

# The oriented chromatic number of graphs: A short survey

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## Abstract

An oriented graph is a digraph with no opposite arcs. The oriented chromatic number of an oriented graph  $\vec{G}$  is the minimum order of an oriented graph to which  $\vec{G}$  admits a homomorphism. The oriented chromatic number of an undirected graph is then defined as the maximum oriented chromatic number of its orientations.

In this paper, we survey the main results about this graph parameter and propose a few open problems.

**Keywords:** Oriented chromatic number; Oriented coloring; Oriented graph; Homomorphism.

**MSC2010 Classification.** 05C15 (primary), 05C20.

## 1 Introduction

A (proper)  $k$ -coloring of an undirected graph  $G$  is a partition of  $V(G)$  into  $k$  subsets, called color classes, such that no two adjacent vertices belong to the same color class. Such a  $k$ -coloring can be equivalently regarded as a homomorphism of  $G$  to the complete graph  $K_k$  on  $k$  vertices. Therefore, the chromatic number  $\chi(G)$  of  $G$ , defined as the smallest  $k$  such that  $G$  admits a  $k$ -coloring, corresponds to the smallest  $k$  such that  $G$  admits a homomorphism to  $K_k$ .

By considering homomorphisms of oriented graphs (that is digraphs with no opposite arcs), we get a natural extension of the notion of coloring to oriented graphs, leading in particular to the notion of the oriented chromatic number of a graph. This notion was first considered by Courcelle [Cou94] as a tool for encoding graph orientations by means of vertex labels. Since then, oriented colorings and homomorphisms of oriented graphs have attracted very much attention.

The aim of this paper is to survey the main results that have been obtained on this topic and to propose a few open problems. It can thus be considered as an update of the papers [Sop97] and [Sop01] (see also [Sop]), and is organised as follows. In the next section, we introduce the main definitions and notation. We give general results in Section 3, relating the oriented chromatic number to other graph parameters, and focus on the special classes of planar graphs in Section 4 and of graphs with bounded degree in Section 5. We then consider various other classes of graphs in Section 6. We briefly discuss complexity issues in Section 7 and finally propose a few open problems in the last section.

## 2 Preliminaries

If  $G$  is an undirected graph, we denote by  $V(G)$  its set of vertices and by  $E(G)$  its set of edges. If  $\vec{G}$  is a digraph, we denote by  $V(\vec{G})$  its set of vertices and by  $A(\vec{G})$  its set of arcs. The *order* of an undirected graph, or a digraph, is the cardinality of its vertex set. All the graphs we consider are simple and have no loops.

**Definition 1 (Homomorphism)** Let  $G$  and  $H$  be two undirected graphs. A *homomorphism* of  $G$  to  $H$  is a mapping  $f : V(G) \rightarrow V(H)$  that preserves the edges:  $f(x)f(y) \in E(H)$  whenever  $xy \in E(G)$ . Homomorphisms of digraphs are defined in a similar way.

An *orientation* of an undirected graph  $G$  is a digraph obtained from  $G$  by giving to every edge  $uv$  of  $G$  one of its two possible orientations, namely  $\vec{uv}$  or  $\vec{vu}$ . A digraph  $\vec{G}$  is an *oriented graph* if it is an orientation of some undirected graph, called the *underlying graph* of  $\vec{G}$ . In the following, an arc  $\vec{uv}$  will be simply denoted  $uv$ .

As for undirected graphs, homomorphisms of oriented graphs induce a notion of oriented graph coloring as follows.

**Definition 2 (Oriented  $k$ -coloring)** An *oriented  $k$ -coloring* of an oriented graph  $\vec{G}$  is a partition of  $V(\vec{G})$  into  $k$  color classes such that no two adjacent vertices belong to the same color class, and all the arcs linking two color classes have the same direction.

Equivalently, an oriented  $k$ -coloring of  $\vec{G}$  may be regarded as a homomorphism of  $\vec{G}$  to some oriented graph of order  $k$ , or as a mapping  $\gamma$  from  $V(\vec{G})$  to a set of  $k$  colors such that

- (i)  $\gamma(u) \neq \gamma(v)$  for every arc  $uv$  in  $A(\vec{G})$ ,
- (ii)  $\gamma(u) \neq \gamma(x)$  for every two arcs  $uv$  and  $wx$  with  $\gamma(v) = \gamma(w)$ .

**Definition 3 (Oriented chromatic number)** The *oriented chromatic number* of an oriented graph  $\vec{G}$ , denoted by  $\chi_o(\vec{G})$ , is the smallest  $k$  such that  $\vec{G}$  admits an oriented  $k$ -coloring or, equivalently, the minimum order of an oriented graph  $\vec{H}$  such that  $\vec{G}$  admits a homomorphism to  $\vec{H}$ .

