

# Homomorphisms of sparse signed graphs

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## Abstract

The notion of homomorphism of signed graphs, introduced quite recently, provides better interplay with the notion of minor and is thus of high importance in graph coloring. A newer, but equivalent, definition of homomorphisms of signed graphs, proposed jointly by the authors of this paper and Tom Zaslavsky, leads to a basic no-homomorphism lemma. According to this definition, a signed graph  $(G, \sigma)$  maps to a signed graph  $(H, \pi)$  if there is a mapping  $\phi$  from the vertices and edges of  $G$  to the vertices and edges of  $H$  (respectively) which preserves adjacencies, incidences, and signs of closed walks. For  $ij = 00, 01, 10, 11$ , let  $g_{ij}(G, \sigma)$  be the length of a shortest closed walk of  $(G, \sigma)$  which is, positive and of even length for  $ij = 00$ , positive and of odd length for  $ij = 01$ , negative and of even length for  $ij = 10$ , negative and of odd length for  $ij = 11$ . For each  $ij$ , if there is no closed walk of the corresponding type, we let  $g_{ij}(G, \sigma) = \infty$ . If  $G$  is bipartite, then  $g_{01}(G, \sigma) = g_{11}(G, \sigma) = \infty$ . In this case,  $g_{10}(G, \sigma)$  is certainly realized by a cycle of  $G$ , and it will be referred to as the *unbalanced-girth* of  $(G, \sigma)$ .

It then follows that if  $(G, \sigma)$  admits a homomorphism to  $(H, \pi)$ , then  $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$  for  $ij \in \{00, 01, 10, 11\}$ .

Studying the restriction of homomorphisms of signed graphs on sparse families, we first prove in this paper that for any given signed graph  $(H, \pi)$ , there is a positive value of  $\epsilon$  such that, if  $G$  is a connected graph of maximum average degree less than  $2 + \epsilon$ , and if  $\sigma$  is a signature of  $G$  such that  $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$  for all  $ij \in \{00, 01, 10, 11\}$ , then  $(G, \sigma)$  admits a homomorphism to  $(H, \pi)$ .

For  $(H, \pi)$  being the signed graph on  $K_4$  with exactly one negative edge, we show that  $\epsilon = \frac{2}{3}$  works and that this is the best possible value of  $\epsilon$ . For  $(H, \pi)$  being the negative cycle of length  $2g$ , denoted  $UC_{2g}$ , we show that  $\epsilon = \frac{1}{2g-1}$  works.

As a bipartite analogue of Jaeger-Zhang conjecture, it was conjectured in [13] that every signed bipartite planar graph  $(G, \sigma)$  satisfying  $g_{ij}(G, \sigma) \geq 4g - 2$  admits a homomorphism to  $UC_{2g}$ . We show that  $4g - 2$  cannot be strengthened, and, supporting the conjecture, we prove it for  $(G, \sigma)$  satisfying the weaker condition  $g_{ij}(G, \sigma) \geq 8g - 2$ .

In the course of our work, we also provide a duality theorem to decide whether a 2-edge-colored graph admits a homomorphism to a certain class of 2-edge-colored signed graphs or not.

Keywords: Signed graph; homomorphism; sparse graph; planarity.

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# 1 Introduction

One of the earliest observations in the theory of (proper) vertex coloring of graphs is that if every subgraph of a graph  $G$  has a vertex of degree at most  $k$ , then  $G$  is  $(k+1)$ -colorable. Many efforts are devoted to improve, generalize, and strengthen this early observation. Brook's theorem is one such result which classifies the case when  $k+1$  colors are actually needed. The four color theorem could also be seen as an improvement as, by Euler's formula, there is a vertex of degree at most five in every planar graph, and thus the trivial bound of 6 is decreased to 4. So is Grötzsch's theorem, as every triangle-free planar graph has a vertex of degree at most 4, and is thus 5-colorable, but the theorem decreases this bound to 3.

Further studies and recent developments are based on notions such as minor, *maximum average degree* of graph, denoted  $\text{mad}(G)$ , various coloring notions such as circular and fractional coloring, and more generally the notion of graph homomorphism.

Considering the general notion of homomorphisms of graphs, the following is easily observed (we denote by  $C_n$  the cycle of order  $n$ ).

**Lemma 1.** *There exists a homomorphism of  $C_{2\ell+1}$  to  $C_{2k+1}$  if and only if  $\ell \geq k$ .*

Therefore, if there is a homomorphism of  $G$  to  $H$ , then the odd-girth of  $G$  is at least the odd-girth of  $H$ . This provides an easy condition upon which one can guarantee the non-existence of a homomorphism from  $G$  to  $H$ . While this condition is not always sufficient, under certain conditions it might be. Some of the well-known coloring results or conjectures in graph theory can be viewed in this context. A notable example is the following conjecture, whose first case, and the only case proven so far, is Grötzsch's theorem.

**Conjecture 2** (Jaeger-Zhang [9]). *Every planar graph of odd-girth  $4k+1$ ,  $k \geq 1$ , admits a homomorphism to  $C_{2k+1}$ .*

For required definitions we refer to Section 2, but we would like to note that the same conjecture with a girth condition rather than an odd-girth condition is the dual statement of a conjecture of Jaeger dealing with flows on graphs, when restricted to the class of planar graphs. That the girth condition can be relaxed to a condition on the odd-girth is proposed by C.Q. Zhang. The folding lemma of W. Klostermeyer and C.Q. Zhang [9] is supporting evidence of this suggestion and implies a similar condition on the average degree. For partial results we refer to [5].

Results of this type can normally be rephrased in the following general framework: "Given a non bipartite graph  $H$ , any sufficiently enough sparse graph  $G$  either admits a homomorphism to  $H$ , or has a small odd cycle".

For each specific problem, working on a given family of graphs (such as planar graphs, or any minor-closed class of graphs) and with a fixed  $H$  or a fixed family of  $H$ 's, one would like to know precisely "how sparse is enough". For a specific conjecture in this regard when  $H$  is just an odd cycle and some related questions we refer to [5, Section 2].

A notion of graph theory which provides room for extensions and strengthening of classical graph theory results is the notion of *signed graph*. This notion, among others, provides a stronger and more natural connection between theories of minors and coloring. Using this notion, colorings and homomorphisms of (signed) bipartite graphs are no longer

a matter of triviality. Indeed, as we will describe later, the notion of homomorphism restricted to signed bipartite graphs captures the classical theory of homomorphisms of graphs.

In this work, we initiate the study of coloring and homomorphism problems for sparse signed graphs, with particular emphasis on the bipartite case. Among other things, we provide partial results supporting the bipartite analogue of Jaeger-Zhang conjecture proposed in [13].

## 2 Notation and terminology

Graphs are finite and simple, i.e., without loops and without parallel edges. For classical graph theory we use the standard notation, mainly following [1].

The main objects of this study are signed graphs. As the theories on signed graphs are under development from a wide range of interests, there are some differences in terminology in the existing literature. In this paper we adopt an improved notation introduced in [4] which fits well with respect to the theory of homomorphisms and matches most other views (compared to the recent introduction of the theory in [13]). We start with basic notation of graphs for clarity.

A graph is a pair  $(V, E)$  where  $V$  is a set, which normally and certainly in this paper, is finite, and  $E$  is a collection of 2-subsets of  $V$  ( $E$  could be considered as a multiset when we speak of a multigraph). Elements of  $V$  are referred to as *vertices* and elements of  $E$  are *edges*. Thus, using this definition, we do not allow loops. We follow the standard terminology of graphs. Some points are as follows. A vertex of degree  $k$  may be referred to as a *k-vertex*. A *walk* of length  $k$  in a graph  $G$  is a sequence of (not necessarily distinct) vertices  $v_0 \dots v_k$ , such that  $v_i v_{i+1}$  is an edge in  $G$  for every  $i$ ,  $0 \leq i < k$ . If  $v_0 = v_k$ , we say the walk is a *closed walk*. A *path* is a walk with the additional property that no vertex appears twice. A closed walk where the first vertex is the only repeated vertex is a *cycle*.

A *thread* is a path whose internal vertices are all of degree 2 in  $G$ . If  $x$  and  $y$  are the end vertices of the walk, the path or the thread, we call it an *xy-walk*, *xy-path* or *xy-thread*, respectively. Observe that every edge in  $G$  is a walk, a path, and a thread of length 1.

A *2-edge-colored graph* is a graph whose edges are assigned one of the two possible colors (this coloring is not necessarily proper). Assuming  $E_1$  and  $E_2$  are the color classes, we will denote the corresponding 2-edge-colored graph by a triplet, namely  $(V, E_1, E_2)$  or  $(G, E_1, E_2)$ , in order to distinguish it from a signed graph.

A *signed graph* is a graph  $G$  together with an assignment of one of the two possible signs (we mean  $+$  or  $-$ ) to each edge of  $G$ . This notion is thus different from that of a 2-edge-colored graph, since  $\{+, -\}$  is a 2-element multiplicative group, which allows us to speak of *positive* or *negative* objects (e.g. subgraphs), using the operation of this group.

One may then use one of the two natural notations to denote a signed graph, namely either  $(G, \sigma)$ , with  $\sigma$  being the function that assigns signs to edges, or  $(G, \Sigma)$ , with  $\Sigma$  being the set of negative edges. We mainly use the former one in this paper.

One notion of importance, which distinguishes signed graphs from 2-edge-colored graphs, is the notion of *resigning* (also called *switching* by some authors). Given an edge-cut  $[X, Y]$  of a signed graph  $(G, \sigma)$ , resigning at  $[X, Y]$  consists of multiplying all edges of  $[X, Y]$  by the negative sign, in other words, it is switching the sign of each edge

with one end in  $X$  and one end in  $Y$ . If  $[X, Y]$  is of the form  $[\{x\}, V(G) \setminus \{x\}]$  for some vertex  $x$  of  $G$ , we will say that we resign *at*  $x$  rather than at  $[\{x\}, V(G) \setminus \{x\}]$ . It is then easy to observe that resigning at any cut  $[X, Y]$  is equivalent to resigning at all vertices of  $X$  or resigning at all vertices of  $Y$ .

Let  $u$  and  $v$  be two (not necessarily distinct) vertices of a signed graph  $(G, \sigma)$ , and  $W$  be a  $uv$ -walk in  $G$  (recall that edges can be used more than once in a walk). The sign of such a walk is simply the product of the sign of its edges, each edge appearing more than once being counted with the corresponding multiplicity.

Observe that resigning at an internal vertex of a walk does not change the sign of the walk. Thus, the sign of a closed walk is invariant under the operation of resigning. However, if  $W$  is a not-closed walk, then resigning at exactly one end will change the sign of  $W$ .

Together with the parity of length, this leads to four distinguishable types of closed walks: positive and of even length (type 00), positive and of odd length (type 01), negative and of even length (10), negative and of odd length (type 11). This type notation is convenient in the following sense: if  $W_1$  is a closed walk of type  $ab$  starting at  $u$  and  $W_2$  is a closed walk of type  $cd$  also starting at  $u$ , then the  $uv$ -walk  $W_1 \cup W_2$ , obtained by concatenating  $W_1$  and  $W_2$ , is of type  $ab + cd$  (where the addition is taken in the additive group  $\mathbb{Z}_2 \times \mathbb{Z}_2$ ).

A positive cycle is said to be *balanced*, whereas a negative cycle is said to be *unbalanced*. An unbalanced cycle of length  $\ell$  is denoted by  $UC_\ell$ .

Structures whose sign is invariant under resigning, specially closed walks, are the key to our study of homomorphisms. Two signatures  $\sigma_1$  and  $\sigma_2$  on a same graph  $G$  are said to be *equivalent* if one is obtained from the other by a resigning. It is easily observed that this induces an equivalence relation on the class of all signatures on  $G$ . For each family of equivalent signatures, the set of balanced cycles is fixed. A key lemma of Zaslavsky is that the converse is also true, i.e., the set of balanced cycles uniquely determines the equivalent classes of signatures.

**Lemma 3** (Zaslavsky [17]). *Given two signatures  $\sigma_1$  and  $\sigma_2$  of a graph  $G$ ,  $\sigma_1$  is a resigning of  $\sigma_2$  if and only if the sets of balanced (or, equivalently, unbalanced) cycles of  $(G, \sigma_1)$  and  $(G, \sigma_2)$  are the same.*

We may now define the key notions of this work: homomorphisms of graphs, of 2-edge-colored graphs and of signed graphs. We emphasize on the difference between homomorphisms of 2-edge-colored graphs and of signed graphs, but we will indeed take advantage of the strong relation between these two notions when proving our results.

