

Acircuitic directed star arboricity of planar graphs with large girth

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Abstract. A directed star forest is a forest all of whose components are stars with arcs emanating from the center to the leaves. The acircuitic directed star arboricity of an oriented graph G is the minimum number of edge-disjoint directed star forests whose union covers all edges of G and such that the union of two such forests is acircuitic. We show that graphs with maximum average degree less than $\frac{7}{3}$ (resp. $\frac{133}{41}$) have acircuitic directed star arboricity at most Δ (resp. $\Delta + 1$); this implies that planar graphs of girth at least 14 (resp. 6) have acircuitic directed star arboricity at most Δ (resp. $\Delta + 1$).

Keywords: Oriented graphs, arboricity, acyclic edge-colorings, planar graphs.

1 Introduction

We consider finite simple loopless undirected or oriented graphs, that is graphs with no cycle of length one or two. For an undirected graph (resp. an oriented graph) G , we denote by $V(G)$ its vertex set and by $E(G)$ its edge set (resp. by $A(G)$ its arc set).

An edge-coloring of an undirected graph G is *proper* if no pair of incident edges receive the same color. A proper edge-coloring of G is *acyclic* if the graph induced by any two color classes is a forest. The minimum number of colors needed to acyclically edge-color G is the *acyclic chromatic index* of G and is denoted by $\chi'_a(G)$. The notion of acyclic coloring was introduced by Grünbaum in [7].

The best known upper bound on acyclic edge-coloring is due to Alon et al. [3] where they proved that every graph has an acyclic (16Δ) -edge-coloring. More recently, Muthu et al. [10] obtained better results for graphs with large girth: in particular, for every undirected graph G with girth $g \geq 9$, $\chi'_a(G) \leq 6\Delta(G)$.

Acyclic chromatic index can be related to the notion of star arboricity. The notion of *star arboricity* was introduced by Akiyama and Kano in [1]. It is defined as the minimum number of edge-disjoint star forests needed to cover $E(G)$. They proved that complete graphs on n vertices can be decomposed into $\lceil \frac{n}{2} \rceil + 1$ star forests. This notion was studied by many authors for some graph families such as d -regular graphs, complete regular multipartite graphs or planar graphs (see [2, 4, 5]).

Considering planar graphs, Algor and Alon showed in [2] that every planar graph can be decomposed into 6 star forests. This bound was decreased to 5 by Hakimi et al. in [9] and this bound is tight.

In [2], Algor and Alon introduced the notion of *directed* star arboricity of a digraph G , defined as the minimum number of edge-disjoint directed star forests needed to cover $A(G)$ (a *directed star* is a star with arcs emanating from the center to the leaves). In [8], Guiduli showed that every digraph with indegree and outdegree both less than D has directed star arboricity at most $D + 20\log D + 84$.

We introduced in [11] the new notion of *acircuitic* directed star arboricity of an oriented graph G , denoted by $\text{adst}(G)$, defined as the minimum number of edge-disjoint directed star forests needed to cover $A(G)$ in such a way that the union of any two such forests contains no circuit. The study of this notion arises from the study of oriented arc-colorings since every oriented graph with acircuitic directed star arboricity at most k admits an oriented $(k \cdot 2^{k-1})$ -arc-coloring (see [11] for more details).

It is easy to see that each color class of an acyclic edge-coloring of G is a matching (therefore a star forest), and the graph induced by any two such matchings is a forest. We thus get that for every orientation \vec{G} of an undirected graph G with $\chi'_a(G) \leq k$, $\text{adst}(\vec{G}) \leq k$.

In this paper, we consider the acyclic chromatic index and the acircuitic directed star arboricity of oriented graphs with bounded maximum average degree.

Definition 1 Let G be a graph. The *maximum average degree* of G , denoted by $\text{mad}(G)$, is given by

$$\text{mad}(G) = \max_{H \subseteq G} \left\{ \frac{2|A(H)|}{|V(H)|} \right\}$$

We prove the following:

Theorem 2 Let G be an undirected graph (which is not a cycle) with $\text{mad}(G) \leq \frac{7}{3}$. Then $\chi'_a(G) = \Delta(G)$.

Therefore, we get:

Corollary 3 Let G be an oriented graph (which is not a cycle) with $\text{mad}(G) \leq \frac{7}{3}$. Then $\text{adst}(G) \leq \Delta(G)$.

For graphs with larger maximum average degree, we prove the following:

Theorem 4 Let G be an oriented graph with $\text{mad}(G) < \frac{133}{41}$. Then $\text{adst}(G) \leq \Delta(G) + 1$.

In case of planar graphs, the maximum average degree parameter is linked to the girth (the shortest cycle of the graph) as follows [6]:

Observation 5 Let G be a planar graph with girth g , then $\text{mad}(G) < \frac{2g}{g-2}$.

Therefore, we get:

Corollary 6

- (1) Let G be an oriented planar graph (which is not a cycle) with girth $g \geq 14$. Then $\text{adst}(G) \leq \Delta(G)$.
- (2) Let G be an oriented planar graph with girth $g \geq 6$. Then $\text{adst}(G) \leq \Delta(G) + 1$.

Moreover, we can easily see that for every undirected graph G , there exist an orientation \vec{G} with acircuitic directed star arboricity at least $\Delta(G)$ (consider \vec{G} containing a star of degree $\Delta(G)$ with arcs emanating from the leaves to the center). Therefore, for oriented planar graphs with girth at least 14, the bound is tight, and for oriented planar graphs G with girth at least 6, we get $\Delta(G) \leq \text{adst}(G) \leq \Delta(G) + 1$.

This paper is organised as follows: we introduce the main definitions and notation in the next section and prove Theorem 2 and 4 in Sections 3 and 4.

2 Definitions and notation

Let G be an oriented graph. We denote by \overrightarrow{uv} the arc from u to v or simply uv whenever its orientation is not relevant (therefore $uv = \overrightarrow{uv}$ or $uv = \overleftarrow{uv}$).

For a vertex v of G , we denote by $d^-(v)$ the indegree of v , by $d^+(v)$ its outdegree and by $d(v)$ its degree, that is $d(v) = d^+(v) + d^-(v)$. A vertex of degree k (resp. at most k , at least k) is called a k -vertex (resp. $\leq k$ -vertex, $\geq k$ -vertex) and a vertex of degree k adjacent to d 2-vertices is called a $k[d]$ -vertex. A neighbor of degree k (resp. at most k , at least k) of a vertex u is called a k -neighbor (resp. $\leq k$ -neighbor, $\geq k$ -neighbor) of u . We denote by $N^-(v)$ the set of *incoming neighbors* of v ($N^-(v) = \{u \in V \mid \overrightarrow{uv} \in A(G)\}$), by $N^+(v)$ the set of *outgoing neighbors* of v ($N^+(v) = \{w \in V \mid \overleftarrow{vw} \in A(G)\}$) and by $N(v)$ the set of *neighbors* of v , that is $N(v) = N^-(v) \cup N^+(v)$. A *source vertex* is a vertex v with $d^-(v) = 0$. The *maximum degree* and *minimum degree* of a graph G are respectively denoted by $\Delta(G)$ and $\delta(G)$.

