

# The acircuitic directed star arboricity of subcubic graphs is at most four

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## Abstract

A directed star forest is a forest all of whose components are stars with arcs emanating from the center to the leaves. The acircuitic directed star arboricity of an oriented graph  $G$  (that is a digraph with no opposite arcs) is the minimum number of edge-disjoint directed star forests whose union covers all edges of  $G$  and such that the union of any two such forests is acircuitic. We show that every subcubic graph has acircuitic directed star arboricity at most four.

## 1 Introduction

We consider finite simple oriented graphs, that is digraphs with no opposite arcs. For an oriented graph  $G$ , we denote by  $V(G)$  its set of vertices and by  $A(G)$  its set of arcs.

In [1], Algor and Alon introduced the notion of the *directed star arboricity* of a digraph  $G$ , defined as the minimum number of edge-disjoint directed star forests needed to cover  $A(G)$ . (A directed star forest is a forest all of whose components are directed stars, that is stars with arcs emanating from the center.) In the same vein, we study here the new notion of the *acircuitic* directed star arboricity of an oriented graph  $G$ , defined as the minimum number of edge-disjoint directed star

forests needed to cover  $A(G)$  in such a way that the union of any two such forests contains no circuit. In [5], Guiduli proved that every oriented graph with indegree and outdegree both less than  $D$  has directed star arboricity at most  $D + 20 \log D + 84$  colors.

In this paper, we prove the following

**Theorem 1** *Every graph with maximum degree at most 3 has acircuitic directed star arboricity at most 4.*

The notion of acircuitic directed star arboricity arises from the study of arc-coloring of oriented graphs. In [4], Courcelle introduced the notion of vertex-coloring of oriented graphs as follows: a  $k$ -vertex-coloring of an oriented graph  $G$  is a mapping  $f$  from  $V(G)$  to a set of  $k$  colors such that (i)  $f(u) \neq f(v)$  whenever  $\vec{uv}$  is an arc in  $G$ , and (ii)  $f(u) \neq f(x)$  whenever  $\vec{uv}$  and  $\vec{wx}$  are two arcs in  $G$  with  $f(v) = f(w)$ . Vertex-coloring of oriented graphs have been studied by several authors in the last past years (see e.g. [2, 6] or [9] for an overview).

Recall that an *acyclic* coloring of an undirected graph  $U$  is a proper coloring of  $U$  such that every cycle in  $U$  uses at least three colors. Raspaud and Sopena proved in [8] that every orientation of an undirected graph that admits an acyclic  $k$ -coloring admits an oriented  $(k \cdot 2^{k-1})$ -coloring.

One can define arc-colorings of oriented graphs in a natural way by saying that, as in the undirected case, an arc-coloring of an oriented graph  $G$  is a vertex-coloring of the line digraph of  $G$ . (Recall that the line digraph  $L(G)$  of  $G$  is given by  $V(L(G)) = A(G)$  and  $(\vec{uv}, \vec{vw}) \in A(L(G))$  whenever  $\vec{uv} \in A(G)$  and  $\vec{vw} \in A(G)$ .) It is not difficult to see that every oriented graph having a  $k$ -vertex-coloring admits a  $k$ -arc-coloring (from a  $k$ -vertex-coloring  $f$ , we obtain a  $k$ -arc-coloring  $g$  by setting  $g(\vec{uv}) = f(u)$ ).

By adapting the proof of the above-mentioned result of Raspaud and Sopena, it is not difficult to prove that every oriented graph with acircuitic directed star arboricity at most  $k$  admits a  $(k \cdot 2^{k-1})$ -arc-coloring.

This paper is organised as follows: we introduce the main definitions and notation in the next section and prove our main result in Section 3.

## 2 Definitions and notation

In the rest of the paper, oriented graphs will be simply called graphs. For a vertex  $v$ , we denote by  $d^-(v)$  the indegree of  $v$ , by  $d^+(v)$  its outdegree and by  $d(v)$  its degree, that is  $d(v) = d^+(v) + d^-(v)$ . A *source vertex* is a vertex  $v$  with  $d^-(v) = 0$ . The *maximum degree* and *minimum degree* of a graph  $G$  are respectively denoted

by  $\Delta(G)$  and  $\delta(G)$ . A graph  $G$  is said to be *cubic* if  $\Delta(G) = \delta(G) = 3$  and *subcubic* if  $\Delta(G) \leq 3$ .

We denote by  $\overrightarrow{uv}$  the arc from  $u$  to  $v$  or simply  $uv$  whenever its orientation is not relevant (therefore  $uv = \overrightarrow{uv}$  or  $uv = \overrightarrow{vu}$ ). If  $a = \overrightarrow{uv}$  is an arc, then  $u$  is the *tail* and  $v$  is the *head* of  $a$ .

For a graph  $G$  and a vertex  $v$  of  $V(G)$ , we denote by  $G \setminus v$  the graph obtained from  $G$  by removing  $v$  together with the set of its incident arcs; similarly, for an arc  $a$  of  $A(G)$ ,  $G \setminus a$  denotes the graph obtained from  $G$  by removing  $a$ . These two notions are extended to sets in a standard way: for a set of vertices  $V'$ ,  $G \setminus V'$  denotes the graph obtained from  $G$  by successively removing all vertices of  $V'$  and their incident arcs, and for a set of arcs  $A'$ ,  $G \setminus A'$  denotes the graph obtained from  $G$  by removing all arcs of  $A'$ .

