

The amazing mixed polynomial closure and its applications to two-variable first-order logic

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ABSTRACT

Polynomial closure is a standard operator which is applied to a class of regular languages. In this paper, we investigate three restrictions called left (*LPol*), right (*RPol*) and mixed polynomial closure (*MPol*). The first two were known while *MPol* is new. We look at two decision problems that are defined for every class \mathcal{C} . Membership takes a regular language as input and asks if it belongs to \mathcal{C} . Separation takes two regular languages as input and asks if there exists a third language in \mathcal{C} including the first one and disjoint from the second. We prove that *LPol*, *RPol* and *MPol* preserve the decidability of membership under mild hypotheses on the input class, and the decidability of separation under much stronger hypotheses. We apply these results to natural hierarchies.

First, we look at several language theoretic hierarchies that are built by applying *LPol*, *RPol* and *MPol* recursively to a single input class. We prove that these hierarchies can actually be defined using almost exclusively *MPol*. We also consider quantifier alternation hierarchies for *two-variable* first-order logic (FO^2) and prove that one can climb them using *MPol*. The result is generic in the sense that it holds for most standard choices of signatures. We use it to prove that for most of these choices, membership is decidable for all levels in the hierarchy. Finally, we prove that separation is decidable for the hierarchy of two-variable first-order logic equipped with only the linear order ($\text{FO}^2(<)$).

CCS CONCEPTS

• Theory of computation → Regular languages; Finite Model Theory.

KEYWORDS

polynomial closure, two-variable first-order logic, quantifier alternation, deterministic hierarchies, separation

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1 INTRODUCTION

This paper is part of a research program whose aim is to investigate natural subclasses of the regular languages of finite words. We are interested in the classes associated to a piece of syntax which is used to define their languages (such as regular expressions or logic). For each class \mathcal{C} , we look at two decision problems. First, \mathcal{C} -membership takes a regular language L as input and asks if $L \in \mathcal{C}$. Also, \mathcal{C} -separation takes two regular languages H, L as input and asks if there exists $K \in \mathcal{C}$ such that $H \subseteq K$ and $K \cap L = \emptyset$. In practice, getting algorithms for these problems requires techniques that cannot be developed without a solid understanding of \mathcal{C} .

We consider generic families of classes. Let us use logic to clarify. Each logical fragment is associated to several classes defined by choosing a *signature* (i.e., a set of predicates one may use in formulas). For instance, in the literature, several classes are associated to first-order logic (FO) by considering natural predicates such as the linear order “ $<$ ” [19, 32], successor “ $+1$ ” [4] or modular predicates “*MOD*” [3]. Hence, a generic approach is desirable. This typically involves two independent steps. The first one consists in characterizing a particular fragment by an *operator* on classes. For example, first-order logic corresponds to *star-free closure*: which builds the least class $\text{SF}(\mathcal{C})$ containing an input class \mathcal{C} and closed under union, complement and concatenation. It was shown [19, 27, 36] that if \mathcal{C} is a *Boolean algebra closed under quotients* (we call this a prevariety), there exists a signature $\mathbb{I}_{\mathcal{C}}$ such that $\text{SF}(\mathcal{C})$ corresponds to $\text{FO}(\mathbb{I}_{\mathcal{C}})$. This captures most of the natural signature choices. The second step consists in proving that the operator preserves the decidability of membership/separation. It was proved that SF [28] preserves the decidability of separation when applied to a prevariety containing only *group languages*. These are the languages recognized by a finite group, or equivalently by a permutation automaton (i.e., a complete, deterministic and co-deterministic automaton). This implies that separation is decidable for variants of FO such as $\text{FO}(<)$ or $\text{FO}(<, \text{MOD})$.

We investigate restrictions of a well-known operator: *polynomial closure*. For an input class \mathcal{C} , it builds the least class $\text{Pol}(\mathcal{C})$ containing all finite unions of marked products $K_0 a_1 K_1 \cdots a_n K_n$ with a_1, \dots, a_n letters and $K_0, \dots, K_n \in \mathcal{C}$. We look at variants obtained by imposing semantic restrictions on the products. A marked product $K_0 a_1 K_1 \cdots a_n K_n$ is *unambiguous* if for all $w \in K_0 a_1 K_1 \cdots a_n K_n$, the decomposition of w witnessing this membership is *unique*. This defines *unambiguous polynomial closure* (*UPol*) which is well-understood [20, 22, 26]. We look at stronger restrictions. For a

marked product $K_0 a_1 K_1 \cdots a_n K_n$, we let $L_i = K_0 a_1 K_1 \cdots a_{i-1} K_{i-1}$ and $R_i = K_i a_{i+1} \cdots K_{n-1} a_n K_n$ for all $i \leq n$. The whole marked product is *left* (resp. *right*) deterministic if for all $i \leq n$, $L_i a_i A^*$ (resp. $A^* a_i R_i$) is unambiguous. It is *mixed deterministic* if for all $i \leq n$, either $L_i a_i A^*$ or $A^* a_i R_i$ is unambiguous. This leads to three operators: *left*, *right* and *mixed polynomial closure* ($LPol$, $RPol$ and $MPol$). Historically, $LPol$ and $RPol$ are well-known. They were first investigated by Schützenberger [33] and Pin [20, 21]. On the other hand, $MPol$ is new. We first prove that these operators have robust properties which are similar to those proved for $UPol$ in [26]. First, they preserve the closure properties of input classes: if \mathcal{C} is a prevariety, then so are $LPol(\mathcal{C})$, $RPol(\mathcal{C})$ and $MPol(\mathcal{C})$. Moreover, we prove that if \mathcal{C} has decidable membership, then this is also the case for $LPol(\mathcal{C})$, $RPol(\mathcal{C})$ and $MPol(\mathcal{C})$.

We look at hierarchies that are built with these operators. In general, $LPol(\mathcal{C})$ and $RPol(\mathcal{C})$ are incomparable. Thus, given an input class \mathcal{C} , two hierarchies can be built. The first levels are $LPol(\mathcal{C})$ and $RPol(\mathcal{C})$, then for all $n > 1$, the levels $LP_n(\mathcal{C})$ and $RP_n(\mathcal{C})$ are defined as $LPol(RP_{n-1}(\mathcal{C}))$ and $RPol(LP_{n-1}(\mathcal{C}))$. One may also define combined levels $LP_n(\mathcal{C}) \cap RP_n(\mathcal{C})$ (the languages belonging to both classes) and $LP_n(\mathcal{C}) \vee RP_n(\mathcal{C})$ (the least Boolean algebra containing both classes). It follows from results of [26] that the union of all levels is $UPol(\mathcal{C})$. In the literature, this construction is well-known for a specific input class: the piecewise testable languages PT [34] (*i.e.*, the Boolean combinations of marked products $A^* a_1 A^* \cdots a_n A^*$). This hierarchy is strict and has characterizations based on algebra [15, 37] and logic [16, 17]. By definition, each hierarchy contains four distinct kinds of levels. Yet, we prove that their construction process can be unified: all four kinds can be climbed from the first level by using only $MPol$. For example, we show that for every $n > 1$, $MPol(LP_{n-1}(\mathcal{C}) \vee RP_{n-1}(\mathcal{C})) = LP_n(\mathcal{C}) \vee RP_n(\mathcal{C})$. This makes the investigation of such hierarchies easier.

In the second part of the paper, we investigate the quantifier alternation hierarchies of *two-variable* first-order logic (FO^2). The fragment FO^2 contains the first-order formulas using at most two distinct reusable variables. For all $n \geq 1$, we let $\mathcal{B}\Sigma_n^2$ as the set of all FO^2 formulas such that each branch in their parse trees contains at most n blocks of alternating quantifiers “ \exists ” and “ \forall ”. There are important classes associated to these fragments and several of them are prominent in the literature. Historically, the full logic FO^2 was first considered. It is known that membership is decidable for the variants $FO^2(<)$ and $FO^2(<, +1)$ equipped with the linear order and successor [35], as well as for $FO^2(<, MOD)$ equipped with modular predicates [6]. For quantifier alternation, it is known that membership is decidable for all levels $\mathcal{B}\Sigma_n^2(<)$ [11, 16, 17] and $\mathcal{B}\Sigma_n^2(<, +1)$ [14]. Note that while the arguments are related these results involve tailored proofs for each particular choice of signature. In this paper, we develop a generic approach based on $MPol$ and look at a family of signatures. Given a prevariety \mathcal{G} containing only group languages, we associate a generic set of predicates $\mathbb{P}_{\mathcal{G}}$. For every $L \in \mathcal{G}$, it contains a unary predicate $P_L(x)$: it checks if the prefix preceding a given position belongs to L . We consider all signatures of the form $\{<, \mathbb{P}_{\mathcal{G}}\}$ or $\{<, +1, \mathbb{P}_{\mathcal{G}}\}$. This captures most of the natural examples such as $\{<\}$, $\{<, +1\}$, $\{<, MOD\}$, or $\{<, +1, MOD\}$ (we present other examples in this

