Rainbow Connections in Digraphs

Elżbieta Sidorowicz[a] and Éric Sopena[b]

[a] Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Zielona Góra, Poland

e-mail: e.sidorowicz@wmie.uz.zgora.pl

[b] Univ. Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR5800, F-33400 Talence, France

e-mail: eric.sopena@labri.fr

December 8, 2017

Abstract

An arc-coloured path in a digraph is rainbow if its arcs have distinct colours. A vertex-coloured path is vertex rainbow if its internal vertices have distinct colours. A totally-coloured path is total rainbow if its arcs and internal vertices have distinct colours. An arc-coloured (resp. vertex-coloured, totally-coloured) digraph D is rainbow connected (resp. rainbow vertex-connected, total rainbow connected) if any two vertices of D are connected by a rainbow (resp. vertex rainbow, total rainbow) path. The rainbow connection number (resp. rainbow vertex-connection number, total rainbow connection number) of a digraph D is the smallest number of colours needed to make D rainbow connected (resp. rainbow vertex-connected, total rainbow connected).

In this paper, we study the rainbow connection, rainbow vertex-connection and total rainbow connection numbers of digraphs. We give some properties of these parameters and establish relations between them. The rainbow connection number and the rainbow vertex-connection number of a digraph D are both upper bounded by the order of D, while its total rainbow connection number is upper bounded by twice of its order. In particular, we prove that a digraph of order n has rainbow connection number n if and only if it is Hamiltonian and has three vertices with eccentricity $n-1$, that it has rainbow vertex-connection number n if and only if it has a Hamiltonian cycle C and three vertices with eccentricity $n-1$ such that no two of them are consecutive on C, and that it has total rainbow connection number $2n$ if and only if it has rainbow vertex-connection number n.

1 Introduction

We consider finite and simple graphs only, and refer to [5] for terminology and notation not defined here.

In an edge-coloured graph G, a path is said to be *rainbow* if it does not use two edges with the same colour. Then the graph G is said to be *rainbow connected* if any two vertices are connected by a rainbow path. The *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the smallest possible number of colours in a rainbow connected colouring of G.

The concept of rainbow connection in graphs was introduced by Chartrand, Johns, McKeon and Zhang in [8]. Since then, the rainbow connection number has attracted much attention. The rainbow connection number of some special graph classes was determined in [6, 12, 13] and the rainbow connection number of graphs with fixed minimum degree was considered in [6, 14, 21, 22]. Also, different other parameters similar to rainbow connection were introduced. Krivelevich and Yuster [14] introduced the concept of rainbow vertex-connection. Liu, Mestre and Sousa [20] proposed the concept of total rainbow connection. A vertex-coloured path is *vertex rainbow* if its internal vertices have distinct colours. A vertex-coloured graph G is *rainbow vertex-connected* if any two vertices are connected by a vertex rainbow path. In a totally-coloured graph G a path is *total rainbow* if its edges and internal vertices have distinct colours. A totally-coloured graph G is *total rainbow connected* if any two vertices are connected by a total rainbow path. The *rainbow vertex-connection number* (resp. *total rainbow connection number*) of a connected graph G, denoted by $\mathrm{rvc}(G)$ (resp. $\mathrm{trc}(G)$), is the smallest number of colors needed to make G rainbow vertex-connected (resp. total rainbow connected).

The computational complexity of rainbow connectivity, rainbow vertex-connectivity and total rainbow connectivity was studied in [4, 7, 9, 10, 15]. It was shown that computing the rainbow connection number, the rainbow vertex-connection number and the total rainbow connection number of an arbitrary graph is NP-hard. Moreover, it was proved that it is already NP-complete to decide whether $\mathrm{rc}(G) = 2$, or $\mathrm{rvc}(G) = 2$, or $\mathrm{trc}(G) = 3$. See [18] or [19] for a survey about these different parameters.

The notions of rainbow connection and strong rainbow connection readily extend to digraphs, using arc-colouring instead of edge-colourings and directed paths (simply called paths in this paper) instead of paths. Note that in order to have bounded rainbow connection number, a digraph must be strongly connected.

Let D be a digraph. A *k-vertex-colouring*, a *t-arc-colouring* and a *p-total-colouring* of D is a mapping $c : V(D) \to \{1, \ldots, k\}$, $\varphi : A(D) \to \{1, \ldots, t\}$, $f : V(D) \cup A(D) \to \{1, \ldots, p\}$, respectively. A *vertex-coloured* digraph, an *arc-coloured* digraph, a *totally-coloured* digraph, is then a pair (D, c), (D, φ), (D, f), respectively, where D is a digraph and c is a vertex-colouring, φ is an arc-colouring, and f is a total-colouring of D, respec-

tively.

A path P in (D, φ) is *rainbow* if no two arcs of P are coloured with the same colour. If any two vertices in an arc-coloured digraph (D, φ) are connected by a rainbow path, then (D, φ) is said to be *rainbow connected* (or, equivalently, φ is a *rainbow arc-colouring* of G).

A path P in a vertex-coloured digraph (D, c) is *vertex rainbow* if its internal vertices have distinct colours. A vertex-coloured digraph (D, c) is *rainbow vertex-connected* (or, equivalently, c is a *rainbow vertex-colouring* of D) if any two vertices in (D, c) are connected by a rainbow path.

A path P in a totally-coloured digraph (D, f) is *total rainbow* if its edges and internal vertices have distinct colours. A totally-coloured digraph (D, f) is *total rainbow connected* (or, equivalently, f is a *rainbow total-colouring* of D) if any two vertices in (D, f) are connected by a rainbow path.

The *rainbow connection number* (*rainbow vertex-connection number*, *total rainbow connection number*) of a strong digraph D, is the minimum number of colours in a rainbow colouring (a rainbow vertex-colouring, a rainbow total-colouring). The rainbow connection number, the rainbow vertex-connection number, the total rainbow connection number are denoted by $\overrightarrow{\mathrm{rc}}(D)$, $\overrightarrow{\mathrm{rvc}}(D)$ and $\overrightarrow{\mathrm{trc}}(D)$, respectively.

The study of rainbow connection in oriented graphs (that is, antisymmetric digraphs) was initiated by Dorbec, Schiermeyer and the authors in [11] and then studied by Alva-Samos and Montellano-Ballesteros in [1, 2, 3]. Lei, Li, Liu and Shi [16] introduced the rainbow vertex-connection of digraphs. The total rainbow connection was studied in [17]. The strong version of rainbow connection was considered in [1, 16, 17, 23].

In [11] it has been observed that the rainbow connection number of a digraph is upper bounded by its order and a characterization of all oriented graphs with rainbow connection number equal to their order has been given. However, this characterization does not lead to a polynomial time algorithm for the corresponding decision problem. In this paper we propose a new characterization of digraphs with rainbow connection number equal to their order. The obtained characterization shows that the problem of deciding whether a digraph of order n has rainbow connection number n can be solved in polynomial time. In contrast, Ananth, Nasre and Sarpatwar in [4] proved that the problem of deciding whether a digraph has rainbow connection number 2 is NP-complete.

Whereas it is easily observed that a graph G of order $n \geq 2$ has rainbow vertex-connection number at most $n-2$, the rainbow vertex-connection number of a digraph can be equal to its order. We prove that a digraph D of order n has rainbow vertex-connection number n if and only if it has a Hamiltonian cycle C and three vertices with eccentricity $n-1$ such that no two of them are consecutive vertices of C. For a digraph D of order n, the total rainbow connection number is at most $2n$ (see Section 3). We prove that a

digraph D of order n has total rainbow connection number $2n$ if and only if it has rainbow vertex-connection number n.

This paper is organized as follows. After some preliminaries given in the next section, we give in Section 3 some basic results about the rainbow vertex-connection number and the total rainbow connection number of digraphs and some relations between these two parameters. In Section 4 we give some sufficient conditions for a digraph of order n to have rainbow connection number less than n, rainbow vertex-connection number less than n and total rainbow connection number less than $2n$. Finally, in Section 5, we propose a characterization of all digraphs of order n with rainbow connection number n, with rainbow vertex-connection number n, and with total rainbow connection $2n$, respectively.

2 Preliminaries

For a given digraph D, we denote by $V(D)$ and $A(D)$ its set of vertices and set of arcs, respectively. Given an arc xy in D, we say that y is an *out-neighbour* of x, while x is an *in-neighbour* of y. Moreover, we call x the *tail* of xy and y the *head* of xy. We denote by $N_D^+(x)$ the set of out-neighbours of x in D, and by $N_D^-(x)$ the set of in-neighbours of x in D. The *out-degree* $\deg_D^+(x)$ of x in D is the cardinality of its out-neighbourhood, that is $\deg_D^+(x) = |N_D^+(x)|$, and the *in-degree* $\deg_D^-(x)$ of x in D is the cardinality of its in-neighbourhood, that is $\deg_D^-(x) = |N_D^-(x)|$.

Given two digraphs D_1 and D_2, not necessarily vertex disjoint, we denote by $D_1 \cup D_2$ the digraph with vertex set $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$ and arc set $A(D_1 \cup D_2) = A(D_1) \cup A(D_2)$.

A *path* of length $k \geq 1$ in a digraph D is a sequence $x_0 \dots x_k$ of vertices such that $x_i x_{i+1} \in A(D)$ for every i, $0 \leq i \leq k-1$. Such a path P, going from x_0 to x_k, is referred to as an *x_0x_k-path*. For every i, j, $0 \leq i < j \leq k$, we say that x_i *precedes* x_j *in* P. Any vertex in $V(P) \setminus \{x_0, x_k\}$ is an *internal vertex* of P. For every i, j, $0 \leq i < j \leq k$, $P[x_i, x_j]$ denotes the subpath of P induced by the vertices $x_i, \dots, x_j$. A path P is *elementary* if no vertex appears twice in P. An elementary path *induced by* a path Q is the unique elementary path P obtained from Q by repeatedly deleting cycles, that is replacing a sequence of the form $u_1 \dots u_k x v_1 \dots v_\ell x w_1 \dots w_m$ by $u_1 \dots u_k x w_1 \dots w_m$ as many times as necessary.

A digraph D is *strongly connected* (*strong* for short) if there exists an xy-path in D for every two vertices x and y. A *strong component* of a digraph D is a maximal strong induced subdigraph of D.

The *distance* from a vertex x to a vertex y in a digraph D, denoted by $\mathrm{dist}_D(x, y)$, is the length of a shortest xy-path in D (if there is no such path, we let $\mathrm{dist}_D(x, y) = \infty$).

The *diameter* of D, denoted by $\mathrm{diam}(D)$, is the maximum distance between any two vertices in D. Two vertices at distance $\mathrm{diam}(D)$ are *antipodal vertices*. The *eccentricity* of a vertex x in D, denoted by $\mathrm{ecc}_D(x)$, is the maximum distance from x to any other vertex of D.

A *cycle* of length $k \geq 1$ in a digraph D is a sequence $x_0 \dots x_k x_0$ of vertices such that $x_i x_{i+1} \in A(D)$ for every i, $0 \leq i \leq k$ (subscripts are taken modulo k). Since cycles are denoted similarly all along the paper, it is taken for granted that subscripts are always taken modulo the length of the cycle. By $C[x_i, x_j]$ we denote the $x_i x_j$-path contained in C (that is, $C[x_i, x_j] = x_i x_{i+1} \dots x_j$) and $|C[x_i, x_j]|$ stands for the length of this path.

Recall that a *Hamiltonian path* in a digraph D is a path P in D such that $V(P) = V(D)$, and that a *Hamiltonian cycle* in D is a cycle C in D such that $V(C) = V(D)$. A *Hamiltonian digraph* is then a digraph containing a Hamiltonian cycle.

