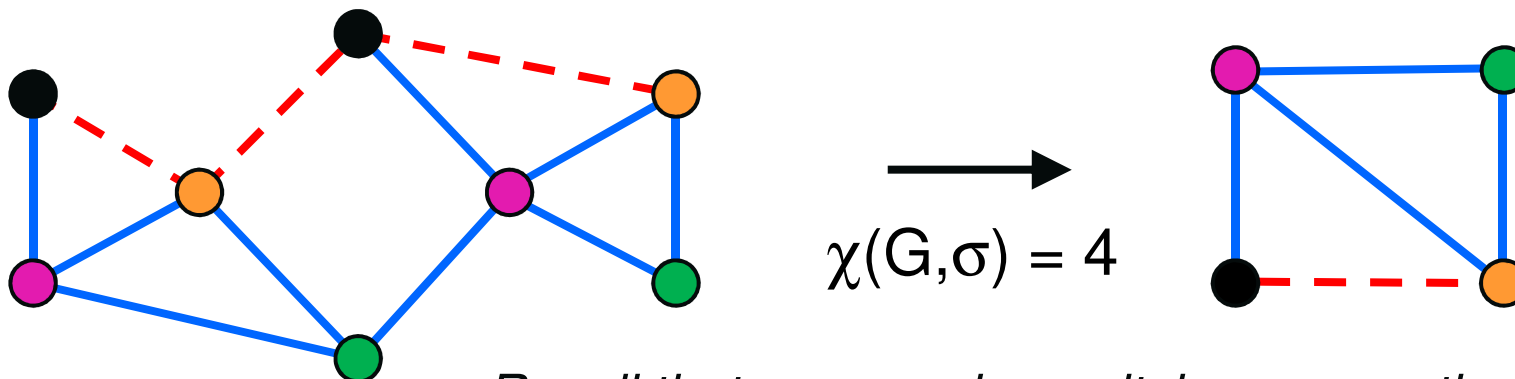
χ of signed graphs – 1. via homomorphisms

Colouring and homomorphisms

Recall that the **chromatic number** $\chi(G)$ of an ordinary graph G can be viewed as the **smallest order** of a (complete) graph H to which G admits a homomorphism.

The **chromatic number** $\chi(G, \sigma)$ of a signed graph (G, σ) can be similarly defined as the **smallest order** of a signed graph (H, π) to which (G, σ) admits a homomorphism.

*The target graph (H, π) can be required to be a “signed clique”:
every two non-adjacent vertices in (H, π) belong to some UC_4 .*

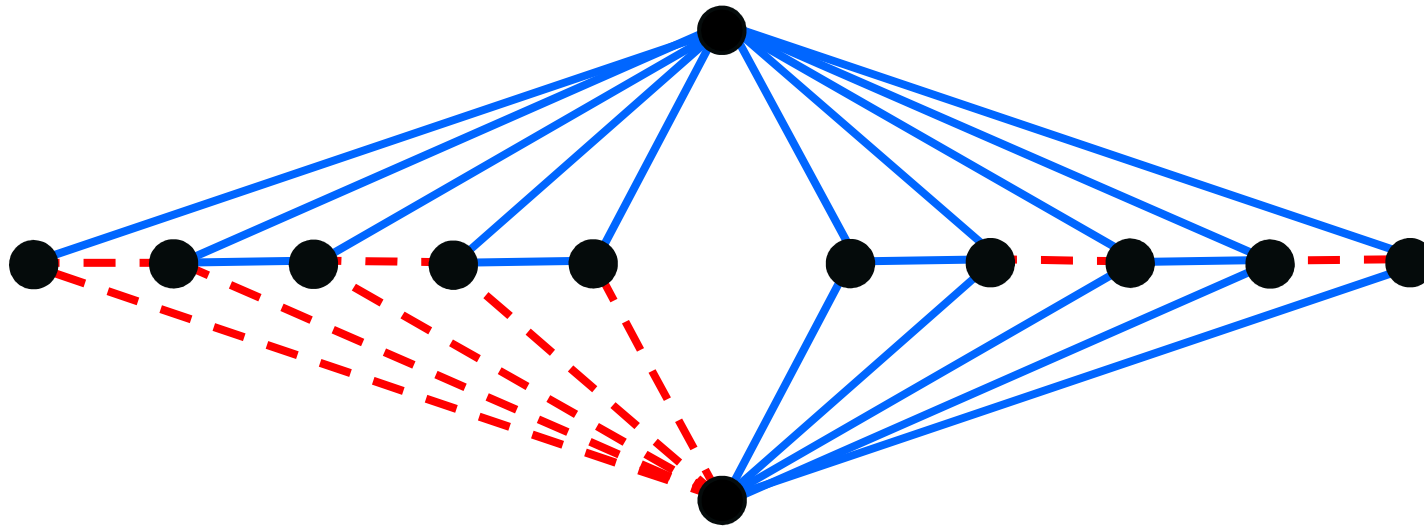


χ of signed graphs – 1. via homomorphisms

Planar graphs

Theorem (Ochem, Pinlou, Sen, 2017). For every *planar* graph G and every signature σ on G , $\chi(G, \sigma) \leq 40$.
Moreover, $\chi(G, \sigma) \leq 25$ if G is *triangle-free*.