**Definition 4.** Given two graphs  $G$  and  $H$ , a homomorphism of  $G$  to  $H$  is a mapping  $\phi$  which maps vertices of  $G$  to vertices of  $H$ , edges of  $G$  to edges of  $H$ , in such a way that adjacencies and incidences are preserved, i.e.,  $\phi(x)\phi(y) \in E(H)$  whenever  $xy \in E(G)$ , and  $\phi(x) \in \phi(e)$  (in  $H$ ) whenever  $x \in e$  (in  $G$ ).

If  $G$  and  $H$  are both 2-edge-colored graphs, then a homomorphism of  $G$  to  $H$  is a homomorphism of the underlying (uncolored) graphs which preserves the colors as well.

Given signed graphs  $(G, \sigma)$  and  $(H, \pi)$ , a homomorphism of  $(G, \sigma)$  to  $(H, \pi)$  is a homomorphism of  $G$  to  $H$  which preserves the balance of all closed walks.

In the three definitions above, if the underlying graphs are simple (which will be the case in this study), then we can define a homomorphism to be a mapping of vertices

which preserves adjacency. For 2-edge-colored graphs, we require furthermore that the mapping preserves the color of edges and, for signed graphs, we require that the mapping preserves the signs of closed walks. Next we show a strong connection between these two conditions.

**Theorem 5.** *A signed graph  $(G, \sigma)$  admits a homomorphism to a signed graph  $(H, \pi)$  if and only if there is a resigning  $(G, \sigma')$  of  $(G, \sigma)$  and a homomorphism of  $G$  to  $H$  which preserves the sign of the edges with respect to  $\sigma'$  and  $\pi$ .*

One direction of this theorem is easy, since a sign preserving mapping of  $(G, \sigma')$  to  $(H, \pi)$  certainly preserves the signs of closed-walks and these signs are the same in  $(G, \sigma)$  and  $(G, \sigma')$ . On the other hand, if there is a homomorphism  $\phi$  of  $(G, \sigma)$  to  $(H, \pi)$ , defining  $\sigma'(e) = \pi(\phi(e))$  we get a signature  $\sigma'$  which induces the same sign as  $\sigma$  on all closed walks. Thus, by Lemma 3,  $\sigma'$  is a resigning of  $\sigma$ .

We note that one can then take the condition of this theorem as the definition of homomorphisms of signed graphs. This is indeed the original definition given in [13], and in the rest of this work we will use this definition. Thus a homomorphism of a signed (simple) graph  $(G, \sigma)$  to  $(H, \pi)$  will be denoted as a function  $f = (f_1, f_2) : \{+, -\} \times V(G) \rightarrow V(H)$ , where for each vertex  $x$  of  $G$ ,  $f_1$  specifies if a resigning is done at  $x$  or not and  $f_2$  specifies to which vertex of  $V(H)$  the vertex  $x$  is mapped.

A rather surprising result of [13] is that the restriction of homomorphisms on signed bipartite graphs captures the classic notion of homomorphisms of graphs as a special case. This is shown through the following construction. Given a graph  $G$ , let  $S(G)$  be the signed bipartite graph built as follows. For each edge  $uv$  of  $G$ , first add a parallel edge, and then subdivide both edges in order to form a 4-cycle (if  $G$  has  $n$  vertices and  $m$  edges, then  $S(G)$  has  $n + 2m$  vertices and  $4m$  edges). Finally, for each such 4-cycle, we assign one negative and three positive signs to its edges. With this construction in mind, the following is proved in [13].

**Theorem 6** (Naserasr, Rollová and Sopena [13]). *For every two graphs  $G$  and  $H$ , there is a homomorphism of  $G$  to  $H$  if and only if there is a homomorphism of  $S(G)$  to  $S(H)$ .*

It is thus of special interest to study the homomorphism relation on the subclass of signed bipartite graphs.

A common notion in the theory of homomorphisms is the notion of *core* which is defined analogously for each of the structural objects. A core is a graph (analogously, a signed or a 2-edge-colored graph) which does not admit any homomorphism to any of its proper subgraphs. The *core of a graph  $G$*  is then the smallest subgraph of  $G$  (with respect to subgraph inclusion) to which  $G$  admits a homomorphism. It is not difficult to show, in each case, that the core of a graph is unique up to isomorphism.

An *automorphism* of a signed graph  $(G, \sigma)$  is a homomorphism of  $(G, \sigma)$  to itself which is one-to-one. A signed graph is *vertex transitive* if, for each pair  $x, y$  of vertices, there is an automorphism which maps  $x$  to  $y$ . The most important examples of vertex transitive signed graphs for this work are unbalanced cycles.

### 3 Girth and maximum average degree conditions

An advantage of the new definition of homomorphisms of signed graphs is an immediate no-homomorphism lemma which is based on the following definition.

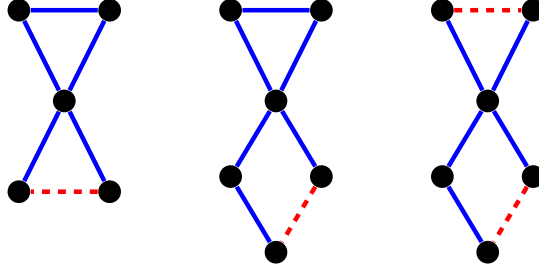


Figure 1: Signed graphs where  $g_{ij}$  is given by a closed walk which is not a cycle.

**Definition 7.** Given a signed graph  $(G, \sigma)$  and  $ij \in \mathbb{Z}_2^2$ , the  $ij$ -closed-walk-girth of  $(G, \sigma)$ , denoted  $g_{ij}(G, \sigma)$ , is the length of a shortest closed walk of type  $ij$  in  $(G, \sigma)$ . When there is no such closed walk, we let  $g_{ij}(G, \sigma) = \infty$ .

Observe that  $g_{00}(G, \sigma) = 2$  unless  $G$  has no edges, in which case  $g_{00}(G, \sigma) = \infty$ . We note that  $g_{ij}(G, \sigma)$  for  $ij \neq 00$  might not be realized by a cycle of  $G$  as shown by examples of Figure 1. However, it can be shown that for each connected graph, at least two of the three values will be realized by a cycle. Furthermore, as shown in [4], in a connected signed graph  $(G, \sigma)$ , among the three values  $g_{01}(G, \sigma)$ ,  $g_{10}(G, \sigma)$  and  $g_{11}(G, \sigma)$ , we cannot have exactly one value being  $\infty$ . This leads to three special subclasses where exactly two of the values are  $\infty$ . Among these three cases, the case  $g_{01}(G, \sigma) = g_{11}(G, \sigma) = \infty$  corresponds to the class of signed bipartite graphs, which is of special interest for this work.

We now state a basic no-homomorphism lemma for signed graphs.

**Lemma 8.** *If a signed graph  $(G, \sigma)$  admits a homomorphism to a signed graph  $(H, \pi)$ , then  $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$  for each  $ij \in \mathbb{Z}_2^2$ .*

It is easy to observe that the conditions  $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$  for each  $ij \in \mathbb{Z}_2^2$  are not sufficient for  $(G, \sigma)$  to admit a homomorphism to  $(H, \pi)$ . However, as we will show in this work, for graphs of small maximum average degree, these conditions are also sufficient. The bound on the maximum average degree will be provided as a function of  $(H, \pi)$ . This is stated more precisely in the next theorem, but we first recall a folklore lemma on the structure of graphs with small maximum average degree.

**Lemma 9.** *If  $G$  is a connected graph with minimum degree at least 2 and maximum average degree less than  $2 + \frac{2}{2+3(d-1)}$ , then either  $G$  is a cycle or  $G$  contains a thread of length  $d$ .*

**Theorem 10.** *For every connected signed graph  $(H, \pi)$ , there exists an  $\epsilon$  such that, for each graph  $G$  with  $\text{mad}(G) < 2 + \epsilon$ , and any signature  $\sigma$  on  $G$  such that  $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$  for each  $ij \in \mathbb{Z}_2^2$ ,  $(G, \sigma) \rightarrow (H, \pi)$ .*

*Proof.* To prove Theorem 10, one may first observe that vertices of degree 0 or 1 are of no importance. Thus, by Lemma 9, either  $G$  is a cycle or it contains a long thread.

If  $G$  is a cycle, then  $(G, \sigma)$  is a cycle of type, say,  $ij$ . As  $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ , this cycle can be mapped to a walk of type  $ij$  in  $(H, \pi)$ . Therefore, by Lemma 9, we may assume that  $G$  has a thread of length  $d$  where  $d$  is a function of  $\epsilon$  which increases when  $\epsilon$  tends to zero.

For the remaining part of the proof we assume  $g_{ij}(H, \pi) < \infty$  for all  $ij \in \mathbb{Z}_2^2$ . The other cases are similar and simpler, since in each case one may consider  $(H, \pi)$  to be just a cycle and then apply a simpler version of the following arguments. Recall that  $\text{diam}(H)$  is the largest possible distance between two vertices of  $H$ . Since  $H$  is a connected graph, this is a finite number and a function of  $H$ .

Take an integer  $d \geq 2\text{diam}(H) + \max\{g_{ij}(H, \pi)\}$  and, using Lemma 9, choose an  $\epsilon$  so that  $\text{mad}(G) < 2 + \epsilon$  implies the existence of a thread  $P$  of length  $d$  in  $G$ . Let  $x$  and  $y$  be the two ends of this thread and consider the signed graph  $(G', \sigma')$  obtained from  $(G, \sigma)$  by removing the internal vertices of  $P$ .

We may map  $(G', \sigma')$  to  $(H, \pi)$  by induction. Let  $\phi$  be such a homomorphism. We would like to extend  $\phi$  to a homomorphism of  $(G, \sigma)$  to  $(H, \pi)$ . We pay attention that  $x$  and  $y$  are already mapped and we are not allowed to resign at these two vertices, but we are allowed to resign at internal vertices of  $P$  after which we may choose where to map them. Recall that resigning at any internal vertex of  $P$  does not change the sign of  $P$ . Furthermore, since the parity of  $P$  is given, employing the terminology of types of closed walks, we may say  $P$  is of type  $ij$  for some  $ij \in \mathbb{Z}_2^2$ . We may now consider a shortest path  $Q$  in  $(H, \sigma)$  connecting  $\phi(x)$  and  $\phi(y)$ . If  $Q$  is of the same type as  $P$ , then we map  $P$  to  $Q$ , which is possible because the parity and sign of  $P$  permits this. Otherwise, we choose a shortest closed walk  $W$  of type  $i'j'$  starting at a vertex  $v$  of  $Q$  such that the walk  $Q'$ , starting from  $\phi(x)$ , going to  $v$  on the path  $Q$ , then traversing  $W$ , and then moving to  $\phi(y)$  on the path  $Q$ , is of the same type as  $P$ . Since we have assumed that  $(H, \pi)$  contains closed walks of each type, and since  $H$  is connected, this is possible. Furthermore, by taking a shortest path and a shortest closed walk, we have ensured that  $Q'$  has length at most  $d$ . Therefore, we may now extend the homomorphism  $\phi$  to a homomorphism of  $P$  to  $Q'$ .  $\square$

A challenging question is then to determine the best value of  $\epsilon$  for a given  $(H, \pi)$ . This value may be improved by further restriction on the graph  $(G, \sigma)$ . For example what if we consider only planar graphs? Note that planarity already imposes the condition of maximum average degree being strictly less than 6. Further conditions on lengths of facial cycles of a planar graph may improve this bound on the average degree. A sort of dual question is then the following one, of high interest.

**Problem 11.** *Given integers  $c_{ij}$  and  $\ell_{ij}$ ,  $ij \in \{01, 10, 11\}$ , satisfying  $c_{ij} \geq \ell_{ij}$  for every  $ij$ , what is a smallest signed graph  $(H, \pi)$  with  $g_{ij}(H, \pi) \geq \ell_{ij}$  such that every planar signed graph  $(G, \sigma)$  with  $g_{ij}(G, \sigma) \geq c_{ij}$  admits a homomorphism to  $(H, \pi)$ ?*

For example, for  $c_{10} = c_{11} = \ell_{10} = \ell_{11} = \infty$ ,  $c_{01} = \ell_{01} = 3$ , it is a restatement of the four color theorem, which gives that the signed graph  $(K_4, \emptyset)$  works. Similarly, if we take  $c_{10} = c_{11} = \ell_{10} = \ell_{11} = \infty$ ,  $c_{01} = 5$  and  $\ell_{01} = 3$ , it is a restatement of Grötzsch's theorem, which gives that the signed graph  $(K_3, \emptyset)$  works. Furthermore, the Jaeger-Zhang conjecture can be restated to the claim that if  $c_{10} = c_{11} = \ell_{10} = \ell_{11} = \infty$ ,  $c_{01} = 4k + 1$  and  $\ell_{01} = 2k + 1$ , then  $(C_{2k+1}, \emptyset)$  is the answer.