Lemma 2.1 *If D is a digraph and P is a Hamiltonian path in D, then the vertices of any strong component of D induce a subpath of P.*

Proof. Suppose to the contrary that D_1 is a strong component of D such that $P[V(D_1)]$ is not connected, which implies, in particular, $2 \leq |V(D_1)| < |V(D)|$. Let $P = v_1 \dots v_n$, p be the smallest index such that $v_{p-1} \in V(D_1)$ and $v_p \notin V(D_1)$, and q be the smallest index such that $q > p$, $v_{q-1} \notin V(D_1)$ and $v_q \in V(D_1)$ (since $P[V(D_1)]$ is not connected, such indices p and q necessarily exist). Thus the internal vertices of the path $P' = v_{p-1} v_p \dots v_q$ are not in D_1. Consider the digraph $D_1 \cup P'$. Since D_1 is a strong digraph, it follows that $D_1 \cup P'$ is also strong, contradicting the assumption that D_1 is a strong component. □

Lemma 2.2 *Let D be a Hamiltonian digraph of order n with $\mathrm{diam}(D) = n - 1$. For every two vertices u and v of D with $\mathrm{dist}_D(u, v) = n - 1$, $vu \in A(D)$ and vu belongs to each Hamiltonian cycle of D.*

Proof. Suppose to the contrary that either $vu \notin A(D)$ or there is a Hamiltonian cycle C not containing vu. In both cases, this implies that the vertices v and u are not consecutive along the cycle C, and thus $\mathrm{dist}_D(u, v) \leq |C[u, v]| < n - 1$, a contradiction. □

Corollary 2.3 *If D is a Hamiltonian digraph, u and v two vertices of D with $\mathrm{dist}_D(u, v) = n - 1$, then $vu \in A(D)$ and the digraph $D - vu$ is not Hamiltonian.*

In [11], a property called Head-Tail-Property was introduced and used for characterizing oriented digraphs with rainbow connection number equal to their order. We will also use this definition and the following proposition in this paper.

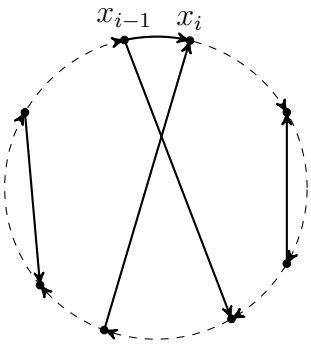


Figure 1: The vertex x_i has the Head-Tail-Property with respect to the cycle C.

Definition 2.4 (Head-Tail-Property) Let $C = x_0 \dots x_{k-1}x_0$ be a cycle in a digraph D. A vertex $x_i \in V(C)$ has the *Head-Tail-Property with respect to* C if, when going along the cycle C from x_i to x_{i-1}, we meet the head of each chord before its tail (see Figure 1 for an example). In particular, if D is itself a cycle, then every vertex of D has the Head-Tail-Property with respect to D.

Proposition 2.5 (Dorbec, Schiermeyer, Sidorowicz and Sopena [11]) *Let D be a digraph and $C = x_0 \dots x_{n-1}x_0$ be a Hamiltonian cycle of D. If a vertex x_i, $0 \leq i \leq n-1$, has the Head-Tail-Property with respect to C, then* $\mathrm{dist}_D(x_i, x_{i-1}) = n - 1$.

3 Some basic results

Recall that the *converse* D^c of a digraph D is obtained by reversing all the arcs of D, that is, $V(D^c) = V(D)$ and $A(D^c) = \{vu \mid uv \in A(D)\}$. Clearly, every rainbow vertex-colouring, rainbow arc-colouring or rainbow total-colouring of D is a rainbow vertex-colouring, rainbow arc-colouring or rainbow total-colouring of D^c, respectively.

Observation 3.1 *For every strong digraph D, $\overrightarrow{\mathrm{rc}}(D^c) = \overrightarrow{\mathrm{rc}}(D)$, $\overrightarrow{\mathrm{rvc}}(D^c) = \overrightarrow{\mathrm{rvc}}(D)$ and $\overrightarrow{\mathrm{trc}}(D^c) = \overrightarrow{\mathrm{trc}}(D)$.*

Note that in a rainbow connected digraph D, there must be a path with at least $\mathrm{diam}(D)$ colours between antipodal vertices. In [11], it was proved that the rainbow connection number of every strong digraph is upper bounded by its order. Similarly, in a rainbow vertex-connected digraph D, there must be a path with at least $\mathrm{diam}(D) - 1$

colours and the vertex-colouring that assigns different colours to each vertex of D is clearly a rainbow vertex-colouring. We thus have the following proposition.

Proposition 3.2 *For every strong digraph D,*

$$\mathrm{diam}(D) \leq \overrightarrow{\mathrm{rc}}(D) \leq n \quad \textit{and} \quad \mathrm{diam}(D) - 1 \leq \overrightarrow{\mathrm{rvc}}(D) \leq n.$$

Observe that these upper bounds are tight since $\overrightarrow{\mathrm{rc}}(C_n) = n$ and $\overrightarrow{\mathrm{rvc}}(C_n) = n$ for the cycle C_n of order n. In [17], it was proved that $\overrightarrow{\mathrm{rvc}}(D) = 0$ if and only if D is a symmetric complete digraph, and that $\overrightarrow{\mathrm{rvc}}(D) = 1$ if and only if $\mathrm{diam}(D) = 2$.

The next proposition deals with strong digraphs whose underlying undirected graph is not 2-connected.

Proposition 3.3 *If S is the set of cut vertices of a rainbow vertex-coloured digraph D, then all vertices of S are assigned distinct colours.*

Proof. Let u, u' be any two distinct vertices of S, $D_1, \ldots, D_p$ and $D'_1, \ldots, D'_q$ be the connected components (not necessarily strong) of $D - u$ and $D - u'$, respectively. Let D_i be a component that does not contain u', and D'_j be a component that does not contain u. Thus each path that joins a vertex of D_i to a vertex of D'_j must contain both vertices u and u'. Therefore, in any rainbow vertex-colouring of D, vertices u and u' must get different colours. □

Using Proposition 3.3, we can easily construct a digraph with rainbow vertex-connection number 2. Indeed, let K^1, K^2 and K^3 be three vertex-disjoint symmetric complete digraphs of order at least 2. Let $v_1 \in V(K^1)$, v_2, $v'_2 \in V(K^2)$ and $v_3 \in V(K^3)$. Let D be the strong digraph obtained by identifying v_1 with v_2, and v_3 with v'_2. In any rainbow vertex-colouring c of D, vertices v_2 and v'_2 must be coloured differently, while all other vertices can get either the colour $c(v_2)$ or $c(v'_2)$. Thus $\overrightarrow{\mathrm{rvc}}(D) = 2$.

By Proposition 3.3, each digraph with rainbow vertex-connection number 2 has at most two cut vertices. However, 2-connected digraphs might also have rainbow vertex-connection number 2.

Proposition 3.4 *If D is a strong bipartite digraph with $\mathrm{diam}(D) = 3$, then $\overrightarrow{\mathrm{rvc}}(D) = 2$.*

Proof. Let (V_1, V_2) be a vertex partition of $V(D)$. Let us colour the vertices of V_1 with colour 1 and the vertices of V_2 with colour 2. Each path of length 1 or 2 is clearly rainbow. Each path of length 3 has two internal vertices whose colours are obviously 1 and 2. Since

$\mathrm{diam}(D) = 3$, the proposed colouring is a rainbow vertex-colouring of D and the result follows. □

However, deciding whether a strong digraph has rainbow vertex-connection number 2 seems not so easy and we thus propose the following problem.

Problem 1 *Characterize strong digraphs with rainbow vertex-connection number* 2.

The rainbow connection number and the rainbow vertex-connection number of a digraph D are lower bounded by $\mathrm{diam}(D)$ and $\mathrm{diam}(D) - 1$, respectively. However there is no upper bound on these two parameters which is a function of $\mathrm{diam}(D)$. To see that, let $C_3^1, \ldots, C_3^s$ be s vertex-disjoint directed 3-cycles with $V(C_3^i) = \{x_i, y_i, z_i\}$ and $A(C_3^i) = \{x_iy_i, y_iz_i, z_ix_i\}$ for every i, $1 \leq i \leq s$. Let D be the digraph obtained by identification all vertices x_i, $1 \leq i \leq s$. We clearly have $\mathrm{diam}(D) = 4$. On the other hand, all arcs y_iz_i, $1 \leq i \leq s$ must get distinct colours in any rainbow colouring of D since any two such arcs, say y_az_a and y_bz_b, belong to any path from y_a to z_b. This gives $\overrightarrow{\mathrm{rc}}(D) \geq s$. Now, let K_s be the symmetric complete digraph of order n and let $V(K_s) = \{v_1, \ldots, v_s\}$. Let D' be the digraph obtained from K_s by adding s new vertices $w_1, \ldots w_s$ together with arcs v_iw_i and w_iv_i for every i, $1 \leq i \leq s$. We clearly have $\mathrm{diam}(D) = 3$ and, by Proposition 3.3, $\overrightarrow{\mathrm{rvc}}(D) \geq s$ since $\{v_1, \ldots, v_s\}$ is a set of s cut vertices in D'.

The next theorem shows that the rainbow vertex-connection number of 2-connected digraphs cannot be bounded by a the function of the diameter.

Theorem 3.5 *For every integers d and k, $4 \leq d \leq k$, there exists a 2-connected strong oriented graph $D_{d,k}$ with* $\mathrm{diam}(D_{d,k}) = d$ *and* $\overrightarrow{\mathrm{rvc}}(D_{d,k}) \geq k$.

Proof. We first consider the case $5 \leq d \leq k$ and let $t = d - 5$. Let $D_{d,k}$ be the oriented graph with vertex set

$$V(D_{d,k}) = \{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_k\} \cup \{z_1^1, \ldots, z_1^{1+t}\} \cup \{z_2, \ldots, z_k\},$$

and set of arcs

$$A(D_{d,k}) = \{x_1z_1^1, z_1^1z_1^2, \ldots, z_1^{1+t}y_1\} \cup \{x_iz_i, z_iy_i \mid 2 \leq i \leq k\}$$

$$\cup \{y_ix_j \mid 1 \leq i, j \leq k,\ i \neq j\}.$$

Hence, each vertex x_i, $2 \leq i \leq k$, is joined to y_i by a 2-path $x_iz_iy_i$, the vertex x_1 is joined to y_1 by a $(t+2)$-path $x_1z_1^1 \ldots z_1^{1+t}y_1$ and any two vertices x_i and y_j, $1 \leq i, j \leq k$, $i \neq j$, are linked by the arc y_jx_i.

We first claim that $\mathrm{diam}(D_{d,k}) = t + 5 = d$. Indeed, the unique shortest path from x_1 to y_j, $2 \leq j \leq k$, is $x_1 z_1^1 \dots z_1^{1+t} y_1 x_j z_j y_j$, of length $t + 5$, and thus $\mathrm{ecc}_{D_{d,k}}(x_1) = t + 5$. Moreover, any two vertices x_i and x_j, $1 \leq i, j \leq k$, $i \neq j$, are linked by a path of length 3 if $i > 1$, or of length $t + 3$ if $i = 1$. Similarly, any two vertices y_i and y_j, $1 \leq i, j \leq k$, $i \neq j$, are linked by a path of length 3 if $j > 1$, or of length $t + 3$ if $j = 1$. Any other two vertices are linked by a path of length at most $t + 2$.

Observe now that the $t + 2$ vertices $z_1^1, \dots, z_1^{1+t}, z_j$ lie on every $x_1 y_j$-path. Moreover, any two vertices z_i and z_j, $2 \leq i < j \leq k$, clearly lie on every $x_i y_j$-path. Therefore, the $k + t$ vertices $z_1^1, \dots, z_1^{1+t}, z_2, \dots, z_k$ must be assigned distinct colours in any rainbow vertex-colouring of $D_{d,k}$, which implies $\overrightarrow{\mathrm{rvc}}(D_{d,k}) \geq k + t \geq k$.

We finally consider the case $d = 4$. Let $D_{4,k}$, $k \geq 4$, be the oriented graph with vertex set

$$V(D_{4,k}) = \{x_1, \dots, x_k\} \cup \{y_1, \dots, y_k\} \cup \{z_1, \dots, z_k\},$$

and set of arcs

$$A(D_{4,k}) = \{x_i z_i, z_i y_i \mid 1 \leq i \leq k\} \cup \{z_i x_j, y_i x_j \mid 1 \leq i, j \leq k,\ i \neq j\}.$$

We clearly have $\mathrm{diam}(D_{4,k}) = 4$. Moreover, since every two vertices z_i and z_j, $i \neq j$, lie on every $x_i y_j$-path, the k vertices $z_1, \dots, z_k$ must be assigned distinct colours in any rainbow vertex-colouring of $D_{4,k}$, which implies $\overrightarrow{\mathrm{rvc}}(D_{4,k}) \geq k$. □

Theorem 3.5 provides examples of 2-connected oriented graphs with large rainbow vertex-connection number and given diameter at least 4. The situation for diameter 3 is less clear and we propose the following problem.