The notions of arboricity discussed in the previous section may be defined in terms of arc-colorings or partitions of the set of arcs. More precisely, a *k-directed-star-coloring* (or simply *k-dst-coloring*) of a graph  $G$  is a partition of  $A(G)$  into  $k$  directed star forests  $\{F_1, F_2, \dots, F_k\}$ . Equivalently, a *k-dst-coloring* of  $G$  is a  $k$ -coloring  $f$  of  $A(G)$  such that (i)  $\overrightarrow{uv}, \overrightarrow{vw} \in A(G) \Rightarrow f(\overrightarrow{uv}) \neq f(\overrightarrow{vw})$ , and (ii)  $\overrightarrow{uv}, \overrightarrow{tv} \in A(G) \Rightarrow f(\overrightarrow{uv}) \neq f(\overrightarrow{tv})$ . The *directed star arboricity* of  $G$ , denoted by  $dst(G)$ , is then the smallest  $k$  for which  $G$  admits a  $k$ -dst-coloring.

A graph  $G$  is *acircuitic* if it does not contain any circuit. A *k-acircuitic-directed-star-coloring* (or simply *k-adst-coloring*) of a graph  $G$  is a partition of  $A(G)$  into  $k$  directed star forests  $\{F_1, F_2, \dots, F_k\}$  such that for all  $i, j \in [1, k]$ ,  $F_i \cup F_j$  is acircuitic. Equivalently, a *k-adst-coloring* of  $G$  is a  $k$ -dst-coloring of  $G$  such that no circuit in  $G$  is bichromatic. The *acircuitic directed star arboricity* of  $G$ , denoted by  $adst(G)$ , is the smallest  $k$  for which  $G$  admits a  $k$ -adst-coloring.

Note that from the above definitions we get that every edge-coloring of an undirected graph  $H$  is a dst-coloring of any orientation of  $H$ . Similarly, every acyclic edge-coloring of  $H$  is an adst-coloring of any orientation of  $H$ .

The following notation will be extensively used in the rest of the paper. Consider a graph  $G$  and let  $A' = \{a_1, a_2, \dots, a_n\}$  be a subset of  $A(G)$ . We denote by  $C_G(a_1, a_2, \dots, a_n)$ , or simply  $C_G(A')$ , the set of circuits of  $G$  that contain all the arcs  $a_1, a_2, \dots, a_n$ .

**Drawing conventions.** In all the figures, we shall use the following convention: a vertex whose neighbors are totally specified will be black, whereas a vertex whose neighbors are partially specified will be white. Moreover, an edge will represent an arc with any of its two possible orientations.

### 3 Proof of Theorem 1

Suppose that Theorem 1 is false and consider a minimal counter-example  $G$ . We prove a series of lemmas. In each of them, we reduce  $G$  to a smaller graph  $G'$  (that is  $|A(G)| > |A(G')|$ ) which admits a 4-adst-coloring  $f'$  which is also a *partial* adst-coloring of  $G$  (that is an adst-coloring only defined on some subset  $A'$  of  $A(G)$ ). We extend such a partial adst-coloring  $f'$  to an adst-coloring  $f$  of  $G$ . In this case, it should be understood that we set  $f(a) = f'(a)$  for every arc  $a \in A(G')$ . We then explain how to set  $f(a)$  for every uncolored  $a \in A(G)$ . The existence of  $f$  proves that  $G$  does not contain some specific configurations. This set of configurations will finally lead to a contradiction.

Consider a circuit  $C$  and let  $u, v \in V(C)$ . We denote by  $P_C(u, v)$  the directed path from  $u$  to  $v$  in  $C$ .

The following observation will be extensively used in the sequel:

**Observation 2** *Let  $C$  be a circuit,  $f$  an adst-coloring of  $C$ , and  $C'$  the circuit obtained from  $C$  by replacing  $P_C(u, v)$  by a directed path  $P_{C'}(u, v)$ . If  $f'$  is a dst-coloring of  $C'$  such that  $f'(a) = f(a)$  for every  $a \notin P_{C'}(u, v)$  and  $\{f(a); a \in P_C(u, v)\} \subseteq \{f'(a'); a' \in P_{C'}(u, v)\}$  then  $f'$  is an adst-coloring of  $C'$ .*

This directly follows from the fact that  $|f'(C')| \geq |f(C)| \geq 3$ .

We first show that a minimal counter-example to Theorem 1 is necessarily a cubic graph.

**Lemma 3** *If  $G$  is a minimal counter-example to Theorem 1, then  $\delta(G) \geq 3$ .*

**Proof.** Let  $v \in V(G)$  with  $d(v) \leq 2$ . We consider two cases:

*Case 1 :  $d_G(v) = 1$ .*

Consider the dangling arc  $uv$  in  $G$  and let  $f'$  be any 4-adst-coloring of the graph  $G' = G \setminus \{v\}$ . We extend  $f'$  to a 4-adst-coloring  $f$  of  $G$  by setting  $f(uv) = a$  for some color  $a$  distinct from the colors of the at most two arcs incident to  $uv$ .

*Case 2 :  $d_G(v) = 2$ .*

Consider the two arcs  $uv$  and  $wv$  in  $G$  and let  $f'$  be any 4-adst-coloring of the graph  $G'$  obtained from  $G$  by contracting  $uv$  in a single vertex  $x$ . We extend  $f'$  to a 4-adst-coloring  $f$  of  $G$  by setting  $f(wv) = f'(wx)$  and  $f(uv) = a$  for any  $a$  distinct from the colors of the three arcs incident to  $uv$ . (By Observation 2, no circuit in  $G$  can be bichromatic.)