For an undirected (resp. oriented) graph G and a vertex v of $V(G)$, we denote by $G \setminus v$ the graph obtained from G by removing v together with the set of its incident edges (resp. arcs); similarly, for an edge (resp. arc) a of $E(G)$ (resp. $A(G)$), $G \setminus a$ denotes the graph obtained from G by removing a . These two notions are extended to sets in a standard way: for a set of vertices $V' \subseteq V(G)$, $G \setminus V'$ denotes the graph obtained from G by successively removing all vertices of V' and their incident edges (resp. arcs), and for a set of edges (resp. arcs) $W \subseteq E(G)$ (resp. $W \subseteq A(G)$), $G \setminus W$ denotes the graph obtained from G by removing all edges (resp. arcs) of W .

The notions of arboricity discussed in the previous section may be defined in terms of arc-coloring or partitions of the set of the arcs. More precisely, a k -*directed-star-coloring* (or simply k -*dst-coloring*) of an oriented graph G is a partition of $A(G)$ into k directed star forests $\{F_1, F_2, \dots, F_k\}$. Equivalently, a k -*dst-coloring* of G is a k -coloring f of $A(G)$ such that (i) $\overrightarrow{uv}, \overleftarrow{vw} \in A(G) \Rightarrow f(\overrightarrow{uv}) \neq f(\overleftarrow{vw})$, and (ii) $\overrightarrow{uv}, \overleftarrow{vw} \in A(G) \Rightarrow f(\overrightarrow{uv}) \neq f(\overleftarrow{vw})$. The *directed star arboricity* of G , denoted by $\text{dst}(G)$, is the smallest k for which G admits a k -*dst-coloring*.

An oriented graph G is *acircuitic* if it does not contain any circuit (that is a cycle with all arcs having the same direction). A k -*acircuitic-directed-star-coloring* (or simply k -*adst-coloring*) of a graph G is a partition of $A(G)$ into k directed star forests $\{F_1, F_2, \dots, F_k\}$ such that for all $i, j \in [1, k]$, $F_i \cup F_j$ is acircuitic. Equivalently, a k -*adst-coloring* of G is a k -*dst-coloring* of G such that no circuit in G is bichromatic. The *acircuitic directed star arboricity* of G , denoted by $\text{adst}(G)$, is the smallest k for which G admits a k -*adst-coloring*.

For an undirected graph G , a given edge-coloring f of G and a vertex $v \in V(G)$, we denote by $C_f(v)$ the set of *incident colors* of v (i.e. $C_f(v) = \{f(uv) \mid u \in N(v)\}$). For an oriented graph G , a given arc-coloring f of G and a vertex $v \in V(G)$, we denote by $C_f^-(v)$ the set of *incoming colors* of v (i.e. $C_f^-(v) = \{f(\overrightarrow{uv}) \mid u \in N^-(v)\}$) and by $C_f^+(v)$ the set of *outgoing colors* of v (i.e. $C_f^+(v) = \{f(\overleftarrow{vu}) \mid u \in N^+(v)\}$) (therefore $C_f(v) = C_f^-(v) \cup C_f^+(v)$).

3 Proof of Theorem 2

We use the method of reducible configurations and discharging procedures to prove Theorem 2.

In the rest of this section, let H be a minimal (with respect to inclusion as a subgraph) counter-example to Theorem 2 and $\Delta = \Delta(H)$. We shall show that H does not contain a set of reducible configurations.

The drawing conventions for a forbidden configuration are the following: a vertex whose neighbors are totally specified will be black (i.e. vertex of fixed degree), whereas a vertex whose neighbors are partially specified will be white. Moreover, an edge will represent an arc with any of its two possible orientations.

In all the proofs which follow, we shall proceed similarly. We suppose that H contains some configurations and, for each of them, we consider a reduction H' of H obtained from H by removing vertices and/or arcs. Since H' is a subgraph of H , $\text{mad}(H') \leq \text{mad}(H)$. Therefore, due to the minimality of H , there exists an acyclic $\Delta(H')$ -edge-coloring f of H' ; moreover, $\Delta(H') \leq \Delta$ and thus f is an acyclic Δ -edge-coloring of H' . The coloring f is therefore a partial acyclic Δ -edge-coloring of H , that is an acyclic edge-coloring on some subset S of $A(H)$, and we show how to extend it to an acyclic Δ -edge-coloring of H .

The existence of such a coloring f of H showed that H does not contain each considered configuration.

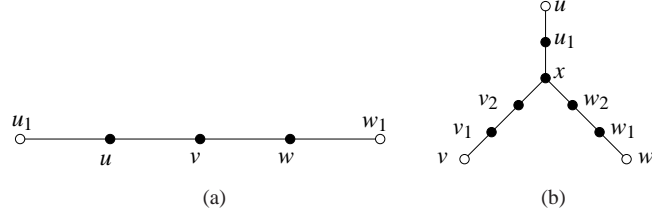


Figure 1: Forbidden configurations for Theorem 2

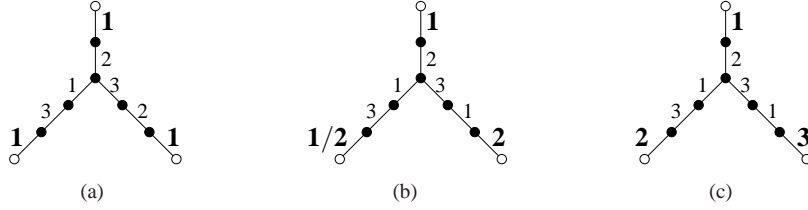


Figure 2: Extended acyclic edge-coloring

Lemma 7 *The graph H does not contain a 1-vertex.*

Proof : Suppose that H contains a dangling edge uv and let $H' = H \setminus uv$. We can easily extend f since uv is adjacent to at most $\Delta - 1$ colors. \square

Since H is not a circuit and does not contains 1-vertices, we get $\Delta \geq 3$ (therefore, we have at least three colors).

Lemma 8 *The graph H does not contain the configuration (a) of Figure 1.*

Proof : Let $H' = H \setminus v$. We can set $f(uv) = c_1$ for some $c_1 \notin \{f(u_1u), f(w_1w)\}$ since $|\{f(u_1u), f(w_1w)\}| \leq 2$. We can finally set $f(vw) = c_2$ for some $c_2 \notin \{f(u_1u), f(w_1w)\}$ since we have at least one available color (f is acyclic since $C_f(v) \cap C_f(w) = \emptyset$). \square

Lemma 9 *The graph H does not contain the configuration (b) of Figure 1.*

Proof : Let $H' = H \setminus \{x, v_2, w_2\}$. Let us consider the two graphs of Figure 2. The bold colors are the colors of the partial acyclic Δ -edge-coloring f , and the notation **a/b** means that we consider two cases on the same configuration.

We consider four distinct cases depending on the coloring of uu_1, vv_1 and ww_1 : $(1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 2)$ and $(1, 2, 3)$. Figure 2 shows the four possible cases and gives a possible extension of the required acyclic edge-coloring. \square

We now prove that every graph with no 1-vertex that does not contain the configurations of Figure 1 has $\text{mad} \geq \frac{7}{3}$.