Problem 2 *Does there exist a constant t such that, for every 2-connected strong oriented graph (or digraph) D with* $\mathrm{diam}(D) = 3$, $\overrightarrow{\mathrm{rvc}}(D) \leq t$*?*

In a rainbow total-coloring, any pair of vertices is joined by a total rainbow path. Since each total rainbow path is also rainbow and vertex rainbow, both the rainbow connection number and the rainbow vertex-connection number are upper bounded by the total rainbow connection number.

Proposition 3.6 *For every strong digraph D,*

$$\overrightarrow{\mathrm{trc}}(D) \geq \max\{\overrightarrow{\mathrm{rc}}(D), \overrightarrow{\mathrm{rvc}}(D)\}.$$

On the other hand, an upper bound on the total rainbow connection number of a strong digraph is the following.

Lemma 3.7 *For every strong digraph D of order n,*

$$\overrightarrow{\mathrm{trc}}(D) \leq n + \min\{\overrightarrow{\mathrm{rc}}(D), \overrightarrow{\mathrm{rvc}}(D)\}.$$

Proof. Suppose first that $\min\{\overrightarrow{\mathrm{rc}}(D), \overrightarrow{\mathrm{rvc}}(D)\} = \overrightarrow{\mathrm{rc}}(D) = k$. Let φ be a rainbow k-arc-colouring of D and c be a n-vertex-colouring of D such that no two vertices are assigned the same colour. Consider the total-colouring f of D given by $f(v) = c(v)$ for every vertex v and $f(uv) = \varphi(uv)$ for every arc uv. Obviously every elementary path in (D, f) is vertex rainbow. Since φ is a rainbow colouring, there exists a total rainbow path between any two vertices and thus f is a total rainbow colouring of D, which implies $\overrightarrow{\mathrm{trc}}(D) \leq n + k$.

Suppose now that $\min\{\overrightarrow{\mathrm{rc}}(D), \overrightarrow{\mathrm{rvc}}(D)\} = \overrightarrow{\mathrm{rvc}}(D) = t$ and let $V(D) = \{v_1, \dots v_n\}$. Let c be a rainbow t-vertex-colouring of D, φ be the n-arc-colouring of D given by $\varphi(x_i x_j) = i$ for every arc $x_i x_j$, and f be the total-colouring of D defined as above. Again, every elementary path in (G, f) is total rainbow, and thus $\overrightarrow{\mathrm{trc}}(D) \leq n + t$. □

From Proposition 3.2 and Lemma 3.7, we get the following result.

Corollary 3.8 *Let D be a strong digraph with n vertices. Then*

$$\max\{\overrightarrow{\mathrm{rc}}(D), \overrightarrow{\mathrm{rvc}}(D)\} \leq \overrightarrow{\mathrm{trc}}(D) \leq 2n.$$

For many strong digraphs D, $\overrightarrow{\mathrm{rc}}(D) + \overrightarrow{\mathrm{rvc}}(D)$ colours are enough to make D total rainbow connected. The next theorem gives an example of a strong digraph that needs more colours for any total rainbow colouring.

Theorem 3.9 *There exists a strong digraph D with $\overrightarrow{\mathrm{trc}}(D) > \overrightarrow{\mathrm{rc}}(D) + \overrightarrow{\mathrm{rvc}}(D)$.*

Proof. Let T_k, $k \geq 7$, be the tournament with vertex set

$$V(T_k) = \{x_1, \dots, x_k\} \cup \{y_1, \dots, y_k\},$$

and arc set $A(T_k) = A_k^1 \cup A_k^2 \cup A_k^3 \cup A_k^4$, with

$$A_k^1 = \{x_i x_j \mid 1 \leq i < j \leq k\},$$

$$A_k^2 = \{y_i y_j \mid 1 \leq i < j \leq k\},$$

$$A_k^3 = \{y_i x_j \mid 1 \leq i, j \leq k,\ i \neq j\}, \text{ and}$$

$$A_k^4 = \{x_i y_i \mid 1 \leq i \leq k\}.$$

Therefore, both sets $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ induce a transitive subtournament of T_k, and all arcs between X and Y are directed from Y to X except for the perfect matching $\{x_iy_i \mid 1 \leq i \leq k\}$.

Observe that $\mathrm{dist}_{T_k}(x_i, y_j) = 3$ for $i > j$ and any two other vertices are at distance either 1 or 2 from each other. Thus $\mathrm{diam}(T_k) = 3$. If we colour each vertex of X with colour 1 and each vertex of Y with colour 2, then we clearly obtain a rainbow 2-vertex-colouring of T_k, which implies $\overrightarrow{\mathrm{rvc}}(T_k) = 2$.

Consider now the 4-arc-colouring φ given by (i) $\varphi(e) = 1$ for every $e \in A_k^1 \cup A_k^2$, (ii) $\varphi(e) = 2$ for every $e \in A_k^3$, and (iii) $\varphi(x_1y_1) = 3$ and $\varphi(x_iy_i) = 4$ for every i, $2 \leq i \leq k$ (arcs in A_k^4). For every two vertices a and b in T_k with $ab \notin A(T_k)$, there exists a rainbow ab-path $P_{a,b}$ in (T_k, φ) given as follows:

- if $(a, b) = (x_j, x_i)$, $1 \leq i < j \leq k$, then $P_{a,b} = x_jy_jx_i$,
- if $(a, b) = (y_j, y_i)$, $1 \leq i < j \leq k$, then $P_{a,b} = y_jx_iy_i$,
- if $(a, b) = (y_1, x_1)$, then $P_{a,b} = y_1y_2x_1$,
- if $(a, b) = (y_i, x_i)$, $2 \leq i \leq k$, then $P_{a,b} = y_ix_{i-1}x_i$,
- if $(a, b) = (x_1, y_j)$, $2 \leq j \leq k$, then $P_{a,b} = x_1y_1y_j$,
- if $(a, b) = (x_i, y_j)$, $2 \leq i, j \leq k$, $i \neq j$, then $P_{a,b} = x_iy_ix_1y_1y_j$.

Thus (T_k, φ) is rainbow connected, which implies $\overrightarrow{\mathrm{rc}}(T_k) \leq 4$.

We now claim that the total rainbow connection number of T_k is at least 7. Let f be a total rainbow colouring of T_k using $\overrightarrow{\mathrm{trc}}(T_k) = t$ colours. Suppose first that there are two arcs x_iy_i and x_jy_j, $i < j$, with $f(x_iy_i) = f(x_jy_j)$. By our assumption there exists a total-rainbow x_jy_i-path P. Since both arcs x_iy_i and x_jy_j have the same colour, P contains at most one of them. However, each such path has at least four arcs, so that we need at least 7 colours to colour P. On the other hand, if no two arcs from X to Y are assigned the same colour, then $t \geq k \geq 7$.

Therefore, for every $k \geq 7$, $\overrightarrow{\mathrm{trc}}(T_k) > \overrightarrow{\mathrm{rc}}(T_k) + \overrightarrow{\mathrm{rvc}}(T_k)$. □

Lei, Liu, Magnant and Shi [17] asked whether there exists an infinite family of digraphs D such that $\overrightarrow{\mathrm{trc}}(D)$ is unbounded, while $\max\{\overrightarrow{\mathrm{rc}}(D), \overrightarrow{\mathrm{rvc}}(D)\}$ is bounded.

Problem 3 *For any integer d, does there exist an infinite family of digraphs with $\overrightarrow{\mathrm{trc}}(D) - (\overrightarrow{\mathrm{rc}}(D) + \overrightarrow{\mathrm{rvc}}(D)) > d$?*

Finally, the following proposition directly follows from the definitions.

Proposition 3.10 *If D' is a strong spanning subdigraph of a digraph D, then*

$$\overrightarrow{\mathrm{rc}}(D) \leq \overrightarrow{\mathrm{rc}}(D'),\ \overrightarrow{\mathrm{rvc}}(D) \leq \overrightarrow{\mathrm{rvc}}(D'),\ \textit{and}\ \overrightarrow{\mathrm{trc}}(D) \leq \overrightarrow{\mathrm{trc}}(D').$$

4 Digraphs with non-maximum connection numbers

In this section, we give several sufficient conditions for a digraph D of order n to satisfy the inequalities $\overrightarrow{\mathrm{rc}}(D) < n$, $\overrightarrow{\mathrm{rvc}}(D) < n$ and $\overrightarrow{\mathrm{trc}}(D) < 2n$. We first define two useful properties of pairs of distinct vertices in a strong digraph.

Definition 4.1 (Vertex-Rainbow-In-Property) Let D be a strong digraph. Two distinct vertices $u, v \in V(D)$ have the *Vertex-Rainbow-In-Property* if

(i) for any two vertices $a, b \in V(D) \setminus \{u, v\}$, there is an ab-path that does not contain both u and v,

(ii) for any vertex $a \in V(D)\setminus\{u, v\}$, there is an au-path not containing v and an av-path not containing u.

Definition 4.2 (Vertex-Rainbow-Out-Property) Let D be a strong digraph. Two distinct vertices $u, v \in V(D)$ have the *Vertex-Rainbow-Out-Property* if

(i) for any two vertices $a, b \in V(D) \setminus \{u, v\}$, there is an ab-path that does not contain both u and v,

(ii) for any vertex $b \in V(D) \setminus \{u, v\}$, there is a ub-path not containing v and a vb-path not containing u.

Note here that two vertices u and v have the Vertex-Rainbow-In-Property (resp. the Vertex-Rainbow-Out-Property) in D if and only if u and v have the Vertex-Rainbow-Out-Property (resp. the Vertex-Rainbow-In-Property) in the converse D^c of D.

The two following lemmas prove that each of these properties provides a sufficient condition for a strong digraph D to satisfy the inequalities $\overrightarrow{\mathrm{rc}}(D) < n$, $\overrightarrow{\mathrm{rvc}}(D) < n$ and $\overrightarrow{\mathrm{trc}}(D) < 2n$.

Lemma 4.3 *If D is a strong digraph of order n which contains two vertices having the Vertex-Rainbow-In-Property, then $\overrightarrow{\mathrm{rvc}}(D) \leq n-1$, $\overrightarrow{\mathrm{rc}}(D) \leq n-1$ and $\overrightarrow{\mathrm{trc}}(D) \leq 2n-2$.*

Proof. Let $V(D) = \{x_1, x_2, \dots x_n\}$ and suppose that x_p and x_q, $1 \leq p < q \leq n$, have the Vertex-Rainbow-In-Property. Let c be the $(n-1)$-vertex-colouring of D defined by $c(x_i) = i$ for every i, $1 \leq i < q$, $c(x_q) = p$, and $c(x_i) = i - 1$ for every i, $q < i \leq n$, so that both vertices x_p and x_q are assigned the colour p and all other vertices are assigned distinct colours. Since x_p and x_q have the Vertex-Rainbow-In-Property, c is a rainbow $(n-1)$-vertex-colouring of D, and thus $\overrightarrow{\text{rvc}}(D) \leq n-1$.

Let now φ be the $(n-1)$-arc-colouring of D defined by $\varphi(x_ix_j) = c(x_j)$ for every arc x_ix_j in $A(D)$. Observe that the only colour that can appear more than once on an elementary path P is p. In such a case, both x_p and x_q are internal vertices of P, or P is an x_ax_b-path with $x_b \in \{x_p, x_q\}$. However, since x_p and x_q have the Vertex-Rainbow-In-Property, there is an x_ax_b-path not containing x_p or not containing x_q so that this path is rainbow. Therefore, (D, φ) is rainbow connected and thus $\overrightarrow{\text{rc}}(D) \leq n-1$.

Finally, the $(2n-2)$-total-colouring f of D defined by $f(x_i) = c(x_i)$ for every $x_i \in V(D)$ and $f(x_ix_j) = n-1+\varphi(x_ix_j)$ for every $x_ix_j \in A(D)$ is obviously a rainbow total-colouring of D, so that $\overrightarrow{\text{trc}}(D) \leq 2n-2$. □

Lemma 4.4 *If D is a strong digraph of order n which contains two vertices having the Vertex-Rainbow-Out-Property, then $\overrightarrow{\text{rvc}}(D) \leq n-1$, $\overrightarrow{\text{rc}}(D) \leq n-1$ and $\overrightarrow{\text{trc}}(D) \leq 2n-2$.*

Proof. This directly follows from Observation 3.1 and Lemma 4.3 since any two vertices having the Vertex-Rainbow-Out-Property in D have the Vertex-Rainbow-In-Property in D^c. □

We now provide a sufficient condition for a Hamiltonian digraph to contain two vertices having the Vertex-Rainbow-In-Property.