In both cases we thus obtain a 4-adst-coloring  $f$  of  $G$ , a contradiction. ■

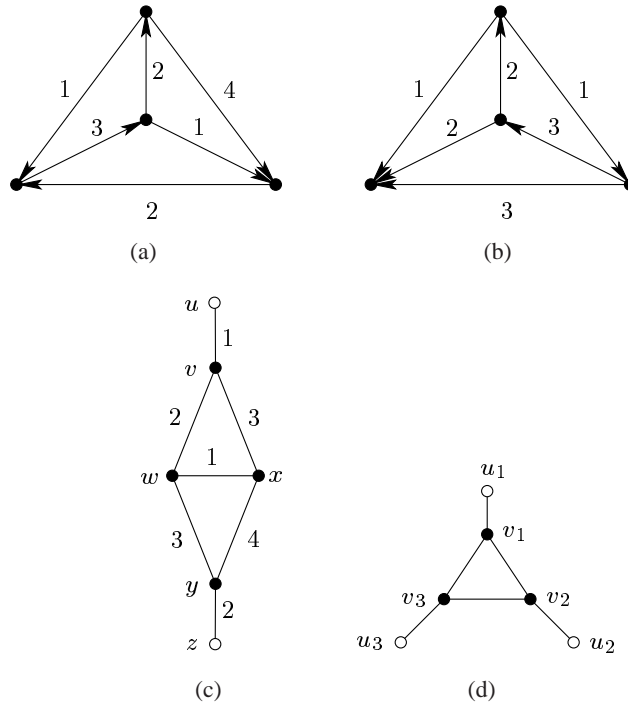


Figure 1: Configurations of Lemma 5

**Lemma 4** *If  $G$  is a minimal counter-example to Theorem 1, then  $G$  does not contain any source vertex.*

**Proof.** Let  $v \in V(G)$  be a source vertex,  $u_1$ ,  $u_2$  and  $u_3$  be the three neighbors of  $v$  and  $f'$  be any 4-adst-coloring of the graph  $G' = G \setminus v$ . By Lemma 3, we know that  $d^+(v) = 3$ . Each of the arcs  $\overrightarrow{vu_1}$ ,  $\overrightarrow{vu_2}$  and  $\overrightarrow{vu_3}$  has at least two available colors. Since they can get the same color, we can extend  $f'$  to a 4-adst-coloring  $f$  of  $G$ , a contradiction. ■

We now prove that a minimal counter-example to Theorem 1 contains no triangle.

**Lemma 5** *If  $G$  is a minimal counter-example to Theorem 1, then  $G$  is triangle-free.*

**Proof.** If  $G$  contains three pairwise adjacent triangles, then  $G$  is an orientation of the complete graph  $K_4$ . By Lemma 4 we only have to consider the two orientations of  $K_4$  depicted on Figures 1(a) and 1(b) that both admit a 4-adst-coloring.

If  $G$  contains two adjacent triangles, then  $G$  contains the configuration of Figure 1(c). Consider the graph  $G' = G \setminus \{w, x\}$  and let  $f'$  be a 4-adst-coloring of  $G'$  such that  $f'(uv) \neq f'(yz)$  (this can be done since we have two possible choices for coloring each of  $uv$  and  $uz$ ). Suppose without loss of generality that  $f'(uv) = 1$  and  $f'(yz) = 2$ . In this case, we can produce an acyclic 4-edge-coloring as depicted on Figure 1(c). Indeed, this coloring is a proper edge-coloring and no path linking  $u$  and  $z$  is bichromatic. Hence, for all possible orientations of the arcs of the configuration, this coloring gives a 4-adst-coloring  $f$  of  $G$ .

Suppose finally that  $G$  contains the configuration of Figure 1(d), and let  $f'$  be any 4-adst-coloring of the graph  $G'$  obtained from  $G$  by contracting the triangle  $v_1v_2v_3$  in a single vertex  $v$ . Therefore, every circuit  $C \in C_G(\overrightarrow{u_i v_i}, \overrightarrow{v_j u_j})$  corresponds to a circuit  $C' \in C_{G'}(\overrightarrow{u_i v}, \overrightarrow{v u_j})$ .

We now extend the partial adst-coloring  $f'$  to a 4-adst-coloring  $f$  of  $G$  as follows. We distinguish two cases:

*Case 1 :*  $f'(vu_1) \neq f'(vu_2) \neq f'(vu_3) \neq f'(vu_1)$ .

Without loss of generality, suppose that  $f'(v_1u_1) = 1$ ,  $f'(v_2u_2) = 2$  and  $f'(v_3u_3) = 3$ . We then set  $f(v_3v_1) = 2$ ,  $f(v_1v_2) = 3$  and  $f(v_2v_3) = 1$ .

*Case 2 :*  $\exists i, j \in \{1, 2, 3\}, i \neq j, f'(v_iu_i) = f'(v_ju_j) = a$ .

In this case we necessarily have  $\overrightarrow{v_iu_i}, \overrightarrow{v_ju_j} \in A(G)$ . Let  $k \in \{1, 2, 3\}, k \neq i, j$ . We then set  $f(v_iv_k)$ ,  $f(v_jv_k)$  and  $f(v_iv_j)$  as follows:

1.  $f(v_iv_k) = b$  for any  $b \notin \{a, f'(u_kv_k)\}$ ,
2.  $f(v_jv_k) = c$  for any  $c \notin \{a, b, f'(u_kv_k)\}$ ,
3.  $f(v_iv_j) = d$  for any  $d \notin \{a, b, c\}$ .

This can be done since we have four available colors.

In both cases, thanks to Observation 2, we obtain a 4-adst-coloring  $f$  of  $G$ , a contradiction. ■

Let  $G$  be a graph and  $C$  a circuit in  $G$ . An arc having exactly one of its endpoints in  $C$  is said to be *incident* to  $C$ . Moreover, two such incident arcs are *neighboring* if their endpoints in  $C$  are linked by an arc of  $C$ .

The four next lemmas will allow us to prove that a minimal counter-example  $G$  to Theorem 1 is necessarily acircuic.