We assign to each vertex v an initial charge $w(v) = d(v)$. Now, we use the following discharging rules :

Rule 1. Each 3-vertex gives $\frac{1}{3}$ to each of its 3-neighbors having a 2-neighbor;

Rule 2. Each 3-vertex gives $\frac{1}{6}$ to each of its other 2-neighbors.

We denote by $w^*(v)$ the new charge of the vertex v after the discharging procedure.

Let v be a k -vertex; we shall prove that the new charge $w^*(v)$ of each vertex v is at least $\frac{7}{3}$. We consider the following cases (recall that H has no 1-vertices):

- if $k = 2$, then $w(v) = 2$ and $w^*(v) \geq 2 + \min\{\frac{1}{3}; 2 \cdot \frac{1}{6}\} = \frac{7}{3}$, since by Lemma 8 a 2-vertex has at least one ≥ 3 -neighbor;
- if $k = 3$, then $w(v) = 3$ and $w^*(v) \geq 3 - \max\{2 \cdot \frac{1}{3}; \frac{1}{3} + 2 \cdot \frac{1}{6}; 3 \cdot \frac{1}{6}\} = \frac{7}{3}$;
- if $k \geq 4$, then $w(v) = k$ and $w^*(v) \geq k - k \cdot \frac{1}{3} > \frac{7}{3}$.

Observe that

$$\sum_{v \in V(H)} w(v) = \sum_{v \in V(H)} w^*(v) = \sum_{v \in V(H)} d(v),$$

and

$$\text{mad}(H) \geq \frac{2|A(H)|}{|V(H)|} = \frac{\sum_{v \in V(H)} d(v)}{|V(H)|} = \frac{\sum_{v \in V(H)} w^*(v)}{|V(H)|} \geq \frac{\frac{7}{3}|V(H)|}{|V(H)|} = \frac{7}{3}.$$

The contradiction with the hypothesis $\text{mad}(H) < \frac{7}{3}$ completes the proof of Theorem 2.

Therefore, every graph (which is not a cycle) with $\text{mad} \leq \frac{7}{3}$ admits an acyclic Δ -edge-coloring and thus a Δ -adst-coloring.

4 Proof of Theorem 4

To prove this theorem, we use the same technique as before.

Let H be a minimal (with respect to inclusion as a subgraph) counter-example to Theorem 4 and $\Delta = \Delta(H)$. We introduce here two new notions. For an uncolored arc \vec{uv} of H , a color c is called an *available color for \vec{uv}* if it satisfies the constraints of a *dst-coloring* for \vec{uv} ; we denote by $F^f(\vec{uv}) = C_f^-(u) \cup C_f^+(v)$ the set of *forbidden colors of \vec{uv}* ; we can easily see that any color $c \notin F^f(\vec{uv})$ is an available color for \vec{uv} . A color c is called a *feasible color for \vec{uv}* if assigning color c to \vec{uv} results in a partial adst-coloring. Note that a feasible color is necessarily an available color whereas the contrary is false. We can also note that for a given partial adst-coloring f of H and an uncolored arc \vec{uv} , if $C_f^-(u) \cap C_f^+(v) = \emptyset$, then any available color for \vec{uv} is also feasible for \vec{uv} (indeed, any circuit which contains \vec{uv} will not be bichromatic).

In almost all cases, the completion process of f will be the following: for an uncolored arc \vec{uv} , we shall initially determine its available colors, and then ensure, by recoloring some arcs if necessary, that any such color is also feasible for \vec{uv} .

We shall show in the following lemmas that H contains none of the configurations depicted in Figure 3.

In [11], we proved that subcubic graphs (graphs of maximum degree at most three) admit a 4-adst-coloring. Hence, we may assume that $\Delta \geq 4$ in our arguments.

Lemma 10 *The graph H does not contain any of the configurations (a) and (b) of Figure 3.*

Proof : Let $H' = H \setminus uu_1$. We can set $f(uu_1) = c$ for some $c \notin F^f(uu_1)$ since each available color for uu_1 is also feasible (no circuit in H contains uu_1) and $|F^f(uu_1)| = |C_f^-(u_1)| \leq \Delta - 1$. \square

Lemma 11 *The graph H does not contain the configuration (c) of Figure 3.*

Proof : Let $H' = H \setminus u_1\vec{u}$. We can set $f(u_1\vec{u}) = c$ for some $c \notin F^f(u_1\vec{u})$ since each available color for $u_1\vec{u}$ is also feasible (no circuit in H contains $u_1\vec{u}$) and $|F^f(u_1\vec{u})| = |C_f^-(u_1) \cup C_f^+(u)| \leq \Delta - 1 + 1 = \Delta$. \square

Note that by the two previous lemmas, if u is a 2-vertex in H , then u has one incoming and one outgoing arc ($\delta^-(u) = \delta^+(u) = 1$).

Lemma 12 *The graph H does not contain the configuration (d) of Figure 3.*

Proof : Let $H' = H \setminus \vec{vu}$. We can set $f(\vec{vu}) = c$ for some $c \notin F^f(\vec{vu})$ since each available color for \vec{vu} is also feasible (no circuit in H contains \vec{vu}) and $|F^f(\vec{vu})| = |C_f^-(v) \cup C_f^+(u)| \leq 1 + \Delta - 1 = \Delta$. \square