Lemma 4.5 *Let D be a Hamiltonian digraph of order n, $C = x_0x_1 \dots x_{n-1}x_0$ a Hamiltonian cycle of D, and x_p, x_q, $0 \leq p, q \leq n-1$, two vertices with $x_p \neq x_q$ (we may have $p = q+1$ or $q = p+1$). If there is an x_px_{q+1}-path not containing x_{p+1} and there is an x_qx_{p+1}-path not containing x_{q+1}, then the vertices x_{p+1} and x_{q+1} have the Vertex-Rainbow-In-Property (subscripts are taken modulo n).*

Proof. Let P_p be an x_px_{q+1}-path not containing x_{p+1} and P_q be an x_qx_{p+1}-path not containing x_{q+1}. We first prove that for any two vertices $x_a, x_b \notin \{x_{p+1}, x_{q+1}\}$, there exists an x_ax_b-path that does not contain both x_{p+1} and x_{q+1}. We may assume that either $x_a, x_b \in C[x_{p+2}, x_q]$ and x_b precedes x_a, or $x_a, x_b \in C[x_{q+2}, x_p]$ and x_b precedes x_a, since otherwise the x_ax_b-subpath of C does not contain both x_{p+1} and x_{q+1}. In the former case, the path $C[x_a, x_q] \cup P_q \cup C[x_{p+1}x_b]$ does not contain x_{q+1}. In the latter case, the path $C[x_a, x_p] \cup P_p \cup C[x_{q+1}x_b]$ does not contain x_{p+1}.

We now prove that for any $x_a \notin \{x_{p+1}, x_{q+1}\}$, there is an $x_a x_{p+1}$-path not containing x_{q+1} and an $x_a x_{q+1}$-path not containing x_{p+1}. If $x_a \in C[x_{p+2}, x_q]$, then $C[x_a, x_{q+1}]$ is an $x_a x_{q+1}$-path not containing x_{p+1}, and $C[x_a, x_q] \cup P_q$ is an $x_a x_{p+1}$-path not containing x_{q+1}. On the other hand, if $x_a \in C[x_{q+2}, x_p]$, then $C[x_a, x_{p+1}]$ and $C[x_a, x_p] \cup P_p$ are the required paths. □

In [11], it was proved that an oriented graph whose rainbow connection number equals its order must be Hamiltonian. We will prove that this result can be extended to all digraphs, and that it also holds for the rainbow vertex-connection number. Moreover, we will prove that the total rainbow connection number of a non-Hamiltonian strong digraph of order n is at most $2n - 2$.

We first prove these results for digraphs with diameter $n - 1$.

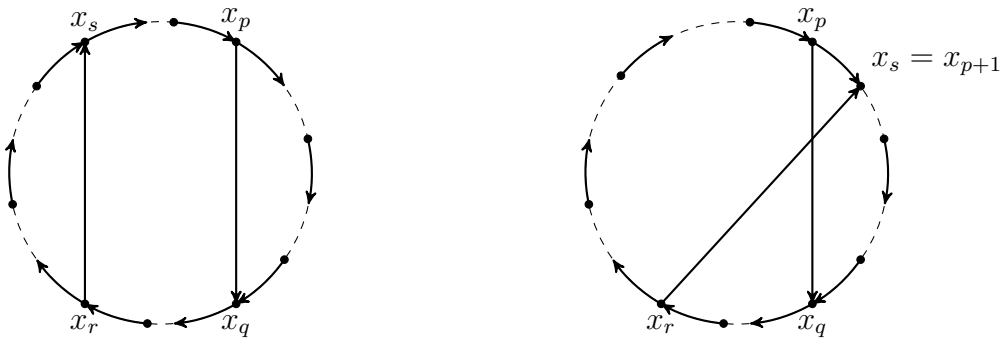
Lemma 4.6 *Let D be a non-Hamiltonian strong digraph of order n. If $\mathrm{diam}(D) = n-1$, then $\overrightarrow{\mathrm{rvc}}(D) \leq n - 1$, $\overrightarrow{\mathrm{rc}}(D) \leq n - 1$, and $\overrightarrow{\mathrm{trc}}(D) \leq 2n - 2$.*

Proof. Let D be a non-Hamiltonian strong digraph with $\mathrm{diam}(D) = n - 1$. Let v_1 and v_n be two antipodal vertices and $P = v_1 v_2 \dots v_n$ be a $v_1 v_n$-path of length $n - 1$. Since P is a shortest path, all arcs of D that are not in P are of the form $v_i v_j$ with $1 \leq j < i \leq n$. Furthermore, $v_n v_1 \notin A(G)$ since D is non-Hamiltonian. Let Q be a shortest $v_n v_1$-path and $v_n v_i$ be the first arc in Q.

Suppose first that there is a vertex v_k such that $v_k \notin Q$, $v_{k+1} \in Q$ and $k \neq n - 1$. We claim that the vertices v_k and v_n have the Vertex-Rainbow-Out-Property. We first prove that for any two vertices $v_a, v_b \in V(D) \setminus \{v_k, v_n\}$, there is a $v_a v_b$-path that does not contain both v_k and v_n. Observe that if v_a precedes v_b in P, then $P[v_a, v_b]$ does not contain v_n. Otherwise, we consider two cases, $k < a < n$ and $1 < a < k$. In the former case, there is a $v_a v_b$-path not containing v_k. Indeed, if $1 \leq b \leq k - 1$, then $P[v_a, v_n] \cup Q \cup P[v_1, v_b]$ is the required path, and if $k + 1 \leq b < a$, then $P[v_a, v_n] \cup Q[v_n, v_{k+1}] \cup P[v_{k+1}, v_b]$ is the required path. In the latter case, we necessarily have $1 \leq b < a$, and thus $P[v_a, v_{k+1}] \cup Q[v_{k+1}, v_1] \cup P[v_1, v_b]$ (or $P[v_a, v_{k+1}] \cup Q[v_{k+1}, v_1]$ if $v_b = v_1$) does not contain v_n.

Now, let $v_b \in V(D) \setminus \{v_k, v_n\}$. If $1 \leq b \leq k - 1$, then $Q \cup P[v_1, v_b]$ is a $v_n v_b$-path not containing v_k, and $v_k v_{k+1} \cup Q[v_{k+1}, v_1] \cup P[v_1, v_b]$ is a $v_k v_b$-path not containing v_n. Thus the vertices v_k and v_n have the Vertex-Rainbow-Out-Property and, thanks to Lemma 4.4, we get $\overrightarrow{\mathrm{rvc}}(D) \leq n - 1$, $\overrightarrow{\mathrm{rc}}(D) \leq n - 1$ and $\overrightarrow{\mathrm{trc}}(D) \leq 2n - 2$.

Suppose now that there is no vertex v_k such that $v_k \notin Q$, $v_{k+1} \in Q$ and $k \neq n - 1$. This implies that Q contains all vertices $v_1, \dots, v_i$ (recall that $v_n v_i$ is the first arc in Q). In this case, we claim that the vertices v_1 and v_n have the Vertex-Rainbow-Out-Property. Let $v_a, v_b \in V(D) \setminus \{v_1, v_n\}$. If v_a precedes v_b, then $P[v_a, v_b]$ contains neither v_1 nor v_n.



(a) (b)

Figure 2: (a) A pair of opposite chords (we may have $x_s = x_p$ or $x_r = x_q$). (b) A pair of almost opposite chords (we may have $x_r = x_q$ or $x_r = x_{p-1}$).

Otherwise, the $v_a v_b$-path $P[v_a, v_n] \cup Q[v_n, v_2] \cup P[v_2, v_b]$ does not contain v_1. Moreover, $P[v_1, v_b]$ is a $v_1 v_b$-path that does not contain v_n. If $2 \leq b \leq i-1$, then $Q[v_n, v_b]$ is a $v_n v_b$-path not containing v_1. Otherwise, $v_n v_i \cup P[v_i, v_b]$ is a $v_n v_b$-path not containing v_1. Thus the vertices v_1 and v_n have the Vertex-Rainbow-Out-Property and, thanks to Lemma 4.4, again we get $\overrightarrow{\text{rvc}}(D) \leq n-1$, $\overrightarrow{\text{rc}}(D) \leq n-1$ and $\overrightarrow{\text{trc}}(D) \leq 2n-2$. □

It remains to consider the case of strong digraphs of order n with diameter at most $n-2$. We will thus consider the case of digraphs having a Hamiltonian cycle with some "special" chords. Let $C = x_0 \dots x_{n-1} x_0$ be a cycle of length n. A *chord* of C is an arc $x_p x_q$, $0 \leq p, q \leq n-1$, with $x_q \neq x_{p+1}$. A pair of chords $(x_p x_q, x_r x_s)$ is a *pair of opposite chords* if $x_s, x_r \in C[x_q, x_p]$ and $x_p, x_q \in C[x_s, x_r]$ (see Figure 2 (a)), and *a pair of almost opposite chords* if $s = p+1$ and $x_r \in C[x_q, x_{p-1}]$ (see Figure 2 (b)). A pair of chords $(x_p x_q, x_r x_s)$ is a *pair of crossing chords* if $x_s \in C[x_{p+2}, x_{q-1}]$ and $x_r \in C[x_{q+1}, x_{p-1}]$ (see Figure 3). A triple of chords $(x_p x_q, x_r x_s, x_t x_u)$ is a *triple of crossing chords* if $(x_p x_q, x_r x_s)$ is a pair of crossing chords, $x_t \in C[x_s, x_{q-1}]$ and $x_u \in C[x_{r+2}, x_{p+1}]$ (see Figure 4).

Lemma 4.7 *If D is a Hamiltonian digraph of order n and C a Hamiltonian cycle of D having a pair of opposite chords, then $\overrightarrow{\text{rvc}}(D) \leq n-1$, $\overrightarrow{\text{rc}}(D) \leq n-1$ and $\overrightarrow{\text{trc}}(D) \leq 2n-2$.*

Proof. Let $C = x_0 \dots x_{n-1} x_0$, and $(x_p x_q, x_r x_s)$ be a pair of opposite chords of C. Since $x_r x_s \cup C[x_s, x_{p+1}]$ is an $x_r x_{p+1}$-path not containing x_{r+1} and $x_p x_q \cup C[x_q, x_{r+1}]$ is an $x_p x_{r+1}$-path not containing x_{p+1}, we get by Lemma 4.5 that the vertices x_{r+1} and x_{p+1} have the Vertex-Rainbow-In-Property. The result then follows by Lemma 4.3. □

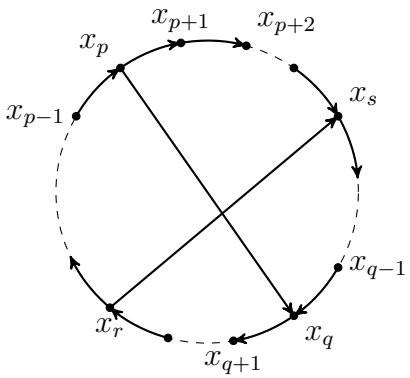


Figure 3: A pair of crossing chords (we may have $x_s = x_{p+2}$, or $x_s = x_{q-1}$, or $x_r = x_{q+1}$, or $x_s = x_{p-1}$).

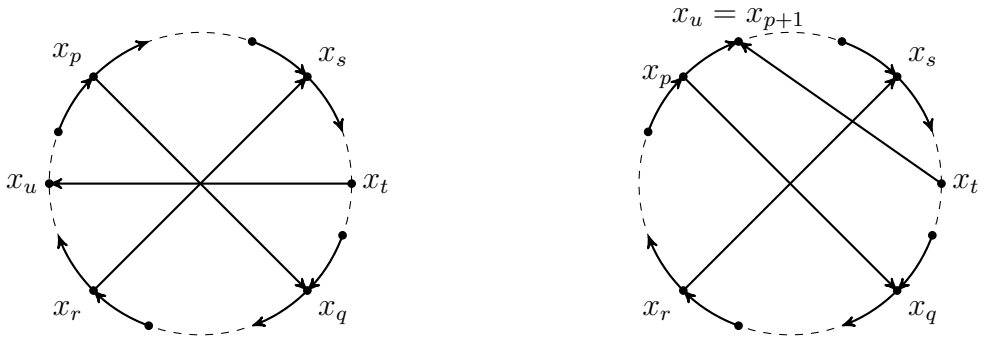


Figure 4: A triple of crossing chords (we may have $x_u = x_p$ or $x_u = x_{p+1}$).