This notion can be extended to undirected graphs as follows: the oriented chromatic number of an undirected graph  $G$  is defined as the maximum of the oriented chromatic numbers of its orientations. Similarly, the oriented chromatic number of a class of undirected or oriented graphs is defined as the maximum of the oriented chromatic numbers of its members.

Figure 1 shows an oriented 4-coloring of an oriented graph  $\vec{G}$ , viewed as a homomorphism to an oriented graph  $\vec{H}$  of order 4. It can easily be checked that  $\vec{G}$  admits no oriented 3-coloring, so that  $\chi_o(\vec{G}) = 4$ .

Observe that conditions (i) and (ii) above imply that any two vertices linked by a directed path of length 1 or 2 must be assigned distinct colors in any oriented coloring. We then get for instance that the oriented chromatic number of the directed cycle  $\vec{C}_5$  on five vertices is 5. This property can also be expressed by saying that homomorphisms of oriented graphs “preserve” directed paths of length at most 2 (the image of such a path is still a directed path of the same length). Homomorphisms preserving other types of configurations, namely paths of length  $k$  or trees, have been considered in [NSV97].

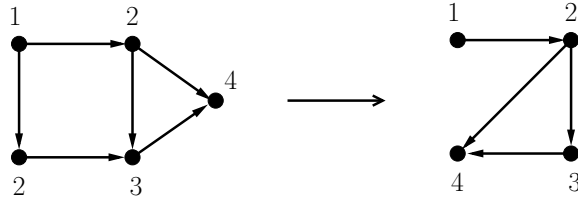


Figure 1: An oriented 4-coloring of an oriented graph  $\vec{G}$

An oriented graph  $\vec{G}$  is an *oriented clique* (or, simply, an *o-clique*) if  $\chi_o(\vec{G}) = |V(\vec{G})|$ . Therefore, an oriented graph is an o-clique if and only if every two of its vertices are linked (in either direction) by a directed path of length 1 or 2. Every tournament is obviously an o-clique but being a tournament is not a necessary condition for being an o-clique. Figure 2 depicts some sample o-cliques of order up to 7.

Let  $\vec{O}_n$  be an o-clique of order  $n$ . Take now two copies of  $\vec{O}_n$ , a new vertex  $v$ , and add all possible arcs from vertices of the first copy towards  $v$  and all possible arcs from  $v$  towards vertices of the second copy. The so-obtained oriented graph is indeed an o-clique of order  $2n+1$ . (The o-clique  $\vec{O}_7$  given in Figure 2 is obtained in this way from the o-clique  $\vec{O}_3$ .)

An undirected graph  $G$  is a *k-tree* if either  $G = K_k$  or  $G$  is obtained from a *k-tree*  $H$  by adding a new vertex linked to a subgraph of  $H$  isomorphic to  $K_k$ . A subgraph of a *k-tree* is a *partial k-tree*. Every forest is thus a 1-tree and every outerplanar graph is a partial 2-tree (recall that a planar graph is *outerplanar* if it admits a planar embedding such that every vertex lies on the outerface). In [KlMa04a], Klostermeyer and MacGillivray proved that the maximum order of an o-clique whose underlying graph is a partial 2-tree is 7 which implies, in particular, that the maximum order of an outerplanar o-clique is 7 (observe that the o-clique  $\vec{O}_7$  of order 7 depicted in Figure 2 is indeed outerplanar). They also proved that every planar o-clique has order at most 36 and conjectured that the maximum order of a planar o-clique is 15. Notice that applying the above-described construction to the outerplanar o-clique  $\vec{O}_7$  produces a planar o-clique of order 15. This conjecture has been recently settled by Sen:

**Theorem 4 (Sen [Sen12])**

*The maximum order of a planar o-clique is 15.*

It was proved in [FHPZ98] and [KLSS99] that the minimum number of edges in an o-clique of order  $n$  is  $(1 + o(1))n \log_2 n$ .

### 3 General results

One of the main difficulties when dealing with oriented colorings is that, on the contrary to the undirected case, we cannot decide “locally” the availability of a color for a given vertex  $u$ : if  $u$  has some already colored neighbor  $v$  then we need to check the neighborhood of all other vertices in the oriented graph having the same color as  $v$ . Therefore, it is not surprising that we cannot bound the oriented chromatic number of an oriented graph by a function of the ordinary chromatic number of its underlying undirected graph. To see that, consider for instance the orientation  $\vec{K}_{n,n}$  of the complete bipartite undirected graph  $K_{n,n}$  obtained by orienting all the edges from the first part of the bipartition to the other