paper). We prove that if \mathbb{S} is one of the two above kinds of signatures, one may climb the quantifier alternation hierarchy of $FO^2(\mathbb{S})$ using $MPol$: $\mathcal{B}\Sigma_{n+1}^2(\mathbb{S}) = MPol(\mathcal{B}\Sigma_n^2(\mathbb{S}))$ for all $n \geq 1$. This also implies that $FO^2(\mathbb{S}) = UPol(\mathcal{B}\Sigma_1^2(\mathbb{S}))$. Hence, we obtain a *generic* language theoretic characterization of FO^2 and its quantifier alternation hierarchy which holds for many natural signature choices. Moreover, $MPol$ and $UPol$ preserve the decidability of membership and independent results [31] state that if \mathbb{S} is a signature built from a group prevariety \mathcal{G} as above, then membership for $\mathcal{B}\Sigma_1^2(\mathbb{S})$ boils down to *separation* for \mathcal{G} (these results exploit the fact that $\mathcal{B}\Sigma_1^2(\mathbb{S}) = \mathcal{B}\Sigma_1(\mathbb{S})$, the level one in the hierarchy of *full* first-order logic). Altogether, it follows that membership is decidable for all classes captured by our results as soon as *separation* is decidable for the input group prevariety \mathcal{G} . This improves the aforementioned results and yields new ones.

In the last part of the paper, we come back to $LPol$, $RPol$ and $MPol$. We look at *separation* and prove that if \mathcal{C} is a *finite* prevariety and \mathcal{D} is a prevariety with decidable separation such that $\mathcal{C} \subseteq \mathcal{D} \subseteq UPol(\mathcal{C})$, then separation is decidable for $LPol(\mathcal{D})$, $RPol(\mathcal{D})$ and $MPol(\mathcal{D})$. This is weaker than what we have for membership since \mathcal{C} must be *finite*. Yet, we detail a key application: the prevariety PT of piecewise testable languages. While PT is infinite, it is known and simple to verify that $AT \subseteq PT \subseteq UPol(AT)$ where AT is the finite prevariety of alphabet testable languages (*i.e.*, the Boolean combinations of languages $A^* a A^*$). Since PT-separation is decidable [5, 24], a simple induction yields the decidability of separation for all classes that can be built recursively from PT by applying $LPol$, $RPol$ and $MPol$. This includes all levels $LP_n(AT)$ and $RP_n(AT)$. Moreover, since $PT = \mathcal{B}\Sigma_1^2(<)$, this can be combined with our generic language theoretic characterization of quantifier alternation for FO^2 to prove that $\mathcal{B}\Sigma_n^2(<)$ -separation is decidable for every $n \geq 1$.

In Section 2, we introduce the definitions and standard tools that we shall need. In Section 3, we present $LPol$, $RPol$ and $MPol$ and their properties. In Section 4, we present algebraic characterizations of these operators which imply that they preserve the decidability of membership. We discuss the language theoretic hierarchies that can be built with our operators in Section 5. We turn to logic in Section 6 and use $MPol$ to characterize the quantifier alternation hierarchies of two-variable first-order logic. Finally, Section 7 is devoted to the separation problem. Due to space limitations, several proofs are postponed to the full version of the paper.

2 PRELIMINARIES

We present terminology that we use throughout the paper. The statements are proved in the full version of the paper.

2.1 Finite words and classes of languages

We fix an arbitrary finite alphabet A for the whole paper. As usual, A^* denotes the set of all words over A , including the empty word ε . We let $A^+ = A^* \setminus \{\varepsilon\}$. For $u, v \in A^*$, we write uv the word obtained by concatenating u and v . Also, if $w \in A^*$, we write $|w| \in \mathbb{N}$ for its length. We also consider *positions*. A word $w = a_1 \cdots a_{|w|} \in A^*$ is viewed as an *ordered set* $P(w) = \{0, 1, \dots, |w|, |w| + 1\}$ of $|w| + 2$ positions. A position i such that $1 \leq i \leq |w|$ carries the label $a_i \in A$. We write $P_c(w) = \{1, \dots, |w|\}$ for this set of labeled positions. On

the other hand, the positions 0 and $|w| + 1$ are *artificial* leftmost and rightmost positions which carry *no label*. Finally, given a word $w = a_1 \cdots a_{|w|} \in A^*$ and $i, j \in P(w)$ such that $i < j$, we write $w(i, j) = a_{i+1} \cdots a_{j-1} \in A^*$ (i.e., the infix obtained by keeping the letters carried by the positions that are *strictly* between i and j). Note that $w(0, |w| + 1) = w$.

A *language* is a subset of A^* . We lift concatenation to languages: for $K, L \subseteq A^*$, $KL = \{uv \mid u \in K \text{ and } v \in L\}$.

Regular languages and morphisms. All languages that we consider in this paper are *regular*. These are the languages which can be defined by a finite automaton or a morphism into a finite monoid. We work with the latter definition which we recall now. A *semigroup* is a pair (S, \cdot) where S is a set and “ \cdot ” is an associative binary operation on S (often called multiplication). It is standard to abuse terminology and make the binary operation implicit: one simply says that “ S is a semigroup”. A *monoid* M is a semigroup whose multiplication has a neutral element denoted by “ 1_M ”. Recall that an idempotent of a semigroup S is an element $e \in S$ such that $ee = e$. A standard result in semigroup theory states that when S is *finite*, there exists $\omega(S) \in \mathbb{N}$ (written ω when S is understood) such that s^ω is idempotent for every $s \in S$.

Clearly, A^* is a monoid whose multiplication is concatenation (ε is the neutral element). Thus, given a monoid M , we may consider morphisms $\alpha : A^* \rightarrow M$. A language $L \subseteq A^*$ is *recognized* by such a morphism α when there exists $F \subseteq M$ such that $L = \alpha^{-1}(F)$. It is well-known that the regular languages are exactly those which can be recognized by a morphism $\alpha : A^* \rightarrow M$ where M is a *finite* monoid.

REMARK 2.1. *Since the only infinite monoid that we consider is A^* , we implicitly assume that every arbitrary monoid M, N, \dots that we consider is finite from now on.*

We shall also consider the standard Green relations that one may associate to every monoid M . Given $s, t \in M$, we write $s \leq_{\mathcal{R}} t$ if there exists $r \in M$ such that $s = tr$. Moreover, $s \leq_{\mathcal{L}} t$ if there exists $q \in M$ such that $s = qt$. Finally, $s \leq_{\mathcal{J}} t$ if there exist $q, r \in M$ such that $s = qtr$. One may verify that these are preorders. We write \mathcal{R} , \mathcal{L} and \mathcal{J} for the equivalences associated to $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{J}}$ (for example, $s \mathcal{R} t$ when $s \leq_{\mathcal{R}} t$ and $t \leq_{\mathcal{R}} s$). Finally, we write $<_{\mathcal{R}}$, $<_{\mathcal{L}}$ and $<_{\mathcal{J}}$ for the strict variants of these preorders (for example, $s <_{\mathcal{R}} t$ when $s \leq_{\mathcal{R}} t$ and $s \neq t$).

Classes and decision problems. A *class of languages* \mathcal{C} is a set of languages. A *lattice* is a class which is closed under both union and intersection, and containing the languages \emptyset and A^* . Moreover, a *Boolean algebra* is a lattice closed under complement. Finally, a class \mathcal{C} is *quotient-closed* when for all $L \in \mathcal{C}$ and all $u, v \in A^*$, the language $\{w \in A^* \mid u w v \in L\}$ belongs to L . Finally, we say that a class \mathcal{C} is a *prevariety* to indicate that it is a quotient-closed Boolean algebra containing only *regular languages*. In this paper, we investigate specific prevarieties. For this purpose, we rely on two decision problems that one may associate to a fixed class \mathcal{C} . The key idea is that finding an algorithm for each of these problems provides a solid understanding of \mathcal{C} .