**Lemma 6** *If  $G$  is a minimal counter-example to Theorem 1, then  $G$  does not contain a circuit all of whose vertices have indegree one and outdegree two.*

**Proof.** Suppose that there exists a circuit  $C = \{\overrightarrow{v_0v_1}, \overrightarrow{v_1v_2}, \dots, \overrightarrow{v_{k-2}v_{k-1}}, \overrightarrow{v_{k-1}v_0}\}$  in  $G$  such that  $d^+(v_i) = 1$  and  $d^-(v_i) = 2$  for  $i \in [0, k-1]$  and let  $f'$  be any 4-adst-coloring of the graph  $G' = G \setminus C$ . Let  $\{\overrightarrow{v_iu_i} \mid i \in [0, k-1]\}$  be the set of arcs incident to  $C$ .

We extend the partial coloring  $f'$  to a 4-adst-coloring  $f$  of  $G$  as follows. Due to the orientation of  $G$ ,  $C$  is the only circuit of  $G$  that does not belong to  $G'$ . Therefore, we only need to color the arcs of  $C$  in such a way that  $C$  is not bichromatic. We distinguish two cases depending on the colors of the arcs incident to  $C$ .

1. All arcs incident to  $C$  are colored with the same color. In this case, we color the arcs of  $C$  using the three other remaining colors.
2. Two neighboring arcs incident to  $C$  have distinct colors. Suppose without loss of generality that  $f'(\overrightarrow{v_0u_0}) = c_0$  and  $f'(\overrightarrow{v_1u_1}) \neq c_0$ . In this case, we set
  - (a)  $f(\overrightarrow{v_0v_1}) = c_0$ ,
  - (b)  $\forall i \in [1, k-2], f(\overrightarrow{v_iv_{i+1}}) = c_i$  for any  $c_i \notin \{c_{i-1}, f(\overrightarrow{v_{i+1}u_{i+1}})\}$ ,
  - (c)  $f(\overrightarrow{v_{k-1}v_0}) = c_{k-1}$  for any  $c_{k-1} \notin \{c_0, c_1, c_{k-2}\}$ .

The circuit  $C$  is clearly not bichromatic since  $c_0 \neq c_1 \neq c_{k-1} \neq c_0$ .

In both cases, we obtain a 4-adst-coloring  $f$  of  $G$ , a contradiction. ■

**Lemma 7** *If  $G$  is a minimal counter-example to Theorem 1, then  $G$  does not contain a circuit all of whose vertices have indegree two and outdegree one.*

**Proof.** Suppose that there exists a circuit  $C = \{\overrightarrow{v_0v_1}, \overrightarrow{v_1v_2}, \dots, \overrightarrow{v_{k-2}v_{k-1}}, \overrightarrow{v_{k-1}v_0}\}$  in  $G$  (see Figures 2(a) or 3(a)) such that  $d^+(v_i) = 2$  and  $d^-(v_i) = 1$  for  $i \in [0, k-1]$ . Let  $\{\overrightarrow{u_iv_i} \mid i \in [0, k-1]\}$  be the set of arcs incident to  $C$ . By Lemma 5, the tails of two neighboring arcs incident to  $C$  are necessarily distinct.

We consider two cases depending on whether the vertices  $u_0$  and  $u_2$  are distinct or not. We first show that in both cases there exists a reduction  $G'$  of  $G$  (see Figures 2(b) and 3(b)) which admits a 4-adst-coloring  $f'$  such that  $f'(\overrightarrow{u_0v_0}) \neq f'(\overrightarrow{u_1v_1}) \neq f'(\overrightarrow{u_2v_2}) \neq f'(\overrightarrow{u_0v_0})$ .

*Case 1 :  $u_0 \neq u_2$  (see Figure 2(a)).*

Let  $f'$  be any 4-adst-coloring of the graph  $G'$  obtained from  $G \setminus C$  by identifying  $v_0, v_1$  and  $v_2$  in a single vertex  $v$  (see Figure 2(b)). We clearly have  $f(\overrightarrow{u_0v}) \neq f(\overrightarrow{u_1v}) \neq f(\overrightarrow{u_2v}) \neq f(\overrightarrow{u_0v})$ .

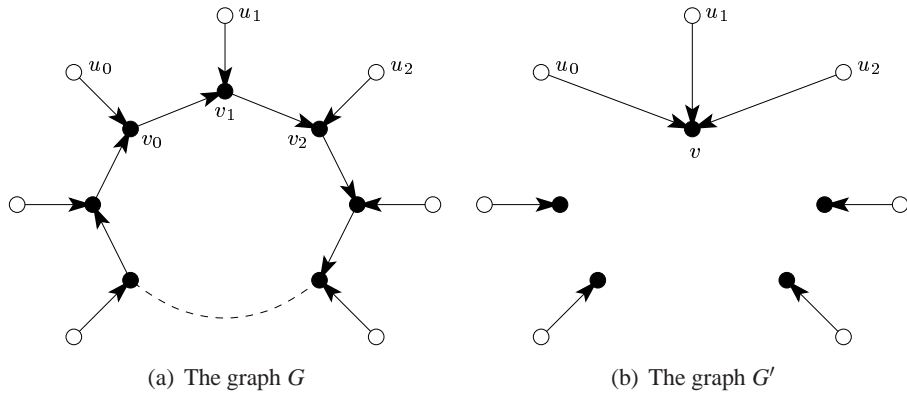


Figure 2: The configuration of Case 1 of Lemma 7 and its reduction.