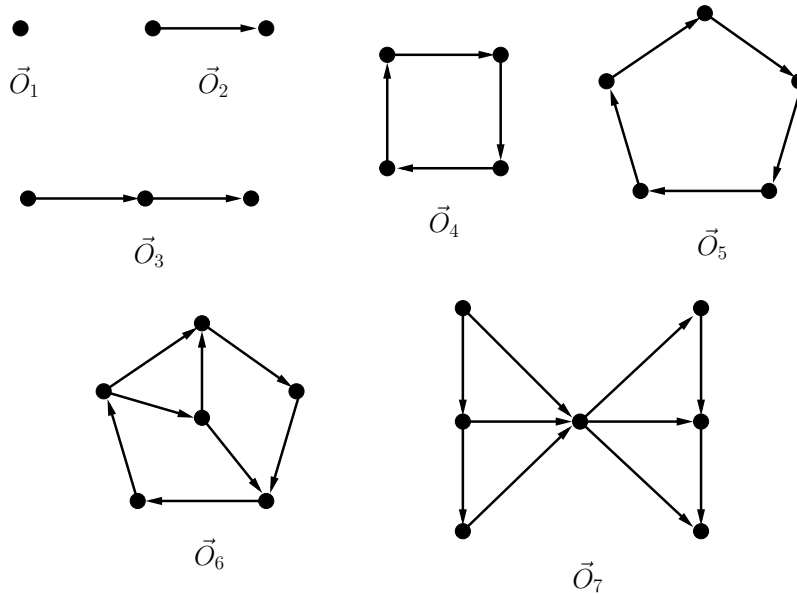


Figure 2: Sample o-cliques

part, and then reversing the orientation of all the edges of any perfect matching of  $K_{n,n}$ . It can then easily be checked that every two vertices are then linked by a directed path of length at most 2, so that  $\chi_o(\vec{K}_{n,n}) = 2n$  while  $\chi(K_{n,n}) = 2$ .

In this section, we give some general relations between the oriented chromatic number of an undirected graph and its acyclic chromatic number or its maximum average degree. We then consider oriented graphs that are critical with respect of oriented colorings.

### 3.1 Graphs with bounded acyclic chromatic number

A proper vertex coloring of an undirected graph  $G$  is *acyclic* if, for every two colors  $a$  and  $b$ , the subgraph of  $G$  induced by the vertices with color  $a$  or  $b$  is a forest. In other words, every cycle of  $G$  uses at least three colors. The *acyclic chromatic number* of  $G$ , denoted  $\chi_a(G)$ , is then the smallest number of colors needed by an acyclic coloring of  $G$ .

Raspaud and Sopena proved that every graph with bounded acyclic chromatic number has bounded oriented chromatic number:

**Theorem 5 (Raspaud and Sopena [RaSo94])**

*If  $G$  is an undirected graph with  $\chi_a(G) \leq k$ , then  $\chi_o(G) \leq k \cdot 2^{k-1}$ .*

In [Och05], Ochem proved that this bound is tight for every  $k \geq 3$ . Kostochka, Sopena and Zhu proved in [KSZ97] that, conversely, every class of graphs with bounded oriented chromatic number has bounded acyclic chromatic number.

Since every planar graph has acyclic chromatic number at most 5 (a celebrated result of Borodin, see [Bor79]), we get by Theorem 5 that every planar graph has oriented chromatic number at most  $5 \cdot 2^4 = 80$ . This upper bound is indeed the best known bound for planar graphs up to now (see Subsection 4.3 and Problem 1 in Section 8).

<b>mad</b>	<b>girth</b>	$\chi_o$	<b>Author(s)</b>
$< 10/3$		$\leq 19$	Borodin, Kostochka, Nešetřil, Raspaud and Sopena [BKN+99]
$< 10/3$	$\geq 5$	$\leq 16$	Pinlou [Pin09]
$< 3$		$\leq 11$	Borodin, Kostochka, Nešetřil, Raspaud and Sopena [BKN+99]
$< 11/4$	$\geq 5$	$\leq 7$	Borodin, Kostochka, Nešetřil, Raspaud and Sopena [BKN+99]
$< 12/5$	$\geq 5$	$\leq 5$	Borodin, Ivanova and Kostochka [BIK07]

Table 1: Oriented chromatic number of graphs with bounded maximum average degree

### 3.2 Graphs with bounded maximum average degree

The *maximum average degree*  $\text{mad}(G)$  of an undirected graph  $G$  is defined as the maximum of the average degrees of its subgraphs, that is

$$\text{mad}(G) = \max_{H \subseteq G} \left\{ \frac{2|E(H)|}{|V(H)|} \right\}.$$

Intuitively speaking, the maximum average degree of an undirected graph thus provides a measure of its “local density”. Recall that the *girth* of an undirected graph  $G$  is the length of a shortest cycle in  $G$ . If  $G$  is a planar graph with girth at least  $g$  then, using Euler’s formula, it can be proved that  $G$  has maximum degree less than  $2g/(g-2)$ . Therefore, several authors have considered the problem of determining the oriented chromatic number of graphs with bounded maximum average degree (and, sometimes, with high girth), in order to get upper bounds on the oriented chromatic number of planar graphs with high girth [BIK07, BKN+99, Pin09]. The best known upper bounds are summarized in Table 1, some of them giving the best known upper bounds for planar graphs with high girth (see Subsection 4.3).