The most simple problem, *\mathcal{C} -membership*, takes as input a regular language L and asks whether $L \in \mathcal{C}$. We turn to the second problem. Given two languages L_0 and L_1 , we say that L_0 is *\mathcal{C} -separable*

from L_1 if there exists $K \in \mathcal{C}$ such that $L_0 \subseteq K$ and $L_1 \cap K = \emptyset$. The *\mathcal{C} -separation* problem takes as input two regular languages L_0 and L_1 and asks whether L_0 is *\mathcal{C} -separable* from L_1 . Note that *\mathcal{C} -membership* can be reduced to *\mathcal{C} -separation*: clearly, $L \in \mathcal{C}$ if and only if L is *\mathcal{C} -separable* from $A^* \setminus L$ (here, we use the fact that the regular languages are closed under complement).

Group languages. When applying our results to logic, we shall consider a particular kind of class. A *group* is a monoid G such that each $g \in G$ has an inverse $g^{-1} \in G$, i.e., $gg^{-1} = g^{-1}g = 1_G$. A “*group language*” is a language which is recognized by a morphism into a *finite group*. In Section 6, we shall consider classes \mathcal{G} that are *group prevarieties* (i.e., containing group languages only).

Additionally, we shall consider “extensions” of the group prevarieties. One may verify from the definition that $\{\varepsilon\}$ and A^+ are *not* group languages. This motivates the following notion: for a class \mathcal{C} , the *well-suited extension* of \mathcal{C} , denoted by \mathcal{C}^+ , consists of all languages of the form $L \cap A^+$ or $L \cup \{\varepsilon\}$ where $L \in \mathcal{C}$. One may verify that when \mathcal{C} is a prevariety, \mathcal{C}^+ is a prevariety as well.

2.2 \mathcal{C} -morphisms

Let \mathcal{C} be a prevariety. A *\mathcal{C} -morphism* is a *surjective* morphism $\eta : A^* \rightarrow N$ such that every language recognized by η belongs to \mathcal{C} . This notion serves as a key mathematical tool in this paper. First, we use it for the membership problem.

Given a regular language L , one may associate a canonical morphism recognizing L . Let us briefly recall the definition. We associate a relation \equiv_L on A^* to L . Given $u, v \in A^*$, we have $u \equiv_L v$ if and only if $xuy \in L \Leftrightarrow xvy \in L$ for every $x, y \in A^*$. It can be verified that \equiv_L is a congruence of A^* and, since L is regular, that it has finite index. Therefore, the map $\alpha : A^* \rightarrow A^*/\equiv_L$ which associates its \equiv_L -class to each word is a morphism into a finite monoid. It is called the *syntactic morphism of L* and it can be computed from any representation of L . We have the following standard result which connects it to *\mathcal{C} -membership*.

PROPOSITION 2.2. *Let \mathcal{C} be a prevariety. A regular language belongs to \mathcal{C} iff its syntactic morphism is a \mathcal{C} -morphism.*

By Proposition 2.2, getting an algorithm for *\mathcal{C} -membership* boils down to finding a procedure which decides if some input morphism $\alpha : A^* \rightarrow M$ is a *\mathcal{C} -morphism*. This is how we approach the question in this paper. We shall also use *\mathcal{C} -morphisms* as mathematical tools in proof arguments. In this context, we shall use the following statement which is a simple corollary of Proposition 2.2.

PROPOSITION 2.3. *Let \mathcal{C} be a prevariety and consider finitely many languages $L_1, \dots, L_k \in \mathcal{C}$. There exists a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ such that L_1, \dots, L_k are recognized by η .*

2.3 Canonical relations

Given a prevariety \mathcal{C} and a morphism $\alpha : A^* \rightarrow M$, we associate two relations on M . The definitions are adapted from notions introduced in [26, 27]. For the first one, we say that (s, t) is a *\mathcal{C} -pair* (for α) if and only if $\alpha^{-1}(s)$ is *not* *\mathcal{C} -separable* from $\alpha^{-1}(t)$. The *\mathcal{C} -pair* relation is not very robust. It is symmetric (this is tied to closure under complement for \mathcal{C}) and reflexive (if α is surjective). However, it is *not* transitive in general.

We turn to the second relation. Let $s, t \in M$. We write $s \sim_{\mathcal{C}, \alpha} t$ if and only if $s \in F \Leftrightarrow t \in F$ for all $F \subseteq M$ such that $\alpha^{-1}(F) \in \mathcal{C}$. It is immediate by definition that $\sim_{\mathcal{C}, \alpha}$ is an equivalence. For the sake of avoiding clutter, we shall abuse terminology when the morphism α is understood and write $\sim_{\mathcal{C}}$ for $\sim_{\mathcal{C}, \alpha}$. Additionally, for every element $s \in M$, we write $[s]_{\mathcal{C}} \in M/\sim_{\mathcal{C}}$ for the $\sim_{\mathcal{C}}$ -class of s . Observe that by definition, computing $\sim_{\mathcal{C}, \alpha}$ boils down to computing the sets $F \subseteq M$ such that $\alpha^{-1}(F) \in \mathcal{C}$, i.e. to \mathcal{C} -membership.

FACT 2.4. *Let \mathcal{C} be a prevariety with decidable membership. Given as input a morphism $\alpha : A^* \rightarrow M$, one may compute the equivalence $\sim_{\mathcal{C}, \alpha}$ on M .*

We now connect our two relations in the following lemma.

LEMMA 2.5. *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ be a morphism. The equivalence $\sim_{\mathcal{C}, \alpha}$ on M is the reflexive transitive closure of the \mathcal{C} -pair relation associated to α .*

It can be verified that when the morphism α is surjective, the equivalence $\sim_{\mathcal{C}, \alpha}$ is a congruence of the monoid M .

LEMMA 2.6. *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ be a surjective morphism. Then, $\sim_{\mathcal{C}, \alpha}$ is a congruence of M .*

In view of Lemma 2.6, when $\alpha : A^* \rightarrow M$ is surjective, the map $[\cdot]_{\mathcal{C}} : M \rightarrow M/\sim_{\mathcal{C}}$ which associates its $\sim_{\mathcal{C}}$ -class to every element in M is a morphism. A key property is that the composition $[\cdot]_{\mathcal{C}} \circ \alpha : A^* \rightarrow M/\sim_{\mathcal{C}}$ is a \mathcal{C} -morphism.

LEMMA 2.7. *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ be a surjective morphism. The languages recognized by $[\cdot]_{\mathcal{C}} \circ \alpha : A^* \rightarrow M/\sim_{\mathcal{C}}$ are exactly those which are simultaneously in \mathcal{C} and recognized by α .*

3 OPERATORS

We introduce the operators that we investigate in this paper. We first recall the definition of standard polynomial closure. Then, we define four semantic restrictions

3.1 Polynomial closure

Given finitely many languages $L_0, \dots, L_n \subseteq A^*$, a *marked product* of L_0, \dots, L_n is a product of the form $L_0 a_1 L_1 \cdots a_n L_n$ where $a_1, \dots, a_n \in A$. Note that a single language L_0 is a marked product (this is the case $n = 0$). In the case $n = 1$ (i.e., there are two languages), we speak of *marked concatenations*.

The *polynomial closure* of a class \mathcal{C} , denoted by $Pol(\mathcal{C})$, is the class containing all *finite unions* of marked products $L_0 a_1 L_1 \cdots a_n L_n$ such that $L_0, \dots, L_n \in \mathcal{C}$. If \mathcal{C} is a prevariety, $Pol(\mathcal{C})$ is a quotient-closed lattice (this is due to Arfi [2], see also [21, 27] for recent proofs). On the other hand, $Pol(\mathcal{C})$ need not be closed under complement. Hence, it is natural to combine Pol with another operator. The Boolean closure of a class \mathcal{D} , denoted by $Bool(\mathcal{D})$, is the least Boolean algebra containing \mathcal{D} . Finally, we write $BPol(\mathcal{C})$ for $Bool(Pol(\mathcal{C}))$. The following proposition is standard (see [27] for example).

PROPOSITION 3.1. *If \mathcal{C} is a prevariety, then so is $BPol(\mathcal{C})$.*

We do not investigate $BPol$ itself. Yet, we use the classes $BPol(\mathcal{C})$ as inputs for the operators that we do investigate.

3.2 Deterministic restrictions

We define weaker operators from Pol by restricting the marked products to those satisfying specific semantic conditions and the finite unions to *disjoint* ones.