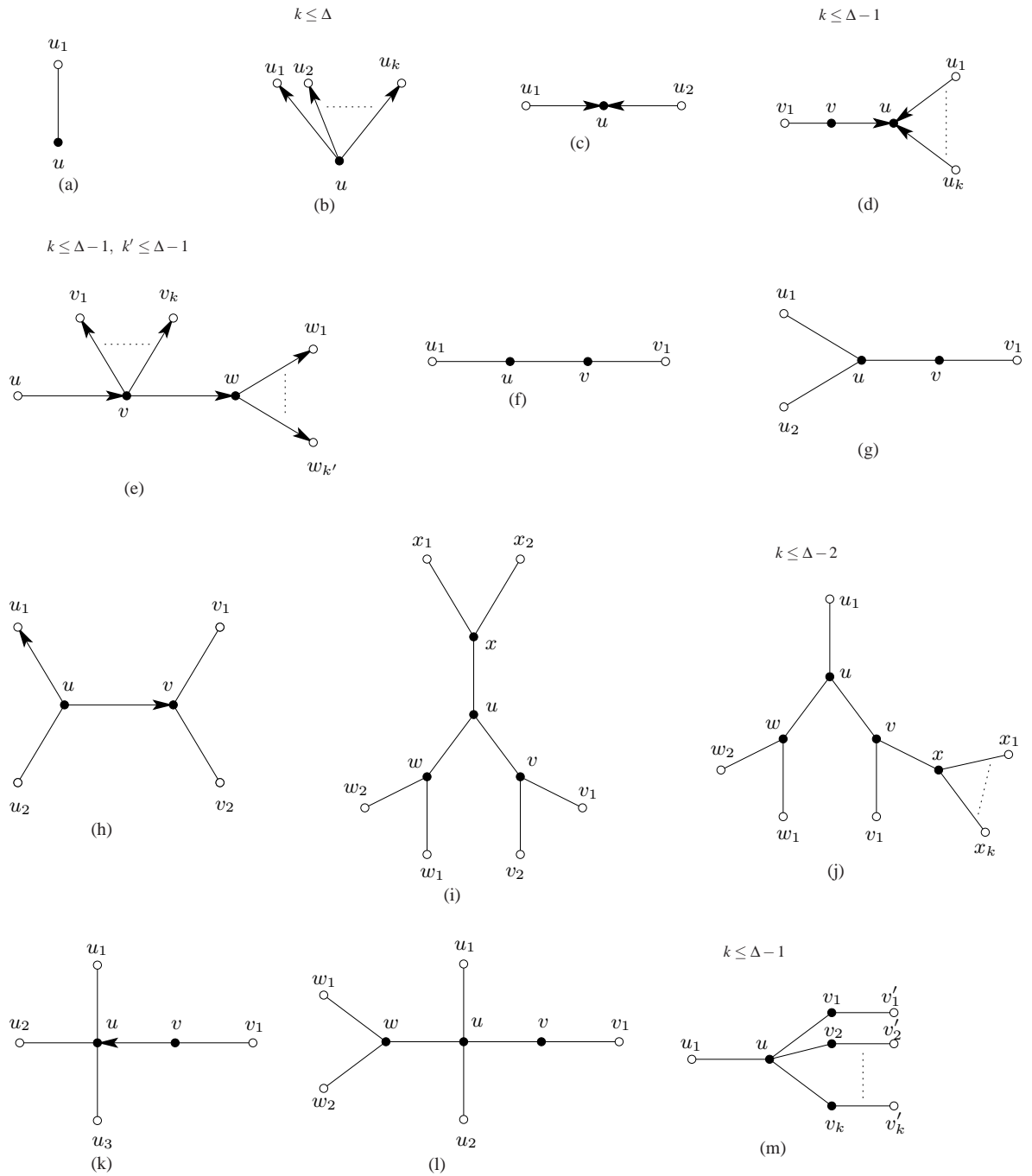


Figure 3: Forbidden configurations for Theorem 4(2)

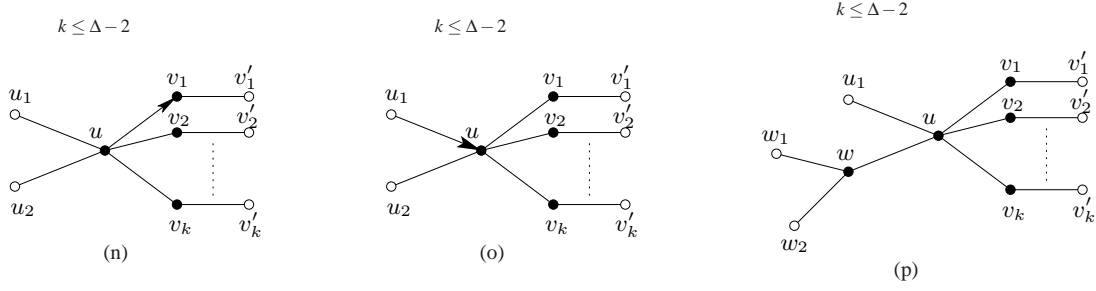


Figure 3: Forbidden configurations for Theorem 4(2)(cont'd)

Lemma 13 *The graph H does not contain the configuration (e) of Figure 3.*

Proof : Let $H' = H \setminus \overrightarrow{vw}$. Suppose first that $f(\overrightarrow{uv}) \notin C_f^+(w)$; therefore $C_f^-(v) \cap C_f^+(w) = \emptyset$ and thus any available color for \overrightarrow{vw} is also feasible: we can set $f(\overrightarrow{vw}) = c$ for some $c \notin F^f(vw)$ since $|F^f(\overrightarrow{vw})| = |C_f^-(v) \cup C_f^+(w)| \leq 1 + \Delta - 1 = \Delta$.

Suppose now that $f(\overrightarrow{uv}) \in C^+(w)$. W.l.o.g., let $f(\overrightarrow{uv}) = f(\overrightarrow{vw_1}) = \dots = f(\overrightarrow{vw_n})$ for some n , $1 \leq n \leq k$. Since $|F^f(\overrightarrow{vw_i})| \leq \Delta - 1$ for $1 \leq i \leq n$, we can recolor $f(\overrightarrow{vw_i}) = c_i$ for some $c_i \notin F^f(\overrightarrow{vw_i}) \cup f(\overrightarrow{uv})$, that leads us to the previous case. \square

Note that by the previous lemma, if \overrightarrow{uvw} is a directed path in H , then one of v and w must have at least two incoming arcs ($\delta^-(v) \cdot \delta^-(w) \geq 2$).

Lemma 14 *The graph H does not contain the configuration (f) of Figure 3.*

Proof : Let $H' = H \setminus uv$. Suppose first that $f(uu_1) \neq f(vv_1)$; we can set $f(uv) = c$ for some $c \notin F^f(uv)$ since each available color for \overrightarrow{uv} is also feasible (we have $C_f(u) \cap C_f(v) = \emptyset$) and $|F^f(uv)| = 2 < \Delta + 1$.

Suppose now that $f(uu_1) = f(vv_1)$. Let $c \notin F^f(uv) \cup C_f^+(v_1) = C_f^+(v_1)$; we have $|C_f^+(v_1)| \leq \Delta$, and thus such a color exists and is an available color for uv ; moreover, since $c \notin C_f^+(v_1)$, c is also a feasible color for uv and therefore we can set $f(uv) = c$. \square

Lemma 15 *The graph H does not contain the configuration (g) of Figure 3.*

Proof : Let $H' = H \setminus uv$. Let $f(uu_1) = a$, $f(uu_2) = b$ and $f(vv_1) = c$. By Lemmas 10, 11 and 13, there exists $i \in \{1, 2\}$ such that $\overrightarrow{u_i u} \in A(H)$; w.l.o.g., let $\overrightarrow{u_1 u} \in (H)$; that implies $a \neq b$. We have two cases to consider:

1. $|\{a, b, c\}| = 3$.

We have $C_f(u) \cap C_f(v) = \emptyset$ and therefore any available color for \overrightarrow{uv} is feasible for $\overrightarrow{u_1 v}$ (no bichromatic circuit in H contains $\overrightarrow{u_1 v}$). We can thus set $f(uv) = d$ for some $d \notin F^f(uv)$ since $|F^f(uv)| \leq 3 < \Delta$.

2. $|\{a, b, c\}| = 2$.

W.l.o.g. we assume that $a = c$. Suppose that $\overrightarrow{uvv_1}$ is a directed path; thus, by Lemma 13, we have $|d^-(v_1)| \geq 2$. Let $d \notin S = F^f(\overrightarrow{uv}) \cup C_f^+(v_1)$; we have $|S| \leq 2 + \Delta - 2 = \Delta$, and thus such a color exists and is an available color for uv ; moreover, since $d \notin C_f^+(v_1)$, d is also a feasible color for uv and therefore we can set $f(uv) = d$.

Suppose now that $\overrightarrow{v_1 v u}$ is directed path; we have $d^-(u) = 2$ by Lemma 12. We can recolor $f(\overrightarrow{v_1 v}) = d$ for some $d \notin T = F^f(\overrightarrow{v_1 v}) \cup \{c\}$ since $|T| \leq \Delta - 1 + 1 = \Delta$; this implies $C_f^-(u) \cap C_f^+(v) = \emptyset$ and therefore any available color for \overrightarrow{vu} is feasible for $\overrightarrow{v_1 u}$ (no bichromatic circuit in H contains $\overrightarrow{v_1 u}$): so, we set $f(\overrightarrow{vu}) = e$ for some $e \notin F^f(\overrightarrow{vu})$ since $|F^f(\overrightarrow{vu})| \leq 3 < \Delta$.