### 3.3 Critical graphs

An oriented graph  $\vec{G}$  is *vertex-critical* if, for every vertex  $u$  in  $V(\vec{G})$ ,  $\chi_o(\vec{G} - u) < \chi_o(\vec{G})$ . Similarly, an oriented graph  $\vec{G}$  is *arc-critical* if, for every arc  $uv$  in  $A(\vec{G})$ ,  $\chi_o(\vec{G} - uv) < \chi_o(\vec{G})$ .

Let  $\vec{G}$  be an oriented graph,  $\gamma$  an oriented coloring of  $\vec{G}$  and  $u$  and  $v$  two non-adjacent vertices of  $\vec{G}$ . By giving two new colors to  $u$  and  $v$ , we clearly get an oriented coloring  $\gamma'$  of  $\vec{G} + uv$ . Hence,  $\chi_o(\vec{G} + uv) - \chi_o(\vec{G}) \leq 2$ . On the other hand, such a universal bound does not exist when we add new vertices:

**Theorem 6 (Borodin *et al.* [BFK+01])**

*For every  $k > 0$ , there exists an oriented graph  $\vec{G}_k$  such that  $\chi_o(\vec{G}_k) - \chi_o(\vec{G}_k - u) \geq k$  for every vertex  $u$  in  $V(\vec{G}_k)$  and  $\chi_o(\vec{G}_k) - \chi_o(\vec{G}_k - uv) = 2$  for every arc  $uv$  in  $A(\vec{G}_k)$ .*

## 4 Planar graphs

Due to the famous Four-Colour Theorem for ordinary colorings of undirected planar graphs, it is not surprising that the study of oriented colorings of various classes of planar graphs has attracted very much attention in the literature.

Since every oriented forest  $\vec{F}$  contains a vertex with degree 1, it clearly admits a homomorphism to  $\vec{C}_3$ , the directed cycle of order 3 (every vertex in  $\vec{C}_3$  has indegree 1 and outdegree 1). Hence, for every undirected forest  $F$ ,  $\chi_o(F) \leq 3$ .

As observed in Section 2, the directed cycle  $\vec{C}_5$  on 5 vertices is an o-clique ( $\vec{O}_5$  in Figure 2) and, therefore, has oriented chromatic number 5. It can be proved that every other oriented cycle has oriented chromatic number at most 4. Let  $\vec{C}$  be an oriented cycle. We define the *level*  $\lambda(\vec{C})$  of  $C$  as the number of forward arcs minus the number of backward arcs in  $\vec{C}$ . We then have:

**Proposition 7** *If  $\vec{C}$  is an oriented cycle then:*

- (1) *if the orientation of  $\vec{C}$  is alternating then  $\chi_o(\vec{C}) = 2$ ,*
- (2) *if the orientation of  $\vec{C}$  is not alternating and either  $\lambda(\vec{C}) \equiv 0 \pmod{3}$  or  $\vec{C}$  does not contain three consecutive arcs with the same direction then  $\chi_o(\vec{C}) = 3$ ,*
- (3) *otherwise,  $\chi_o(\vec{C}) = 4$  if  $\vec{C}$  is not the directed cycle  $\vec{C}_5$  on 5 vertices.*

## 4.1 Outerplanar graphs

Recall that a graph is outerplanar if it admits a planar embedding such that every vertex lies on the outerface. The maximum oriented chromatic number of an outerplanar undirected graph is given by:

**Theorem 8 (Sopena [Sop97])**

*If  $G$  is an outerplanar undirected graph, then  $\chi_o(G) \leq 7$  and this bound is tight.*

In fact, this result was shown to hold for the larger class of partial 2-trees. The tightness of the bound comes from the outerplanar o-clique  $\vec{O}_7$  depicted in Figure 2.