Consider a marked product $L_0 a_1 L_1 \cdots a_n L_n$. For $1 \leq i \leq n$, we define $L'_i = L_0 a_1 L_1 \cdots a_{i-1} L_{i-1}$ (in particular, $L'_1 = L_0$) and $L''_i = L_i a_{i+1} L_{i+1} \cdots a_n L_n$ (in particular, $L''_n = L_n$). We say that,

- $L_0 a_1 L_1 \cdots a_n L_n$ is *left deterministic* if and only if for all i such that $1 \leq i \leq n$, we have $L'_i \cap L'_i a_i A^* = \emptyset$.
- $L_0 a_1 L_1 \cdots a_n L_n$ is *right deterministic* if and only if for all i such that $1 \leq i \leq n$, we have $L''_i \cap A^* a_i L''_i = \emptyset$.
- $L_0 a_1 L_1 \cdots a_n L_n$ is *mixed deterministic* if and only if for all i such that $1 \leq i \leq n$, either $L'_i \cap L'_i a_i A^* = \emptyset$ or $L''_i \cap A^* a_i L''_i = \emptyset$.
- $L_0 a_1 L_1 \cdots a_n L_n$ is *unambiguous* if and only if for every word $w \in L_0 a_1 L_1 \cdots a_n L_n$, there exists a *unique* decomposition $w = w_0 a_1 w_1 \cdots a_n w_n$ with $w_i \in L_i$ for $1 \leq i \leq n$.

These notions depend on the product itself and not only on the resulting language. For example, the marked concatenations $A^* a A^*$ (which is not unambiguous) and $(A \setminus \{a\})^* a A^*$ (which is left deterministic) evaluate to the same language. By definition, a left or right deterministic marked product is also mixed deterministic. It is also simple to verify that mixed deterministic marked products are unambiguous.

REMARK 3.2. *A mixed deterministic product need not be left or right deterministic. For example, let $L_1 = (ab)^+$, $L_2 = c^+$ and $L_3 = (ba)^+$. The product $L_1 c L_2 c L_3$ is mixed deterministic since $L_1 \cap L_1 c A^* = \emptyset$ and $L_3 \cap A^* c L_3 = \emptyset$. However, it is neither left deterministic nor right deterministic. Similarly, an unambiguous product need not be mixed deterministic. If $L_4 = (ca)^+$, the product $L_1 a L_4$ is unambiguous but it neither left nor right deterministic.*

The *left polynomial closure* of a class \mathcal{C} , written $LPol(\mathcal{C})$, contains the *finite disjoint unions* of *left deterministic marked products* $L_0 a_1 L_1 \cdots a_n L_n$ such that $L_0, \dots, L_n \in \mathcal{C}$ (by “disjoint” we mean that the languages in the union must be pairwise disjoint). The *right polynomial closure* of \mathcal{C} ($RPol(\mathcal{C})$), the *mixed polynomial closure* of \mathcal{C} ($MPol(\mathcal{C})$) and the *unambiguous polynomial closure* of \mathcal{C} ($UPol(\mathcal{C})$) are defined analogously by replacing the “left deterministic” requirement on marked products by the appropriate one. The following lemma can be verified from the definition.

LEMMA 3.3. *Let \mathcal{C} be a class. Then, we have $LPol(\mathcal{C}) \subseteq MPol(\mathcal{C})$, $RPol(\mathcal{C}) \subseteq MPol(\mathcal{C})$ and $MPol(\mathcal{C}) \subseteq UPol(\mathcal{C}) \subseteq Pol(\mathcal{C})$.*

The operators $LPol$, $RPol$ and $UPol$ are fairly standard. See for example [20, 22, 33]. In particular, they admit the following standard alternate definition (see [21] for a proof).

LEMMA 3.4. *Let \mathcal{C} be a class. Then, $LPol(\mathcal{C})$ (resp. $RPol(\mathcal{C})$, $UPol(\mathcal{C})$) is the least class containing \mathcal{C} which is closed under disjoint union and left deterministic (resp. right deterministic, unambiguous) marked concatenation.*

On the other hand, $MPol$ is new. It is arguably the key notion of the paper. In particular, the application to two-variable first-order logic is based on it (see Section 6). Unfortunately, it is less robust than the other operators: there is no equivalent to Lemma 3.4 for $MPol$. In particular, it is not idempotent: in general $MPol(\mathcal{C})$

is *strictly included* in $MPol(MPol(\mathcal{C}))$. This is because a mixed product of mixed products is *not* a mixed product itself in general.

Example 3.5. Consider the alphabet $A = \{a, b, c\}$. We let $L_0 = b^+$, $L_1 = a^+$ and $K = (a + b + c)^+$. Clearly, L_0bL_1 and K are defined by mixed deterministic products. Moreover, if $L = L_0bL_1$, then LcK is mixed deterministic. However, the combined product L_0bL_1cK is not mixed deterministic itself. Indeed, the marked concatenation $(L_0b)(L_1cK)$ is neither left deterministic nor right deterministic.

Note that $UPol$ is well-understood. In particular, we shall use two key results from [26]. First, while this is not apparent on the definition, $UPol(\mathcal{C})$ has robust properties.

THEOREM 3.6 ([26]). *If \mathcal{C} is a prevariety, then so is $UPol(\mathcal{C})$.*

Moreover, we have the following generic characterization of the $UPol(\mathcal{C})$ -morphisms.

THEOREM 3.7 ([26]). *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ a surjective morphism. The following are equivalent:*

- α is a $UPol(\mathcal{C})$ -morphism.
- $s^{\omega+1} = s^\omega t s^\omega$ for all \mathcal{C} -pairs $(s, t) \in M^2$.
- $s^{\omega+1} = s^\omega t s^\omega$ for all $s, t \in M$ such that $s \sim_{\mathcal{C}} t$.

By Fact 2.4, the equivalence $\sim_{\mathcal{C}}$ can be computed from α as soon as \mathcal{C} -membership is decidable. Hence, in view of Proposition 2.2, Theorem 3.7 implies that when \mathcal{C} -membership is decidable, then so is $UPol(\mathcal{C})$ -membership. We prove similar results for $LPol$, $RPol$ and $MPol$ in Section 4.

3.3 Framework

We present a general framework and use it to characterize the languages in $LPol(\mathcal{C})$, $RPol(\mathcal{C})$ and $MPol(\mathcal{C})$ for a prevariety \mathcal{C} (the statements are proved in the full version of the paper). We rely on it whenever we manipulate these classes in proof arguments. In particular, we apply it at the end of the section to generalize Theorem 3.6 to $LPol$, $RPol$ and $MPol$.

Sets of positions associated to a morphism. Given an arbitrary surjective morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$, we use the Green relations of N to associate three sets of positions to every $w \in A^*$. Let $w = a_1 \cdots a_\ell \in A^*$. We define two sets $P_{\triangleright}(\eta, k, w) \subseteq P_c(w)$ and $P_{\triangleleft}(\eta, k, w) \subseteq P_c(w)$ by induction on k . When $k = 0$, we define $P_{\triangleright}(\eta, 0, w) = P_{\triangleleft}(\eta, 0, w) = \emptyset$. Assume now that $k \geq 1$ and let $i \in P_c(w)$. We let,

- $i \in P_{\triangleright}(\eta, k, w)$ if there is $j \in P_{\triangleright}(\eta, k-1, w) \cup \{0\}$ such that $j < i$ and $\eta(w(j, i)a_i) <_{\mathcal{G}} \eta(w(j, i))$.
- $i \in P_{\triangleleft}(\eta, k, w)$ if there is $j \in P_{\triangleleft}(\eta, k-1, w) \cup \{|w|+1\}$ such that $i < j$ and $\eta(a_i w(i, j)) <_{\mathcal{G}} \eta(w(i, j))$.

Finally, we let $P_{\bowtie}(\eta, k, w) = P_{\triangleright}(\eta, k, w) \cup P_{\triangleleft}(\eta, k, w)$ for every $k \in \mathbb{N}$. We turn to an important lemma which we shall use when considering $UPol(\mathcal{C})$ -morphisms. The proof is based on Theorem 3.7.

LEMMA 3.8. *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ a $UPol(\mathcal{C})$ -morphism. For every $h \in \mathbb{N}$ and every word $w \in A^*$, $P_{\triangleright}(\alpha, h, w) \subseteq P_{\triangleright}([\cdot]_{\mathcal{C}} \circ \alpha, h|M|, w)$ and $P_{\triangleleft}(\alpha, h, w) \subseteq P_{\triangleleft}([\cdot]_{\mathcal{C}} \circ \alpha, h|M|, w)$.*

We turn to an independent definition that we shall use conjointly with the first one. Consider a surjective morphism $\eta : A^* \rightarrow N$. Given a word $w = a_1 \cdots a_\ell \in A^*$ and a set $P \subseteq P_c(w)$, we use η to

associate a tuple in $N \times (A \times N)^{|P|}$ that we call the η -snapshot of (w, P) . For the definition, we write $k = |P|$ and let $i_1 < \cdots < i_k$ be the positions such that $P = \{i_1, \dots, i_k\}$. Finally, we let $i_0 = 0$ and $i_{k+1} = |w| + 1$. For $0 \leq h \leq k$, we let $s_h = \eta(w(i_h, i_{h+1})) \in N$. The η -snapshot of (w, P) , denoted by $\sigma_\eta(w, P)$, is the following tuple:

$$\sigma_\eta(w, P) = (s_0, a_{i_1}, s_1, \dots, a_{i_k}, s_k) \in N \times (A \times N)^k.$$

We connect the two notions with the following key lemma.