For  $c_{10} = \ell_{10} = 4$ ,  $c_{11} = \ell_{11} = c_{01} = \ell_{01} = 3$ , while a lower bound of 10 on the order of  $H$  is given in [13], an upper bound of 40 is proved in [15]. Moreover, the authors of [15] conjecture that 10 should also be an upper bound, and propose a specific signed graph of order 10 as a candidate.

For  $c_{10} = 4$ ,  $c_{01} = c_{11} = 3$ ,  $\ell_{10} = 2$ ,  $\ell_{01} = 3$  and  $\ell_{11} = 1$ , it is a conjecture of Naserasr and Raspaud that the signed graph of order 2, consisting of a positive edge and a negative edge between two vertices and a negative loop on each vertex works.

The main results of this work can then be seen as providing optimal values in Theorem 10 for specific cases. More precisely, we prove in Section 6 the following result.

**Theorem 12.** *If  $G$  is a graph with  $\text{mad}(G) < \frac{8}{3}$  then, for every signature  $\sigma$  on  $G$ ,  $(G, \sigma) \rightarrow (K_4, \{e\})$ . Moreover, the bound of  $\frac{8}{3}$  is best possible.*

In Section 7, studying the case of signed bipartite graphs, we prove the following theorem.

**Theorem 13.** *If  $G$  is a bipartite graph with  $\text{mad}(G) < 2 + \frac{1}{2g-1}$  then, for every signature  $\sigma$  on  $G$  such that  $g_{01}(G, \sigma) \geq 2g$ ,  $(G, \sigma) \rightarrow UC_{2g}$ .*

Furthermore, as a bipartite analogue of Jaeger-Zhang conjecture, the following conjecture was proposed in [13], for which we will provide some supporting results. Recall that  $g_{10}(G, \sigma)$  is called the unbalanced-girth of  $(G, \sigma)$ .

**Conjecture 14** (Naserasr, Rollová and Sopena [13]). *Every signed bipartite planar graph  $(G, \sigma)$  of unbalanced-girth at least  $4g - 2$  admits a homomorphism to  $UC_{2g}$ .*

The rest of the paper is organized as follows. In the next section, we provide a duality theorem for mapping 2-edge-colored graphs to  $(C_{2g}, \{e\}, E - e)$ . In Section 5, we prove some properties of minimum counterexamples for our statements. We prove Theorem 12 in Section 6, and Theorem 13 in Section 7. In Section 8, applying our results on planar graphs, we provide some support for Conjecture 14. Finally, in Section 9, we give some examples proving lower claims on some of our results and conjectures being the best possible.

## 4 A duality theorem for homomorphisms to a 2-edge-colored cycle

As a 2-edge-colored homomorphism problem, given a fixed 2-edge-colored cycle  $C$ , the corresponding  $C$ -coloring problem, by Feder-Vardi dichotomy conjecture [6], is expected to be either solvable in polynomial time or to be NP-complete, see [2, 3].

For the case of  $(C_{2g}, \{e\}, E - \{e\})$  we show that a duality theorem holds, which in particular implies a polynomial time algorithm. A polynomial time algorithm is also given in [2], but our interest is in theoretical applications of the simple and nice duality theorem which we provide here.

Let  $(P_{n+1}, \{e_1, e_n\}, E - \{e_1, e_n\})$  be a path of length  $n$  whose first and last edges are colored red (or 1), while all other edges are colored blue (or 2). It is easily observed that there exists a homomorphism of this 2-edge-colored graph to the 2-edge-colored cycle  $(C_{2g}, \{e\}, E - \{e\})$  if and only if either  $n$  is even, or  $n$  is odd and  $n \geq 2g + 1$ . Therefore, if a 2-edge-colored graph  $(G, E_1, E_2)$  admits a homomorphism from  $(P_{2\ell-1}, \{e_1, e_{2\ell-1}\}, E - \{e_1, e_{2\ell-1}\})$ , with  $\ell \leq g$ , then it does not admit any homomorphism to  $(C_{2g}, \{e\}, E - \{e\})$ . Our claim is that this necessary condition is also sufficient for an input whose underlying graph is bipartite.



**Theorem 15.** *Given a bipartite graph  $G$ , a 2-edge-colored graph  $(G, E_1, E_2)$  admits a homomorphism to  $(C_{2g}, \{e\}, E - \{e\})$  if and only if there is no homomorphism of  $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$  to  $(G, E_1, E_2)$ .*

*Proof.* Observe that for every  $i, i \leq g$ ,  $(P_{2i-1}, \{e_1, e_{2i-1}\}, E - \{e_1, e_{2i-1}\})$  and  $(C_{2i}, \{e\}, E - \{e\})$  are both homomorphic images of  $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$ . Moreover, any bipartite image of  $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$  must contain either  $(P_{2i-1}, \{e_1, e_{2i-1}\}, E - \{e_1, e_{2i-1}\})$  or  $(C_{2i}, \{e\}, E - \{e\})$  as a subgraph for some  $i \leq g$ .

Thus  $(G, E_1, E_2)$  does not admit any homomorphism from  $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$  if and only if it contains neither  $(P_{2i-1}, \{e_1, e_{2i-1}\}, E - \{e_1, e_{2i-1}\})$  nor  $(C_{2i}, \{e\}, E - \{e\})$  as a subgraph, for every  $i \leq g$ . In other words,  $(G, E_1, E_2)$  does not admit any homomorphism from  $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$  if and only if no two vertices, each incident with a red edge, are connected by a blue path of odd length  $2\ell + 1$  where  $\ell < g$ . In particular, this implies that  $(C_{2g}, \{e\}, E - \{e\})$  does not admit any homomorphism from  $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$ , which proves the “only if” part of our theorem.

For the converse, denote by  $x_0y_0x_1y_1 \dots x_{g-1}y_{g-1}$  the cycle  $C_{2g}$  and assume  $x_0y_0$  is the red edge of  $(C_{2g}, \{e\}, E - \{e\})$ . Consider a 2-edge-colored graph  $(G, E_1, E_2)$ , where  $G$  is a bipartite graph with bipartition  $(X, Y)$  and such that  $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$  does admit a homomorphism to  $(G, E_1, E_2)$ . We need to find a homomorphism of  $(G, E_1, E_2)$  to  $(C_{2g}, \{e\}, E - \{e\})$ . Let  $X_0$  (respectively  $Y_0$ ) be vertices in  $X$  (respectively  $Y$ ) each of which is incident with a red edge. Let  $X_i, i \leq g - 2$ , be the set of vertices in  $X$  at distance  $i$  from  $X_1 \cup Y_1$ . Thus a vertex  $v$  of  $X$  is in  $X_i$  if a closest vertex to it in  $X_0 \cup Y_0$ , say  $f(v)$ , is at distance  $i$  from it. Observe that, because of the bipartition of  $G$ , for odd values of  $i$ ,  $f(v)$  is in  $Y_0$  and for even values of  $i$ ,  $f(v)$  is in  $X_0$ . Let  $X_{g-1}$  be all the remaining vertices in  $X$ , thus vertices in  $X_g$  are at distance  $g - 1$  or more from all vertices in  $X_0 \cup Y_0$ . We similarly define  $Y_i, i \leq g - 1$ .

We may now define a homomorphism of  $(G, E_1, E_2)$  to  $(C_{2g}, \{e\}, E - \{e\})$  as follows: all vertices in  $X_i$  are mapped to the vertex  $x_i$  and all vertices in  $Y_i$  are mapped to the vertex  $y_i$ . It remains to show that red edges of  $(G, E_1, E_2)$  (edges in  $E_1$ ) are mapped to red edges of  $(C_{2g}, \{e\}, E - \{e\})$  (the edge  $e = x_1y_1$ ) and that blue edges are mapped to blue edges. A red edge in  $(G, E_1, E_2)$  must have, by definition and because of the bipartition, one end in  $X_0$  and the other end in  $Y_0$ . Thus it maps to  $x_0y_0$  which is the red edge of  $(C_{2g}, \{e\}, E - \{e\})$ . Let  $e = uv$  be a blue edge, and assume  $u \in X_i, i < g - 1$ . Let  $f(u)$  be a closest vertex in  $X_0 \cup Y_0$  to  $u$ . Recall that whether  $f(u)$  is in  $X_0$  or in  $Y_0$  only depends on the parity of  $i$ . Observe that  $v$  is a vertex in the  $Y$  part of  $G$  because of the bipartition of  $G$ . We claim that  $v$  cannot be in  $Y_i$ . Otherwise,  $f(v)$ , a closest vertex to  $v$  in  $X_0 \cup Y_0$ , and  $f(u)$  must be in distinct parts. Then the walk composed of a shortest connection from  $f(u)$  to  $u$ , the edge  $uv$  and a shortest connection from  $v$  to  $f(v)$ , fits the parity condition to be the image of  $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$ , but this path cannot map to  $(G, E_1, E_2)$ . Thus, considering the triangular inequality, the vertex  $v$  has to be either in  $Y_{i-1}$  or in  $Y_{i+1}$ . Since both  $y_{i-1}x_i$  and  $x_iy_{i+1}$  are blue edges of  $(C_{2g}, \{e\}, E - \{e\})$ ,  $uv$  is mapped to a blue edge and this type of edges are fine as well. It only remains to consider a blue edge  $uv$  where  $u \in X_{g-1}$ . In this case, again by the triangular inequality, either  $v \in Y_{g-2}$  or  $v \in Y_{g-1}$ , since  $y_{g-2}x_{g-1}$  and  $x_{g-1}y_{g-1}$  are both blue edges of  $(C_{2g}, \{e\}, E - \{e\})$ , and we are done with this case as well.  $\square$

In relation to signed graph homomorphisms, we observed that every unbalanced cycle

can be resigned in such a way that it contains a single negative edge. Therefore, the question of deciding whether a signed bipartite graph admits a homomorphism to an unbalanced cycle  $UC_{2g}$  amounts to finding an equivalent signature which does not induce a  $(P_{n+1}, \{e_1, e_n\}, E - \{e_1, e_n\})$  structure.

However, unless  $P = NP$ , this would not be an easy task as it is shown in [3, 7] that this homomorphism question, in contrast to its 2-edge-colored counterpart, is an NP-complete problem.

As an application of Theorem 15, we have the following result.

**Theorem 16.** *A signed bipartite graph  $(G, \sigma)$  maps to  $(C_4, \{e\})$  if and only if it maps to  $(K_4, \{e\})$ .*

*Proof.* Since  $(C_4, \{e\})$  is a subgraph of  $(K_4, \{e\})$ , if  $(G, \sigma)$  maps to  $(C_4, \{e\})$  then it also maps to  $(K_4, \{e\})$ . Conversely, suppose that  $(G, \sigma)$  maps to  $(K_4, \{e\})$  under the signature  $\sigma$ . It follows that  $(G, \sigma)$  cannot contain any positive edge  $xy$  such that both  $x$  and  $y$  are incident with a negative edge. Assuming  $G$  is bipartite, this property corresponds to the hypothesis of Theorem 15 for  $g = 2$ , and thus  $(G, \sigma)$ , with this particular signature, maps to  $(C_4, \{e\})$ .  $\square$

## 5 Minimal elements

A standard technique to prove results of the type we consider in this work is to consider a minimum counterexample, prove some properties that such a graph must satisfy, and finally derive a contradiction. The minimality of such a counterexample can be viewed in two ways. The first one is to say that no subgraph of our minimal counterexample is a counterexample, which, in particular, says that a minimal counterexample is a core. The second one is to say that no smaller member of the class of graphs we are working with is a counterexample. This, in particular, implies that no (proper) homomorphic image of our minimal counterexample belongs to the considered class of graphs.

In view of the former case, i.e., a minimum counterexample necessarily being a core, we will show here that certain subdivisions of  $K_4$  cannot be a core. In view of the latter case, Klostermyer and Zhang developed in [9] a so-called “folding lemma” for homomorphism problems on the class of planar graphs, which implies that every planar graph of odd-girth  $2k + 1$  has a planar homomorphic image for which, in every planar embedding, every face is a  $(2k + 1)$ -cycle. An analogue of this lemma for the class of signed bipartite planar graphs is developed in [12], which would also be of importance for the part of our work which deals with this subclass of signed graphs. We thus restate this lemma and its corollary below.