Pinlou and Sopena later determined the oriented chromatic number of outerplanar graphs (in fact, again more generally, of partial 2-trees) of given girth:

**Theorem 9 (Pinlou and Sopena [PiSo06])**

*If  $G$  is an outerplanar undirected graph with girth  $g$ , then*

- (1)  *$\chi_o(G) \leq 6$  if  $g \geq 4$ , and this bound is tight,*
- (2)  *$\chi_o(G) \leq 5$  if  $g \geq 5$ , and this bound is tight.*

A graph is 2-outerplanar if it has a planar embedding such that the subgraph obtained by removing the vertices of the outerface is outerplanar. In [EsOc07], Esperet and Ochem proved that every 2-outerplanar graph has oriented chromatic number at most 67. This bound was later decreased to 40 by Ochem and Pinlou:

**Theorem 10 (Ochem and Pinlou [OcPi11])**

*If  $G$  is a 2-outerplanar undirected graph, then  $\chi_o(G) \leq 40$ .*

## 4.2 Rectangular grids

The *rectangular grid*  $G_{m,n}$  is the Cartesian product (see Subsection 6.3) of  $P_m$  by  $P_n$ , the two paths with  $m$  and  $n$  vertices, respectively. Fertin, Raspaud and Roychowdhury proved the following:

**Theorem 11 (Fertin, Raspaud and Roychowdhury [FRR03])**

*For every integers  $m, n \geq 1$ ,  $\chi_o(G_{m,n}) \leq 11$ .*

Moreover, they proved that there exists an orientation of the grid  $G_{4,5}$  with oriented chromatic number 7. They also conjectured that the oriented chromatic number of every rectangular grid is at most 7, and that every such grid admits a homomorphism to the

<b>girth</b>	$\chi_o$	<b>Author(s)</b>
$\geq 3$	$18 \leq \chi_o \leq 80$	Marshall [Mar12a] - Raspaud and Sopena [RaSo94]
$\geq 4$	$11 \leq \chi_o \leq 40$	Ochem [Och04] - Ochem and Pinlou [OcPi11]
$\geq 5$	$7 \leq \chi_o \leq 16$	Marshall [Mar12b] - Pinlou [Pin09]
$\geq 6$	$7 \leq \chi_o \leq 11$	Marshall [Mar12b] - Borodin <i>et al.</i> [BKN+99]
$\geq 7$	$6 \leq \chi_o \leq 7$	Nešetřil <i>et al.</i> [NRS97] - Borodin and Ivanova [BoIv05a]
$\geq 8$	$5 \leq \chi_o \leq 7$	Nešetřil <i>et al.</i> [NRS97] - Borodin and Ivanova [BoIv05a]
$\geq 11$	$5 \leq \chi_o \leq 6$	Nešetřil <i>et al.</i> [NRS97] - Ochem and Pinlou [OcPi08]
$\geq 12$	$\chi_o = 5$	Nešetřil <i>et al.</i> [NRS97] - Borodin, Ivanova and Kostochka [BIK07]

Table 2: Oriented chromatic number of planar graphs with large girth

Paley tournament  $\vec{T}_7$ , given by  $V(\vec{T}_7) = \{0, 1, \dots, 6\}$  and  $uv \in A(\vec{T}_7)$  if and only if  $v \equiv u+x \pmod{7}$  for some  $x \in \{1, 2, 4\}$ . They proved that this conjecture is true when  $m = 2, 3$ .

Szepietowski and Targan [SzTa04] proved that this conjecture is also true when  $m = 4$  but that the second part of the conjecture fails to be true whenever  $m \geq 5$  by providing an orientation of the grid  $G_{5,33}$  which admits no homomorphism to the tournament  $\vec{T}_7$ . The first part of the conjecture was finally disproved by Dybizbański and Nenca [DyNe12] who exhibited an orientation of the grid  $G_{7,212}$  with oriented chromatic number at least 8.

Bielak studied hexagonal grids  $H_{m,n}$ , made of  $m$  rows of  $n$  hexagons and she proved the following:

**Theorem 12 (Bielak [Bie06])**

For every integers  $m, n \geq 1$ ,  $\chi_o(H_{m,n}) \leq 6$ . Moreover,  $\chi_o(H_{1,n}) = 5$  for every  $n \geq 3$ .

### 4.3 Planar graphs with given girth

In [Bor79], Borodin proved that every undirected planar graph is acyclically 5-colorable. Therefore, using Theorem 5, we get:

**Theorem 13 (Raspaud and Sopena [RaSo94])**

If  $G$  is a planar undirected graph, then  $\chi_o(G) \leq 80$ .

Despite many efforts, this upper bound has not been improved yet. As observed in Section 3, there exists a planar o-clique of order 15. This lower bound on the oriented chromatic number of the class of planar graphs has successively been improved to 16 by Sopena [Sop02], to 17 by Marshall [Mar07], and to 18, again by Marshall [Mar12a].

Several authors have considered the problem of determining the oriented chromatic number of classes of planar graphs with given girth, successively improving various lower or upper bounds (see [RaSo94], [NRS97], [BKN+98], [BKN+99], [Och04], [BKKW04], [BoIv05a], [BoIv05b], [EsOc07], [BIK07], [MOP08], [OcPi08], [Pin09], [OcPi11], [Mar12a], [Mar12b]). The best known lower and upper bounds are summarized in Table 2 (first reference is for the lower bound, second reference is for the upper bound).