There exist signed planar graphs with chromatic number 10:



Is it true that $\chi(G, \sigma) \leq 10$ for every signed *planar* graph (G, σ) ?

χ of signed graphs – 1. via homomorphisms

Planar graphs

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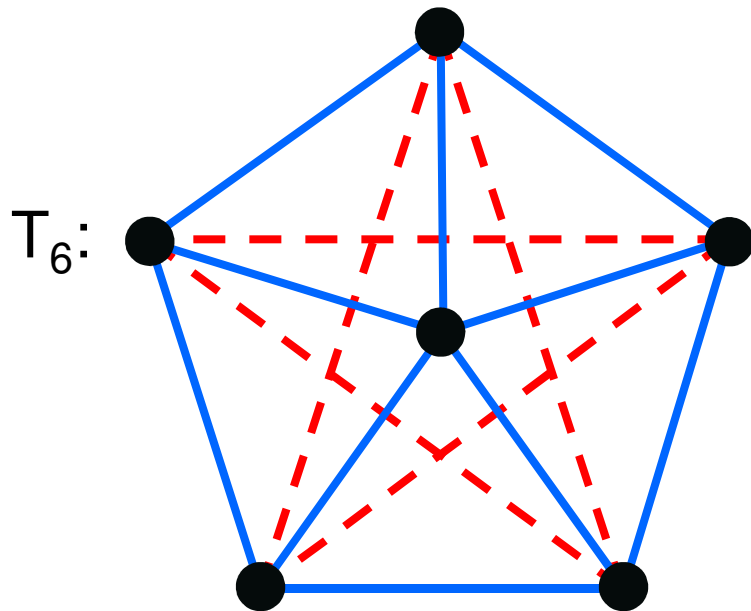
Outerplanar graphs

Theorem (Naserasr, Rollová, S., 2015). For every *outerplanar* graph G and every signature σ on G , $\chi(G, \sigma) \leq 5$, and this bound is *tight*.
This bound also holds, more generally, for K_4 -minor free graphs.

χ of signed graphs – 1. via homomorphisms

Graphs with bounded degree

Theorem (Bensmail, Pierron, S., 2019+). *Every signed graph (G, σ) , with $\Delta(G) \leq 3$, $G \neq K_4$, admits a homomorphism to the signed tournament T_6 , which implies $\chi(G, \sigma) \leq 6$ (tight bound).*



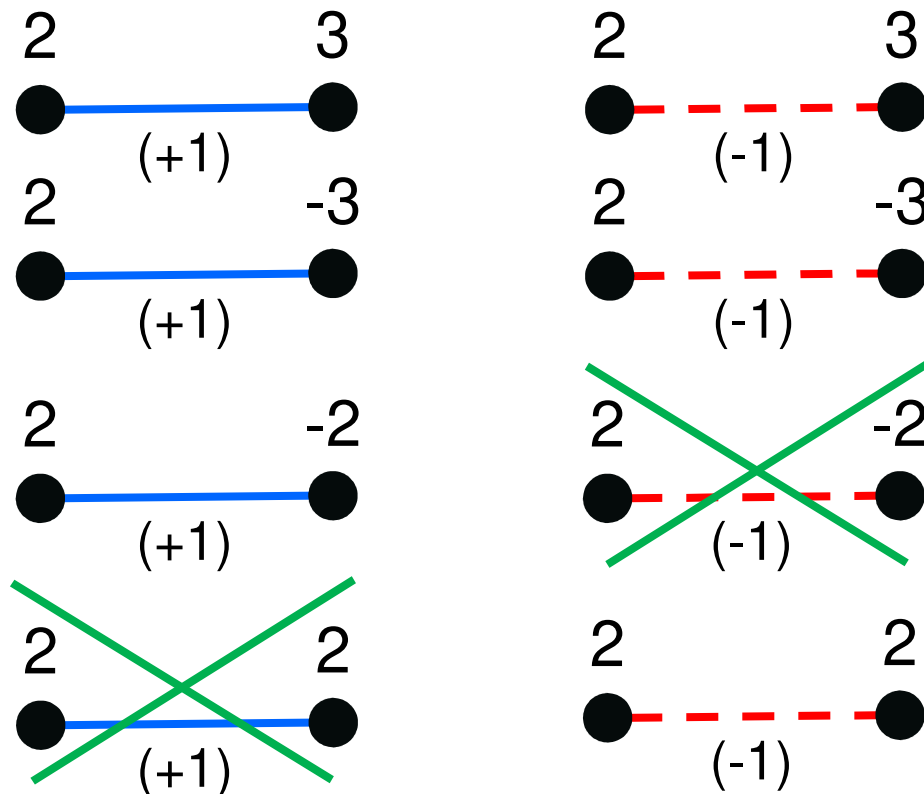
(Jacques, 2019+)