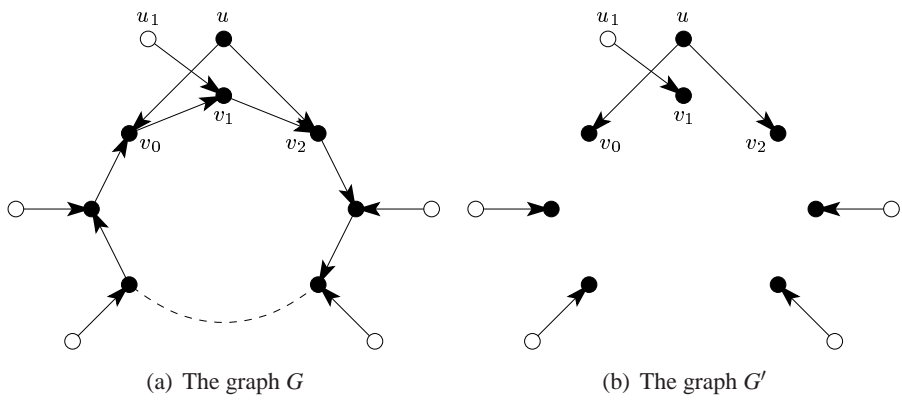


Figure 3: The configuration of Case 2 of Lemma 7 and its reduction.



Case 2 :  $u_0 = u_2 = u$  (see Figure 3(a)).

Note that by Lemma 4 we have  $u_1 \neq u$ . Let  $f'$  be any 4-adst-coloring of the graph  $G' = G \setminus C$  (see Figure 3(b)). Since we have at least three available colors for the arcs  $\overrightarrow{uv_0}$  and  $\overrightarrow{uv_2}$ , we can choose  $f'$  in such a way that  $f'(\overrightarrow{uv_0}) \neq f'(\overrightarrow{u_1v_1}) \neq f'(\overrightarrow{uv_2}) \neq f'(\overrightarrow{uv_0})$ .

Assume now that  $f(\overrightarrow{u_0v_0}) = c_1$ ,  $f(\overrightarrow{u_1v_1}) \neq c_1$  and  $f(\overrightarrow{u_2v_2}) \neq c_1$ . As in the previous lemma,  $C$  is the only circuit of  $G$  that does not belong to  $G'$ . Therefore, we only need to color the arcs of  $C$  in such a way that  $C$  is not bichromatic. We then set  $f$  as follows:

1.  $f(\overrightarrow{v_1v_2}) = c_1$ ,
2.  $\forall i \in [2, k-1]$ ,  $j = i+1 \pmod{k}$ ,  $f(\overrightarrow{v_iv_j}) = c_i$   
for any  $c_i \notin \{c_{i-1}, f(\overrightarrow{u_iv_i}), f(\overrightarrow{u_jv_j})\}$ ,
3.  $f(\overrightarrow{v_0v_1}) = c_0$  for any  $c_0 \notin \{c_{k-1}, c_1, f(\overrightarrow{u_1v_1})\}$ .

Note that  $c_{k-1} \neq f(\overrightarrow{u_0v_0}) = c_1$ . Therefore,  $c_{k-1} \neq c_0 \neq c_1 \neq c_{k-1}$  and  $C$  is not bichromatic. We thus obtain a 4-adst-coloring  $f$  of  $G$ , a contradiction. ■

From the two previous lemmas, we get that if  $C$  is a circuit in a minimal counter-example to Theorem 1, there exist two neighboring arcs incident with  $C$  having opposite directions (with respect to  $C$ ). The next two lemmas will show that this situation is also not possible.

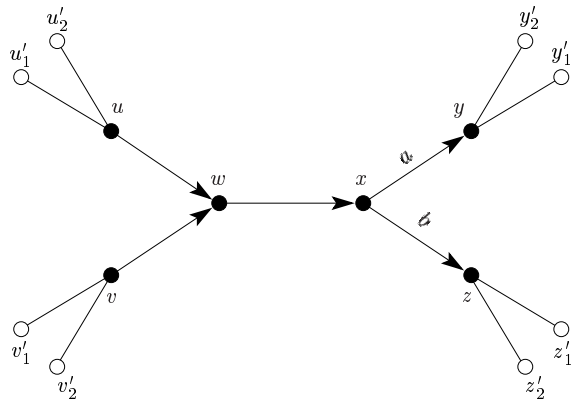
**Lemma 8** *If  $G$  is a minimal counter-example to Theorem 1, then  $G$  does not contain the configuration depicted on Figure 4(a).*

**Proof.** Suppose that the graph  $G$  contains the configuration of Figure 4(a), with the arcs  $u'_1u$ ,  $u'_2u$ ,  $y'_1y$ ,  $y'_2y$ ,  $v'_1v$ ,  $v'_2v$ ,  $z'_1z$  and  $z'_2z$  being pairwise distinct, and let  $f'$  be any 4-adst-coloring of the graph  $G'$  obtained from  $G \setminus \{w, x\}$  by adding the arcs  $\overrightarrow{uy}$  and  $\overrightarrow{vz}$  (see Figure 4(b)). Suppose that  $f'(\overrightarrow{uy}) = a$  and  $f'(\overrightarrow{vz}) = b$ .

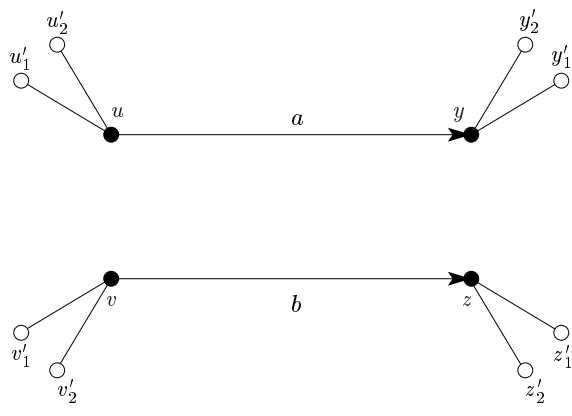
We extend the partial 4-adst-coloring  $f'$  to a 4-adst-coloring  $f$  of  $G$  as follows.