**Lemma 17** (Naserasr, Rollová and Sopena [12]). *Let  $(G, \sigma)$  be a signed bipartite graph of unbalanced-girth  $2g$ , together with a planar embedding. If  $(G, \sigma)$  has a facial cycle  $C$  which is not an unbalanced cycle of length  $2g$ , then there exist two vertices  $x$  and  $y$  of  $C$ , at distance two from each other, such that the planar signed graph obtained from  $(G, \sigma)$  by identifying  $x$  and  $y$  is a homomorphic image of  $(G, \sigma)$  with unbalanced-girth  $2g$ .*

**Corollary 18.** *Every signed planar bipartite graph of unbalanced-girth  $2g$  admits a planar homomorphic image of unbalanced-girth  $2g$  where, in every planar embedding, every face is an unbalanced cycle of length exactly  $2g$ .*

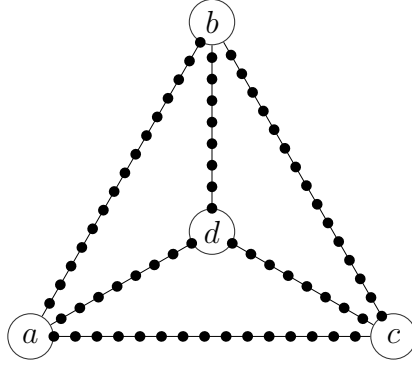


Figure 2: A subdivision of  $K_4$

Thus, for instance, a minimal counterexample to Conjecture 14 should be a signed bipartite planar graph such that every facial cycle of any of its planar embeddings is an unbalanced cycle of length exactly  $4g - 2$ .

A *subdivision of  $K_4$*  is a graph obtained by replacing some or all edges of  $K_4$  by threads connecting their end points. A *signed subdivision of  $K_4$*  is any signed graph based on a subdivision of  $K_4$ . A planar drawing of a subdivision of  $K_4$  has four facial cycles. As the number of unbalanced faces must be even (since each negative edge changes the balance of two incident faces), there are essentially three different types of signed subdivisions of  $K_4$ , namely those having two, four, or no unbalanced faces. We are interested in the case where the number of unbalanced faces is two, and we want to determine when such a signed subdivision of  $K_4$  is a core. The next two lemmas give an answer to this question for two particular cases, and will be used in Section 7 to prove Theorem 13.

Let  $\overline{\overline{K_4}}$  be a subdivision of  $K_4$  whose (main) vertices are  $a, b, c$  and  $d$  (see Figure 2). For every  $x, y \in \{a, b, c, d\}$ ,  $x \neq y$ , let  $P_{xy}$  be the path which represents the edge  $xy$ , and let  $L_{xy}$  be the length of  $P_{xy}$ . Moreover, let  $C_{xyz} = P_{xy} + P_{yz} + P_{zx}$ .

In [14], all subdivisions of  $K_4$  which are cores are classified. The following case of that result is used in this work. (The proof of this result is given in Appendix A. The next Lemma has a similar proof.)

**Lemma 19.** *If  $\overline{\overline{K_4}}$  is a bipartite subdivision of  $K_4$  such that  $|C_{abc}| = 2g$  and  $L_{ad} + L_{bd} + L_{cd} \geq 4g$ , and  $\sigma$  is a signature of  $\overline{\overline{K_4}}$  such that  $C_{abc}$  is an unbalanced cycle and  $(\overline{\overline{K_4}}, \sigma)$  is of unbalanced-girth  $2g$ , then the core of  $(\overline{\overline{K_4}}, \sigma)$  is  $UC_{2g}$ .*

Let  $K_3^{++}$  be the multigraph obtained from  $K_3$ , with vertices  $a, b$  and  $c$ , by adding a parallel edge to each of the edges  $ab$  and  $ac$ . Similarly to the proof of the previous lemma, we can prove the following result which claims that certain signed graphs, build upon specific subdivisions of  $K_3^{++}$  (see Figure 3) are not cores. The following has also been proved in [14].

**Lemma 20.** *If  $\overline{\overline{K_3^{++}}}$  is a bipartite subdivision of  $K_3^{++}$  such that the outer cycle is of length  $2g$  and the total length of the three threads incident to  $b$  is at least  $4g$ , and  $\sigma$  is a signature of  $\overline{\overline{K_3^{++}}}$  such that  $(\overline{\overline{K_3^{++}}}, \sigma)$  is of unbalanced-girth  $2g$ , then the core of  $(\overline{\overline{K_3^{++}}}, \sigma)$  is  $UC_{2g}$ .*

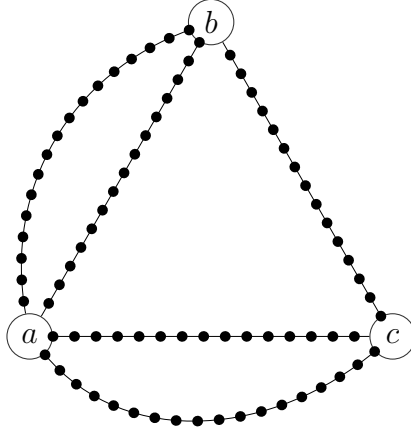


Figure 3: A subdivision of  $K_3^{++}$

## 6 Mapping sparse signed graphs to $(K_4, \{e\})$

In this section, we prove Theorem 12, using standard discharging technique. A reducible configuration for this theorem is a signed graph  $(F, \sigma_1)$  with the following property: given any signed graph  $(G, \sigma)$  containing  $(F, \sigma_1)$  as a subgraph, every possible homomorphism of the signed graph induced by  $G - F$  to  $(K_4, \{e\})$  can be extended to a homomorphism of  $(G, \sigma)$  to  $(K_4, \{e\})$ .

To prove Theorem 12, we first exhibit a set of reducible configurations (Lemmas 24 to 27). Then, to complete the proof, we show that if a graph  $G$  has maximum average degree less than  $\frac{8}{3}$ , then  $G$  must contain at least one of these reducible configurations. This clearly implies that Theorem 12 does not admit any counterexample.

We start with some notation and terminology. In the rest of this section,  $(G, \sigma)$  is considered to be a minimal counterexample to Theorem 12, that is, a counterexample with the smallest possible number of vertices and, among all such counterexamples, having a smallest number of edges. That is to say,  $\text{mad}(G) < \frac{8}{3}$ ,  $(G, \sigma) \not\rightarrow (K_4, \{e\})$ , and every signed graph with less vertices than  $G$  having maximum average degree less than  $\frac{8}{3}$  maps to  $(K_4, \{e\})$ . Note that, since the maximum average degree is taken over all subgraphs, every subgraph of  $G$  has also maximum average degree less than  $\frac{8}{3}$ . Thus, any proper signed subgraph of  $(G, \sigma)$  maps to  $(K_4, \{e\})$ , which implies that  $G$  must be connected.

Recall that a  $k$ -vertex is a vertex of degree  $k$ .

**Definition 21** (Weak vertex). A *weak vertex* is a 3-vertex adjacent to two 2-vertices.

**Definition 22** (3-subgraph, 3-subtree). A *3-subgraph* of a graph  $G$  is a connected subgraph  $S$  of  $G$  all of whose vertices have degree 3 in  $G$  and which is maximal for this property. Every vertex of  $G - S$  adjacent to a vertex of  $S$  is a *neighbor* of  $S$ . (Observe that no 3-vertex can be a neighbor of  $S$ .) If  $S$  is a tree, then  $S$  is a *3-subtree* of  $G$ .

We will label vertices of the signed graph  $(K_4, \{e\})$  with  $V(K_4) = \{1, 2, 3, 4\}$ , where  $e = 14$  is the negative edge (see Figure 4). This signed graph has the following easy to observe properties.

**Observation 23.** For every two (not necessarily distinct) vertices  $x$  and  $y$  of  $(K_4, \{e\})$ ,

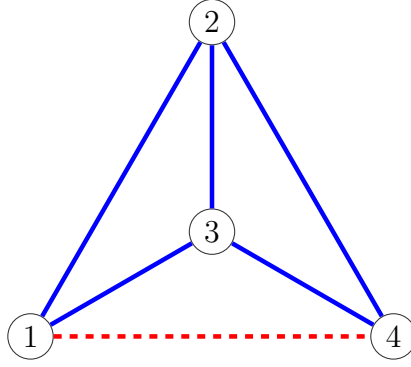


Figure 4: The signed graph  $(K_4, \{e\})$

1. there exists a vertex  $z$  such that the walk  $xzy$  is positive,
2. if  $x \in \{a, c\}$  and  $y \in \{b, d\}$ , then there exists a vertex  $z$  such that the path  $xzy$  is negative,
3. there exist vertices  $z$  and  $z'$  such that the walk  $xzz'y$  is positive,
4. there exist vertices  $z$  and  $z'$  such that the walk  $xzz'y$  is negative.

We now prove a series of lemmas giving a set of reducible configurations.

**Lemma 24** (Reducible configuration 1). *The graph  $G$  has no 1-vertex, i.e.,  $\delta(G) \geq 2$ .*

*Proof.* Assume to the contrary that  $u$  is a 1-vertex of  $G$ , and let  $v$  be the neighbor of  $u$ . By the minimality of  $G$ , there is a homomorphism  $\phi$  of  $(G - u, \sigma - \{uv\})$  to  $(K_4, \{e\})$ . We can then extend this homomorphism to a homomorphism of  $(G, \sigma)$  to  $(K_4, \{e\})$  as follows. If  $uv \in \sigma$ , we first resign at  $u$ , so that the edge  $uv$  is positive. We can then map  $u$  to any vertex of  $(K_4, \{e\})$  connected to  $\phi(v)$  by a positive edge. We thus obtain a homomorphism of  $(G, \sigma)$  to  $(K_4, \{e\})$ , a contradiction.  $\square$

**Lemma 25** (Reducible configuration 2). *The graph  $G$  has no pair of adjacent 2-vertices.*

*Proof.* Assume to the contrary that  $u$  and  $u'$  are two adjacent 2-vertices in  $G$ , and let  $G' = G - \{u, u'\}$ . By the minimality of  $G$ , there is a homomorphism  $\phi$  of  $(G', \sigma')$  to  $(K_4, \{e\})$ , where  $\sigma'$  is the signature induced by  $\sigma$  on  $G'$ . We prove that this homomorphism can be extended to a homomorphism of  $G$  to  $(K_4, \{e\})$ .

Let  $v$  and  $v'$  denote the other neighbor of  $u$  and  $u'$ , respectively. By Observation 23(3,4), there exist a positive 3-walk and an negative 3-walk from  $\phi(v)$  to  $\phi(v')$  in  $(K_4, \{e\})$ . If necessary, we can resign at  $u$ , or  $u'$ , or both, in such a way that the signs of the edges of the path  $vu u'v'$  and the corresponding, positive or negative, walk in  $(K_4, \{e\})$  coincide. By mapping  $u$  and  $u'$  to the internal vertices of the corresponding walk, we get a homomorphism of  $(G, \sigma)$  to  $(K_4, \{e\})$ , a contradiction.  $\square$

**Lemma 26** (Reducible configuration 3). *The graph  $G$  has no 3-vertex adjacent to three 2-vertices.*

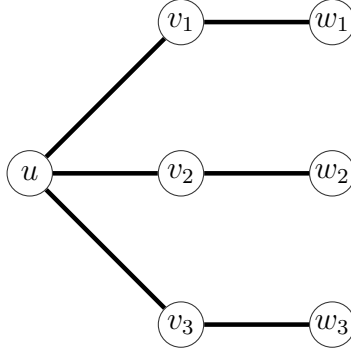


Figure 5: The reducible configuration 3

*Proof.* Assume to the contrary that  $u$  is a 3-vertex, adjacent to three 2-vertices  $v_1$ ,  $v_2$  and  $v_3$  (see Figure 5), and let  $G' = G - \{u, v_1, v_2, v_3\}$ . By the minimality of  $G$ , there is a homomorphism  $\phi$  of  $(G', \sigma')$  to  $(K_4, \{e\})$ , where  $\sigma'$  is the signature induced by  $\sigma$  on  $G'$ . Again, we prove that this homomorphism can be extended to a homomorphism of  $G$  to  $(K_4, \{e\})$ , leading to a contradiction.

Let  $w_1$ ,  $w_2$  and  $w_3$  denote the other neighbor of  $v_1$ ,  $v_2$  and  $v_3$ , respectively. We consider two cases.

1. *The three paths  $uv_1w_1$ ,  $uv_2w_2$  and  $uv_3w_3$  are all positive or all negative.*

In this case, by resigning at  $v_1$ ,  $v_2$ ,  $v_3$  and  $u$ , if necessary, we may get an equivalent signature where all the six edges incident to  $v_1$ ,  $v_2$  and  $v_3$  are positive. We then map  $u$  to any vertex of  $(K_4, \{e\})$ , and extend the homomorphism  $\phi$  to  $v_1$ ,  $v_2$  and  $v_3$  thanks to Observation 23(1).

2. *The three paths  $uv_1w_1$ ,  $uv_2w_2$  and  $uv_3w_3$  do not have the same balance.*

In that case, we may resign at  $u$  in such a way that only one path, say  $uv_1w_1$ , is negative. If  $\phi(w_1) \in \{a, c\}$  (respectively,  $\phi(w_1) \in \{b, d\}$ ), then we map  $u$  to  $b$  (respectively, to  $a$ ). The homomorphism can then be extended to  $v_1$ , thanks to Observation 23(2), and to  $v_2$  and  $v_3$ , thanks to Observation 23(1).