□

Lemma 16 *The graph H does not contain the configuration (h) of Figure 3.*

Proof : Let $H' = H \setminus \vec{uv}$. By Lemma 10, we have $\vec{u_2u} \in A(H)$, and by Lemma 13, $d^-(v) \geq 2$ (w.l.o.g. assume $\vec{v_1v} \in A(H)$).

Suppose that either $\vec{v_2v} \in A(H)$ or $f(\vec{u_2u}) \neq f(\vec{vv_2})$; therefore, $C_f^-(u) \cap C_f^+(v) = \emptyset$ and thus any available color for \vec{uv} is also feasible; we can thus set $f(\vec{uv}) = c$ for some $c \notin F^f(\vec{uv})$ since $|F^f(\vec{uv})| \leq 3 < \Delta$.

Suppose now that $f(\vec{u_2u}) = f(\vec{vv_2})$. Consider the set of colors $S = F^f(\vec{uv}) \cup (C_f^-(u_2) \cap C_f^+(v_2))$. Clearly, any color $c \notin S$ is feasible for \vec{uv} . If $|S| \leq \Delta$, we can set $f(\vec{uv}) = c$ for some $c \notin S$. If $|S| = \Delta + 1$, it implies $C_f^-(u_2) = C_f^+(v_2)$, $d^-(u_2) = d^+(v_2) = \Delta - 1$ and $f(\vec{v_1v}) \notin C_f^+(v_2)$. In this case, we erase $f(\vec{vv_2})$ and pick one outgoing arc from v_2 , denoted by $\vec{v_2v'_2}$. Denote $f(\vec{v_2v'_2}) = d$; since $|F^f(v_2v'_2)| = |C_f^+(v_2)| \leq \Delta - 1$, we can recolor $f(\vec{v_2v'_2}) = e$ for some $e \notin F^f(v_2v'_2) \cup \{d\}$. Thus, the color d is feasible for $\vec{vv_2}$; indeed, $f(\vec{v_1v}) \notin \{f(v_2w) \mid w \in N^+(v_2) \setminus v'_2\}$ and $d \notin C_f^+(v'_2)$. Thus, we set $f(\vec{vv_2}) = d$, which leads us to the previous case ($f(\vec{u_2u}) \neq f(\vec{vv_2})$). □

Lemma 17 *The graph H does not contain the configuration (i) of Figure 3.*

Proof : Let $H' = H \setminus u$. By Lemma 16, u has two incoming arcs and one outgoing arc (w.l.o.g. assume $\vec{vu}, \vec{wu}, \vec{ux} \in A(H)$). We first set $f(\vec{vu}) = c$ for some $c \notin S = F^f(\vec{vu}) \cup \{f(xx_1), f(xx_2)\}$ since $|S| \leq 2 + 2 \leq \Delta$. Let $T = F^f(\vec{wu}) \cup \{f(xx_1), f(xx_2)\}$. We consider two cases:

1. If $|T| \leq \Delta$, we can set $f(\vec{wu}) = d$ for some $d \notin T$. Therefore $C_f^-(u) \cap C_f^+(x) = \emptyset$, and thus any available color for \vec{ux} is feasible and we can set $f(\vec{ux}) = e$ for some $e \notin F^f(\vec{ux})$ since $|F^f(\vec{ux})| \leq 4 \leq \Delta$.
2. If $|T| = \Delta + 1$, this implies $\Delta = 4$. Assume w.l.o.g. that $C_f^-(v) = C_f^-(w) = \{1, 2\}$ and $C_f^+(x) = \{3, 4\}$. We can recolor in this case $f(\vec{vu}) = 3$, and set $f(\vec{wu}) = 4$; finally, the color 5 is clearly feasible for \vec{ux} and we set $f(\vec{ux}) = 5$.

□

Lemma 18 *The graph H does not contain the configuration (j) of Figure 3.*

Proof : By Lemmas 10 and 16, we have $d^+(u) \leq 1$. We consider three cases:

1. Suppose first that u is a sink and let $H' = H \setminus wu$. Since $C_f^-(w) \cap C_f^+(u) = \emptyset$, any available color for \vec{wu} is feasible and thus we can set $f(\vec{wu}) = c$ for some $c \notin F^f(wu)$ since $|F^f(wu)| \leq 4 \leq \Delta$.
2. Suppose now that one of the arcs \vec{uv}, \vec{wv} belongs to $A(H)$ (w.l.o.g. assume $\vec{uv} \in A(H)$) and let $H' = H \setminus wu$. By Lemma 16, we have $\vec{w_1w}, \vec{w_2w} \in A(H)$. Let $H' = H \setminus \vec{wu}$.

If $C_f^-(w) \cap C_f^+(u) = \emptyset$, any available color for \vec{wu} is also feasible, and therefore, we can set $f(\vec{wu}) = c$ for some $c \notin F^f(\vec{wu})$ since $|F^f(\vec{wu})| \leq 4 \leq \Delta$.

Otherwise, $f(\vec{uv}) \in C_f^-(w)$ (w.l.o.g. assume $f(\vec{uv}) = f(\vec{w_1w})$). Let $S = F^f(\vec{wu}) \cup C_f^+(v)$. Any color $c \notin S$ is feasible for \vec{wu} since $c \notin C_f^+(v)$ and thus, if $|S| \leq \Delta$, we can set $f(\vec{wu}) = c$. Otherwise, we have $|S| = \Delta + 1$, which implies $\Delta = 4$ and $d^+(v) = 2$; therefore, we may thus assume w.l.o.g. that $f(\vec{w_1w}) = f(\vec{uv}) = 1, f(\vec{w_2w}) = 2, f(\vec{u_1u}) = 3, f(\vec{v_1v}) = 4$ and $f(\vec{vx}) = 5$. Then, we erase the colors $f(\vec{uv})$ and $f(\vec{v_1v})$ and set $f(\vec{uv}) = 4$; now, any color $c \notin F^f(\vec{v_1v})$ is feasible since $C_f^-(v) \cap C_f^+(v_1) = \emptyset$, and we can set $f(\vec{v_1v}) = c$ since $|F^f(\vec{v_1v})| \leq \Delta$. This leads us to the previous case ($f(\vec{uv}) \notin \{f(w_1w), f(w_2w)\}$).

3. Suppose finally that $\overrightarrow{uu_1} \in A(H)$; by Lemmas 10 and 16, we have $\overrightarrow{v_1v}, \overrightarrow{xv}, \overrightarrow{w_1w}, \overrightarrow{w_2w} \in A(H)$. Since $k \leq \Delta - 2$, we have $d(x) \leq \Delta - 1$. Let $H' = H \setminus v$. We first set $f(\overrightarrow{v_1v}) = c$ for some $c \notin S = F^f(\overrightarrow{v_1v}) \cup f(\overrightarrow{uu_1})$ since $|S| \leq \Delta - 1 + 1 = \Delta$ and $f(\overrightarrow{xv}) = d$ for some $d \notin T = F^f(xv) \cup f(\overrightarrow{uu_1}) = C_f^-(x) \cup \{f(\overrightarrow{v_1v}), f(\overrightarrow{uu_1})\}$ since $|T| \leq \Delta - 2 + 2 = \Delta$. Therefore, any available color for \overrightarrow{vu} is also feasible since $C_f^-(v) \cap C_f^+(u) = \emptyset$; we can thus set $f(\overrightarrow{vu}) = e$ for some $e \notin F^f(\overrightarrow{vu})$.