- Max. deg. 4: $\chi(G, \sigma) \leq 16$
- Max. deg. k: $\chi(G, \sigma) \leq (k - 1)^2 2^{k+1}$

χ of signed graphs – 2. via vertex colourings

Signed graph colouring

[Zaslavsky, 1982] A **vertex-colouring** of a signed graph (G, σ) is a mapping $\varphi : V(G) \rightarrow \mathbb{Z}$ such that, for every **edge** uv with sign $\sigma(uv) \in \{-1, +1\}$, $\varphi(u) \neq \sigma(uv)\varphi(v)$.



Remark.

Switching at vertex v



Replace $\varphi(v)$ by $-\varphi(v)$

Corresponds to a homomorphism to some special graph with negative loops at each vertex...

χ of signed graphs – 2. via vertex colourings

Signed graph colouring

[Zaslavsky, 1982] A **vertex-colouring** of a signed graph (G, σ) is a mapping $\varphi : V(G) \rightarrow \mathbb{Z}$ such that, for every **edge** uv with sign $\sigma(uv) \in \{-1, +1\}$, $\varphi(u) \neq \sigma(uv)\varphi(v)$.

Chromatic number

[Máčajová, Raspaud, Škoviera, 2016+] The **chromatic number** $\chi_{\pm}(G, \sigma)$ of a signed graph (G, σ) is the **smallest integer** n such that (G, σ) admits a proper colouring using **colours from** M_n .

- if $n = 2k$, $M_n = \{\pm 1, \dots, \pm k\}$,
- if $n = 2k + 1$, $M_n = \{0, \pm 1, \dots, \pm k\}$.

*If (G, σ) is loopless and **all-positive**, then $\chi_{\pm}(G, \sigma) = \chi(G)$.*

For every loopless graph G and every signature σ on G , $\chi_{\pm}(G, \sigma) \leq 2\chi(G) - 1$.

χ of signed graphs – 2. via vertex colourings

A Brooks-type Theorem

Theorem (Máčajová, Raspaud, Škoviera, 2016+).

For every signed graph (G, σ) , $\chi_{\pm}(G, \sigma) \leq \Delta(G)$, unless (G, σ) is an all-positive complete graph, an all-positive odd cycle, or an all-negative even cycle.

Also, if G is k -degenerate, then $\chi_{\pm}(G, \sigma) \leq k + 1$.

Planar graphs

Theorem (Máčajová, Raspaud, Škoviera, 2016+).

For every signed planar graph (G, σ) , $\chi_{\pm}(G, \sigma) \leq 5$.

Moreover, $\chi_{\pm}(G, \sigma) \leq 4$ if $g(G) \geq 4$, and $\chi_{\pm}(G, \sigma) \leq 3$ if $g(G) \geq 5$.