Borodin, Kim, Kostochka and West considered other surfaces and proved the following:

**Theorem 14 (Borodin, Kim, Kostochka and West [BKKW04])**

If  $G$  is an undirected graph with girth at least 13, embeddable on the torus or the Klein bottle, then  $\chi_o(G) \leq 5$ .

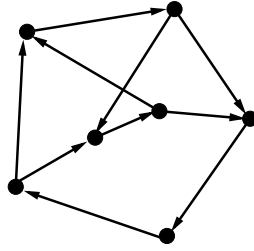


Figure 3: An o-clique of order 7 with maximum degree 3

## 5 Graphs with bounded degree

It was first proved in [Sop97] that every undirected graph with maximum degree  $\Delta$  has oriented chromatic number at most  $(2\Delta - 1) \cdot 2^{2\Delta-2}$ . This bound was further decreased as follows:

**Theorem 15 (Kostochka, Sopena and Zhu [KSZ97])**

*If  $G$  is an undirected graph with maximum degree  $\Delta$ , then  $\chi_o(G) \leq 2 \cdot \Delta^2 \cdot 2^\Delta$ .*

These authors also proved that for every  $\Delta \geq 2$ , there exists an oriented graph with maximum degree  $\Delta$  and oriented chromatic number at least  $2^{\frac{\Delta}{2}}$ .

The upper bound of Theorem 15 has been specialised by Aravind and Subramanian. Recall that the *degeneracy* of an undirected graph  $G$  is the smallest integer  $d$  such that every subgraph of  $G$  contains a vertex of degree at most  $d$ . Then we have:

**Theorem 16 (Aravind and Subramanian [ArSu09])**

*If  $G$  is an undirected graph with maximum degree  $\Delta$  and degeneracy  $d$ , then*

$$\chi_o(G) \leq 16 \cdot \Delta \cdot d \cdot 2^d.$$

In [Woo07], Wood proved that every oriented graph of order  $n$  with maximum degree  $\Delta$  has oriented chromatic number at most  $2 \cdot \Delta \cdot \sqrt{n-1}$  and constructed such graphs with oriented chromatic number at least  $\Delta \cdot \sqrt{\frac{n}{8}}$ .

It was proved in [Sop97] that every oriented graph with maximum degree 3 has oriented chromatic number at most 16. This upper bound was decreased to 11 by Sopena and Vignal [SoVi96] and has not been improved yet.

The following conjecture was proposed in [Sop97]:

**Conjecture 17** *Every connected oriented graph with maximum degree 3 has oriented chromatic number at most 7.*

If the conjecture is true then the bound of 7 cannot be improved, as shown by the oriented graph depicted in Figure 3.

## 6 Other graph classes

### 6.1 Halin graphs

A *Halin graph* is a planar graph which admits a planar embedding such that deleting the edges of its external face gives a tree with at least three vertices. The first result for this class of graphs was the following:

**Theorem 18 (Hosseini Dolama and Sopena [HoSo06b])**

*If  $G$  is a Halin graph, then  $\chi_o(G) \leq 9$ . Moreover, there exist oriented Halin graphs with oriented chromatic number at least 8.*

The exact bound for the oriented chromatic number of Halin graphs has been recently proved:

**Theorem 19 (Dybizbański and Szepietowski [DySz14])**

*If  $G$  is a Halin graph, then  $\chi_o(G) \leq 8$ .*

### 6.2 Hypercubes

Let  $Q_d$  denote the  $d$ -dimensional hypercube. Wood proved the following:

**Theorem 20 (Wood [Woo07])**

*For every integer  $d \geq 1$ ,  $0.8007\dots\sqrt{2^d} \leq \chi_o(Q_d) \leq 2d \cdot \sqrt{2^d - 1}$ .*

### 6.3 Cartesian products

The *Cartesian product* of two undirected graphs  $G$  and  $H$ , denoted  $G \square H$ , is the graph defined by  $V(G \square H) = V(G) \times V(H)$  and  $(u, v)(u', v') \in E(G \square H)$  if and only if either  $u = u'$  and  $vv' \in E(H)$  or  $v = v'$  and  $uu' \in E(G)$ .

In [ANS11], Aravind, Narayanan and Subramanian proved that for every undirected graph  $G$ :

- $\chi_o(G \square P_k) \leq (2k - 1) \cdot \chi_o(G)$ , for every  $k \geq 2$ ,
- $\chi_o(G \square C_k) \leq 2k \cdot \chi_o(G)$ , for every  $k \geq 3$ ,

where  $P_k$  and  $C_k$  denote the path and the cycle on  $k$  vertices, respectively. These results were further improved as follows:

**Theorem 21 (Sopena [Sop12])**

*For every undirected graph  $G$ , we have*

- (1)  $\chi_o(G \square T) \leq 6 \cdot \chi_o(G)$ , for every undirected tree  $T$ ,
- (2)  $\chi_o(G \square C_k) \leq 12 \cdot \chi_o(G)$ , for every  $k \geq 3$ ,  $k \neq 5$ ,
- (3)  $\chi_o(G \square C_5) \leq 15 \cdot \chi_o(G)$ .