This concludes the proof. □

**Lemma 27** (Reducible configuration 4). *The graph  $G$  has no 3-subtree  $T$  all of whose neighbors (in  $G - T$ ) are 2-vertices in  $G$ .*

*Proof.* Consider a 3-subtree  $T$  of  $G$  which is only adjacent in  $G - T$  to 2-vertices. Let  $T_2$  be the subgraph of  $G$  containing  $T$  and all its adjacent 2-vertices, and  $G' = G - T_2$ . By the minimality of  $G$ , there is a homomorphism  $\phi$  of  $(G', \sigma')$  to  $(K_4, \{e\})$ , where  $\sigma'$  is the signature induced by  $\sigma$  on  $G'$ . Again, we want to extend this homomorphism to a homomorphism of  $G$  to  $(K_4, \{e\})$ , leading to a contradiction. In what follows, we show that  $\phi$  can be partially extended, with some degree of freedom which is made more precise by the following definitions.

Let  $Q$  be a connected (proper) subtree of  $T$  such that  $T - Q$  is connected, and let  $x = \text{end}(Q)$  be the unique vertex of  $Q$  having a neighbor in  $T - Q$ , called the *end-vertex* of  $Q$ . (For instance, any leaf vertex  $\ell$  of  $T$  can be chosen as  $Q$ , in which case  $\text{end}(Q) = \ell$ .) Let  $Q_2$  be the subgraph of  $G$  containing the vertices of  $Q$  and the 2-vertices of  $G$  adjacent

to vertices of  $Q$ . Recall that the homomorphism  $\phi = (\phi_1, \phi_2)$  decides, for each vertex  $v$  of  $G'$ , if there must be a resigning at  $v$  (determined by the component  $\phi_1$ ), and to which of the four vertices of  $(K_4, \{e\})$  the vertex  $v$  is mapped to (determined by the component  $\phi_2$ ).

Based on the following notation we show that one can extend the homomorphism  $\phi$  to  $Q_2$ , with a degree of freedom at  $x$ . We define the six following sets of images of the homomorphism  $\phi$ .

$$\begin{aligned} A_1 &= \{(+, 1), (+, 4), (-, 1), (-, 4)\}, & A_2 &= \{(+, 2), (+, 3), (-, 2), (-, 3)\}, \\ A_3 &= \{(+, 1), (+, 4), (-, 2), (-, 3)\}, & A_4 &= \{(+, 2), (+, 3), (-, 1), (-, 4)\}, \\ A_5 &= \{(+, 1), (+, 4), (+, 2), (+, 3)\}, & A_6 &= \{(-, 1), (-, 4), (-, 2), (-, 3)\}. \end{aligned}$$

We now prove the following claim.

**Claim 1.** *Given a connected subtree  $Q$  of  $T$  with end-vertex  $x$ , and a homomorphism  $\phi$  of  $G' = G - T_2$  to  $(K_4, \{e\})$ , there exists a set  $A_i$ ,  $1 \leq i \leq 6$ , such that for any of the four choices  $(\alpha, \beta) \in A_i$ , the homomorphism  $\phi$  can be extended to  $Q_2$  in such a way that  $\phi(x) = (\alpha, \beta)$ .*

*Proof.* We will prove this claim using dynamic programming technique. We first show that the claim holds if  $Q$  is just a leaf vertex of  $T$ . Then we show that we can add a vertex  $u$  (with degree 2 in  $T$ ) to  $Q$  as a new end-vertex so that the so-obtained tree satisfies the claim. Finally we show that two end-vertices can be merged to form a larger tree with a new end-vertex that also satisfies the claim.

Suppose first that  $Q$  is any leaf  $u$  of  $T$ , in particular we have  $\text{end}(Q) = u$ . Since all vertices of  $T$  are of degree 3 in  $G$  and all neighbors of  $T$  outside  $T$  are of degree 2,  $u$  is a weak vertex of  $G$ . Let  $v_1$  and  $v_2$  denote the two neighbors of  $u$  not in  $T$ . Let  $H = \{u, v_1, v_2\}$  and let  $w_1, w_2$  be the other neighbors of  $v_1$  and  $v_2$  in  $G$ , respectively. Depending on the signs of the paths  $uv_1w_1$  and  $uv_2w_2$ , we consider two cases.

1. *The paths  $uv_1w_1$  and  $uv_2w_2$  are both positive or both negative.*

In the former case, by resigning at  $v_1$ , or at  $v_2$ , or at both  $v_1$  and  $v_2$ , we may assume that the four edges of these two paths are positive. Then, any choice of  $\phi_2(u)$  (without resigning at  $u$ ) can be extended to  $v_1$  and  $v_2$  by Observation 23(1). Thus,  $\phi(u)$  can be any member of  $A_5$ . In the latter case, after resigning at  $u$ , we are in the former case, and thus  $\phi(u)$  can be any member of  $A_6$ .

2. *The paths  $uv_1w_1$  and  $uv_2w_2$  do not have the same sign.*

Without loss of generality, assume  $uv_1w_1$  is negative. If  $w_1$  is mapped to 1 or 4, then each choice of  $\phi(u) = (+, 2)$  and  $\phi(u) = (+, 3)$  can be extended to  $v_1$  and  $v_2$ . If  $w_1$  is mapped to 2 or 3, then each choice of  $\phi(u) = (+, 1)$  and  $\phi(u) = (-, 1)$  can be extended to  $v_1$  and  $v_2$ . But then, after a resigning at  $u$ , we may exchange the role of  $w_1$  with  $w_2$  to get two more possibilities for  $u$ . Therefore, depending on the values of  $\phi_2(w_1)$  and  $\phi_2(w_2)$ , one of the four sets  $A_1, A_2, A_3$  or  $A_4$  works.

To proceed with our goal, given a subtree  $Q$  with end-vertex  $x$  which satisfies our claim, we want to extend  $Q$  by adding a new vertex  $u$  that will be the new end-vertex and will also satisfy our claim. This is done in the following setting. Let  $u$  be a 2-vertex of  $T$  which is a neighbor of  $x$ . Let  $Q$  be the subtree of  $T$  such that  $\text{end}(Q) = x$  and

$u \notin Q$ . Observe that  $u$  is the end-vertex of the subtree  $Q^+ = Q + \{u\}$ . We now prove the following.

**Claim 2.** *If  $x$  satisfies Claim 1 as the end-vertex of  $Q$ , then  $u$  also satisfies Claim 1 as the end-vertex of  $Q^+$ .*

*Proof.* Since all vertices of  $T$  are 3-vertices of  $G$ ,  $u$  has one neighbor in  $G$ , say  $v$ , which is not in  $T$ . As we have assumed that all neighbors of  $T$  are 2-vertices,  $v$  is a vertex of degree 2 in  $G$ . Thus, it has another neighbor, say  $w$ . We note that  $w$  could be a vertex of  $T$  or  $Q$ , but even then we will assume, in the rest of the proof, that  $\phi(w)$  is already given. Depending on the sign of the path  $xuvw$ , we have two possible cases.

1. *The path  $xuvw$  is positive.*

We first resign at  $u$  or  $v$  or both, if needed, in such a way that all the three edges of the path  $xuvw$  are positive. Assume first that, among the four choices for  $x$ , two are without a resigning at  $x$ , i.e., choices of the form  $(+, i)$ . For any such choice of  $(+, i)$  all three choices of  $(+, j)$ ,  $j \neq i$  for  $u$  are then extendable at  $v$ . Therefore, if we have not resigned at  $u$ , then  $A_5$  works, whereas if we have resigned at  $u$ , then  $A_6$  works. If our assumption does not hold, then  $A_6$  is the set of four choices for  $x$ . In this case, for each choice of  $(-, i)$ , the choice of color  $i$  for  $u$  is also extendable to  $v$ . Thus, again depending on whether we have already resigned at  $u$  or not, the set  $A_5$  or  $A_6$  works.

2. *The path  $xuvw$  is negative.*

In this case, we may resign at  $u$  or  $v$  or both, if needed, in such a way that  $ux$  is a negative edge and it is the only negative edges of the path  $xuvw$ . Consider a set of four possible choices for  $x$ . If for some  $i \in \{1, 2, 3, 4\}$ , we have  $(-, i)$  as a choice, then every choice of  $(+, j)$ ,  $j \neq i$  for  $u$  can be extended to  $v$  as well. Observe that if for some  $j$  we have  $(-, j) \in A_i$ , then there is a  $j' \neq j$  such that  $(-, j')$  is also in  $A_i$ . In this case, depending on whether we have originally resigned at  $u$  or not,  $A_5$  or  $A_6$  is a collection of four possible choices for  $u$ . Finally, if there are no negative choice available for  $x$ , then the corresponding set of four possible choices for  $x$  are the four members of  $A_5$ . In this case, with respect to the current signature, for each choice of  $(+, i)$  for  $x$ , assigning  $u$  to the same vertex, i.e.,  $i$ , can be extended to  $v$ . Thus, again depending on whether we have already resigned at  $u$  or not, the set  $A_5$  or  $A_6$  provides a list of four possible choice for  $u$ .

This concludes the proof of Claim 2. □

Now, we want to merge two subtrees with distinct end-vertices to get a larger subtree with a new end-vertex, and to show that if we had the four choices (from sets  $A_i$  and  $A_j$ ) for each of the two previous end-vertices, then we have such a set of four choices for the new end-vertex as well. This is done with the following precise notation. Let  $y$  be a 3-vertex of  $T$ , with vertices  $x_1$  and  $x_2$  of  $T$  being two of its neighbors. Let  $Q_1$  and  $Q_2$  be the subtrees of  $T$  with  $\text{end}(Q_1) = x_1$ ,  $\text{end}(Q_2) = x_2$ ,  $y \notin Q_1$  and  $y \notin Q_2$ . Observe that  $y$  is an end-vertex of the subtree  $Q$  induced by  $Q_1$ ,  $Q_2$  and  $y$ . Assume that for a coloring  $\phi$  of  $G'$  (recall that  $G' = G - T_2$ ), there is set of  $A_i$  ( $i \in \{1, 2, \dots, 6\}$ ) of four possible choices for  $x_1$ , with which  $\phi$  can be extended to  $Q'_1$  and a set  $A_j$  ( $j \in \{1, 2, \dots, 6\}$ ) of four possible



choices for  $x_2$ , with which  $\phi$  can be extended to  $Q'_2$ . Then we claim that there is a set  $A_l$ ,  $1 \leq l \leq 6$ , such that any choice of an element of  $A_l$  for  $\phi(y)$  can be extended to the  $Q'$ .

To prove this, suppose first that the corresponding set for one of  $x_1$  or  $x_2$ , say  $x_1$ , is  $A_i$  for some  $i \leq 4$ . That means that we have two choices of colors for  $x_1$  without resigning at  $x_1$ , and two choices together with resigning. Thus, over all, we have two choices of distinct colors for each of  $x_1$  and  $x_2$  such that the path  $x_1yx_2$  is positive. Hence, for each choice of color for  $y$ , we can make it distinct from a choice of color for  $x_1$  and a choice of color for  $x_2$ , and such that the signs of the edges  $x_1y$  and  $yx_2$  remain the same. Thus, if the signs of these edges are positive, then  $A_5$  is a set of possible choices for  $y$ , otherwise  $A_6$  is such a set for  $y$ .

We may thus assume that the set of four choices for each of  $x_1$  and  $x_2$  is either  $A_5$  or  $A_6$ . In such a case, after each of such an assignment to  $x_1$  and  $x_2$ , each of the two edges  $x_1y$  and  $yx_2$  has a fixed sign. If these fixed signs matches, i.e., if the path  $x_1yx_2$  is positive, then  $A_5$  or  $A_6$  is a set of possible choices for  $y$  as before. Otherwise, one of the two edges, say  $x_1y$ , is negative and the other,  $yx_2$ , remains positive. Then, for each choice of  $(\alpha, i)$  for  $x_1$ , the choice of  $(+, i)$  works with some choice for  $x_2$ , and in this case  $A_5$  is a set of choices for  $y$ .

This concludes the proof of Claim 1.  $\square$

We are finally ready to prove that a 3-subtree of  $G$ , all whose neighbors are 2-vertices of  $G$ , is a reducible configuration.

