http://www.labri.fr/~sutre/Teaching/B

## 1 Assertions on Natural Numbers

Specify the following properties in a B abstract machine that only contains an ASSERTIONS clause, for instance:

```
MACHINE NatAsserts
ASSERTIONS
/* Exercise 1 */
[To be filled]
&
/* Exercise 2 */
[To be filled]
[...]
END
```

For each exercise, check with the type checker of Atelier B that formulas are correctly typed, and launch the automatic prover on the machine.

Exercise 1 There exists a least natural number.

**Exercise 2** For every natural number, there exists a (strictly) larger natural number.

**Exercise 3** For every pair (n,m) of natural numbers such that  $n \neq m$ , it holds that either n < m or n > m.

**Exercise 4** Every natural number is either even or odd.

Recall that divisibility on natural numbers is defined by:  $d \text{ divides } n \text{ if } n = k \cdot d$  for some  $k \in \mathbb{N}$ . For convenience, add a definition of the divisibility relation to your machine. This definition acts as a macro that can be used in the rest of the machine.

```
MACHINE NatAsserts
DEFINITIONS
Divides (d, n) == (#kk . (kk : NATURAL & (n) = kk*(d)))
ASSERTIONS
END
```

Exercise 5 Every natural number divides itself and zero.

**Exercise 6** There exists a natural number that divides all natural numbers.

**Exercise 7** For every natural numbers d and n such that  $d \neq 0$ , d divides n if and only if n mod d = 0.

**Exercise 8** The divisibility relation is a partial order on  $\mathbb{N}$ .

## 2 Assertions on Sets and Functions

As for natural numbers, specify the following properties in a B abstract machine that only contains an ASSERTIONS clause.

**Exercise 9** Every finite set of integers is strictly contained in another finite set of integers.

**Exercise 10** Every finite set S of integers is contained in an interval of integers.

**Exercise 11** For every natural number  $n \neq 0$ , the set of all divisors of n is finite.

**Exercise 12** Every total injective function from  $\mathbb{N}$  to  $\mathbb{N}$  is bijective.

**Exercise 13** For every finite subset S of  $\mathbb{N}$ , the sum of all elements of S is larger or equal to each element of S.

The following exercises specify properties on an arbitrary set. To this end, add the following declaration in your abstract machine:

MACHINE SetAsserts SETS UNIVERSE ASSERTIONS END

In the following exercises, the set UNIVERSE is shortly written U. Recall that the *cardinal* of a finite set S is the number of elements of S. We let Card(S) denote the cardinal of S.

**Exercise 14** For every finite subsets S, T of  $\mathbb{U}$ , if  $S \subseteq T$  then  $Card(S) \leq Card(T)$ 

**Exercise 15** For every finite subsets S, T of  $\mathbb{U}$ ,  $Card(S) \leq Card(T)$  if and only if there exists a total injection from S to T.

**Exercise 16** For every finite subsets S, T of  $\mathbb{U}$ ,  $Card(S) \ge Card(T)$  if and only if there exists a total surjection from S to T.

**Exercise 17** The relation  $\subseteq$  is a partial order on  $\mathbb{P}(\mathbb{U})$ .

**Exercise 18** Every non-decreasing function  $f : \mathbb{P}(\mathbb{U}) \to \mathbb{P}(\mathbb{U})$  has a fixed point.