□

Lemma 19 *The graph H does not contain the configuration (k) of Figure 3.*

Proof: Let $H' = H \setminus v$. By Lemmas 10 and 11, we have $\overrightarrow{v_1v} \in A(H)$ and by Lemma 12 we have $d^+(u) > 0$ (w.l.o.g. assume $\overrightarrow{uu_1} \in A(H)$). We shall consider three cases depending on the cardinality of the color set $C_f^+(u)$.

1. $|C_f^+(u)| = 1$.

We set $f(\overrightarrow{v_1v}) = c$ for some $c \notin S = F^f(\overrightarrow{v_1v}) \cup C_f^+(u)$ since $|S| \leq \Delta - 1 + 1 = \Delta$. Then, we have $C_f^-(v) \cap C_f^+(u) = \emptyset$ and thus any available color for \overrightarrow{vu} is also feasible. We then set $f(\overrightarrow{vu}) = d$ for some $d \notin F^f(\overrightarrow{vu})$ since $|F^f(\overrightarrow{vu})| = 2 < \Delta$.

2. $|C_f^+(u)| = 2$.

W.l.o.g., assume $f(\overrightarrow{uu_1}) = 1$ and $f(\overrightarrow{uu_2}) = 2$. Let $S = F^f(\overrightarrow{v_1v}) \cup C_f^+(u) = C_f^-(v_1) \cup C_f(v) \cup C_f^+(u) = C_f^-(v_1) \cup C_f^+(u)$. Clearly, if $|S| \leq \Delta$, we can proceed as in the previous case.

If $|S| = \Delta + 1$, $C_f^-(v_1) = \{3, 4, \dots, \Delta + 1\}$. Suppose first that $\overrightarrow{uu_3} \in A(H)$ and w.l.o.g. assume $f(\overrightarrow{uu_3}) = 2$. We can recolor $f(\overrightarrow{uu_1}) = c$ for some $c \notin F^f(\overrightarrow{uu_1}) \cup \{1\}$ since $|F^f(\overrightarrow{uu_1})| \leq \Delta - 1$. This implies $|S| \leq \Delta$ and therefore, we can also proceed as in the previous case.

Suppose now that $\overrightarrow{u_3u} \in A(H)$ and assume w.l.o.g. $f(\overrightarrow{u_3u}) = 3$. Let $T = \{4, 5, \dots, \Delta + 1\}$. If there exists some $i \in \{1, 2\}$ such that $T \not\subseteq C_f^+(u_i)$, we have $|T \setminus C_f^+(u_i)| \neq \emptyset$ and therefore, we can pick a color $c \in T \setminus C_f^+(u_i)$ and set $f(\overrightarrow{v_1v}) = i$ and $f(\overrightarrow{vu}) = c$: the color c is feasible for \overrightarrow{vu} since $f(\overrightarrow{uu_{3-i}}) \neq f(\overrightarrow{v_1v})$ and $c \notin C_f^+(u_i)$.

Finally, we can suppose that for all $i \in \{1, 2\}$, $T \subseteq C_f^+(u_i)$. We first erase $f(\overrightarrow{v_1v})$ and consider four subcases:

- $C_f^+(u_1) = T$ and $2 \notin C_f^-(u_1)$.

We can recolor $f(\overrightarrow{uu_1}) = 2$ (the color 2 is feasible for $\overrightarrow{uu_1}$ since $3 \notin C_f^+(u_1)$) and set $f(\overrightarrow{v_1v}) = 1$. We thus have $C_f^-(v) \cap C_f^+(u) = \emptyset$ which implies that any available color for \overrightarrow{vu} is also feasible; we can set $f(\overrightarrow{vu}) = c$ for some $c \notin F^f(\overrightarrow{vu})$ since $|F^f(\overrightarrow{vu})| = 3 < \Delta$.

- $C_f^+(u_1) = T \cup \{2\}$.

Let $\overrightarrow{u_1w}$ denote the outgoing arc of u_1 with $f(\overrightarrow{u_1w}) = 2$. We first erase $f(\overrightarrow{uu_1})$ and we recolor $f(\overrightarrow{u_1w}) = c$ for some $c \notin U = F^f(\overrightarrow{u_1w}) \cup \{2\}$ (we have $|U| \leq \Delta - 1 + 1 = \Delta$); then, 2 is a feasible color since $f(\overrightarrow{u_3u}) \notin \{f(u_1x) \mid x \in N^+(u_1) \setminus w\}$ and $2 \notin C_f^+(w)$ and we set $f(\overrightarrow{uu_1}) = 2$. We can then set $f(\overrightarrow{v_1v}) = 1$, which implies $C_f^-(v) \cap C_f^+(u) = \emptyset$. Therefore any available color for \overrightarrow{vu} is also feasible and we can set $f(\overrightarrow{vu}) = c$ for some $c \notin F^f(\overrightarrow{vu})$ since $|F^f(\overrightarrow{vu})| = 3 < \Delta$.

- $C_f^+(u_1) = T \cup \{3\}$.

Let $\overrightarrow{u_1w}$ denote the outgoing arc of u_1 with $f(\overrightarrow{u_1w}) = 3$. We can set $f(\overrightarrow{u_1w}) = c$ for some $c \notin U = F^f(\overrightarrow{u_1w}) \cup \{3\}$ (we have $|U| \leq \Delta - 1 + 1$). At this point, we have $C_f^+(u_1) = T$ and $2 \notin C_f^-(u_1)$, or $C_f^+(u_1) = T \cup \{2\}$: these two subcases were already considered before.

- $C_f^+(u_1) = T$ and $C_f^-(u_1) = \{2\}$.

By considering the three previous subcases on the vertex u_2 , we deduce that we have $C_f^+(v_2) = T$ and $C_f^-(v_2) = \{1\}$. In this case, we erase $f(\overrightarrow{uu_1})$ and $f(\overrightarrow{uu_2})$ and recolor $f(\overrightarrow{u_3u}) = c$ for some $c \notin V = F^f(\overrightarrow{u_3u}) \cup \{3\}$ (we have $|V| \leq \Delta - 1 + 1$). We thus have $C_f^-(u) \cap C_f^+(u_i) = \emptyset$ for $i = 1, 2$; therefore the color 3 is feasible for $\overrightarrow{uu_1}$ and $\overrightarrow{uu_2}$ since $3 \notin F^f(\overrightarrow{uu_i})$ and we can set $f(\overrightarrow{uu_1}) = 3, f(\overrightarrow{uu_2}) = 3$. Finally, we set $f(\overrightarrow{v_1v}) = 1$, which implies $C_f^-(v) \cap C_f^+(u) = \emptyset$: we can set $f(\overrightarrow{v\bar{u}}) = d$ for some $d \notin F^f(\overrightarrow{v\bar{u}})$ since $|F^f(\overrightarrow{v\bar{u}})| \leq 3 < \Delta$.