If  $T$  has only one vertex, then this is done in reducible configuration 3. We may thus assume that  $T$  has at least one edge  $xy$ . Suppose that  $x$  is an end-vertex of  $Q_x$ ,  $y \notin V(Q_x)$ , and that  $y$  is an end-vertex of  $Q_y$ ,  $x \notin V(Q_y)$ . Note that each of  $Q_x$  and  $Q_y$  can be constructed, starting from leaf vertices of  $T$ , by the operations of adding 2-vertices of  $T$  or merging at 3-vertices as described above. Thus, there are sets  $A_i$  and  $A_j$  such that, for each choice of  $\phi(x)$  from  $A_i$  and  $\phi(y)$  from  $A_j$ , we have an extension which works for all edges except possibly for the edge  $xy$ . Observe that, for each element  $(\alpha, l)$  in  $A_i$ , there is an element  $(\alpha, l')$ ,  $l \neq l'$ , in  $A_i$ . Therefore, if for any such choice of  $\phi(x)$  and  $\phi(y)$  the edge  $xy$  is of positive sign, then we may simply change the color of one of its end, if necessary, to have all edges satisfied. Otherwise,  $xy$  is a negative edge and we must have  $i, j \in 5, 6$ . In such a case, it is enough to choose  $\phi(x)$  and  $\phi(y)$  as vertices 1 and 4 of the  $K_4$ .  $\square$

We now prove that every graph with maximum average degree less than  $\frac{8}{3}$  necessarily contains one of the four configurations introduced before.

**Lemma 28.** *If  $G$  is a graph with  $\text{mad}(G) < \frac{8}{3}$ , then  $G$  contains one of the four following configurations:*

C1: *a 1-vertex,*

C2: *two adjacent 2-vertices,*

C3: *a 3-vertex adjacent to three 2-vertices, or*

C4: *a 3-subtree  $S$  whose neighbors are all 2-vertices of  $G$ .*

*Proof.* We prove this lemma with a discharging argument. Suppose that the lemma is false, and let  $G$  be a counterexample, i.e.,  $\text{mad}(G) < \frac{8}{3}$  and  $G$  has no subgraph isomorphic to any of the configurations C1, C2, C3 or C4. We assign to each vertex  $v$  of  $G$  an *initial charge*  $\mu(v) = d(v) - \frac{8}{3}$ . Since  $\text{mad}(G) < \frac{8}{3}$ , the sum of the initial charges of the graph is negative.

We then redistribute these charges, according to the two rules given below, in such a way that the total amount of charge on the graph does not change. After this discharging, we compute the *final charge*  $\mu^*(v)$  at each vertex  $v$ , and prove that the sum of all the final charges on the graph  $\mu^*(G)$  is non-negative. This contradiction completes the proof.

The discharging rules are defined as follows.

R1: Each  $3^+$ -vertex gives  $\frac{1}{3}$  to each of its 2-neighbors.

R2: Each  $4^+$ -vertex gives  $\frac{1}{3}$  to each of its 3-neighbors.

We now consider the final charge of vertices in  $G$ . Let  $v$  be any vertex of  $G$ .

1. If  $d(v) = 2$ , then  $\mu(v) = -\frac{2}{3}$ . Since  $G$  does not contain the configuration C2,  $v$  has two  $3^+$ -neighbors and thus receives  $2 \times \frac{1}{3}$  by R1, so that  $\mu^*(v) = 0$ .
2. If  $d(v) = 3$  and  $v$  has no neighbor of degree 3 (that is, the 3-subgraph containing  $v$  consists of  $v$  only), then  $\mu(v) = \frac{1}{3}$ . Since  $G$  does not contain the configuration C3,  $v$  has at most two neighbors of degree 2 and thus gives at most  $\frac{2}{3}$  by R1. Moreover,  $v$  receives at least  $\frac{1}{3}$  by R2, so that  $\mu^*(v) \geq 0$ .
3. If  $d(v) \geq 4$ , then  $\mu^*(v) \geq d(v) - \frac{8}{3} - \frac{d(v)}{3}$  by R2, so that  $\mu^*(v) \geq \frac{2d(v)-8}{3} \geq 0$ .

It remains to consider the case of 3-vertices having at least one 3-neighbor. Recall that a 3-subgraph is a connected component of the subgraph induced by 3-vertices of  $G$ . We have already seen that the charge of each vertex not in this subgraph and the charge of each isolated vertex of this subgraph are positive. We will prove that the sum of charges on each component of this subgraph is also positive, which will imply that the total charge is positive, as required.

Let  $S$  be such a component having at least two vertices. For every vertex  $v$  in  $S$ , we denote by  $d_S(v)$  the degree of  $v$  in  $S$ . Consider now any fixed vertex  $v$  in  $S$  (recall that  $\mu(v) = \frac{1}{3}$ ). We have three cases to consider.

1. If  $d_S(v) = 3$ , then  $v$  is adjacent to three 3-vertices in  $G$ , and thus  $\mu^*(v) = \mu(v) = \frac{1}{3}$ .
2. If  $d_S(v) = 2$ , then  $v$  is adjacent to two 3-vertices in  $G$ . If the third neighbor of  $v$  in  $G$  has degree 2, then  $\mu^*(v) = \frac{1}{3} - \frac{1}{3} = 0$  by R1. Otherwise,  $\mu^*(v) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$  by R2.
3. If  $d_S(v) = 1$ , then  $\mu^*(v) = \frac{1}{3} - 2 \times \frac{1}{3} = -\frac{1}{3}$  by R1 if  $v$  has two 2-neighbors, or  $\mu^*(v) \geq \frac{1}{3} - \frac{1}{3} = 0$  otherwise ( $v$  has at most one 2-neighbor).

For  $i \in \{1, 2, 3\}$ , we denote by  $n_i(S)$  the number of vertices  $v$  with  $d_S(v) = i$ , and by  $\mu_i^*(S)$  the sum of the charges on those vertices. Since  $G$  does not contain the configuration C4, we have three possible cases: either  $S$  is not a tree (case 1), or one leaf of  $S$  is not a weak vertex (case 2), or at least one internal vertex of  $S$  has a  $4^+$ -neighbor (case 3).

We claim that  $n_3(S) \geq n_1(S) - 2$  for every connected (multi)-graph  $S$  of maximum degree at most 3, and, moreover, that  $n_3(S) \geq n_1(S)$  if  $S$  is not a tree. This claim can be proved by induction on  $|S|$ . It is easily verified for small values of  $|S|$ . Assuming it is true for every graph of order at most  $k$ , consider a graph  $S$  on  $k + 1$  vertices. If  $S$  has a 2-vertex  $v$ , we contract one of the edges incident to  $v$ . The resulting graph has the same number of vertices of degree 3 and 1 as the original graph, and it is still of maximum degree at most 3, so we are done. Otherwise, we only have vertices of degree 1 and 3. Since  $S$  is connected and not a  $K_2$ , each vertex of degree 1 is adjacent to a vertex of degree 3. If each vertex of degree 3 has at most one 1-neighbor, then  $n_3(S) \geq n_1(S)$ . Otherwise, a 3-vertex  $x$  has two 1-neighbors  $u$  and  $v$ . Delete both  $u$  and  $v$ , so that  $x$  is now a 1-vertex. In this new graph, we thus have one 3-vertex less and one 1-vertex less than in the original graph. The claim then follows thanks from the induction hypothesis.

Recall that  $\mu^*(S) = \mu_3^*(S) + \mu_2^*(S) + \mu_1^*(S)$ , and that  $\mu_2^*(S) = 0$ . We consider the three possible cases above mentioned.

1.  $S$  is not a tree.

In this case,  $\mu^*(S) \geq \frac{1}{3}n_3(S) + 0 - \frac{1}{3}n_1(S)$  and, since  $S$  is not a tree, we get  $\mu^*(S) \geq 0$ .

2.  $S$  is a tree and one leaf of  $S$  is not a weak vertex.

In this case, a neighbor  $u$  of  $S$  is not a 2-vertex. Since  $S$  is a connected component induced by 3-vertices,  $u$  is of degree at least 4. Thus, the vertices in  $S$  send a maximum charge of  $\frac{1}{3}(n_1(S) - 1)$  to their neighbors by R1, but they receive a charge of  $\frac{1}{3}$  from  $u$ . Therefore, we get

$$\mu^*(S) \geq \frac{1}{3}n_3(S) + 0 - \frac{1}{3}(n_1(S) - 1) + \frac{1}{3} = \frac{1}{3}(n_3(S) - n_1(S) + 2),$$

which is non-negative since  $n_3(S) \geq n_1(S) - 2$ .

3.  $S$  is a tree and one internal vertex of  $S$  has a  $4^+$ -neighbor.

Let  $u$  be the internal vertex of  $S$  which has a  $4^+$ -neighbor. We thus have  $d_S(u) = 2$ , so that  $u$  does not lose any charge and receives at least  $\frac{1}{3}$  by R2. Therefore,

$$\mu^*(S) \geq \frac{1}{3}n_3(S) - \frac{1}{3}n_1(S) + \frac{2}{3} = \frac{1}{3}(n_3(S) - n_1(S) + 2),$$

which, again, is non-negative since  $n_3(S) \geq n_1(S) - 2$ .

This concludes the proof of Lemma 28. □

We are now able to prove Theorem 12.

*Proof of Theorem 12.* By contradiction, let  $(G, \sigma)$  be a counterexample to the theorem having the smallest possible number of vertices. Since  $\text{mad}(G) < \frac{8}{3}$ , and by Lemma 28,  $G$  must contain one of the reducible configurations, which contradicts one of the lemmas 24, 25, 26 or 27. □

## 7 Mapping sparse signed bipartite graphs to unbalanced even cycles

Recall that the study of homomorphisms of signed bipartite graphs captures the study of homomorphisms of graphs as a special case. Hence, the restriction of the study on the class of bipartite graphs is of high interest. In the world of signed bipartite graphs, for  $(H, \pi)$  to admit a homomorphism from any signed bipartite graph  $(G, \sigma)$  with  $G$  having maximum average degree less than  $2 + \epsilon$ , where  $\epsilon > 0$  depends on  $H$ , it is necessary and sufficient for  $H$  to have an unbalanced cycle. Here, proving Theorem 13, we give a lower bound on the best value of  $\epsilon$ . Let us first recall the theorem.

**Theorem 29.** *If  $G$  is a bipartite graph with  $\text{mad}(G) < 2 + \frac{1}{2g-1}$ , and  $\sigma$  a signature of  $G$  such that the unbalanced-girth of  $(G, \sigma)$  is at least  $2g$ , then  $(G, \sigma) \rightarrow (C_{2g}, \{e\})$ .*

*Proof.* Note that as a signed graph but not a 2-edge-colored graph, the unbalanced-cycle  $UC_\ell$  of length  $\ell$  is vertex-transitive. Let  $(G, \sigma)$  be a minimum counterexample to the theorem, that is,  $G$  is a bipartite graph,  $\text{mad}(G) < 2 + \frac{1}{2g-1}$ , the unbalanced-girth of  $(G, \sigma)$  is at least  $2g$ , but  $(G, \sigma)$  does not map to  $UC_{2g}$ . Since  $(G, \sigma)$  is of unbalanced-girth at least  $2g$ , and by Lemma 9,  $G$  is not just a cycle. Our next conclusion is that  $G$  must be 2-connected. Indeed, if this is not the case, then either  $G$  has a connected component  $G_1$  which is a proper subgraph, or  $G$  has a 2-connected end-block  $G_2$  which is connected to the rest of the graph at a cut-vertex  $x$ . By minimality of  $G$ , the signed graph induced on  $G_1$  or on  $G_2$  maps to  $UC_{2g}$ . Similarly, if we remove all vertices of  $G_1$ , or if we remove all vertices of  $G_2$  except  $x$ , then the induced signed graph also maps to  $UC_{2g}$ . Note that to resign at a set  $X$  of vertices of any signed graph is the same as resigning at the complement of  $X$ . Since  $G_2$  has only one common vertex with the rest of the graph and  $G_1$  has none, in any homomorphism of the two parts we may choose resignings that coincide on the possible common vertex. Finally, since  $UC_{2g}$  is vertex-transitive, by composing one of the two homomorphisms with an automorphism of  $UC_{2g}$ , we may assume that  $x$  is mapped to a same vertex by the two homomorphisms. Hence, they can be composed to a homomorphism of  $(G, \sigma)$  to  $UC_{2g}$ .

The fact that  $G$  must be 2-connected implies, in particular, that  $G$  has minimum degree at least 2. A 2-vertex is then an internal vertex of a maximal thread. Our next claim is that no such thread can have a length  $2g - 1$  or higher. Assume to the contrary that  $G$  has a thread  $T$  of length exactly  $2g - 1$  (not necessarily a maximal thread), with  $x$  and  $y$  as its endpoints. Let  $G'$  be the subgraph obtained by deleting all internal vertices of  $T$  and let  $\sigma'$  be the signature induced by  $\sigma$  on  $G'$ . Observe that since  $G$  was 2-connected,  $G'$  is a connected graph. By minimality of  $(G, \sigma)$ , the signed bipartite graph  $(G', \sigma')$  maps to  $UC_{2g}$ . Let  $\phi$  be such a homomorphism. Since  $G'$  is connected,  $\phi$ , as a homomorphism of the underlying bipartite graph  $G'$  to the bipartite graph  $C_{2g}$ , must preserve the bipartition. Thus  $\phi(x)$  and  $\phi(y)$  partition  $UC_{2g}$  into two paths of odd length: one containing the negative edge, the other only composed of positive edges. Then, depending on the parity of the number of negative edges of  $T$ , and after a suitable resigning of its internal vertices, we may extend the homomorphism  $\phi$  to the vertices of  $T$ , thus mapping  $(G, \sigma)$  to  $UC_{2g}$ , which contradicts the choice of  $(G, \sigma)$ .