Let  $E = C_G(\overrightarrow{uw}, \overrightarrow{wx}, \overrightarrow{xy}) \cup C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xz})$  and  $F = C_G(\overrightarrow{uw}, \overrightarrow{wx}, \overrightarrow{xz}) \cup C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xy})$ . We first set  $f(\overrightarrow{xy}) = a$  and  $f(\overrightarrow{xz}) = b$ . Clearly, all circuits in  $G$  not belonging to  $E \cup F$  also belong to  $G'$ , and thus are already not bichromatic. Moreover, by Observation 2, the circuits in  $E$  will not be bichromatic. Therefore, we only have to pay attention to the circuits in  $F$ .

We consider two cases depending on the colors  $a$  and  $b$ :



(a) The graph  $G$



(b) The reduction  $G'$

Figure 4: The configuration of Lemma 8 and its reduction.

Case 1 :  $a \neq b$ .

We set  $f(\overrightarrow{uw}) = a$ ,  $f(\overrightarrow{vw}) = b$  and  $f(\overrightarrow{wx}) = c$  for any  $c \notin \{a, b\}$ . Since  $|\{f(\overrightarrow{uw}), f(\overrightarrow{wx}), f(\overrightarrow{xz})\}| = |\{a, c, b\}| = 3$  and  $|\{f(\overrightarrow{vw}), f(\overrightarrow{wx}), f(\overrightarrow{xy})\}| = |\{b, c, a\}| = 3$ , no circuit in  $F$  is bichromatic.

Case 2 :  $a = b$ .

We consider three subcases.

1.  $\{\overrightarrow{uu'_1}, \overrightarrow{uu'_2}, \overrightarrow{vv'_1}, \overrightarrow{vv'_2}\} \cap A(G) \neq \emptyset$ .

We assume without loss of generality that  $\overrightarrow{vv'_1} \in A(G)$ . In this case, we first set  $f(\overrightarrow{uw}) = c$  for any  $c \notin \{a, f(\overrightarrow{uu'_1}), f(\overrightarrow{uu'_2})\}$  and  $f(\overrightarrow{vw}) = d$  for any  $d \notin \{a, c, f(\overrightarrow{vv'_2})\}$ . Now, we can color the arc  $\overrightarrow{wx}$  with the fourth color  $e \notin \{a, c, d\}$ . Therefore,  $|\{f(\overrightarrow{uw}), f(\overrightarrow{wx}), f(\overrightarrow{xz})\}| = |\{c, e, a\}| = 3$ ,  $|\{f(\overrightarrow{vw}), f(\overrightarrow{wx}), f(\overrightarrow{xy})\}| = |\{d, e, a\}| = 3$ , and so no circuit in  $F$  is bichromatic.

2.  $\overrightarrow{u'_1u}, \overrightarrow{u'_2u}, \overrightarrow{v'_1v}, \overrightarrow{v'_2v} \in A(G)$

and  $\{f(\overrightarrow{u'_1u}), f(\overrightarrow{u'_2u})\} \neq \{f(\overrightarrow{v'_1v}), f(\overrightarrow{v'_2v})\}$ .

Note that since  $a \notin \{f(\overrightarrow{u'_1u}), f(\overrightarrow{u'_2u}), f(\overrightarrow{v'_1v}), f(\overrightarrow{v'_2v})\}$ , we necessarily have  $\{f(\overrightarrow{u'_1u}), f(\overrightarrow{u'_2u})\} \cap \{f(\overrightarrow{v'_1v}), f(\overrightarrow{v'_2v})\} \neq \emptyset$ . Therefore, we can assume without loss of generality that  $f(\overrightarrow{u'_1u}) = f(\overrightarrow{v'_1v}) = c$ ,  $f(\overrightarrow{u'_2u}) = d$  and  $f(\overrightarrow{v'_2v}) = e$ , with  $a, c, d, e$  being pairwise distinct. In this case, we set  $f(\overrightarrow{uw}) = e$  and  $f(\overrightarrow{vw}) = d$ . Now, we can color the arc  $\overrightarrow{wx}$  with the color  $c$ . Therefore,  $|\{f(\overrightarrow{uw}), f(\overrightarrow{wx}), f(\overrightarrow{xz})\}| = |\{e, c, a\}| = 3$ ,  $|\{f(\overrightarrow{vw}), f(\overrightarrow{wx}), f(\overrightarrow{xy})\}| = |\{d, c, a\}| = 3$ , and so no circuit in  $F$  is bichromatic.

3.  $\overrightarrow{u'_1u}, \overrightarrow{u'_2u}, \overrightarrow{v'_1v}, \overrightarrow{v'_2v} \in A(G)$

and  $\{f(\overrightarrow{u'_1u}), f(\overrightarrow{u'_2u})\} = \{f(\overrightarrow{v'_1v}), f(\overrightarrow{v'_2v})\}$ .

We assume without loss of generality that  $f(\overrightarrow{u'_1u}) = f(\overrightarrow{v'_1v}) = c$  and  $f(\overrightarrow{u'_2u}) = f(\overrightarrow{v'_2v}) = d$ ,  $a \neq c \neq d \neq a$ . We then set  $f(\overrightarrow{vw}) = a$  and  $f(\overrightarrow{uw}) = e$  with  $e \notin \{a, c, d\}$ .

Since  $\overrightarrow{uw}$  and  $\overrightarrow{xz}$  are colored with distinct colors, no circuit in  $C_G(\overrightarrow{uw}, \overrightarrow{wx}, \overrightarrow{xz})$  is bichromatic.

We still have to set the color of the arc  $\overrightarrow{wx}$ . We consider three subcases.