LEMMA 3.9. *Let $\eta : A^* \rightarrow N$ be a surjective morphism, $k \in \mathbb{N}$ and $x \in \{\triangleright, \triangleleft, \bowtie\}$. Consider two words $w, w' \in A^*$ and $P' \subseteq P_c(w')$. If $\sigma_\eta(w, P_x(\eta, k, w)) = \sigma_\eta(w', P')$, then $P' = P_x(\eta, k, w')$.*

We may now connect these notions to our three operators $LPol$, $RPol$ and $MPol$.

LEMMA 3.10. *Let $\eta : A^* \rightarrow N$ be a morphism, $w \in A^*$ and $k \in \mathbb{N}$. Let P be the set $P_{\triangleright}(\eta, k, w)$ (resp. $P_{\triangleleft}(\eta, k, w)$, $P_{\bowtie}(\eta, k, w)$) and $(s_0, a_1, s_1, \dots, a_n, s_n) = \sigma_\eta(w, P)$. Then, the marked product $\eta^{-1}(s_0)a_1\eta^{-1}(s_1) \cdots a_n\eta^{-1}(s_n)$ is left (resp. right, mixed) deterministic.*

Finally, we present an important property of these sets which is specific to the case when $\eta : A^* \rightarrow N$ is either a $BPol(\mathcal{G})$ - or a $BPol(\mathcal{G}^+)$ -morphism for some group prevariety \mathcal{G} . We prove that for such morphisms, we may restrict ourselves to the special case when $k = 1$. This property will be crucial in Section 6 in order to prove the characterization of quantifier alternation for two-variable first-order logic with mixed polynomial closure.

PROPOSITION 3.11. *Let \mathcal{G} be a group prevariety and $\mathcal{C} \in \{\mathcal{G}, \mathcal{G}^+\}$. If $\eta : A^* \rightarrow N$ is a $BPol(\mathcal{C})$ -morphism and $k \in \mathbb{N}$, there exists a $BPol(\mathcal{C})$ -morphism, $\gamma : A^* \rightarrow Q$ such that $P_{\triangleright}(\eta, k, w) \subseteq P_{\triangleright}(\gamma, 1, w)$ and $P_{\triangleleft}(\eta, k, w) \subseteq P_{\triangleleft}(\gamma, 1, w)$.*

Equivalence relations. We define equivalences and use them to characterize the classes built with $LPol$, $RPol$ and $MPol$. Consider a surjective morphism $\eta : A^* \rightarrow N$. For every $k \in \mathbb{N}$, we associate three equivalence relations $\triangleright_{\eta, k}$, $\triangleleft_{\eta, k}$ and $\bowtie_{\eta, k}$ on A^* . Consider $u, v \in A^*$. We define,

- $u \triangleright_{\eta, k} v$ if $\sigma_\eta(u, P_{\triangleright}(\eta, k, u)) = \sigma_\eta(v, P_{\triangleright}(\eta, k, v))$.
- $u \triangleleft_{\eta, k} v$ if $\sigma_\eta(u, P_{\triangleleft}(\eta, k, u)) = \sigma_\eta(v, P_{\triangleleft}(\eta, k, v))$.
- $u \bowtie_{\eta, k} v$ if $\sigma_\eta(u, P_{\bowtie}(\eta, k, u)) = \sigma_\eta(v, P_{\bowtie}(\eta, k, v))$.

It is immediate by definition that $\triangleright_{\eta, k}$, $\triangleleft_{\eta, k}$ and $\bowtie_{\eta, k}$ are equivalence relations. It turns out that they are actually *congruences of finite index*.

LEMMA 3.12. *If $\eta : A^* \rightarrow N$ is a surjective morphism and $k \in \mathbb{N}$, then $\triangleright_{\eta, k}$, $\triangleleft_{\eta, k}$ and $\bowtie_{\eta, k}$ are congruences of finite index.*

We are ready to characterize the classes built with $LPol$, $RPol$ and $MPol$ with these three equivalences.

PROPOSITION 3.13. *Let \mathcal{C} be a prevariety and $L \subseteq A^*$. Then, $L \in LPol(\mathcal{C})$ (resp. $L \in RPol(\mathcal{C})$, $L \in MPol(\mathcal{C})$) if and only if there exist a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$ such that L is a union of $\triangleright_{\eta, k}$ -classes (resp. $\triangleleft_{\eta, k}$ -classes, $\bowtie_{\eta, k}$ -classes).*

We complete the statement with a useful corollary. It strengthens the “only if” implication in Proposition 3.13.

COROLLARY 3.14. *Let \mathcal{C} be a prevariety and L_1, \dots, L_m finitely many languages in $LPol(\mathcal{C})$ (resp. $RPol(\mathcal{C})$, $MPol(\mathcal{C})$). There is a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$ such that L_1, \dots, L_m are unions of $\triangleright_{\eta,k}$ -classes (resp. $\triangleleft_{\eta,k}$ -classes, $\bowtie_{\eta,k}$ -classes).*

Let us present a first application. We use Proposition 3.13 to investigate closure properties.

THEOREM 3.15. *Let \mathcal{C} be a prevariety. Then, $LPol(\mathcal{C})$, $RPol(\mathcal{C})$ and $MPol(\mathcal{C})$ are prevarieties as well.*

PROOF. We present a proof for $MPol$ (the argument is symmetrical for $LPol$ and $RPol$). Let $K, L \in MPol(\mathcal{C})$ and $u, v \in A^*$. We show that $K \cup L$, $A^* \setminus L$ and $H = \{w \mid uwv \in L\}$ belong to $MPol(\mathcal{C})$. By Corollary 3.14, there exist a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$ such that K and L are unions of $\bowtie_{\eta,k}$ -classes. We show that $K \cup L$, $A^* \setminus L$ and H are also unions of $\bowtie_{\eta,k}$ -classes which completes the proof by Proposition 3.13. This is immediate for $K \cup L$ and $A^* \setminus L$. Hence, we concentrate on H . Given $w, w' \in A^*$ such that $w \bowtie_{\eta,k} w'$, we have to show that $w \in H \Leftrightarrow w' \in H$. Since $\bowtie_{\eta,k}$ is a congruence by Lemma 3.12, we have $uwv \bowtie_{\eta,k} uw'v$. Since L is a union of $\bowtie_{\eta,k}$ -classes, this implies that $uwv \in L \Leftrightarrow uw'v \in L$. Therefore, we get $w \in H \Leftrightarrow w' \in H$ by definition of H which completes the proof. \square

4 ALGEBRAIC CHARACTERIZATIONS

For every prevariety \mathcal{C} , we present generic algebraic characterizations for $LPol(\mathcal{C})$, $RPol(\mathcal{C})$ and $MPol(\mathcal{C})$. The statements are similar to the characterization of $UPol(\mathcal{C})$ presented in Theorem 3.7. First, we present symmetrical statements for $LPol$ and $RPol$.

THEOREM 4.1. *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ a surjective morphism. The following properties are equivalent:*

- α is an $LPol(\mathcal{C})$ -morphism.
- $s^{\omega+1} = s^\omega t$ for all \mathcal{C} -pairs $(s, t) \in M^2$.
- $s^{\omega+1} = s^\omega t$ for all $s, t \in M$ such that $s \sim_{\mathcal{C}} t$.

THEOREM 4.2. *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ a surjective morphism. The following properties are equivalent:*

- α is an $RPol(\mathcal{C})$ -morphism.
- $s^{\omega+1} = ts^\omega$ for all \mathcal{C} -pairs $(s, t) \in M^2$.
- $s^{\omega+1} = ts^\omega$ for all $s, t \in M$ such that $s \sim_{\mathcal{C}} t$.

Theorem 4.1 and Theorem 4.2 are proved in the full version of the paper. We concentrate on the more involved characterization of $MPol(\mathcal{C})$ which is as follows.