We next consider 3-vertices. Let  $v$  be a 3-vertex of  $G$  and let  $x$ ,  $y$  and  $z$  be the three ends of the maximal threads whose other end is  $v$ . Let  $\ell_x$ ,  $\ell_y$  and  $\ell_z$  be the corresponding

lengths of these threads. We claim that  $\ell_x + \ell_y + \ell_z \leq 4g - 1$ . Suppose the contrary and let  $G'$  be the subgraph obtained from  $G$  by removing  $v$  and all the internal vertices of the three threads. Let  $\sigma'$  be the induced signature on  $G'$ . By the minimality of  $G$ , we have a homomorphism  $\phi = (\phi_1, \phi_2)$  of  $(G', \sigma')$  to  $UC_{2g}$ . Considering the size of the set  $\{\phi_2(x), \phi_2(y), \phi_2(z)\}$ , we have three possible cases.

If this set is of size one, that is,  $\phi_2(x) = \phi_2(y) = \phi_2(z)$ , we consider the image of  $G$  after applying  $\phi$  on the subgraph  $G'$ . The result is then a graph obtained as an unbalanced cycle of length  $2g$ , and the vertex  $v$  which is joined to a vertex  $(\phi_2(x))$  of the cycle by three internally vertex disjoint paths of length  $\ell_x$ ,  $\ell_y$  and  $\ell_z$  respectively, whose signature is induced by then signature of  $(G, \sigma)$  and  $\phi_1(x) = \phi_1(y) = \phi_1(z)$ . Among these three paths, two are of a same sign and, since our graph is bipartite, all three of them are of same parity. We can thus map the largest of the two to the smallest one. Noting the this largest thread is not of length greater than  $2g$ , and that the sum of the three lengths was at least  $4g$ , the cycle formed by the two remaining threads between  $v$  and  $\phi_2(x)$  must be of length at least  $2g$ . We may now use vertex transitivity of  $UC_{2g}$  to complete our mapping as in the previous case.

If  $|\{\phi_2(x), \phi_2(y), \phi_2(z)\}| \geq 2$ , we similarly use  $\phi$  to build a homomorphic image of  $(G, \sigma)$ , then we show that the image itself can be mapped to  $UC_{2g}$ . In this image, we first apply the resigning of  $\phi_1$ . Next, every two vertices that are identified by the mapping  $\phi_2$  are also identified, and no further resigning or identification is done. Let  $(G^*, \sigma^*)$  be the corresponding image. Since  $G$  is connected and  $\phi_2$  preserves bipartition, the image,  $G^*$  is also bipartite. Furthermore, any cycle using  $v$  must use two of the threads incident to it, and thus is of length at least  $2g$ . The only cycle not using  $v$  is an unbalanced cycle of length exactly  $2g$ . Depending on whether  $\{\phi_2(x), \phi_2(y), \phi_2(z)\}$  is of size 3 or 2, the signed graph  $(G^*, \sigma^*)$  satisfies the conditions of Lemma 19 or Lemma 20 (respectively). Thus, in both cases,  $(G^*, \sigma^*)$ , and therefore  $(G, \sigma)$ , maps to  $UC_{2g}$ .

Finally, to complete the proof, we show that the four conditions of (1)  $G$  is 2-connected, (2)  $G$  is not cycle (3)  $G$  has no thread of length  $2g - 1$  or longer, (4)  $G$  has no 3-vertex whose total length of incident threads is larger than  $4g - 1$ , would imply that the average degree of  $G$  is at least  $2 + \frac{1}{2g-1}$ , contradicting our assumption.

We use the discharging technique for this step. Assign the degree of each vertex of  $G$  as a charge to this vertex. The average charge is then the average degree of  $G$ . For each 2-vertex  $v$  of  $G$ , consider the two ends  $x_v$  and  $y_v$  of the thread to which  $v$  belongs. Note that  $x_v$  and  $y_v$  do exist because  $G$  is not a cycle, that  $x_v$  and  $y_v$  are distinct, and that they both have degree at least 3 because  $G$  is 2-connected (but not a cycle). We then use the following discharging rule.

R: Each 2-vertex receives a charge of  $\frac{1}{2(2g-1)}$  from each end of the thread it belongs to.

We claim that after applying this rule, every vertex  $v$  has a charge of at least  $2 + \frac{1}{2g-1}$ , proving that  $G$  has average degree at least this value. We consider three cases, depending on the degree of  $v$ .

1. If  $v$  is a 2-vertex, then its initial charge is 2. Then, as mentioned above, there are two distinct vertices as the ends of the thread containing  $v$ , and each of them gives a charge of  $\frac{1}{2(2g-1)}$  by R. As  $v$  does not lose any charge, its final charge is increased to  $2 + \frac{1}{2g-1}$ .

2. If  $v$  is a 3-vertex, then its initial charge is 3. The vertex  $v$  may give a charge of  $\frac{1}{2(2g-1)}$  to each vertex which is on a thread with  $v$  as an endpoint. Since the total length of the three threads having  $v$  as an end point is at most  $4g - 1$ , there are at most  $4g - 4$  such 2-vertices. Therefore,  $v$  loses at most  $(4g - 4) \times \frac{1}{2(2g-1)} = \frac{2g-2}{2g-1}$ , so that its final charge is at least  $3 - \frac{2g-2}{2g-1} = 2 + \frac{1}{2g-1}$ .
3. If  $v$  is a  $k$ -vertex,  $k \geq 4$ , then its initial charge is  $k$ . The vertex  $v$  is an endpoint of  $k$  threads, each of length at most  $2g - 1$ . Thus, in total,  $v$  may give a charge of  $\frac{1}{2(2g-1)}$  to at most  $k(2g - 2)$  vertices. Hence,  $v$  loses at most  $k(2g - 2) \times \frac{1}{2(2g-1)} = \frac{k(g-1)}{2g-1}$ , so that its final charge is at least  $k - \frac{k(g-1)}{2g-1} = \frac{kg}{2g-1}$  which is greater than  $2 + \frac{1}{2g-1} = \frac{4g-1}{2g-1}$  for  $k \geq 4$ .

Thus, at the end, the charge of every vertex is at least  $2 + \frac{1}{2g-1}$ , implying that the average charge, or, equivalently, the average degree, is at least  $2 + \frac{1}{2g-1}$ , contradicting our assumption.  $\square$

The bound on the maximum average degree in the statement of Theorem 29 is not tight. For the case of  $g = 1$ , allowing multi-edges, the unbalanced cycle of length 2 is the digon. It is then straightforward to check that any signed bipartite multi-graph maps to  $UC_2$ . For  $g = 2$ , while Theorem 29 implies that every signed bipartite graph  $(G, \sigma)$  with no digon maps to  $UC_4$  as long as  $G$  has maximum average degree less than  $\frac{7}{3}$ , an improved and tight bound of  $\frac{8}{3}$  is proved below. Note that unlike the exceptional case of  $g = 1$ , in general case of the theorem we do not consider multi-graphs since any such parallel edges must be of the same sign to satisfy the unbalanced-girth condition, so that the existence of a homomorphism only depends on the underlying simple signed graph.

**Theorem 30.** *If  $G$  is a bipartite graph with  $\text{mad}(G) < \frac{8}{3}$ , and  $\sigma$  is a signature of  $G$ , then  $(G, \sigma) \rightarrow UC_4$ . Furthermore, the bound of  $\frac{8}{3}$  cannot be improved.*

*Proof.* Since  $\text{mad}(G) < \frac{8}{3}$ , and by Theorem 12,  $(G, \sigma)$  maps to  $(K_4, \{e\})$ . Then, since  $G$  is bipartite, by Theorem 16 it also maps to  $(C_4, \{e\})$ . An example of a bipartite graph with maximum average degree  $\frac{8}{3}$  which does not map to  $(C_4, \{e\})$  is given in Section 9.  $\square$

For larger values of  $g$ , i.e.,  $g \geq 3$ , we do not know the optimal bound on the maximum average degree for which Theorem 13 would be valid, and leave this as an open question.

One may also consider subclasses of signed bipartite graphs. One subclass of special interest is the class of signed bipartite planar graphs. In Conjecture 14, we have introduced a condition on the unbalanced-girth of signed bipartite planar graphs which may imply the existence of a homomorphism to  $UC_{2g}$ . Using the folding lemma, Lemma 17 or its corollary, Corollary 18, and Euler's formula, this can translate a condition on the average degree. We will show in Section 9 that the proposed bound of this conjecture, if true, is tight.

## 8 Application to planar graphs

For the class planar signed graphs, a class of graphs of maximum average degree strictly smaller than 6, the first question is whether there exists or not a signed graph to which

every planar signed graph admits a homomorphism. Using techniques developed in [RS94, AM98], such a signed graph of order 48 is built in [13] where a lower bound of 10 on the number of vertices of such a bound is presented as well. The upper bound of 48 is improved to 40 in [15] where the authors have, furthermore, eliminated all but one graph on 10 vertices as a candidate for a bound on 10 vertices (up to equivalence of signatures). This signed graph is a signed graph on  $K_{10}$  whose set of negative edges induces a graph on 9 vertices which is isomorphic to the Cartesian product  $K_3 \square K_3$ . This signed graph on 10 vertices then seems to be a natural candidate and it is tempting to believe that 10 is the right number.

Here we address the problem for planar graphs of large girth, which, combined with Euler's formula, is a translation of the maximum average degree condition for planar graphs. Recall that by Euler's formula, for any connected planar graph  $G$ , the relation  $n - e + f = 2$  holds, where  $n$  is the number of vertices of  $G$ ,  $e$  is the number of edges of  $G$ , and  $f$  is the number of faces in a planar embedding of  $G$ . For the sake of our study, we may assume that  $G$  is 2-connected, which implies that each edge lies exactly on two distinct faces (counting the outer face). If, in a planar embedding of  $G$ , each face is of length at least  $g$ , then there are at least  $\frac{f \times g}{2}$  edges in  $G$ , or, equivalently,  $f \leq \frac{2e}{g}$ , so that  $n - e + \frac{2e}{g} \geq 2$ . Using the fact that  $\frac{2e}{n}$  is the average degree of  $G$ , we conclude that the average degree of  $G$  is strictly less than  $\frac{2g}{g-2}$ . This proves the following folklore statement.

**Property 31.** *If  $G$  is a 2-connected planar graph whose faces are all of length at least  $g$ , then  $G$  has average degree strictly less than  $\frac{2g}{g-2}$ .*

We then get the following corollary.

**Corollary 32.** *If  $G$  is a planar graph, all of whose faces are of length at least 8, and  $\sigma$  is any signature of  $G$ , then  $(G, \sigma) \rightarrow (K_4, e)$ .*

Furthermore, combined with Corollary 18, we also get the following corollaries.

**Corollary 33.** *If  $G$  is a planar bipartite graph, and  $\sigma$  is a signature of  $G$  such that the unbalanced-girth of  $(G, \sigma)$  is at least 8, then  $(G, \sigma) \rightarrow UC_4$ .*

**Corollary 34.** *If  $G$  be a planar bipartite graph, and  $\sigma$  is a signature of  $G$  such that the unbalanced-girth of  $(G, \sigma)$  is at least  $8k - 2$ , then  $(G, \sigma) \rightarrow UC_{2k}$ .*

This corollary is in support of Conjecture 14, where we conjecture that having unbalanced-girth at least  $4k - 2$  is enough for a signed bipartite planar graph to admit a homomorphism to  $UC_{2k}$ . If true, then the condition of unbalanced-girth at least  $4k - 2$  is the best possible, as shown by the examples in Section 9.