### 6.4 Fully subdivided graphs

Let  $G$  be an undirected graph. We denote by  $G^s$  the undirected graph obtained from  $G$  by subdividing exactly once every edge of  $G$ . Wood proved the following:

**Theorem 22 (Wood [Woo05])**

For every undirected graph  $G$ ,

- (1)  $\chi_o(G^s) \leq 7$  if  $\chi(G) \leq 7$ ,
- (2)  $\chi_o(G^s) \leq 9$  if  $\chi(G) = 8$ ,
- (3)  $\chi_o(G^s) = \chi(G)$  if  $\chi(G) \geq 9$ ,

where  $\chi(G)$  stands for the ordinary chromatic number of  $G$ .

**6.5 Line-digraphs**

An edge-coloring of an undirected graph is nothing but a vertex-coloring of its line-graph, so that the *chromatic index*  $\chi'(G)$  of  $G$  is the chromatic number of the line-graph of  $G$ . The same idea applies to oriented graphs.

Let  $\vec{G}$  be an oriented graph. The *line-digraph* of  $\vec{G}$ , denoted  $\text{LD}(\vec{G})$ , is the oriented graph whose set of vertices is the set of arcs of  $\vec{G}$ , and whose set of arcs is the set of pairs  $(uv, vw)$  of “consecutive” arcs in  $\vec{G}$ . An *oriented  $k$ -arc-coloring* of  $\vec{G}$  is then defined as an oriented (vertex)  $k$ -coloring of  $\text{LD}(\vec{G})$ . In other words, an oriented  $k$ -arc-coloring of  $\vec{G}$  is a mapping  $c$  from  $A(\vec{G})$  to a set of  $k$  colors such that (i)  $c(uv) \neq c(vw)$  for every arcs  $uv$  and  $vw$  in  $A(\vec{G})$  and (ii)  $c(vw) \neq c(xy)$  for every arcs  $uv, vw, xy$  and  $yz$  in  $A(\vec{G})$  with  $c(uv) = c(yz)$ .

The *oriented chromatic index* of  $\vec{G}$ , denoted  $\chi'_o(\vec{G})$ , is then the smallest  $k$  such that  $\vec{G}$  admits an oriented  $k$ -arc-coloring.

Let  $\vec{G}$  be an oriented graph with oriented chromatic number  $k$  and  $\gamma$  be an oriented (vertex)  $k$ -coloring of  $\vec{G}$ . It is not difficult to check that the mapping  $c$  defined by  $c(uv) = \gamma(u)$  for every arc  $uv$  in  $A(\vec{G})$  is an oriented  $k$ -arc-coloring of  $\vec{G}$ . Therefore,  $\chi'_o(\vec{G}) \leq \chi_o(\vec{G})$  for every oriented graph  $\vec{G}$ .

Arc-colorings of oriented graphs have been studied by Ochem, Pinlou and Sopena [PiSo06, Pin06, OcPi07, OPS08]. The main results can be summarized as follows:

**Theorem 23**

- (1) If the underlying undirected graph of  $\vec{G}$  has acyclic chromatic number at most  $k$ , then  $\chi'_o(\vec{G}) \leq 2k(k-1) - \lfloor k \rfloor$ . As a corollary, we get that every oriented planar graph has chromatic index at most 38. Moreover, there exist oriented planar graphs with oriented chromatic index at least 10 [OPS08].
- (2) If  $\vec{G}$  is a planar oriented graph with girth  $g \geq 46$  then  $\chi'_o(\vec{G}) \leq 4$  [OPS08].
- (3) For every partial 2-tree  $\vec{G}$ ,  $\chi'_o(\vec{G}) \leq 7$  and this bound is tight [PiSo06].
- (4) Every outerplanar graph with girth  $g$  has oriented chromatic index at most 7 if  $g \geq 3$ , at most 6 if  $g \geq 4$ , at most 5 if  $5 \leq g \leq 9$  and at most 4 if  $g \geq 10$ . Moreover, all these bounds are tight [PiSo06].
- (5) Every partial 2-tree with girth  $g$  has oriented chromatic index at most 7 if  $g \geq 3$ , at most 6 if  $g \geq 5$ , at most 5 if  $7 \leq g \leq 17$  (tight bound), and at most 4 if  $g \geq 18$  (tight bound) [OcPi07].
- (6) If  $\vec{G}$  is an oriented graph with maximum degree  $\Delta$ , then  $\chi'_o(\vec{G}) \leq 2 \cdot (\lfloor \frac{\Delta^2}{2} \rfloor + \Delta)$  [OPS08].
- (7) If  $\vec{G}$  is an oriented graph with maximum degree 3, then  $\chi'_o(\vec{G}) \leq 7$ , and there exists such oriented graphs with oriented chromatic index at least 6 [Pin06].