Lemma 4.8 *If D is a Hamiltonian digraph of order n and C a Hamiltonian cycle of D having a pair of almost opposite chords, then $\overrightarrow{\mathrm{rvc}}(D) \leq n-1$, $\overrightarrow{\mathrm{rc}}(D) \leq n-1$ and $\overrightarrow{\mathrm{trc}}(D) \leq 2n-2$.*

Proof. Let $C = x_0 \dots x_{n-1}x_0$, and (x_px_q, x_rx_s) be a pair of almost opposite chords of C, so that $x_s = x_{p+1}$. Since the path $x_px_q \cup C[x_q, x_{r+1}]$ does not contain x_s, and x_rx_s is an x_rx_s-path not containing x_{r+1}, we get by Lemma 4.5 that the vertices x_{r+1} and $x_s = x_{p+1}$ have the Vertex-Rainbow-In-Property. The result then follows by Lemma 4.3. □

Lemma 4.9 *If D is a Hamiltonian digraph of order n and C a Hamiltonian cycle of D having a triple of crossing chords, then $\overrightarrow{\mathrm{rvc}}(D) \leq n-1$, $\overrightarrow{\mathrm{rc}}(D) \leq n-1$ and $\overrightarrow{\mathrm{trc}}(D) \leq 2n-2$.*

Proof. Let $C = x_0 \dots x_{n-1}x_0$, and (x_px_q, x_rx_s, x_tx_u) be a triple of crossing chords of C. We claim that the vertices x_{r+1} and x_{p+1} have the Vertex-Rainbow-In-Property. Observe first that $x_px_q \cup C[x_q, x_{r+1}]$ is an x_px_{r+1}-path not containing x_{p+1}. Next $x_rx_s \cup C[x_s, x_t] \cup x_tx_u \cup C[x_u, x_{p+1}]$ is an x_rx_{p+1}-path not containing x_{r+1} (we put $C[x_s, x_t] = \emptyset$ while $x_t = x_s$ and $C[x_u, x_{p+1}] = \emptyset$ while $x_u = x_{p+1}$). Thus, we get by Lemma 4.5 that the vertices x_{r+1} and x_{p+1} have the Vertex-Rainbow-In-Property and the result follows by Lemma 4.3. □

We are now able to extend Lemma 4.6 to strong digraphs of order n with diameter at most $n-2$.

Lemma 4.10 *If D is a strong digraph of order n with $\mathrm{diam}(D) \leq n-2$, then $\overrightarrow{\mathrm{rvc}}(D) \leq n-1$, $\overrightarrow{\mathrm{rc}}(D) \leq n-1$ and $\overrightarrow{\mathrm{trc}}(D) \leq 2n-2$.*

Proof. Let T be a BFS-tree of D with root r and L be the set of leaves of T. For every vertex v, we denote by B_v the *branch of* v, that is, the unique rv-path in T. We say that a vertex u *precedes* v *in* T, denoted $u \preceq_T v$, if $u \in V(B_v)$. (In the literature, u is sometimes called an *ancestor* of v in T.)

Suppose first that ℓ_1 and ℓ_2 are two leaves of T such that there is an $\ell_1 r$-path P_{ℓ_1} in D not containing ℓ_2 and an $\ell_2 r$-path P_{ℓ_2} in D not containing ℓ_1. We claim that the vertices ℓ_1 and ℓ_2 have the Vertex-Rainbow-Out-Property. To see that, let $a, b \in V(D) \setminus \{\ell_1, \ell_2\}$ and Q be any ab-path in D. If Q does not contain both vertices ℓ_1 and ℓ_2 we are done. Suppose now that Q contains both vertices ℓ_1 and ℓ_2. We can assume without loss of generality that ℓ_1 precedes ℓ_2 in Q. In that case, the ab-path $Q[a, \ell_1] \cup P_{\ell_1} \cup B_b$ does not contain ℓ_2.

Furthermore, the paths $P_{\ell_1} \cup B_b$ and $P_{\ell_2} \cup B_b$ do not contain ℓ_2 and ℓ_1, respectively, and thus ℓ_1 and ℓ_2 have the Vertex-Rainbow-Out-Property. By Lemma 4.4, we get that

the inequalities $\overrightarrow{\text{rvc}}(D) \leq n-1$, $\overrightarrow{\text{rc}}(D) \leq n-1$ and $\overrightarrow{\text{trc}}(D) \leq 2n-2$ hold and we are done.

Suppose now that for any two leaves $\ell_1, \ell_2 \in L$, either every $\ell_1 r$-path contains ℓ_2 or every $\ell_2 r$-path contains ℓ_1. For every leaf $\ell \in L$, we denote by P_ℓ a shortest ℓr-path in D. Let $\leq_L$ be the partial order on L defined in the following way: for every $\ell_1, \ell_2 \in L$, $\ell_1 \leq_L \ell_2$ if and only if $\ell_2 \in P_{\ell_1}$. By our assumption, either $\ell_1 \leq_L \ell_2$ or $\ell_2 \leq_L \ell_1$ for any two leaves ℓ_1, ℓ_2, and thus $\leq_L$ is a linear ordering of L.

Let $L = \{\ell_1, \ldots, \ell_s\}$ with $\ell_i \leq_L \ell_j$ for every i, j, $1 \leq i < j \leq s$. Note that for every i, $1 \leq i \leq s$, P_{ℓ_i} contains all vertices $\ell_{i+1}, \ldots, \ell_s$, in that order. Moreover, $P_{\ell_j} \subseteq P_{\ell_i}$ for every i, j, $1 \leq i < j \leq s$. We then let $P = P_{\ell_1}$.

We consider two cases.

1. $V(B_{\ell_2} \cup \ldots \cup B_{\ell_s}) \setminus V(P) \neq \emptyset$.

   Since all leaves are in P, there exists a vertex $x \in V(B_{\ell_2} \cup \ldots \cup B_{\ell_s}) \setminus V(P)$ such that x has an out-neighbour x' with $x' \in V(P)$. We choose the vertex x in such a way that $\text{dist}_D(r, x)$ is maximum. Since every leaf ℓ_i with $x \preceq_T \ell_i$ is in P, the maximality of $\text{dist}_D(r, x)$ implies that every vertex y with $x \preceq_T y$ is in P.

   We claim that the vertices ℓ_1 and x have the Vertex-Rainbow-Out-Property. To see that, let $a, b \in V(D) \setminus \{\ell_1, x\}$.

   Let Q be any ab-path. If Q does not contain both ℓ_1 and x, we are done. So assume that Q contains both vertices ℓ_1 and x. Suppose first that ℓ_1 precedes x in Q. If $x \preceq_T b$, then b belongs to P, and thus $Q[a, \ell_1] \cup P[\ell_1, b]$ does not contain x. Otherwise, B_b does not contain x, and thus $Q[a, \ell_1] \cup P \cup B_b$ does not contain x. Suppose now that x precedes ℓ_1 in Q. Since the out-neighbour x' of x is in P and ℓ_1 is the first vertex of P, $P[x', r]$ does not contain ℓ_1. Furthermore, B_b also does not contain ℓ_1, and thus $Q[a, x] \cup xx' \cup P[x', r] \cup B_b$ does not contain ℓ_1. Hence, in all cases, we can find an ab-path that does not contain both vertices ℓ_1 and x.

   Now we need to show that there exists an $\ell_1 b$-path not containing x and an xb-path not containing ℓ_1. If $x \notin B_b$, then $P \cup B_b$ is an $\ell_1 b$-path not containing x. If $x \in B_b$, which implies $x \preceq_T b$ and thus $b \in P$, then $P[\ell_1, b]$ is an $\ell_1 b$-path not containing x. Finally, $xx' \cup P[x', r] \cup B_b$ is an xb-path not containing ℓ_1.

   Therefore, the vertices ℓ_1 and x have the Vertex-Rainbow-Out-Property and the theorem follows from Lemma 4.4.

2. $V(B_{\ell_2} \cup \ldots \cup B_{\ell_s}) \subseteq V(P)$.

   We consider two subcases, depending on whether the path P contains an internal vertex of B_{ℓ_1} or not.

(a) *P contains an internal vertex of B_{ℓ_1}.*

We consider two subcases.

i. *There exists an arc xx' in B_{ℓ_1} with $x \notin V(P)$, $x' \in V(P)$ and $x' \neq \ell_1$.*
In that case, we claim that the vertices x and ℓ_1 have the Vertex-Rainbow-Out-Property. Let $a, b \in V(D) \setminus \{x, \ell_1\}$. We first prove that there exists an ab-path not containing both x and ℓ_1.
Suppose first that a and b are internal vertices of B_{ℓ_1}. If $a \preceq_T b$, then $B_{\ell_1}[a, b]$ does not contain ℓ_1. Otherwise, we have three possible cases: $x \preceq_T b \preceq_T a$, $b \preceq_T x \preceq_T a$, or $b \preceq_T a \preceq_T x$. In the first case, $B_{\ell_1}[a, \ell_1] \cup P[\ell_1, x'] \cup B_b[x', b]$ does not contain x. In the second case, $B_{\ell_1}[a, \ell_1] \cup P[\ell_1, r] \cup B_b$ does not contain x. In the last case, $B_{\ell_1}[a, x'] \cup P[x', r] \cup B_b$ does not contain ℓ_1.
Suppose now that none of a and b is an internal vertex of B_{ℓ_1}, which implies $a, b \in V(P)$. If a precedes b in P, then $P[a, b]$ contains neither x nor ℓ_1. If b precedes a in P, then $P[a, r] \cup B_b$ does not contain ℓ_1.
Suppose finally that exactly one of the vertices a and b is an internal vertex of B_{ℓ_1}. If b is an internal vertex of B_{ℓ_1}, then $P[a, r] \cup B_b$ does not contain ℓ_1. If a is an internal vertex of B_{ℓ_1}, we consider two cases. If $x \leq_T a \leq_T \ell_1$, then $B_{\ell_1}[a, \ell_1] \cup P[\ell_1, b]$ does not contain x. If $r \leq_T a \leq_T x$, then an ab-path not containing ℓ_1 is $B_{\ell_1}[a, x'] \cup P[x', b]$ if x' precedes b in P, or $B_{\ell_1}[a, x'] \cup P[x', r] \cup B_b$ if b precedes x' in P.
Therefore, in each case, there exists an ab-path not containing both x and ℓ_1.

We now prove that condition (ii) of the Vertex-Rainbow-Out-Property is also satisfied. Suppose first that b belongs to B_{ℓ_1}. If $x \preceq_T b \preceq_T \ell_1$, then $B_{\ell_1}[x, b]$ is an xb-path not containing ℓ_1, and $P[\ell_1, x'] \cup B_{\ell_1}[x', b]$ is an $\ell_1 b$-path not containing x. If $r \preceq_T b \preceq_T x$, then $xx' \cup P[x', r] \cup B_b$ is an xb-path not containing ℓ_1, and $P \cup B_b$ is an $\ell_1 b$-path not containing x. Suppose finally that b belongs to $P \setminus B_{l_1}$. In that case, $P[\ell_1, b]$ is an $\ell_1 b$-path not containing x, and $xx' \cup P[x', r] \cup B_b$ is an xb-path not containing ℓ_1.