THEOREM 4.3. *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ a surjective morphism. The following properties are equivalent:*

- α is an $MPol(\mathcal{C})$ -morphism.
- $(sq)^\omega s(rs)^\omega = (sq)^\omega t(rs)^\omega$ for all \mathcal{C} -pairs $(s, t) \in M^2$ and all $q, r \in M$.
- $(sq)^\omega s(rs)^\omega = (sq)^\omega t(rs)^\omega$ for all $q, r, s, t \in M$ such that $s \sim_{\mathcal{C}} t$.

By Fact 2.4, one may compute the equivalence $\sim_{\mathcal{C}}$ associated to a morphism provided that \mathcal{C} -membership is decidable. Hence, in view of Proposition 2.2, we obtain the following corollary of Theorems 4.1, 4.2 and 4.3.

COROLLARY 4.4. *Let \mathcal{C} be a prevariety. If \mathcal{C} -membership is decidable, then so are $LPol(\mathcal{C})$ -, $RPol(\mathcal{C})$ - and $MPol(\mathcal{C})$ -membership.*

PROOF OF THEOREM 4.3. We fix a prevariety \mathcal{C} and a surjective morphism $\alpha : A^* \rightarrow M$. We start with $a) \Rightarrow b)$. Assume that α is an $MPol(\mathcal{C})$ -morphism. Let $q, r, s, t \in M$ such that (s, t) is a \mathcal{C} -pair. We show that $(sq)^\omega s(rs)^\omega = (sq)^\omega t(rs)^\omega$. Corollary 3.14 yields a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$ such that every language recognized by α is a union of $\bowtie_{\eta,k}$ -classes. Since (s, t) is a \mathcal{C} -pair and η is a \mathcal{C} -morphism, one may verify that there exist $u, v \in A^*$ such that $\eta(u) = \eta(v)$, $\alpha(u) = s$ and $\alpha(v) = t$. Let $x, y \in A^*$ such that $\alpha(x) = q$ and $\alpha(y) = r$. We define $p = \omega(M) \cdot \omega(N)$. Let $w = (ux)^{pk} u (yu)^{pk}$ and $w' = (ux)^{pk} v (yu)^{pk}$. The following lemma can be verified from the definition of $P_{\bowtie}(\eta, k, w)$ and the fact that $(\eta(ux))^p$ and $(\eta(yu))^p$ are idempotents of N .

LEMMA 4.5. *For every $i \in P_{\bowtie}(\eta, k, w)$, either $i \leq |(ux)^{pk}|$ or $i > |(ux)^{pk}u|$.*

Lemma 4.5 states that all positions in $P_{\bowtie}(\eta, k, w)$ belong either to the prefix $(ux)^{pk}$ or to the suffix $(yu)^{pk}$. We consider the set P' made of the corresponding positions in $P_c(w')$:

$$P' = \{i \mid i \in P_{\bowtie}(\eta, k, w) \text{ and } i \leq |(ux)^{pk}|\} \cup \{i - |u| + |v| \mid i \in P_{\bowtie}(\eta, k, w) \text{ and } i > |(ux)^{pk}u|\}.$$

Since $\eta(u) = \eta(v)$, one may verify from the definition that we have $\sigma_\eta(w, P_{\bowtie}(\eta, k, w)) = \sigma_\eta(w', P')$. Thus, $P' = P_{\triangleright}(\eta, k, w')$ by Lemma 3.9 and we get $w \bowtie_{\eta,k} w'$. Since the languages recognized by α are unions of $\bowtie_{\eta,k}$ -classes, we get $\alpha(w) = \alpha(w')$. By definition, this yields $(sq)^\omega s(rs)^\omega = (sq)^\omega t(rs)^\omega$.

We turn to the implication $b) \Rightarrow c)$. Assume that $b)$ holds and consider $q, r, s, t \in M$ such that $s \sim_{\mathcal{C}} t$. We show that $(sq)^\omega s(rs)^\omega = (sq)^\omega t(rs)^\omega$. We start with a preliminary remark. By hypothesis, the second assertion in Theorem 3.7 holds (this is the special case of $b)$ when $q = r = 1_M$). Thus, Theorem 3.7 yields the following property:

$$x^{\omega+1} = x^\omega y x^\omega \quad \text{for all } x, y \in M \text{ such that } x \sim_{\mathcal{C}} y. \quad (1)$$

Since $s \sim_{\mathcal{C}} t$, Lemma 2.5 yields $s_0, \dots, s_n \in M$ such that $s_0 = s$, $s_n = t$ and (s_i, s_{i+1}) is a \mathcal{C} -pair for all $i < n$. We now prove that $(sq)^\omega s_i (rt)^\omega = (sq)^\omega s_{i+1} (rt)^\omega$ for every $i < n$. Since $s = s_0$ and $t = s_n$, this yields the desired result by transitivity. We fix $i < n$. By definition, $s \sim_{\mathcal{C}} t \sim_{\mathcal{C}} s_i$. Hence, since $\sim_{\mathcal{C}}$ is a congruence, we get $sq \sim_{\mathcal{C}} s_i q$ and $rt \sim_{\mathcal{C}} r s_i$. It then follows from (1) that $(sq)^{\omega+1} = (sq)^\omega s_i q (sq)^\omega$ and $(rs)^\omega = (rs)^\omega r s_i (rs)^\omega$. Thus,

$$\begin{aligned} (sq)^\omega &= ((sq)^\omega s_i q (sq)^\omega)^\omega = (sq)^\omega (s_i q (sq)^\omega)^\omega. \\ (rs)^\omega &= ((rs)^\omega r s_i (rs)^\omega)^\omega = ((rs)^\omega r s_i)^\omega (rs)^\omega. \end{aligned}$$

Moreover, since (s_i, s_{i+1}) is a \mathcal{C} -pair and $b)$ holds, we know that $(s_i q (sq)^\omega)^\omega s_i ((rs)^\omega r s_i)^\omega = (s_i q (sq)^\omega)^\omega s_{i+1} ((rs)^\omega r s_i)^\omega$. Hence,

$$\begin{aligned} (sq)^\omega s_i (rs)^\omega &= (sq)^\omega (s_i q (sq)^\omega)^\omega s_i ((rs)^\omega r s_i)^\omega (rs)^\omega \\ &= (sq)^\omega (s_i q (sq)^\omega)^\omega s_{i+1} ((rs)^\omega r s_i)^\omega (rs)^\omega \\ &= (sq)^\omega s_{i+1} (rs)^\omega. \end{aligned}$$

This concludes the proof for the implication $b) \Rightarrow c)$.

It remains to prove $c) \Rightarrow a)$. We assume that $c)$ holds and show that α is an $MPol(\mathcal{C})$ -morphism. Let $N = M / \sim_{\mathcal{C}}$ and recall that N is a monoid since $\sim_{\mathcal{C}}$ is a congruence by Lemma 2.6. We write $\eta = [\cdot]_{\mathcal{C}} \circ \alpha : A^* \rightarrow N$ which is a \mathcal{C} -morphism by Lemma 2.7. We

let $k = |M|$ and consider the equivalence $\vDash_{\eta,k}$ on A^* . We prove the following property:

$$\text{for every } w, w' \in A^*, \quad w \vDash_{\eta,k} w' \Rightarrow \alpha(w) = \alpha(w'). \quad (2)$$

This implies that every language recognized by α is a union of $\vDash_{\eta,k}$ -classes. Together with Proposition 3.13 this yields that every language recognized by α belongs to $MPol(\mathcal{C})$ since η is a \mathcal{C} -morphism. We now concentrate on (2). Let $w, w' \in A^*$ such that $w \vDash_{\eta,k} w'$. We show that $\alpha(w) = \alpha(w')$. We first use our hypothesis to prove the following lemma.