- (a)  $\{y'_1y, y'_2y\} \cap A(G) \neq \emptyset$

We assume without loss of generality that  $y'_1y \in A(G)$ . So, if  $f(y'_1y) = c$  (resp.  $f(y'_2y) = d$ ), we set  $f(\overrightarrow{wx}) = d$  (resp.  $f(\overrightarrow{wx}) = c$ ), otherwise ( $f(y'_2y) = e$ ), we use either  $c$  or  $d$ . Therefore,  $|\{f(\overrightarrow{wx}), f(\overrightarrow{xy}), f(\overrightarrow{yy'_2})\}| = 3$ , and thus no circuit in  $C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xy})$  is bichromatic.

(b)  $\overrightarrow{yy'_1}, \overrightarrow{yy'_2} \in A(G)$  and  $\{f(\overrightarrow{v'_1v}), f(\overrightarrow{v'_2v})\} \neq \{f(\overrightarrow{yy'_1}), f(\overrightarrow{yy'_2})\}$ .

We assume without loss of generality that  $f(\overrightarrow{yy'_2}) = e$ . Now, if  $f(\overrightarrow{yy'_1}) = c$  (resp.  $f(\overrightarrow{yy'_1}) = d$ ) we set  $f(\overrightarrow{w\bar{x}}) = d$  (resp.  $f(\overrightarrow{w\bar{x}}) = c$ ). This implies that for any  $i \in \{1, 2\}$ ,  $|\{f(\overrightarrow{w\bar{x}}), f(\overrightarrow{x\bar{y}}), f(\overrightarrow{yy'_i})\}| = 3$ , and thus no circuit in  $C_G(\overrightarrow{v\bar{w}}, \overrightarrow{w\bar{x}}, \overrightarrow{x\bar{y}})$  is bichromatic.

(c)  $\overrightarrow{yy'_1}, \overrightarrow{yy'_2} \in A(G)$  and  $\{f(\overrightarrow{v'_1v}), f(\overrightarrow{v'_2v})\} = \{f(\overrightarrow{yy'_1}), f(\overrightarrow{yy'_2})\}$ .

We can suppose without loss of generality that  $f(\overrightarrow{yy'_1}) = c$  and  $f(\overrightarrow{yy'_2}) = d$ . We then set  $f(\overrightarrow{w\bar{x}}) = c$ . If there is no arc emanating from  $y'_1$  and colored with  $a$ , no circuit in  $C_G(\overrightarrow{v\bar{w}}, \overrightarrow{w\bar{x}}, \overrightarrow{x\bar{y}})$  is bichromatic. If there exists an arc emanating from  $y'_1$  and colored with  $a$ , then there exists at least one available color distinct from  $c$  that can be used to recolor the arc  $\overrightarrow{yy'_1}$  in such a way that we forbid bichromatic circuits in  $C_G(\overrightarrow{v\bar{w}}, \overrightarrow{w\bar{x}}, \overrightarrow{x\bar{y}})$ .

In all cases, we obtain a 4-adst-coloring  $f$  of  $G$ , a contradiction. ■

In the configuration of the previous Lemma, the arcs  $uu'_i$  and  $vv'_j$  on one hand,  $yy'_i$  and  $zz'_j$  on the other hand, are necessarily distinct since, by Lemma 5, a minimal counter-example to Theorem 1 contains no triangle. The next lemma deals with the case where two arcs  $uu'_i$  (or  $vv'_i$ ) and  $yy'_j$  (or  $zz'_j$ ) are the same. Without loss of generality, we will suppose that the arcs  $v'_1v$  and  $z'_1z$  are the same.

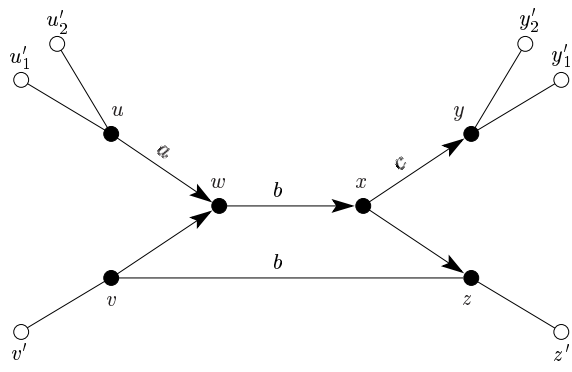
**Lemma 9** *If  $G$  is a minimal counter-example to Theorem 1, then  $G$  does not contain the configuration depicted on Figure 5(a).*

**Proof.** Suppose that the graph  $G$  contains the configuration of Figure 5(a) (in this configuration, two arcs linking a black and a white vertex may be the same provided it does not produce a triangle). We consider two cases depending on the orientation of the arc  $vz$ .

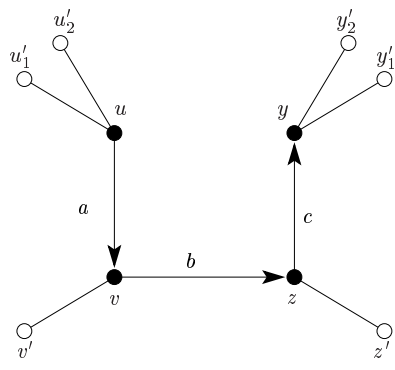
*Case 1 :*  $\overrightarrow{vz} \in A(G)$ .

Consider the graph  $G'_1$  (see Figure 5(b)) obtained from  $G \setminus \{w, x\}$  by adding the arcs  $\overrightarrow{u\bar{v}}$  and  $\overrightarrow{z\bar{y}}$  (see Figure 5(b)) and let  $f'_1$  be any 4-adst-coloring of  $G'_1$ . Assume that  $f'_1(\overrightarrow{u\bar{v}}) = a$ ,  $f'_1(\overrightarrow{v\bar{z}}) = b$  and  $f'_1(\overrightarrow{z\bar{y}}) = c$  (see Figure 5(b)). We extend the partial 4-adst-coloring  $f'_1$  to a 4-adst-coloring  $f$  of  $G$  as follows.