3. $|C_f^+(u)| = 3$.

W.l.o.g. assume $f(\overrightarrow{uu_1}) = 1, f(\overrightarrow{uu_2}) = 2, f(\overrightarrow{uu_3}) = 3$. Let $S = F^f(\overrightarrow{v_1v}) \cup C_f^+(u)$. If $|S| \leq \Delta$, we can set $f(\overrightarrow{v_1v}) = c$ for some $c \notin S$ and $f(\overrightarrow{v\bar{u}}) = d$ for some $d \notin F^f(\overrightarrow{v\bar{u}})$ since any available color for $\overrightarrow{v\bar{u}}$ is also feasible (we have $C_f^-(v) \cap C_f^+(u) = \emptyset$ and $|F^f(\overrightarrow{v\bar{u}})| = 4 \leq \Delta$). If $|S| = \Delta + 1$, this implies $\{4, 5, \dots, \Delta + 1\} \subseteq C_f^-(v_1)$ and therefore at least two colors of $C_f^+(u)$ do not belong to $C_f^-(v_1)$. Suppose w.l.o.g. that $1 \notin C_f^-(v_1)$. We then set $f(\overrightarrow{uu_1}) = c$ for some $c \notin T = F^f(\overrightarrow{uu_1}) \cup \{1\}$ since $|T| \leq \Delta - 1 + 1 = \Delta$. We can then set $f(\overrightarrow{v_1v}) = 1$, which implies $C_f^-(v) \cap C_f^+(u) = \emptyset$. We finally set $f(\overrightarrow{v\bar{u}}) = d$ for some $d \notin F^f(\overrightarrow{v\bar{u}})$ since $|F^f(\overrightarrow{v\bar{u}})| \leq 4 \leq \Delta$.

□

Lemma 20 *The graph H does not contain the configuration (l) of Figure 3.*

Proof: Let $H' = H \setminus v$. By Lemma 19, we have $\overrightarrow{uv} \in A(H)$ and $d(v_1) \geq 5$ (therefore, $\Delta \geq 5$), and by Lemma 13 we have $d^-(v_1) \geq 2$. Let $S = F^f(\overrightarrow{uv}) \cup C_f^+(v_1)$. If $|S| \leq \Delta$, we can set $f(\overrightarrow{uv}) = c$ for some $c \notin S$, which implies $C_f^-(v) \cap C_f^+(v_1) = \emptyset$: any available color for $\overrightarrow{vv_1}$ is also feasible and thus we can set $f(\overrightarrow{vv_1}) = d$ for some $d \notin F^f(\overrightarrow{vv_1})$ since $|F^f(\overrightarrow{vv_1})| \leq \Delta$. If $|S| = \Delta + 1$, it means that $C_f^-(u) = 3$ (w.l.o.g. assume $f(\overrightarrow{u_1u}) = 1, f(\overrightarrow{u_2u}) = 2, f(\overrightarrow{w\bar{u}}) = 3$) and $|C_f^+(v_1)| = \Delta - 2$ (w.l.o.g. assume $C_f^+(v_1) = \{4, 5, \dots, \Delta + 1\}$). In this case, we erase $f(\overrightarrow{w\bar{u}})$, which implies $|S| \leq 2 + \Delta - 2 = \Delta$ and we set $f(\overrightarrow{uv}) = 3$. Then, we set $f(\overrightarrow{w\bar{u}}) = d$ for some $d \notin F^f(\overrightarrow{w\bar{u}})$ since $|F^f(\overrightarrow{w\bar{u}})| \leq 5 \leq \Delta$. Finally, since $3 \notin C_f^+(v_1)$, any available color for $\overrightarrow{vv_1}$ is also feasible: we can set $f(\overrightarrow{vv_1}) = d$ for some $d \notin F^f(\overrightarrow{vv_1})$ since $|F^f(\overrightarrow{vv_1})| \leq \Delta$. □

Note that on the four last configurations (m),(n),(o) and (p) of Figure 3, by Lemmas 10 and 11, we have a directed 2-path linking u and v'_i , for all $i \in [1, k]$.

Lemma 21 *The graph H does not contain the configuration (m) of Figure 3.*

Proof: Let $H' = H \setminus v_1$. We consider two cases:

1. Suppose first that there exists $i \in [1, k]$ such that $\overrightarrow{uv_i v'_i}$ is a directed path in H (w.l.o.g. assume $\overrightarrow{uv_1 v'_1}$). We can set $f(\overrightarrow{v_1 v'_1}) = c$ for some $c \notin F^f(\overrightarrow{v_1 v'_1})$ since $|F^f(\overrightarrow{v_1 v'_1})| \leq \Delta - 1$. Clearly, any feasible color for $\overrightarrow{uv_1}$ does not belong to $f(\overrightarrow{v_1 v'_1}) \cup C_f^-(u) \cup \{f(\overrightarrow{v'_j v_j}) \mid f(\overrightarrow{v'_j \bar{u}}) = c\}$; therefore at most Δ colors are not feasible and it remains at least one feasible color to set $f(\overrightarrow{uv_1})$.
2. Suppose now that $\overrightarrow{v'_i v_i u}$ is a directed path in H for all $i \in [1, k]$. By Lemma 12, we have $\overrightarrow{uu_1} \in A(H)$. We can set $f(\overrightarrow{v'_1 v_1}) = c$ for some $c \notin S = F^f(\overrightarrow{v'_1 v_1}) \cup \{f(\overrightarrow{uu_1})\}$ since $|S| \leq \Delta - 1 + 1 = \Delta$. Thus, we have $C_f^-(v_1) \cap C_f^+(u) = \emptyset$ which implies that any available color for $\overrightarrow{v_1 u}$ is also feasible; we then set $f(\overrightarrow{v_1 u}) = d$ for some $d \notin F^f(\overrightarrow{v_1 u})$ since $|F^f(\overrightarrow{v_1 u})| \leq \Delta$.

□

Lemma 22 *The graph H does not contain the configuration (n) of Figure 3.*

Proof : Suppose first that $\overrightarrow{uv_i} \in A(H)$ for all $i \in [1, k]$. By Lemma 13, we have $d^-(v_i) \geq 2$ for all $i \in [1, k]$. Let $H' = H \setminus v_1$. We can set $f(\overrightarrow{uv_1}) = c$ for some $c \notin S = F^f(\overrightarrow{uv_1}) \cup C_f^+(v_1)$ since $|S| \leq 2 + \Delta - 2 = \Delta$. Then, we have $C_f^-(v_1) \cap C_f^+(v_1) = \emptyset$ and therefore any available color for $\overrightarrow{v_1v_1}$ is also feasible: we can set $f(\overrightarrow{v_1v_1}) = d$ for some $d \notin F^f(\overrightarrow{v_1v_1})$ since $|F^f(\overrightarrow{v_1v_1})| \leq \Delta$.

Suppose now that there exists $i \in [1, k]$ such that $\overrightarrow{v_iu} \in A(H)$ (w.l.o.g. assume $\overrightarrow{v_2u} \in A(H)$). We shall consider two cases depending on the orientations of uu_1 and uu_2 :

- $\overrightarrow{uu_1}, \overrightarrow{uu_2} \in A(H)$.