Therefore, the vertices x and ℓ_1 have the Vertex-Rainbow-Out-Property, and the theorem follows from Lemma 4.4.

ii. *For each arc xx' in B_{ℓ_1} with $x \notin V(P)$, if any, $x' \notin V(P) \setminus \{\ell_1\}$.*
In that case, for every vertex $x \in B_{\ell_1} \setminus P$, no vertex y with $x \preceq_T y$ and $y \neq \ell_1$ is in P. We thus have $B_{\ell_1} = r x_1 x_2 \dots x_p y_1 y_2 \dots y_q \ell_1$, with $x_i \in V(P)$ for every i, $1 \leq i \leq p$, and $y_j \notin V(P)$ for every j, $1 \leq j \leq q$. Observe

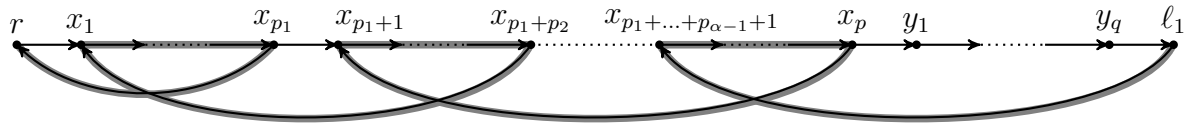


Figure 5: The subpaths $P_1, \ldots, P_\alpha$ induced by vertices $x_1, \ldots, x_p$.

that since P contains an internal vertex of B_{ℓ_1}, we necessarily have $p \geq 1$. Moreover, we also have $q \geq 1$, since otherwise the path P would contain all vertices of D, which would imply $\mathrm{diam}(D) = n - 1$, contradicting the assumption $\mathrm{diam}(D) \leq n - 2$.

We first claim that for every i, $1 \leq i \leq p-1$, if x_i precedes x_{i+1} in P, then $x_i x_{i+1} \in A(P)$. Indeed, if $x_i x_{i+1} \notin A(P)$, then replacing the subpath $P[x_i, x_{i+1}]$ of P by the arc $x_i x_{i+1}$ gives an $\ell_1 r$-path shorter than P, a contradiction. This implies that the vertices $x_1, \ldots, x_p$ induce α subpaths $P_1, \ldots, P_\alpha$ of P, $\alpha \geq 1$, of respective order $p_1, \ldots, p_\alpha \geq 1$, with $P_1 = x_1 \ldots x_{p_1}$, $\ldots$, $P_\alpha = x_{p_1+\cdots+p_{\alpha-1}} \ldots x_p$. Note that for every i, j, $1 \leq i < j \leq \alpha$, each vertex of P_j precedes each vertex of P_i in P (see Figure 5).

Consider now the spanning subdigraph D' of D which contains all the arcs of P and B_{ℓ_1}, that is, $V(D') = V(D)$ and $A(D') = A(P) \cup A(B_{\ell_1})$. Note that since D' is the union of an $\ell_1 r$-path and an $r\ell_1$-path, D' is a strong digraph.

Let v be the out-neighbour of x_{p_1} in D' with $x_{p_1} v \in A(P)$ (observe that $x_u \neq x_{p_1+1}$ and it may happen that $v = r$, or that $p_1 = p$). We claim that the vertices v and y_1 have the Vertex-Rainbow-In-Property in D'. To see that, let $a, b \in V(D') \setminus \{v, y_1\}$ and Q be any ab-path in D'. If Q does not contain both v and y_1 we are done. Otherwise, we consider two cases. Suppose first that v precedes y_1 in Q. Since $d^-_{D'}(v) = 1$, Q also contains x_{p_1}, and thus $Q = a \ldots x_{p_1} v \ldots y_1 \ldots b$. We then obtain an ab-path not containing v by replacing in Q the subpath $Q[x_{p_1}, y_1]$ by $B_{\ell_1}[x_{p_1}, y_1]$. Suppose now that y_1 precedes v in Q. Since $d^-_{D'}(y_1) = 1$, Q also contains x_p, and thus $Q = a \ldots x_p y_1 \ldots v \ldots b$. In that case, we obtain an ab-path not containing y_1 by replacing in Q the subpath $Q[x_p, v]$ by $P[x_p, v]$ (recall that either $x_p = x_{p_1}$ or x_p precedes x_{p_1} in P). Let now Q' be any av-path and Q'' be any ay_1-path. If Q' contains y_1, then it also contains x_p, so that we get an av-path not containing y_1 by replacing $Q'[x_p, v]$ by $P[x_p, v]$. On the other hand, if Q'' contains v, then it also contains x_{p_1}, so that we get

an ay_1-path not containing v by replacing $Q''[x_{p_1}, y_1]$ by $B_{\ell_1}[x_{p_1}, y_1]$. Therefore, the vertices v and y_1 have the Vertex-Rainbow-In-Property in D'. By Lemma 4.3, we get $\overrightarrow{\text{rvc}}(D') \leq n-1$, $\overrightarrow{\text{rc}}(D') \leq n-1$ and $\overrightarrow{\text{trc}}(D') \leq 2n-2$. The theorem then follows from Proposition 3.10.

(b) *P does not contain any internal vertex of B_{ℓ_1}.*

Observe first that since $\text{diam}(D) \leq n-2$ and P is a shortest $\ell_1 r$-path, B_{ℓ_1} contains at least one internal vertex. Similarly, since $\text{diam}(D) \leq n-2$ and B_{ℓ_1} is a shortest $r\ell_1$-path, P contains at least one internal vertex.

Let $P = x_1 x_2 \dots x_t$, with $x_1 = \ell_1$ and $x_t = r$, and $B_{\ell_1} = x_t x_{t+1} \dots x_n x_1$. We thus have $V(P) \cap V(B_{\ell_1}) = \{x_1, x_t\}$ and $V(P) \cup V(B_{\ell_1}) = V(D)$, which implies that $C = P \cup B_{\ell_1} = x_1 x_2 \dots x_n$ is a Hamiltonian cycle of D.

If C has either a pair of opposite chords, a pair of almost opposite chords, or a triple of crossing chords, then the result directly follows from Lemma 4.7, Lemma 4.8, or Lemma 4.9, respectively. Thus we may assume that none of these situations occurs.

We first claim that C has two chords $x_i x_j$ and $x_k x_\ell$, such that

(i) $x_i, x_\ell \in V(P)$, and

(ii) x_j and x_k are internal vertices of B_{ℓ_1}, and $j \leq k$.

Observe first that since B_{ℓ_1} is a shortest $r\ell_1$-path, every arc $x_a x_b \in A(D) \setminus A(B_{\ell_1})$ with $x_a, x_b \in V(B_{\ell_1})$ is such that either $a = 1$ and $t \leq b \leq n$, or $t \leq b < a \leq n$. Similarly, since P is a shortest $\ell_1 r$-path, every arc $x_a x_b \in A(D) \setminus A(P)$ with $x_a, x_b \in V(P)$ is such that $1 \leq b < a \leq t$. Furthermore, since both P and B_{ℓ_1} have internal vertices, $x_t x_1 \notin A(D)$ and $x_1 x_t \notin A(D)$.

Consider the vertices x_1 and x_n. Since $\text{dist}_D(x_1, x_n) \leq \text{diam}(D) \leq n-2$, there must be a chord $x_i x_j$ with $x_i \in \{x_1, \dots, x_{t-1}\}$ and $x_j \in \{x_{t+1}, \dots x_n\}$. Let $x_i x_j$ be such a chord which minimizes $\text{dist}_D(x_t, x_j)$.

Consider now the vertices x_t and x_{t+1}. Again, since $\text{dist}_D(x_{t+1}, x_t) \leq \text{diam}(D) \leq n-2$, there must be a chord $x_k x_\ell$ with $x_k \in \{x_{t+1}, \dots, x_n\}$ and $x_\ell \in \{x_2, \dots, x_t\}$. Let $x_k x_\ell$ be such a chord which minimizes $\text{dist}_D(x_k, x_1)$.

Clearly, the chords $x_i x_j$ and $x_k x_\ell$ satisfy condition (i) and x_j, x_k are internal vertices of B_{ℓ_1}. We now prove that $j \leq k$. Suppose to the contrary that $j > k$ and let x_a be any vertex with $k \leq a < j$. Since x_j and x_k have been chosen in such a way that $\text{dist}_D(x_t, x_j)$ and $\text{dist}_D(x_k, x_1)$ are minimal, and since both P and B_{ℓ_1} are shortest paths, it follows that x_a has the Head-Tail-Property with respect to C. This implies that the path $C[x_{a+1}, x_a]$ is a shortest $x_{a+1}x_a$-path,

so that $\mathrm{dist}_D(x_{a+1}, x_a) = n-1$, contradicting the assumption $\mathrm{diam}(D) \leq n-2$. Thus $j \leq k$ and condition (ii) is satisfied.

Since we assumed that C has neither a pair of opposite chords nor a pair of almost opposite chords, we necessarily have $\ell \geq i+2$. Therefore, (x_ix_j, x_kx_ℓ) is a pair of crossing chords of C.

Consider now any vertex $x_a \in C[x_{i+2}, x_\ell]$. Since $\mathrm{dist}_D(x_a, x_{a-1}) \leq n-2$, x_a cannot have the Head-Tail-Property with respect to C, that is, there is a chord x_px_q of C such that x_p precedes x_q on the path $C[x_a, x_{a-1}]$. Let x_px_q be a chord of C which contradicts the Head-Tail-Property of every vertex in $\{x_\ell, \ldots, x_{\ell-\alpha}\} \subseteq V(C[x_{i+2}, x_\ell])$ with respect to C, and suppose that x_px_q is chosen in such a way that α is maximal.

Since B_{ℓ_1} is a shortest $r\ell_1$-path, at most one of the vertices x_p, x_q belongs to B_{ℓ_1}. Moreover, the minimality of $\mathrm{dist}_D(x_t, x_j)$ implies $x_q \in C[x_j, x_{\ell-\alpha}]$.

If $x_q \in C[x_j, x_{k+1}]$, then (x_px_q, x_kx_ℓ) is a pair of opposite or almost opposite chords of C, contradicting our assumption. If $x_q \in C[x_{k+2}, x_{i+1}]$, then $(x_kx_\ell, x_ix_j, x_px_q)$ is a triple of crossing chords if $x_p \in C[x_\ell, x_{j-1}]$, or (x_ix_j, x_px_q) is a pair of opposite or almost opposite chords if $x_p \in C[x_j, x_{q-2}]$, again contradicting our assumption. Thus we may assume $x_q \in C[x_{i+2}, x_{\ell-1}]$.

The minimality of $\mathrm{dist}_D(x_k, x_1)$ then implies $x_p \in C[x_\ell, x_k]$.

To finish the proof, we will show that the vertices x_{p+1} and x_{i+1} have the Vertex-Rainbow-In-Property. We first prove the following claim.

Claim 1 *There exists an x_px_{i+1}-path in D that does not contain x_{p+1}.*

Proof. Consider the vertex x_q. Since $\mathrm{dist}_D(x_q, x_{q-1}) \leq n-2$, x_q cannot have the Head-Tail-Property with respect to C. Let $x_{p_1}x_{q_1}$ be a chord that contradicts the Head-Tail-Property of every vertex in $\{x_q, \ldots, x_{q-\beta}\} \subseteq C[x_{i+2}, x_q]$ with respect to C, and suppose that $x_{p_1}x_{q_1}$ has been chosen in such a way that β is maximal.

Since the chord x_px_q has been chosen in such a way that α is maximal, we necessarily have $x_{p_1} \in C[x_q, x_{\ell-1}]$. If $x_{q_1} \in C[x_{p_1+2}, x_{p+1}]$, then $(x_px_q, x_{p_1}x_{q_1})$ is a pair of opposite or almost opposite chords of C, contradicting our assumption. Thus, $x_{q_1} \in C[x_{p+2}, x_{q-\beta-1}]$. If $x_{q_1} \in C[x_{p+2}, x_i]$, then the path $x_px_q \cup C[x_q, x_{p_1}] \cup x_{p_1}x_{q_1} \cup C[x_{q_1}, x_{i+1}]$ is an x_px_{i+1}-path not containing x_{p+1}. If $q_1 = i+1$, then the path $x_px_q \cup C[x_q, x_{p_1}] \cup x_{p_1}x_{q_1}$ does not contain x_{p+1}. Therefore, we may assume $x_{q_1} \in C[x_{i+2}, x_{q-1}]$.

Consider now the vertex x_{q_1}. Again, since $\text{dist}_D(x_{q_1}, x_{q_1-1}) \leq n-2$, x_{q_1} cannot have the Head-Tail-Property with respect to C. Let $x_{p_2}x_{q_2}$ be a chord that contradicts the Head-Tail-Property of every vertex in $\{x_{q_1}, \ldots, x_{q_1-\gamma}\} \subseteq C[x_{i+2}, x_{q_1}]$ with respect to C, and suppose that $x_{p_2}x_{q_2}$ has been chosen in such a way that γ is maximal.