## 7 Complexity issues

It is NP-complete to decide whether an undirected graph has ordinary chromatic number at most  $k$ , unless  $k \leq 2$  [GaJo79]. Unsurprisingly, the situation is quite similar in the oriented case. Consider the following decision problem:

**PROBLEM:**  $\text{OCN}_k$  (Oriented chromatic number at most  $k$ )

*Instance:* An oriented graph  $\vec{G}$  and a positive integer  $k$ .

*Question:* Does  $\vec{G}$  admit an oriented  $k$ -coloring?

Klostermeyer and MacGillivray proved the following:

**Theorem 24 (Klostermeyer and MacGillivray [KIMa04b])**

*The problem  $\text{OCN}_k$  is polynomial if  $k \leq 3$  and is NP-complete if  $k \geq 4$ , even if the input graph is connected.*

Culus and Demange [CuDe06] proved that  $\text{OCN}_4$  is NP-complete even if the underlying graph of the input graph is bipartite with bounded degree, or acyclic with bounded degree. Ganian and Hliněný [GaHl10] proved that  $\text{OCN}_4$  is also NP-complete when restricted to graphs with K-width 1 and DAG-depth 3<sup>1</sup>.

Recall that the oriented chromatic index of an oriented graph  $\vec{G}$  is defined as the oriented chromatic number of its line-digraph  $\text{LD}(\vec{G})$  (see Subsection 6.5). The associated decision problem is the following:

**PROBLEM:**  $\text{OCI}_k$  (Oriented chromatic index at most  $k$ )

*Instance:* An oriented graph  $\vec{G}$  and a positive integer  $k$ .

*Question:* Does  $\text{LD}(\vec{G})$  admit an oriented  $k$ -coloring?

Ochem, Pinlou and Sopena proved the following:

**Theorem 25 (Ochem, Pinlou and Sopena [OPS08])**

*The problem  $\text{OCI}_k$  is polynomial if  $k \leq 3$  and is NP-complete if  $k \geq 4$ .*

## 8 Open problems

In this section, we propose some open problems concerning the oriented chromatic number of special classes of graphs.

The most challenging one is certainly the following:

**Problem 1** *Determine the largest possible oriented chromatic number of a planar graph.*

We know from Theorem 13 and [Mar12a] that the answer to Problem 1 lies between 18 and 80.

The situation for cubic graphs is also still unknown. We thus propose the following:

**Problem 2** *Determine the largest possible oriented chromatic number of a cubic graph. Determine the largest possible oriented chromatic number of a connected cubic graph.*

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<sup>1</sup>A digraph  $G$  has K-width  $k$  if  $k$  is the lowest integer such that, for any pair of vertices  $u, v \in V(G)$ , the number of distinct directed paths from  $u$  to  $v$  is at most  $k$ .

For a digraph  $G$  and any  $v \in V(G)$ , let  $G_v$  denote the subdigraph of  $G$  induced by the vertices reachable from  $v$ . The maximal elements of the poset  $\{G_v : v \in V(G)\}$  in the digraph-inclusion order are then called reachable fragments of  $G$ , referred to as  $\text{RF}(G)$ . The DAG-depth  $\text{ddp}(G)$  of a digraph  $G$  is then inductively defined as follows: If  $|V(G)| = 1$ , then  $\text{ddp}(G) = 1$ . If  $G$  has a single reachable fragment, then  $\text{ddp}(G) = 1 + \min\{\text{ddp}(G_v) : v \in V(G)\}$ . Otherwise,  $\text{ddp}(G) = \max\{\text{ddp}(F) : F \in \text{RF}(G)\}$ .

We know from [SoVi96] and Figure 3 that the answer to the first part of Problem 2 lies between 7 and 11. Conjecture 17 claims that the answer to the second part is 7.

The case of rectangular grids is also quite intriguing:

**Problem 3** *Determine the largest possible oriented chromatic number of a rectangular grid.*

We know from Theorem 11 and [DyNe12] that the answer to Problem 3 lies between 8 and 11.

The following question was proposed by André Raspaud and first results on this problem were obtained by Wood [Woo07]:

**Problem 4** *Determine the oriented chromatic number of the  $d$ -dimensional hypercube.*

We finally propose a question concerning the oriented chromatic index of graphs:

**Problem 5** *Determine the largest possible oriented chromatic index of a cubic graph.*

From Theorem 23(7), we know that the answer to Problem 5 lies between 6 and 7.

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