LEMMA 4.6. *There exist $P \subseteq P_c(w)$ and $P' \subseteq P_c(w')$ which satisfy $P_{\triangleright}(\alpha, 1, w) \subseteq P$, $P_{\triangleleft}(\alpha, 1, w') \subseteq P'$ and $\sigma_{\eta}(w, P) = \sigma_{\eta}(w', P')$.*

PROOF. We write $Q = P_{\triangleright}(\eta, k, w)$ and $Q' = P_{\triangleright}(\eta, k, w')$. Since $w \vDash_{\eta,k} w'$, we have $\sigma_{\eta}(w, Q) = \sigma_{\eta}(w', Q')$. In particular, we have $|Q| = |Q'|$ and there is a unique increasing bijection $f: Q \rightarrow Q'$. Since α satisfies c), one may verify from Theorem 3.7 that it is a $UPol(\mathcal{C})$ -morphism. Thus, since $k = |M|$, Lemma 3.8 yields $P_{\triangleright}(\alpha, 1, w) \subseteq P_{\triangleright}(\eta, k, w) \subseteq Q$ and $P_{\triangleleft}(\alpha, 1, w') \subseteq P_{\triangleleft}(\eta, k, w') \subseteq Q'$. Therefore, the set $f(P_{\triangleright}(\alpha, 1, w)) \subseteq Q'$ is well-defined. We define $P' = f(P_{\triangleright}(\alpha, 1, w)) \cup P_{\triangleleft}(\alpha, 1, w') \subseteq Q'$ and $P = f^{-1}(P')$. It is clear from the definition that $P_{\triangleright}(\alpha, 1, w) \subseteq P$ and $P_{\triangleleft}(\alpha, 1, w') \subseteq P'$. Moreover, since $\sigma_{\eta}(w, Q) = \sigma_{\eta}(w', Q')$, it is immediate from the definition that $\sigma_{\eta}(w, P) = \sigma_{\eta}(w', P')$ as well. \square

Consider the α -snapshots $(s_0, a_1, s_1, \dots, a_n, s_n) = \sigma_{\alpha}(w, P)$ and $(t_0, b_1, t_1, \dots, b_m, t_m) = \sigma_{\alpha}(w', P')$. Since $\sigma_{\eta}(w, P) = \sigma_{\eta}(w', P')$, we get $n = m$, $a_i = b_i$ for $1 \leq i \leq n$ and $s_i \sim_{\mathcal{C}} t_i$ for $0 \leq i \leq n$ by definition of η . Therefore, we have $\alpha(w) = s_0 a_1 s_1 \dots a_n s_n$ and $\alpha(w') = t_0 a_1 t_1 \dots a_n t_n$ by definition of α -snapshots (for the sake of avoiding clutter, we abuse terminology and write a_i for $\alpha(a_i)$). We now prove that $s_0 a_1 s_1 \dots a_n s_n = t_0 a_1 t_1 \dots a_n t_n$. For all h such that $0 \leq h \leq n$, we write $q_h = s_0 a_1 \dots s_{h-1} a_h$ and $r_h = a_{h+1} t_{h+1} \dots a_n t_n$ ($q_0 = 1_M$ and $r_n = 1_M$). Since $P_{\triangleright}(\alpha, 1, w) \subseteq P$ and $P_{\triangleleft}(\alpha, 1, w') \subseteq P'$, one may verify from the definitions that $q_h s_h \mathcal{R} q_h$ and $t_h r_h \mathcal{L} r_h$ for $0 \leq h \leq n$. We prove that $q_h s_h r_h = q_h t_h r_h$ for $0 \leq h \leq n$.

Let us first explain why this implies $\alpha(w) = \alpha(w')$. One may verify from the definition that $q_h s_h r_h = q_{h+1} t_{h+1} r_{h+1}$ for $0 \leq h < n$. Together with $q_h s_h r_h = q_h t_h r_h$, this yields $q_h t_h r_h = q_{h+1} t_{h+1} r_{h+1}$. By transitivity, we get $q_0 t_0 r_0 = q_n t_n r_n$. Together with the fact that $q_0 s_0 r_0 = q_0 t_0 r_0$, this yields $q_0 s_0 r_0 = q_n t_n r_n$. By definition, this states that $s_0 a_1 s_1 \dots a_n s_n = t_0 a_1 t_1 \dots a_n t_n$, i.e. we obtain that $\alpha(w) = \alpha(w')$ as desired.

We now fix an index h such that $0 \leq h \leq n$ and show that $q_h s_h r_h = q_h t_h r_h$. Recall that $q_h s_h \mathcal{R} q_h$ and $t_h r_h \mathcal{L} r_h$. Hence, we get two elements $x, y \in M$ such that $q_h = q_h s_h x = q_h (s_h x)^{\omega}$ and $r_h = y t_h r_h = (y t_h)^{\omega} r_h$. Since $s_h \sim_{\mathcal{C}} t_h$ and $\sim_{\mathcal{C}}$ is a congruence we have $y s_h \sim_{\mathcal{C}} y t_h$. By c), this yields $(y t_h)^{\omega+1} = (y t_h)^{\omega} y s_h (y t_h)^{\omega}$. Therefore, $(y t_h)^{\omega} = ((y t_h)^{\omega} y s_h (y t_h)^{\omega})^{\omega} = ((y t_h)^{\omega} y s_h)^{\omega} (y t_h)^{\omega}$. Moreover, since $s_h \sim_{\mathcal{C}} t_h$ and α satisfies c), we have,

$$(s_h x)^{\omega} s_h ((y t_h)^{\omega} y s_h)^{\omega} = (s_h x)^{\omega} t_h ((y t_h)^{\omega} y s_h)^{\omega}.$$

We now multiply by $(y t_h)^{\omega}$ on the right. As shown above, this yields $(s_h x)^{\omega} s_h (y t_h)^{\omega} = (s_h x)^{\omega} t_h (y t_h)^{\omega}$. Hence, since we have $q_h = q_h (s_h x)^{\omega}$ and $r_h = (y t_h)^{\omega} r_h$, it follows that $q_h s_h r_h = q_h t_h r_h$ as desired which completes the proof. \square

5 DETERMINISTIC HIERARCHIES

We present a construction process which take a single input class \mathcal{C} and uses $LPol$ and $RPol$ to build a hierarchy which classifies the languages in $UPol(\mathcal{C})$. Then, we prove the mixed polynomial closure is a key ingredient for investigating these hierarchies.

5.1 Definition

The definition is motivated by a result of [26]. Let \mathcal{C} be a prevariety. We define the alternating polynomial closure of \mathcal{C} ($APol(\mathcal{C})$) as the least class containing \mathcal{C} and closed under both left deterministic and right deterministic marked products and under disjoint union. The following theorem is proved in [26].

THEOREM 5.1. *If \mathcal{C} is a prevariety, then $UPol(\mathcal{C}) = APol(\mathcal{C})$.*

Hence, for a prevariety \mathcal{C} , applying $LPol$ and $RPol$ in alternation builds a classification of $UPol(\mathcal{C})$. For all $n \in \mathbb{N}$, there are two levels $LP_n(\mathcal{C})$ and $RP_n(\mathcal{C})$. We let $LP_0(\mathcal{C}) = RP_0(\mathcal{C}) = \mathcal{C}$. Then, for every $n \geq 1$, we define $LP_n(\mathcal{C}) = LPol(RP_{n-1}(\mathcal{C}))$ and $RP_n(\mathcal{C}) = RPol(LP_{n-1}(\mathcal{C}))$. Clearly, the union of all levels $LP_n(\mathcal{C})$ (or $RP_n(\mathcal{C})$) is exactly the class $APol(\mathcal{C})$, i.e. $UPol(\mathcal{C})$ by Theorem 5.1. In general these are strict hierarchies (we discuss a well-known example below) and the levels $LP_n(\mathcal{C})$ and $RP_n(\mathcal{C})$ are incomparable for every $n \geq 1$. This motivates the introduction of additional intermediary levels “combining” $LP_n(\mathcal{C})$ and $RP_n(\mathcal{C})$.

Consider two classes \mathcal{D}_1 and \mathcal{D}_2 . We write $\mathcal{D}_1 \cap \mathcal{D}_2$ for the class made of all languages which belong simultaneously to \mathcal{D}_1 and \mathcal{D}_2 . Moreover, we write $\mathcal{D}_1 \vee \mathcal{D}_2$ for the least Boolean algebra containing both \mathcal{D}_1 and \mathcal{D}_2 . We consider the additional levels $LP_n(\mathcal{C}) \cap RP_n(\mathcal{C})$ and $LP_n(\mathcal{C}) \vee RP_n(\mathcal{C})$. The following statement can be verified from Theorem 3.15.

COROLLARY 5.2. *Let \mathcal{C} be a prevariety. For every $n \in \mathbb{N}$, $LP_n(\mathcal{C})$, $RP_n(\mathcal{C})$, $LP_n(\mathcal{C}) \cap RP_n(\mathcal{C})$ and $LP_n(\mathcal{C}) \vee RP_n(\mathcal{C})$ are prevarieties.*

A specific hierarchy of this kind is well-known. Its input \mathcal{C} is the class PT of piecewise testable languages: the class $BPol(ST)$ with $ST = \{\emptyset, A^*\}$ as the trivial prevariety. This hierarchy is known to be strict and has many characterizations based on algebra [15, 37] or logic [16, 17] (we come back to the second point in Section 6). It is known [15] that membership is decidable for $LP_n(PT)$, $RP_n(PT)$ and $LP_n(PT) \cap RP_n(PT)$ for every $n \in \mathbb{N}$. This can be reproved using Corollary 4.4 and the decidability of PT -membership [34]. It is also known [1, 12, 13] that for every $n \in \mathbb{N}$, membership is decidable for $LP_n(PT) \vee RP_n(PT)$. We explain below that part of these results can be reproved using Corollary 5.2.