Since the chord $x_{p_1}x_{q_1}$ has been chosen in such a way that β is maximal, we necessarily have $x_{p_2} \in C[x_{q_1}, x_{q-1}]$. If $x_{q_2} \in C[x_{p_2+2}, x_{x_{p_1}+1}]$, then $(x_{p_1}x_{q_1}, x_{p_2}x_{q_2})$ is a pair of opposite or almost opposite chords, contradicting our assumption. If $x_{q_2} \in C[x_{p_1+2}, x_{x_p+1}]$, then $(x_px_q, x_{p_1}x_{q_1}, x_{p_2}x_{q_2})$ is a triple of crossing chords, again contradicting our assumption. Thus, $x_{q_2} \in C[x_{p+2}, x_{q_1-\gamma-1}]$. If $x_{q_2} \in C[x_{p+2}, x_i]$, then the path

$$x_px_q \cup C[x_q, x_{p_1}] \cup x_{p_1}x_{q_1} \cup C[x_{q_1}, x_{p_2}] \cup x_{p_2}x_{q_2} \cup C[x_{q_2}, x_{i+1}]$$

is an x_px_{i+1}-path not containing x_{p+1}. If $q_1 = i+1$, then the path

$$x_px_q \cup C[x_q, x_{p_1}] \cup x_{p_1}x_{q_1} \cup C[x_{q_1}, x_{p_2}] \cup x_{p_2}x_{q_2}$$

does not contain x_{p+1}.

Thus $x_{q_2} \in C[x_{i+2}, x_{q_1-1}]$. Since again x_{q_2} cannot have the Head-Tail-Property with respect to C, we can iterate this process. Since there is a finite number of vertices in $C[x_{i+2}, x_\ell]$, the process eventually ends up with an x_px_{i+1}-path of the form either

$$x_px_q \cup C[x_q, x_{p_1}] \cup \bigcup_{1 \leq \alpha \leq z-1} (x_{p_\alpha}x_{q_\alpha} \cup C[x_{q_\alpha}, x_{p_{\alpha+1}}]) \cup x_{q_z}x_{q_z},$$

or

$$x_px_q \cup C[x_q, x_{p_1}] \cup \bigcup_{1 \leq \alpha \leq z-1} (x_{p_\alpha}x_{q_\alpha} \cup C[x_{q_\alpha}, x_{p_{\alpha+1}}]) \cup x_{q_z}x_{q_z} \cup C[x_{q_z}, x_{i+1}],$$

for some $z \geq 2$. □

Finally, observe that the path $x_ix_j \cup C[x_j, x_k] \cup x_kx_\ell \cup C[x_\ell, x_{p+1}]$ does not contain x_{i+1}. Therefore, by Lemma 4.5, the vertices x_{p+1} and x_{i+1} have the Vertex-Rainbow-In-Property. The result then follows from Lemma 4.3.

This completes the proof. □

Lemmas 4.6 and 4.10 directly imply the following.

Corollary 4.11 *Let D be a strong digraph of order n. If $\overrightarrow{\mathrm{rvc}}(D) = n$, $\overrightarrow{\mathrm{rc}}(D) = n$, or $\overrightarrow{\mathrm{trc}}(D) = 2n$, then D is Hamiltonian and* $\mathrm{diam}(D) = n - 1$.

Recall that a digraph D is *minimally strongly connected* if D is strong and, for every arc uv in D, the digraph $D - uv$ is not strong. In [11], it was proved that a minimally strongly connected oriented graph D of order n has rainbow connection number n if and only if D is a cycle (their proof readily extends to the case of digraphs). Observe that a Hamiltonian digraph D is minimally strongly connected if and only if D is a cycle. Therefore, a similar result holds for the rainbow vertex-connection number and the total rainbow connection number.

Theorem 4.12 *Let D be a minimally strongly connected digraph of order n. If D is not a cycle, then $\overrightarrow{\mathrm{rvc}}(D) \leq n - 1$ and $\overrightarrow{\mathrm{trc}}(D) \leq 2n - 1$.*

5 Digraphs with maximum connection numbers

In this section, we characterize digraphs of order n with rainbow connection number n, rainbow vertex-connection number n, and total rainbow connection number $2n$, respectively.

Theorem 5.1 *If D is a strong digraph of order n, then $\overrightarrow{\mathrm{rc}}(D) = n$ if and only if D is Hamiltonian and has at least three vertices with eccentricity $n - 1$.*

Proof. ($\Leftarrow$) Let $C = x_0 \dots x_{n-1}x_0$ be a Hamiltonian cycle of D and x_i, x_j, x_k be three vertices with eccentricity $n - 1$. Suppose that there is an arc-colouring φ of D using less than n colours such that (D, φ) is rainbow connected.

Since $\mathrm{dist}_D(x_i, x_{i-1}) = n - 1$, $C[x_i, x_{i-1}]$ is the only x_ix_{i-1}-path in D. Thus φ must assign distinct colours to the arcs of $C[x_i, x_{i-1}]$. Since φ uses less than n colours, the arc $x_{i-1}x_i$ must get a colour which also appears on the path $C[x_i, x_{i-1}]$. Furthermore, since $\mathrm{dist}_D(x_j, x_{j-1}) = n - 1$, $C[x_j, x_{j-1}]$ is the only x_jx_{j-1}-path in D. Thus φ must assign distinct colours to the arcs of $C[x_j, x_{j-1}]$. Since φ is rainbow on both $C[x_i, x_{i-1}]$ and $C[x_j, x_{j-1}]$, we get $\varphi(x_{i-1}x_i) = \varphi(x_{j-1}x_j)$. Since $\mathrm{dist}_D(x_k, x_{k-1}) = n - 1$, $C[x_k, x_{k-1}]$ is the only x_kx_{k-1}-path in D. However, $C[x_k, x_{k-1}]$ contains both arcs $x_{i-1}x_i$ and $x_{j-1}x_j$, which implies, since $\varphi(x_{i-1}x_i) = \varphi(x_{j-1}x_j)$, that (D, φ) is not rainbow connected, contrary to our assumption.

($\Rightarrow$) Suppose now that $\overrightarrow{\mathrm{rc}}(D) = n$. Corollary 4.11 implies that D has a Hamiltonian cycle $C = x_0 \dots x_{n-1}x_0$ and that $\mathrm{ecc}_D(x_i) = n - 1$ for some vertex x_i, $0 \leq i \leq n - 1$. Consider

the digraph $D' = D - x_{i-1}x_i$. Suppose first that D' is strong. By Corollary 2.3, D' is not Hamiltonian and thus, using Corollary 4.11 and Proposition 3.10, we get $\overrightarrow{\mathrm{rc}}(D) \leq \overrightarrow{\mathrm{rc}}(D') \leq n-1$, a contradiction.

Hence the digraph D' is not strong. Suppose first that D' has two strong components D_1 and D_2. By Lemma 2.1, we may assume that D_1 is induced by vertices $x_i, \ldots, x_{j-1}$ and that D_2 is induced by vertices $x_j, \ldots, x_{i-1}$ (it may happen that $i = j-1$ or $j = i-1$). We claim that the vertices x_i and x_j have the Vertex-Rainbow-In-Property in D. Since D_1 is strong, $x_{j-1}, x_i \in V(D_1)$, and $x_j \notin V(D_1)$, there is an $x_{j-1}x_i$-path not containing x_j. Similarly, since D_2 is strong, $x_{i-1}, x_j \in V(D_2)$, and $x_i \notin V(D_2)$, there is an $x_{i-1}x_j$-path not containing x_i. Thus, by Lemma 4.5, the vertices x_i and x_j have the Vertex-Rainbow-In-Property in D, which implies $\overrightarrow{\mathrm{rc}}(D) \leq n-1$, contradicting the assumption $\overrightarrow{\mathrm{rc}}(D) = n$.

Therefore, D' has at least three strong components, say $D_1, D_2, \ldots, D_k$, $k \geq 3$. Let $x_{i_1} = x_i$. We then have

$$D_1 = D'[V(C[x_{i_1}x_{i_2-1}])],\ D_2 = D'[V(C[x_{i_2}, x_{i_3-1}])],\ \ldots,\ D_k = D'[V(C[x_{i_k}, x_{i_1-1}])],$$

for some $i_1, i_2, \ldots, i_k$, $i = i_1 < i_2 < \cdots < i_k < i_1$ (in cyclic order). Since $C[x_i, x_{i-1}]$ is a shortest x_ix_{i-1}-path in D, it follows that $\mathrm{dist}_D(x_{i_\alpha}, x_{i_{\alpha+1}-1}) = |C[x_{i_\alpha}, x_{i_{\alpha+1}-1}]|$ for every α, $1 \leq \alpha \leq k-1$, and that $\mathrm{dist}_D(x_{i_k}, x_{i_1-1}) = |C[x_{i_k}, x_{i_1-1}]|$.

Therefore, $\mathrm{dist}_D(x_{i_\alpha}, x_{i_\alpha-1}) = n-1$ for every α, $1 \leq \alpha \leq k$. Since $k \geq 3$, D contains at least three vertices with eccentricity $n-1$. □

Theorem 5.2 *If D is a strong digraph of order $n \geq 6$, then $\overrightarrow{\mathrm{rvc}}(D) = n$ if and only if D has a Hamiltonian cycle C and three vertices with eccentricity $n-1$ such that no two of them are consecutive on C.*

Proof. ($\Leftarrow$) Let $C = x_0 \ldots x_{n-1}x_0$ be a Hamiltonian cycle of D and x_i, x_j, x_k be vertices with eccentricity $n-1$ such that $|C[x_i, x_j]| \geq 2$, $|C[x_j, x_k]| \geq 2$ and $|C[x_k, x_i]| \geq 2$. Suppose that there exists a vertex-colouring c of D using less than n colours such that (D, c) is rainbow vertex-connected.

Since $\mathrm{dist}_D(x_i, x_{i-1}) = n-1$, $C[x_i, x_{i-1}]$ is the only x_ix_{i-1}-path in D. Thus, c must assign distinct colours to all internal vertices of $C[x_i, x_{i-1}]$. Since c uses less than n colours, either $c(x_i) = c(x_{i-1})$ or one of the colours $c(x_i)$, $c(x_{i-1})$ appears on $C[x_i, x_{i-1}]$. Similarly, since $\mathrm{dist}_D(x_j, x_{j-1}) = n-1$, $C[x_j, x_{j-1}]$ is the only x_jx_{j-1}-path in D, and c must assign distinct colours to all internal vertices of $C[x_j, x_{j-1}]$. Since x_i and x_{i-1} are both internal vertices of $C[x_j, x_{j-1}]$, we get $c(x_i) \neq c(x_{i-1})$. Since c uses less than n colours, we necessarily have either $c(x_i) \in \{c(x_j), c(x_{j-1})\}$ or $c(x_{i-1}) \in \{c(x_j), c(x_{j-1})\}$. Using similar arguments, since $C[x_k, x_{k-1}]$ is the only x_kx_{k-1}-path in D, we get that c must assign distinct colours to all internal vertices of $C[x_k, x_{k-1}]$. However, $x_{i-1}, x_i, x_{j-1}, x_j$ are

internal vertices of $C[x_k, x_{k-1}]$, which contradicts the condition $c(x_i) \in \{c(x_j), c(x_{j-1})\}$ or $c(x_{i-1}) \in \{c(x_j), c(x_{j-1})\}$.

($\Rightarrow$) Suppose now that $\overrightarrow{\text{rvc}}(D) = n$. Corollary 4.11 implies that D has a Hamiltonian cycle $C = x_0 \dots x_{n-1}x_0$ and that $\text{ecc}_D(x_i) = n-1$ for some vertex x_i, $0 \leq i \leq n-1$. Consider the digraph $D' = D - x_{i-1}x_i$. As in the proof of Theorem 5.1, D' cannot be strong, since otherwise we would get $\overrightarrow{\text{rvc}}(D) \leq \overrightarrow{\text{rvc}}(D') \leq n-1$.

Similarly as in the proof of Theorem 5.1, if D' has two strong components $D_1 = D'[V(C[x_i, x_{j-1}])]$ and $D_2 = D'[V(C[x_j, x_{i-1}])]$, then we can prove that the vertices x_i and x_j have the Vertex-Rainbow-In-Property in D, so that, by Lemma 4.3, we get $\overrightarrow{\text{rvc}}(D) \leq n-1$, a contradiction.