Let $H' = H \setminus v_1$. We can set $f(\overrightarrow{v_1v_1}) = c$ for some $c \notin F^f(\overrightarrow{v_1v_1})$ since $|F^f(\overrightarrow{v_1v_1})| \leq \Delta - 1$. Clearly, any feasible color for $\overrightarrow{uv_1}$ does not belong to $f(\overrightarrow{v_1v_1}) \cup C_f^-(u) \cup \{f(\overrightarrow{v_jv_j}) \mid f(\overrightarrow{v_ju}) = c\}$; therefore at most Δ colors are not feasible for $\overrightarrow{uv_1}$ and it remains at least one feasible color to set $f(\overrightarrow{uv_1})$.

- $\overrightarrow{u_1u} \in A(H)$.

Let $H' = H \setminus v_2$. We can set $f(\overrightarrow{v_2v_2}) = c$ for some $c \notin S = F^f(\overrightarrow{v_2v_2}) \cup \{f(uu_2)\}$ since $|S| \leq \Delta - 1 + 1 = \Delta$. Clearly, any feasible color for $\overrightarrow{v_2u}$ does not belong to $f(\overrightarrow{v_2v_2}) \cup C_f^-(u) \cup \{f(\overrightarrow{v_jv_j}) \mid f(\overrightarrow{uv_j}) = c\}$; therefore at most Δ colors are not feasible for $\overrightarrow{v_2u}$ and it remains at least 1 feasible color to set $f(\overrightarrow{v_2u})$. \square

Lemma 23 *The graph H does not contain the configuration (o) of Figure 3.*

Proof : Let $H' = H \setminus v_1$. By Lemma 22, we have $\overrightarrow{v_iu} \in A(H)$ for all $i \in [1, k]$. We can set $f(\overrightarrow{v_1v_1}) = c$ for some $c \notin S = F^f(\overrightarrow{v_1v_1}) \cup \{f(uu_2)\}$ since $|S| \leq \Delta - 1 + 1 = \Delta$. Therefore, $C_f^-(v_1) \cap C_f^+(u) = \emptyset$ and any available color for $\overrightarrow{v_1u}$ is also feasible; we thus set $f(\overrightarrow{v_1u}) = d$ for some $d \notin F^f(\overrightarrow{v_1u})$. \square

Lemma 24 *The graph H does not contain the configuration (p) of Figure 3.*

Proof : Let $H' = H \setminus uw$. By Lemmas 22 and 23, we have $\overrightarrow{v_iu} \in A(H)$ for all $i \in [1, k - 2]$, and $\overrightarrow{uu_1}, \overrightarrow{uw} \in A(H)$. Clearly, any feasible color for $\overrightarrow{uu_2}$ does not belong to $F^f(\overrightarrow{uu_2}) \cup \{f(\overrightarrow{v_jv_j}) \mid f(\overrightarrow{v_ju}) \in \{f(\overrightarrow{ww_1}), f(\overrightarrow{ww_2})\}\}$; therefore at most Δ colors are not feasible and thus it remains at least 1 feasible color to set $f(\overrightarrow{uw})$. \square

We now prove that every graph that does not contain the configurations of Figure 3 has $\text{mad} \geq \frac{133}{41}$.

We call a *strong k -vertex* a k -vertex which is adjacent to at most $k - 3$ 2-vertices, and a *weak k -vertex* a $k[k - 2]$ -vertex. We also call *strong 3-vertex* a 3-vertex which is adjacent to at most one 3-vertex, and *weak 3-vertex* a 3-vertex adjacent to two 3-vertices.

Note that by Lemma 22, if H contains a weak k -vertex u , the $k - 2$ 2-vertices v_i adjacent to u , $1 \leq i \leq k - 2$, are directed towards u (i.e. $\overrightarrow{v_iu} \in A(H) \forall i \in [1, k - 2]$). Therefore, since H contains no source vertex (by Lemmas 10 and 11), a 2-vertex cannot be adjacent to two weak k -vertices.

We assign to each vertex v an initial charge $w(v) = d(v)$. Now, we use the following discharging rules :

Rule 1. Each strong 3-vertex gives $\frac{2}{41}$ to each of its 3-neighbors;

Rule 2. Each 4-vertex gives $\frac{27}{41}$ to each of its 2-neighbors and $\frac{6}{41}$ to each of its 3-neighbors;

Rule 3. Each ≥ 5 -vertex gives $\frac{6}{41}$ to each of its 3-neighbors;

Rule 4. Each weak ≥ 5 -vertex gives $\frac{24}{41}$ to each of its 2-neighbors;

Rule 5. Each strong ≥ 5 -vertex gives $\frac{27}{41}$ to each of its 2-neighbors.

We denote by $w^*(v)$ the new charge of the vertex v after the discharging procedure.

Let H be a minimal counter-example to Theorem 4 and v be a k -vertex; we shall prove that the new charge $w^*(v)$ of each vertex v is at least $\frac{133}{41}$. We consider the following cases (recall that H has no 1-vertices):

- if $k = 2$, then $w(v) = 2$ and $w^*(v) \geq 2 + \frac{24}{41} + \frac{27}{41} = \frac{133}{41}$, since by Lemma 22 a 2-vertex is not adjacent to two weak ≥ 5 -vertices;
- if $k = 3$ and v is a strong 3-vertex, then $w(v) = 3$ and $w^*(v) \geq 3 + 2 \cdot \frac{6}{41} - \frac{2}{41} = \frac{133}{41}$;
- if $k = 3$ and v is a weak 3-vertex, then $w(v) = 3$ and $w^*(v) = 3 + 2 \cdot \frac{2}{41} + \frac{6}{41} = \frac{133}{41}$, since by Lemma 18 a weak 3-vertex is adjacent to two strong 3-vertices;
- if $k = 4$ and v is adjacent to a 2-vertex, then $w(v) = 4$ and $w^*(v) \geq 4 - \frac{27}{41} = \frac{137}{41}$, since by Lemma 19, a 4-vertex is linked to a 2-vertex by an outgoing arc and therefore, by Lemmas 22 and 23, a 4-vertex can have only one 2-neighbor; moreover, by Lemma 20, a 4-vertex is not adjacent to a 3-vertex if it is already adjacent to 2-vertices;
- if $k = 4$ and v is not adjacent to a 2-vertex, then $w(v) = 4$ and $w^*(v) \geq 4 - 4 \cdot \frac{6}{41} = \frac{140}{41}$;
- if $k \geq 5$ and v is a strong k -vertex, then $w(v) = k$ and $w^*(v) \geq k - 3 \cdot \frac{6}{41} - (k - 3) \cdot \frac{27}{41} \geq \frac{133}{41}$;
- if $k \geq 5$ and v is a weak k -vertex, then $w(v) = k$ and $w^*(v) \geq k - (k - 2) \cdot \frac{24}{41} \geq \frac{133}{41}$, since by Lemma 24 the two other neighbors of a weak k -vertex are ≥ 4 -vertices.

As in the proof of Theorem 2, we get that $\text{mad}(H) \geq \frac{133}{41}$. The contradiction concludes the proof of Theorem 4.

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