For the special case of  $k = 3$ , Conjecture 14 claims that every planar signed bipartite graph with unbalanced-girth at least 10 admits a homomorphism to  $UC_6$ . Further support for this case, together with connection to the well known Grötzsch's theorem, is as follows. Let  $G$  be a triangle-free planar graph, and let  $\vec{G}$  be the graph obtained from  $G$  by subdividing each edge exactly once. Observe that the result is indeed a planar bipartite graph. Let  $\Sigma$  be a set of edges of  $\vec{G}$  such that, for each original edge of  $G$ , exactly one of the two corresponding edges in  $\vec{G}$  is in  $\Sigma$ . Then, the signed graph  $(\vec{G}, \sigma)$  is a signed planar bipartite graph with unbalanced-girth 10 because each unbalanced cycle of  $(\vec{G}, \sigma)$  is a subdivision of an odd-cycle of  $G$  and  $G$  is assumed to be triangle-free. A

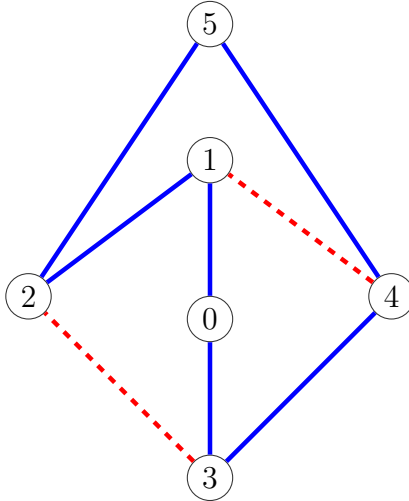


Figure 6: The graph  $\Omega_4$  (Lemma 36)

homomorphism of  $G$  to  $K_3$ , provided by Grötzsch's theorem, can then be extended to a homomorphism of  $(\vec{G}, \sigma)$  to  $UC_6$ . However, it is not clear whether the case  $k = 3$  of our conjecture would imply Grötzsch's theorem or not. We leave this as an open problem.

## 9 Tightness

Consider a  $K_4$  and let  $M_1$ ,  $M_2$  and  $M_3$  be a partition of its edges into three perfect matchings. Subdivide the edges in  $M_1$  each once, noting that the resulting graph is bipartite, and then build a signed graph by assigning to two edges of  $M_2$  the negative sign. Let  $\Omega_4$  be the resulting signed graph which is depicted in Figure 6. In the following, we will use the labeling of vertices depicted in this figure. We prove here that this signed graph has maximum average degree  $\frac{8}{3}$ , and that it does not admit any homomorphism to  $(K_4, e)$  or to  $(C_4, e)$ , thus proving that both Theorem 12 and Theorem 30 are tight.

The proofs of these claims use following simple lemma.

**Lemma 35** (Naserasr, Rollová and Sopena [13]). *For every two unbalanced cycles  $UC_\ell$  and  $UC_k$ ,  $UC_\ell \rightarrow UC_k$  if and only if  $\ell \geq k$  and  $\ell \equiv k \pmod{2}$ .*

**Lemma 36.** *The signed graph  $\Omega_4$ , depicted in Figure 6, has no homomorphism to  $UC_4$ .*

*Proof.* We consider  $UC_4$  as a 4-cycle with exactly one negative edge. The graph  $\Omega_4$  is planar and has four faces, each isomorphic to  $UC_4$ . By Lemma 35, any mapping of each of these faces to  $UC_4$  must be onto, and thus must have exactly one negative and three positive edges. Toward a contradiction, assume there is a homomorphism of  $\Omega_4$  to  $UC_4$  under an equivalent signature  $\Sigma$ . Then,  $\Sigma$  must contain exactly one edge of each facial cycle. The only choices for that are either two edges of  $M_2$ , or two edges of  $M_3$ , or two non adjacent edges obtained from the subdivision of  $M_1$ . But in any of these cases, two end vertices of negative edges are linked by a positive edge, and thus form an obstacle given in Theorem 15 for the existence of a homomorphism to  $UC_4$ .  $\square$

**Lemma 37.** *The graph  $\Omega_4$  has maximum average degree  $\frac{8}{3}$ .*



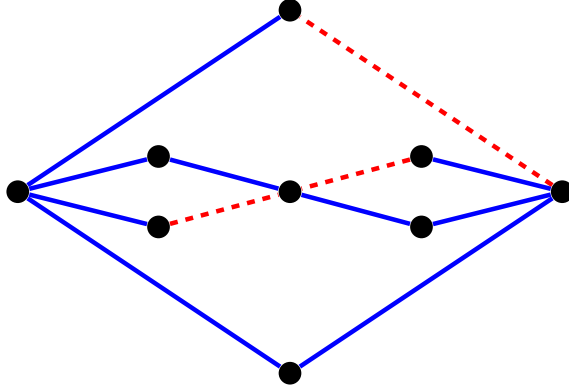


Figure 7: The graph  $W_2$  (Theorem 40)

*Proof.* This graph has six vertices and eight edges, thus the average degree of the whole graph is  $\frac{8}{3}$ . Using the labeling of vertices given in Figure 6, if one of the vertices  $x$  or  $y$  is deleted, the remaining graph has average degree  $\frac{12}{5}$ . We claim that any other proper subgraph of  $\Omega_4$  has average degree at most 2. To this end, note that a vertex of degree 1 cannot contribute for exceeding the maximum average degree above 2. One can then easily check that, by the process of removing vertices of degree one from a proper subgraph which is not one of the two mentioned subgraphs, one either get a cycle or a single vertex.  $\square$

From Lemmas 36 and 37, we get the following.

**Corollary 38.** *There exists a signed bipartite graph with maximum average degree  $\frac{8}{3}$  which does not map to  $(C_4, \{e\})$ .*

Moreover, since  $\Omega_4$  is a signed bipartite graph, by Theorem 16 we also get the following.

**Corollary 39.** *There exists a signed bipartite graph with maximum average degree  $\frac{8}{3}$  which does not map to  $(K_4, \{e\})$ .*

Next we show that the claim of Conjecture 14, if true, is also tight.

**Theorem 40.** *For every  $k \geq 2$ , there exists a signed planar bipartite graph with unbalanced-girth  $4k - 4$  which does not map to  $UC_{2k}$ .*

*Proof.* Label the vertices of the unbalanced cycle  $UC_{4k-4}$  by  $v_1, v_2, v_3, \dots, v_{4k-4}$ , and consider a planar circular drawing of  $UC_{4k-4}$ . Add a vertex  $u$  in the center. Then, join each vertex  $v_{2j}$ ,  $j = 1, 2, \dots, 2k - 2$ , to  $u$  by two internally disjoint paths of length  $2k - 2$  and assign to one of the  $4k - 4$  resulting edges a negative sign. Call the resulting signed graph  $W_k$  (the signed graph  $W_2$  is depicted in Figure 7).

It is easily observed that  $W_k$  is a signed bipartite planar graph of unbalanced-girth  $4k - 4$  (in fact, the girth of its underlying graph is  $4k - 4$ ). We claim that  $W_k$  does not admits any homomorphism to  $UC_{2k}$ . By contradiction, suppose that  $\phi$  is a homomorphism of  $W_k$  to  $UC_{2k}$ . Observe first that  $\phi$  must preserve the bipartition since the image is bipartite. Thus, among the vertices  $v_1, v_2, v_3, \dots, v_{4k-4}$ , those of even index must map to a same part of  $UC_{2k}$ , and those of odd index must map to the other part. Furthermore,

since  $u$  is of even distance from the vertices of even index,  $u$  must map to the same part as vertices of even index.

Another observation is that, since  $UC_{4k-4}$  is an unbalanced cycle, the restriction of  $\phi$  to this cycle must be surjective. Thus, the image  $\phi(u)$  of  $u$  must also be the image of a vertex  $v_{2j}$  for some  $j$ ,  $1 \leq j \leq 2k-2$ . Then, in the image of the unbalanced  $(4k-4)$ -cycle build on  $u$  and  $v_j$ , there must be an unbalanced cycle of length at most  $2k-2$ , which contradicts Lemma 35.  $\square$

## 10 Concluding remarks

This paper is a first work in the study of homomorphisms of sparse signed graphs. A main result here was to show that, given any signed graph  $(H, \pi)$ , there exist a positive  $\epsilon$  such that every signed graph satisfying  $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$  and  $\text{mad}(G) \leq 2 + \epsilon$  maps to  $(H, \pi)$ . We determined the optimal value of  $\epsilon$  for  $(H, \pi)$  being  $(K_4, e)$  or  $(C_4, e)$ . We believe that a geometric condition such as planarity can help proving parallel results.

This study generalizes various questions from graph coloring. For example, one can define a  $(H, \pi)$ -critical graph to be a signed graph satisfying  $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ , and such that  $(G, \sigma)$  does not admit any homomorphism to  $(H, \pi)$  while each of its subgraphs does. Extending the notion of *excess* of  $k$ -critical graphs, a natural question is then to give a lower bound on the number of edges of an  $(H, \pi)$ -critical graph. In particular, considering Theorem 6.2 of [13], the case when  $(H, \pi)$  is a complete bipartite graph with the set of negatives edges being a perfect matching is of special interest.

On the other hand, when  $(H, \pi)$  is a signed projective cube, as discussed in [12], the condition of planarity in place of the maximum average degree condition is conjectured to be a sufficient condition. This conjecture directly generalizes the four-color theorem and is in connection with the study of various notions of coloring of planar graphs as discussed in [11].

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## A Proof of Lemma 19

**Proof of Lemma 19** First of all, we may assume that  $L_{xy} \leq 2g - 1$  for every  $x, y \in \{a, b, c, d\}$ , as otherwise, we may delete internal vertices of  $P_{xy}$  in order to get a signed graph  $(G, \sigma)$  which is of unbalanced-girth  $2g$ . Applying Corollary 18, we get that the unbalanced cycle of length  $2g$  is a homomorphic image of  $(G, \sigma)$ . Since  $G$  is a connected graph, in any such a homomorphism,  $x$  and  $y$  are mapped to the same part of the bipartite graph  $C_{2g}$ . Subject to this condition, and regardless of which vertices of  $UC_{2g}$  these two end vertices of  $P_{xy}$  are mapped to, assuming  $L_{xy} \geq 2g$ , we can extend this mapping to the internal vertices of  $P_{xy}$  and we are done.

Considering a planar embedding of  $\overline{K_4}$ , since  $(\overline{K_4}, \sigma)$  is not balanced, we have two cases to consider.

1. *Exactly two faces are unbalanced.*

Using a repeated application of the folding lemma (Lemma 17) on the two balanced faces, we can map the whole graph to  $C_{abc}$  which is isomorphic to  $UC_{2g}$ .

2. *All four faces are unbalanced.*

Thanks to Lemma 3, we may assume, up to resigning, that exactly one edge on each path  $P_{xy}$ ,  $x, y \in \{a, b, c, d\}$ , is negative.

We first claim that for at least one face incident to  $d$ , say  $C_{abd}$ , we have  $L_{ad} + L_{bd} \geq L_{ab} + 2g$ . To see that, assume to the contrary that  $L_{ad} + L_{bd} < L_{ab} + 2g$ ,  $L_{ad} + L_{cd} < L_{ac} + 2g$ , and  $L_{bd} + L_{cd} < L_{cb} + 2g$ . This implies

$$2(L_{ad} + L_{bd} + L_{cd}) < L_{ab} + L_{bc} + L_{ac} + 6g = 8g,$$

contradicting the assumption of the Lemma.

As  $L_{ad} + L_{bd} \geq L_{ab} + 2g$ , and since  $L_{ad} \leq 2g - 1$ , we have  $L_{bd} > L_{ab}$ . Let  $x_a$  be a vertex of  $C_{abc}$  such that the path  $P_{abx_a}$  (a subgraph of  $C_{abc}$ , connecting  $a$  to  $x_a$  through  $b$ ) is of length  $L_{da}$ . Similarly, it follows that  $L_{bd} > L_{ab}$ , and thus we can choose a vertex  $x_b$  on  $C_{abc}$  such that  $P_{bax_b}$  is of length  $L_{db}$ .

Observe that  $I = P_{bax_b} \cap P_{abx_a} - P_{ab} \neq \emptyset$  and, furthermore, that there is a vertex  $y \in I$  which is in the same part of the bipartition as  $d$ . We claim that it is possible to choose this vertex  $y$  in such a way that  $d_{P_{acb}}(y, c) \leq L_{cd}$ . If  $c \in I$ , then we choose  $y$  to be either  $c$  or a neighbor of  $c$  (depending on whether  $d$  belongs to the same part as  $c$  or not). If  $c \notin I$ , then either  $I$  is a subset of  $P_{ac}$  or a subset of  $P_{bc}$ . By symmetry, suppose  $I$  is a subset of  $P_{ac}$ . In this case, we choose  $y$  to be the vertex of  $I$  that belongs to the same part as  $d$  and which is the closest to  $c$  in  $P_{ac}$ . This means in particular that  $|P_{bay}|$  is of length exactly  $L_{db}$ . Then, since

$$|P_{cy}| + |P_{bay}| = |P_{bac}| \leq |P_{cd} \cup P_{db}| = L_{cd} + L_{db},$$

we get  $|P_{cy}| \leq L_{cd}$ .

Finally, observing that the negative edges of  $P_{ad}$  and  $P_{bd}$  can be chosen to be anywhere in these paths, by resigning at their internal vertices, we conclude that the mapping of  $d$  to  $y$  can be extended to a homomorphism of  $\overline{K_4}$  to  $C_{abc}$ , which is isomorphic to  $UC_{2g}$ .

This completes the proof.