Therefore, D' has at least three strong components, say $D_1, D_2, \dots, D_k$, $k \geq 3$. Let $x_{i_1} = x_i$. We then have

$$D_1 = D'[V(C[x_{i_1}x_{i_2-1}])],\ D_2 = D'[V(C[x_{i_2}, x_{i_3-1}])],\ \dots,\ D_k = D'[V(C[x_{i_k}, x_{i_1-1}])],$$

for some $i_1, i_2, \dots, i_k$, $i = i_1 < i_2 < \cdots < i_k < i_1$ (in cyclic order). Since $C[x_i, x_{i-1}]$ is a shortest x_ix_{i-1}-path in D, it follows that $\text{dist}_D(x_{i_\alpha}, x_{i_{\alpha+1}-1}) = |C[x_{i_\alpha}, x_{i_{\alpha+1}-1}]|$ for every α, $1 \leq \alpha \leq k-1$, and that $\text{dist}_D(x_{i_k}, x_{i_1-1}) = |C[x_{i_k}, x_{i_1-1}]|$. Therefore, $\text{dist}_D(x_{i_\alpha}, x_{i_\alpha-1}) = n-1$ for every α, $1 \leq \alpha \leq k$, which implies $\text{ecc}_D(x_\alpha) = n-1$ for every such vertex x_α.

If $k \geq 6$, then the set $\{x_{i_1}, \dots x_{i_6}\}$ contains three non-consecutive vertices. If $k = 5$, then at least one strong component contains at least two vertices, since $n \geq 6$. Thus, again, the set $\{x_{i_1}, \dots x_{i_5}\}$ contains three non-consecutive vertices.

Suppose now that $k = 4$. If the set $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$ does not contain three non-consecutive vertices, then, without loss of generality, we may assume that either

(i) $|V(D_1)| > 1, |V(D_2)| = 1, |V(D_3)| = 1, |V(D_4)| = 1$, or

(ii) $|V(D_1)| > 1, |V(D_2)| = 1, |V(D_3)| > 1, |V(D_4)| = 1$.

Consider the vertex-colouring $c : V(D) \to \{1, \dots n-1\}$ given by $c(x_{i_2}) = c(x_{i_4})$ and all other vertices are coloured differently. We claim that (D, c) is rainbow vertex-connected. To see that, let a, b be any two vertices of D.

In Case (i), if $a, b \in V(D_1)$, then each path containing only vertices of $V(D_1)$ is rainbow, and thus, since D_1 is strong, there exists a rainbow ab-path. If $a, b \notin V(D_1)$, then either $a \in \{x_{i_2}, x_{i_4}\}$ or $b \in \{x_{i_2}, x_{i_4}\}$, so that no elementary ab-path contains both x_{i_2} and x_{i_4} as internal vertices. If $a \in V(D_1)$ and $b \notin V(D_1)$, then x_{i_4} cannot be an internal vertex of an elementary ab-path. If $b \in V(D_1)$ and $a \notin V(D_1)$, then x_{i_2} cannot be an internal vertex of an elementary ab-path. Therefore, there always exists an ab-path not containing both vertices x_{i_2} and x_{i_4} as internal vertices, which is thus a rainbow ab-path.

In Case (ii), if $a, b \in V(D_1)$ or $a, b \in V(D_3)$, then there exists a rainbow ab-path going through vertices of D_1 or D_3, respectively. If either $a \in \{x_{i_2}, x_{i_4}\}$ or $b \in \{x_{i_2}, x_{i_4}\}$, then no elementary ab-path contains both x_{i_2} and x_{i_4} as internal vertices, and thus every such path is rainbow. If $a \in D_1$ and $b \in D_2$, or $b \in D_1$ and $a \in D_2$, then the path $C[a, b]$ is rainbow.

Suppose finally that $k = 3$. If the set $\{x_{i_1}, x_{i_2}, x_{i_3}\}$ contains two vertices that are consecutive on C, then, without loss of generality, we may assume that either

(i) $|V(D_1)| > 1, |V(D_2)| = 1, |V(D_3)| = 1$ or

(ii) $|V(D_1)| > 1, |V(D_2)| = 1, |V(D_3)| > 1$.

In Case (i), we claim that the vertex-colouring $c : V(D) \to \{1, \dots n-1\}$ given by $c(x_{i_2}) = c(x_{i_3})$ and all other vertices are coloured differently is such that (D, c) is rainbow vertex-connected. Since D_1 is strong and contains neither x_{i_2} nor x_{i_3}, each elementary ab-path is rainbow whenever $a, b \in V(D_1)$. If $|\{a, b\} \cap \{x_{i_2}, x_{i_3}\}| \geq 1$, then no elementary ab-path contains both x_{i_2} and x_{i_3} as internal vertices, and thus every such path is rainbow.

In Case (ii), we claim that the vertex-colouring $c : V(D) \to \{1, \dots n-1\}$ given by $c(x_{i_1}) = c(x_{i_2})$ and all other vertices are coloured differently is such that (D, c) is rainbow vertex-connected. If $a, b \in V(D_1)$ or $a, b \in V(D_3)$, then there exists a rainbow ab-path going through vertices of D_1 or D_3, respectively. If $a \in D_1$ and $b \in D_3$, or $b \in D_1$ and $a \in D_3$, then the path $C[a, b]$ is rainbow. Otherwise, x_{i_2} is either the first or the last vertex of the ab-path, which implies that every elementary ab-path is rainbow. □

Theorem 5.3 *If D is a strong digraph of order $n \geq 6$, then $\overrightarrow{\mathrm{trc}}(D) = 2n$ if and only if $\overrightarrow{\mathrm{rvc}}(D) = n$.*

Proof. ($\Leftarrow$) Let $C = x_0 \dots x_{n-1}x_0$ be a Hamiltonian cycle of D and x_i, x_j, x_k be vertices with eccentricity $n-1$ such that $|C[x_i, x_j]| \geq 2, |C[x_j, x_k]| \geq 2, |C[x_k, x_i]| \geq 2$. Suppose that there exists a total-colouring f using less than $2n$ colours such that (D, f) is total rainbow connected.

Since $\mathrm{dist}_D(x_i, x_{i-1}) = n-1$, $C[x_i, x_{i-1}]$ is the only $x_i x_{i-1}$-path in D. Thus, f must assign distinct colours to all internal vertices and arcs of $C[x_i, x_{i-1}]$. Since f uses less than $2n$ colours, either one of the equalities $f(x_i) = f(x_{i-1})$, $f(x_i) = f(x_{i-1}x_i)$ or $f(x_{i-1}) = f(x_{i-1}x_i)$ holds, or one of the colours $f(x_i)$, $f(x_{i-1})$ or $f(x_{i-1}x_i)$ appears on $C[x_i, x_{i-1}]$. Since $\mathrm{dist}_D(x_j, x_{j-1}) = n-1$, $C[x_j, x_{j-1}]$ is the only $x_j x_{j-1}$-path in D. Thus, f must assign distinct colours to all internal vertices and arcs of $C[x_j, x_{j-1}]$. Since both x_i and x_{i-1} are internal vertices of $C[x_j, x_{j-1}]$, we get $f(x_i) \neq f(x_{i-1})$, $f(x_i) \neq f(x_{i-1}x_i)$, and $f(x_{i-1}) \neq f(x_{i-1}x_i)$. Therefore, we necessarily have either $f(x_i) \in \{f(x_j), f(x_{j-1}), f(x_{j-1}x_j)\}$, or $f(x_{i-1}) \in \{f(x_j), f(x_{j-1}), f(x_{j-1}x_j)\}$, or $f(x_{i-1}x_i) \in \{f(x_j), f(x_{j-1}), f(x_{j-1}x_j)\}$.

Using similar arguments, since $C[x_k, x_{k-1}]$ is the only $x_k x_{k-1}$-path in D, we get that f must assign distinct colours to all internal vertices and arcs of $C[x_k, x_{k-1}]$. However, $x_{i-1}, x_i, x_{j-1}, x_j$ are internal vertices of $C[x_k, x_{k-1}]$, so that $x_{i-1}, x_i, x_{j-1}, x_j, x_{i-1}x_i$ and $x_{j-1}x_j$ must be assigned distinct colours, contradicting the condition obtained above.

($\Rightarrow$) Suppose now that $\overrightarrow{\text{trc}}(D) = 2n$. If $\overrightarrow{\text{rvc}}(D) \leq n-1$, then $\overrightarrow{\text{trc}}(D) \leq 2n-1$ by Lemma 3.7, and thus $\overrightarrow{\text{rvc}}(D) = n$. □

References

[1] J. Alva-Samos and J. J. Montellano-Ballesteros. Rainbow connection in some digraphs. *Graphs Combin.* 32 (6) (2016), 2199–2209.

[2] J. Alva-Samos and J. J. Montellano-Ballesteros. A Note on the Rainbow Connectivity of Tournaments. Preprint (2015), available at `http://arxiv.org/abs/1504.07140`.

[3] J. Alva-Samos and J. J. Montellano-Ballesteros. Rainbow Connectivity of Cacti and of Some Infinity Digraphs. *Discuss. Math. Graph Theory* 37 (2017) 301–313.

[4] P. Ananth, M. Nasre and K. K. Sarpatwar. Rainbow Connectivity: Hardness and Tractability. In: 31st Int'l Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2011), *Leibniz International Proceedings in Informatics*, 241–251.

[5] J. A. Bondy and U.S.R. Murty. *Graph Theory*. Graduate Texts in Mathematics, vol. 244, Springer, London, 2011.

[6] Y. Caro, A. Lev, Y. Roditty, Zs. Tuza and R. Yuster. On rainbow connection. *Electron. J. Combin.* 15 (2008), #R57.

[7] S. Chakraborty, E. Fischer, A. Matsliah and R. Yuster. Hardness and algorithms for rainbow connectivity. *J. Comb. Optim.* 21 (2011), 330–347.

[8] G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang. Rainbow connection in graphs. *Math. Bohem.* 133 (2008), 85–98.

[9] L. Chen, B. Huo and Y. Ma. Hardness results for total rainbow connection of graphs. *Discuss. Math. Graph Theory*, 36 (2016) 355–362.

[10] L. Chen, X. Li and Y. Shi. The complexity of determining the rainbow vertex-connection of graphs. *Theoret. Comput. Sci.* 412, (2011) 4531–4535

[11] P. Dorbec, I. Schiermeyer, E. Sidorowicz and É. Sopena. Rainbow Connection in Oriented Graphs. *Discrete Appl. Math.* 179 (2014), 69–78.

[12] J. Ekstein, P. Holub, T. Kaiser, M. Koch, S. Matos Camacho, Z. Ryjáček and I. Schiermeyer. The rainbow connection number of 2-connected graphs. *Discrete Math.* 313(19) (2013), 1884–1892.

[13] A. Kemnitz and I. Schiermeyer. Graphs with rainbow connection number two. *Discuss. Math. Graph Theory* 31(2) (2011), 313–320.

[14] M. Krivelevich and R. Yuster. The rainbow connection of a graph is (at most) reciprocal to its minimum degree. *J. Graph Theory* 63(3) (2009), 185–191.

[15] V. B. Le and Zs. Tuza. Finding optimal rainbow connection is hard. Technical Report CS-03-09, Universität Rostock, (2009).

[16] H. Lei, S. Li, H. Liu and Y. Shi. Rainbow vertex connection of digraphs. *J. Comb. Optim.* (2017) https://doi.org/10.1007/s10878-017-0156-7

[17] H. Lei, H. Liu, C. Magnant and Y. Shi. Total rainbow connection of digraphs. *Discrete Appl. Math.* (2017) https://doi.org/10.1016/j.dam.2017.10.016

[18] X. Li, Y. Shi and Y. Sun. Rainbow Connections of Graphs: A survey. *Graphs Combin.* 29 (2013), 1–38.

[19] X. Li and Y. Sun. Rainbow Connections of Graphs. *Springer Briefs in Mathematics*, Springer (2012).

[20] H. Liu, Â. Mestre and T. Sousa. Total rainbow k-connection in graphs. *Discrete Appl. Math.* 174 (2014), 92–101.

[21] I. Schiermeyer. Rainbow connection in graphs with minimum degree three. IWOCA 2009, *Lect. Notes Comput. Sci.* 5874 (2009), 432–437.

[22] I. Schiermeyer. Rainbow connection and minimum degree. *Discrete Appl. Math.* 161(12) (2013), 1784–1787.

[23] E. Sidorowicz and É. Sopena. Strong Rainbow Connection in Digraphs. Preprint (2016).