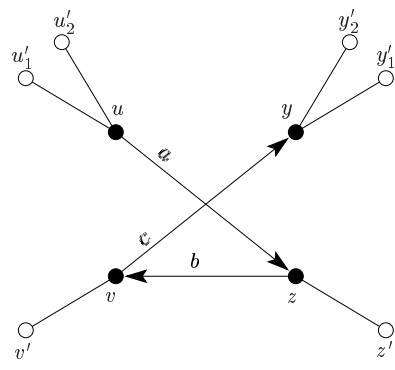
We first set  $f(\overrightarrow{u\bar{w}}) = a$ ,  $f(\overrightarrow{w\bar{x}}) = f(\overrightarrow{v\bar{z}}) = b$  and  $f(\overrightarrow{x\bar{y}}) = c$  (see Figure 5(a)). By Observation 2, no circuit in  $G$  is thus bichromatic. We then color the arcs  $\overrightarrow{v\bar{w}}$  and  $\overrightarrow{x\bar{z}}$  so that  $f(\overrightarrow{v\bar{w}}) \notin \{a, b, f(\overrightarrow{v'v})\}$  and  $f(\overrightarrow{x\bar{z}}) \notin \{b, f(\overrightarrow{z'z})\}$ .



(a) The graph  $G$



(b) The reduction  $G'_1$



(c) The reduction  $G'_2$

Figure 5: The configuration of Lemma 9 and its reductions.

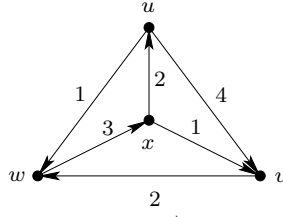


Figure 6: The orientation  $\vec{K}_4$  such that  $adst(\vec{K}_4) = 4$

Case 2 :  $\vec{z}v \in A(G)$ .

Consider the graph  $G'_2$  obtained from  $G \setminus \{w, x\}$  by adding the arcs  $\vec{u}z$  and  $\vec{v}y$  and let  $f'_2$  be any 4-adst-coloring of  $G'_2$ . Assume that  $f'_2(\vec{u}z) = a$ ,  $f'_2(\vec{z}v) = b$  and  $f'_2(\vec{v}y) = c$  (see Figure 5(c)). We extend the partial 4-adst-coloring  $f'$  to a 4-adst-coloring  $f$  of  $G$  as follows.

As in the previous case, we set  $f(\vec{u}w) = a$ ,  $f(\vec{w}x) = f(\vec{v}z) = b$  and  $f(\vec{x}y) = c$  (see Figure 5(a)). By Observation 2, we only have to pay attention to the circuits in  $C_G(\vec{v}w, \vec{w}x, \vec{x}z)$ . We then color the arcs  $\vec{v}w$  and  $\vec{x}z$  in such a way that  $f(\vec{v}w) \notin \{a, b, f(v'v)\}$  and  $f(\vec{x}z) = a$  (this can be done since  $f'(zz') \neq a$ ). Since  $a \neq b \neq f(\vec{v}w) \neq a$ , no circuit in  $C_G(\vec{v}w, \vec{w}x, \vec{x}z)$  is bichromatic.

In both cases we obtain a 4-adst-coloring  $f$  of  $G$ , a contradiction. ■

Using the previous lemmas, we can now prove our main result.

**Proof of Theorem 1.** By Lemmas 6, 7, 8 and 9, a minimal counter-example  $G$  to Theorem 1 does not contain any circuit. Therefore, any 4-dst-coloring of  $G$  is a 4-adst-coloring of  $G$ . Moreover, it follows from the definitions that any  $k$ -edge-coloring of the underlying undirected graph of  $G$  is a  $k$ -dst-coloring of  $G$ . Therefore, by Vizing's theorem [11], the graph  $G$  admits a 4-edge-coloring and thus a 4-adst-coloring, a contradiction. ■

The bound given in Theorem 1 is optimal. To see that, consider the orientation  $\vec{K}_4$  of the complete graph  $K_4$  given on Figure 6. If we want to color this graph with three colors, the only way to color the arcs  $\vec{x}u$ ,  $\vec{u}w$ ,  $\vec{w}x$ ,  $\vec{x}v$  and  $\vec{v}w$  is clearly the one depicted on Figure 6. But in this case, we need one more color for the arc  $\vec{u}v$  and thus,  $adst(\vec{K}_4) = 4$ .

## 4 Discussion

In [3] Burnstein proved that every graph with maximum degree 4 admits an acyclic 5-vertex-coloring. Since the line graph of a subcubic graph has maximum degree at most 4, we get that every subcubic graph admits an acyclic 5-edge-coloring and thus a 5-adst-coloring. Our result shows that this bound can be decreased to 4 when considering oriented graphs and acircuitic arc-colorings.

We also provided an oriented cubic graph with acircuitic directed star arboricity 4. However, we do not know any other example of a cubic oriented graph that does not admit a 3-adst-coloring.

From our result, we get that every oriented graph with maximum degree three admits a  $4 \cdot 2^{4-1} = 32$ -arc-coloring. However, every such graph admits an 11-vertex-coloring [10] and thus an 11-arc-coloring.

In a companion paper [7] we show that every  $K_4$ -minor free oriented graph  $G$  has acircuitic directed star arboricity at most  $\min\{\Delta(G), \Delta^-(G) + 2\}$ , where  $\Delta^-(G)$  stands for the maximum indegree of  $G$ . This class of graphs contains in particular outerplanar graphs. It would thus be interesting to determine the acircuitic directed star arboricity of planar graphs.

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