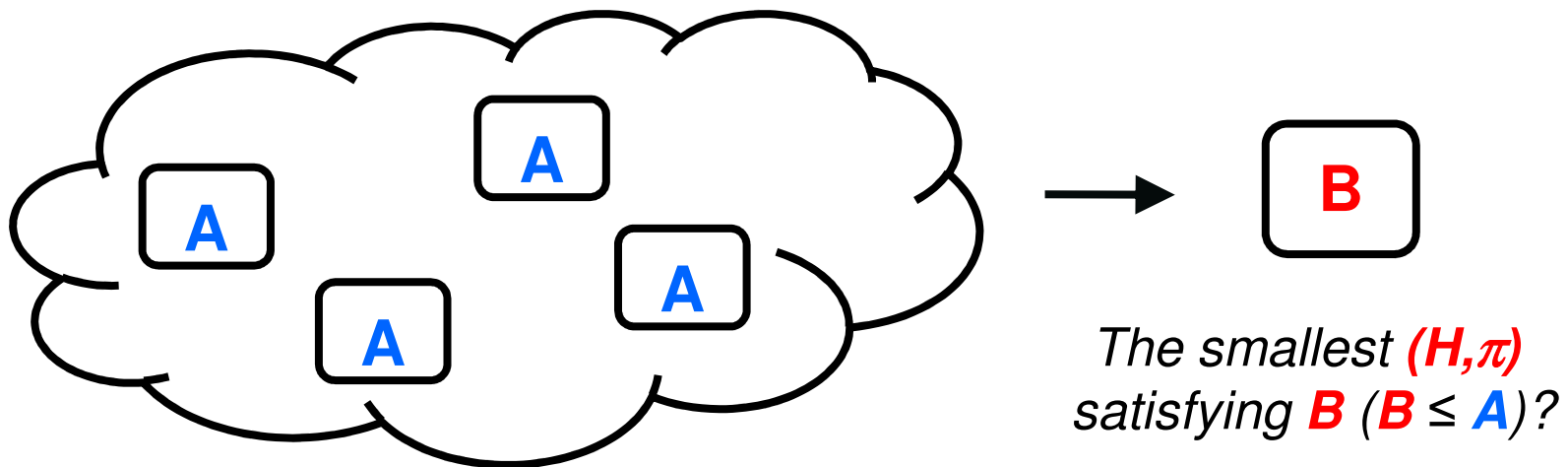
They also conjectured that $\chi_{\pm}(G, \sigma) \leq 4$ for every signed planar graph (G, σ) .

This conjecture has been **disproved** by Kardoš and Narboni (2019+).

χ of signed graphs – 3. with girth constraints

(A,B)-chromatic number

[Naserasr, S., Zaslavsky, 2019+] Given two triples of values from $\mathbb{N}^* \cup \{\infty\}$, $A = (a_{+o}, a_{-e}, a_{-o})$ and $B = (b_{+o}, b_{-e}, b_{-o})$, the (A,B)-chromatic number $\chi_{(A,B)}(G, \sigma)$ of a signed graph (G, σ) satisfying $g_{\alpha\beta}(G, \sigma) \geq a_{\alpha\beta}$ for every $\alpha\beta \in \{+o, -e, -o\}$ is the smallest order of a signed graph (H, π) , with $g_{\alpha\beta}(H, \pi) \geq b_{\alpha\beta}$ for every $\alpha\beta \in \{+o, -e, -o\}$, such that $(G, \sigma) \rightarrow (H, \pi)$.



The set of signed graphs satisfying the girth conditions A

χ of signed graphs – 3. with girth constraints

(A,B)-chromatic number

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- For $A = (a, \infty, \infty)$, $B = (b, \infty, \infty)$, or $A = (\infty, \infty, a)$, $B = (\infty, \infty, b)$, $(G, \sigma) \rightarrow (H, \pi)$ iff $G \rightarrow H$ (all-positive and all-negative signed graphs, respectively).
- G planar, $A = B = (3, \infty, \infty)$: Four-Colour Theorem.
- G planar, $A = (5, \infty, \infty)$ and $B = (3, \infty, \infty)$: Grötschz's Theorem.
- G planar, $A = B = (5, \infty, \infty)$, $\chi_{(A,B)}(G, \sigma) \leq 16$ [Naserasr, 2013, $(G, \sigma) \rightarrow PC(4)$]
- etc.

χ of signed graphs – 3. with girth constraints

Homomorphisms to signed projective cubes

Conjecture (Guenin, 2005 / Naserasr, 2007).

*If (G, σ) is a signed **planar** graph, then $(G, \sigma) \rightarrow SPC(k)$ iff the “girth conditions” $g_{\alpha\beta}(G, \sigma) \geq g_{\alpha\beta}(SPC(k))$ are satisfied.*

An equivalent conjecture

Conjecture (Naserasr, S., Zaslavsky, 2019+).

*For $A = B = (\infty, \infty, 2k+1)$, the (A, B) -chromatic number of every signed **planar** graph is at most 2^{2k} .*

*For $A = B = (\infty, 2k, \infty)$, the (A, B) -chromatic number of every signed **planar** graph is at most 2^{2k-1} .*

The equivalence follows from a result of Naserasr, Sen, Sun (2016).

Summary...

Summary

- Homomorphisms of signed graphs (switching matters...)
- Hom. of signed graphs \Rightarrow hom. of 2-edge-coloured graphs
- Signed bipartite graphs play an important role
- General question on planar graphs under girth constraints
- Several ways to define colourings and chromatic numbers of signed graphs
*Min. order of H s.t., for every signed planar graph (G, σ) , $(G, \sigma) \rightarrow (H, \pi)$?
Lies between 10 and 40...*
- *Other topics about signed graphs not discussed here:
List colouring / DP-colouring, Circular colouring, Signed cliques, Signed minors...*