We complete the definition of deterministic hierarchies with a useful result. We prove that when applying $LPol$, $RPol$ or $MPol$ to some level in a deterministic hierarchy, one may strengthen the requirements on marked products. Let \mathcal{C} be a prevariety. We say that a marked product $L_0 a_1 L_1 \dots a_n L_n$ is *left (resp. right, mixed) \mathcal{C} -deterministic* when there exist $H_0, \dots, H_n \in \mathcal{C}$ such that $L_i \subseteq H_i$ for each $i \leq n$ and $H_0 a_1 H_1 \dots a_n H_n$ is left (resp. right, mixed) deterministic. In other words, $L_0 a_1 L_1 \dots a_n L_n$ can be “over-approximated” by a left (resp. right, mixed) deterministic marked product of languages in \mathcal{C} . We use Lemma 3.8 and Proposition 3.13 to prove the following result.

PROPOSITION 5.3. *Let \mathcal{C}, \mathcal{D} be two prevarieties such that $\mathcal{C} \subseteq \mathcal{D}$ and $\mathcal{D} \subseteq UPol(\mathcal{C})$. Moreover, consider a language L in $LPol(\mathcal{D})$ (resp. $RPol(\mathcal{D})$, $MPol(\mathcal{D})$). Then, L is a finite union of left (resp. right, mixed) \mathcal{C} -deterministic marked products of languages in \mathcal{D} .*

PROOF. We present a proof for the case when $L \in MPol(\mathcal{D})$ (the two other cases are symmetrical). Proposition 3.13 yields a \mathcal{D} -morphism $\alpha : A^* \rightarrow M$ and $k \in \mathbb{N}$ such that L is a union of $\triangleright_{\alpha,k}$ -classes. Thus, it suffices to prove that each $\triangleright_{\alpha,k}$ -class is a finite union of mixed \mathcal{C} -deterministic marked products of languages in \mathcal{D} . Let $w \in A^*$ and $K \subseteq A^*$ its $\triangleright_{\alpha,k}$ -class. For every $u \in A^*$ such that $u \triangleright_{\alpha,k} w$, we build a language $H_u \subseteq A^*$ defined by a mixed \mathcal{C} -deterministic marked product of languages in \mathcal{D} and such that $u \in H_u \subseteq L$. Moreover, we show that while there might be infinitely many words $u \in A^*$ such that $u \triangleright_{\alpha,k} w$, there are only finitely many distinct languages H_u . Altogether, it will follow that K is equal to the *finite* union of all languages H_u for $u \in A^*$ such that $u \triangleright_{\alpha,k} w$ which completes the proof. For the construction, we consider the canonical equivalence $\sim_{\mathcal{C}}$ on M and write $N = M/\sim_{\mathcal{C}}$. We also define η as the morphism $\eta = [\cdot]_{\mathcal{C}} \circ \alpha : A^* \rightarrow N$. We know from Lemma 2.7 that η is a \mathcal{C} -morphism.

We now consider $u \in A^*$ such that $u \triangleright_{\alpha,k} w$ and build H_u . We write $P_u = P_{\triangleright}(\eta, k|M|, u)$. One may verify from the definition that $|P_u| \leq 2|N|^{k|M|}$ (the key point is that this bound is independent from u). We let $(s_0, a_1, s_1, \dots, a_n, s_n) = \sigma_{\alpha}(u, P_u)$ and define $H_u = \alpha^{-1}(s_0)a_1\alpha^{-1}(s_1) \cdots a_n\alpha^{-1}(s_n)$. Since $|P_u| \leq 2|N|^{k|M|}$, we know that H_u is the marked product of at most $2|N|^{k|M|} + 1$ languages recognized by α . Hence, there are only finitely many languages H_u for $u \in A^*$ such that $u \triangleright_{\alpha,k} w$. Moreover, the languages in the product defining H_u belong to \mathcal{D} by hypothesis on α . We now prove that this marked product is mixed \mathcal{C} -deterministic. Let $(t_0, a_1, t_1, \dots, a_n, t_n) = \sigma_{\eta}(u, P_u)$. Since we have $P_u = P_{\triangleright}(\eta, k|M|, u)$ and η is a \mathcal{C} -morphism, Lemma 3.8 implies that $\eta^{-1}(t_0)a_1\eta^{-1}(t_1) \cdots a_n\eta^{-1}(t_n)$ is a mixed deterministic marked product of languages in \mathcal{C} . Moreover, since $\eta = [\cdot]_{\mathcal{C}} \circ \alpha$, we have $\alpha^{-1}(s_i) \subseteq \eta^{-1}(t_i)$ for every $i \leq n$. Therefore, the marked product $\alpha^{-1}(s_0)a_1\alpha^{-1}(s_1) \cdots a_n\alpha^{-1}(s_n)$ which defines H_u is mixed \mathcal{C} -deterministic as desired.

It remains to prove that $u \in H_u \subseteq L$. That $u \in H_u$ is immediate by definition since $(s_0, a_1, s_1, \dots, a_n, s_n) = \sigma_{\alpha}(u, P_u)$. Hence, we let $v \in H_u$ and prove that $v \in L$, i.e. $v \triangleright_{\alpha,k} u$. By definition of H_u , we know that there exists a set $Q \subseteq P(w)$ such that $\sigma_{\alpha}(v, Q) = (s_0, a_1, s_1, \dots, a_n, s_n) = \sigma_{\alpha}(u, P_u)$. Moreover, since $\mathcal{D} \subseteq UPol(\mathcal{C})$ by hypothesis, we know α is a $UPol(\mathcal{C})$ -morphism. Therefore, $P_{\triangleright}(\alpha, k, w) \subseteq P_{\triangleright}(\eta, k|M|, u) = P_u$ by Lemma 3.8. Hence, since $\sigma_{\alpha}(v, Q) = \sigma_{\alpha}(u, P_u)$, one may verify that there exists $Q' \subseteq Q$ such that $\sigma_{\alpha}(v, Q') = \sigma_{\alpha}(u, P_{\triangleright}(\alpha, k, u))$. Finally, Lemma 3.9 implies that $Q' = P_{\triangleright}(\alpha, k, u)$ and we obtain $v \triangleright_{\alpha,k} u$ as desired. \square

5.2 Connection with mixed polynomial closure

The definition associates *four* closely related hierarchies to every prevariety \mathcal{C} . Their construction processes can be unified using $MPol$. As seen in Section 3, $MPol$ is not idempotent: given a prevariety \mathcal{D} , it may happen that $MPol(\mathcal{D})$ is *strictly* included in $MPol(MPol(\mathcal{D}))$. Hence, a hierarchy is obtained by applying $MPol$ iteratively to \mathcal{D} . It turns out that all levels in the above hierarchies can be built in this way. First, the levels $LP_n(\mathcal{C})$ and $RP_n(\mathcal{C})$ can

be built from $LPol(\mathcal{C})$ and $RPol(\mathcal{C})$ using only $MPol$ (this can be verified from the definition).

LEMMA 5.4. *Let \mathcal{C} be a prevariety. For every $n \geq 1$, we have $LP_{n+1}(\mathcal{C}) = MPol(RP_n(\mathcal{C}))$ and $RP_{n+1}(\mathcal{C}) = MPol(LP_n(\mathcal{C}))$.*

Moreover, the levels $LP_n(\mathcal{C}) \cap RP_n(\mathcal{C})$ can all be built from $LPol(\mathcal{C}) \cap RPol(\mathcal{C})$ using only $MPol$ (the proof is based on the algebraic characterizations of $LPol$, $RPol$ and $MPol$).

THEOREM 5.5. *Let \mathcal{C} be a prevariety. For every $n \geq 1$, we have $LP_{n+1}(\mathcal{C}) \cap RP_{n+1}(\mathcal{C}) = MPol(LP_n(\mathcal{C}) \cap RP_n(\mathcal{C}))$.*

A similar result holds for the levels $LP_n(\mathcal{C}) \vee RP_n(\mathcal{C})$: they can all be built from $LPol(\mathcal{C}) \vee RPol(\mathcal{C})$ using only $MPol$. We present the proof below.

THEOREM 5.6. *Let \mathcal{C} be a prevariety. For every $n \geq 1$, we have $LP_{n+1}(\mathcal{C}) \vee RP_{n+1}(\mathcal{C}) = MPol(LP_n(\mathcal{C}) \vee RP_n(\mathcal{C}))$.*

Theorem 5.6 has an interesting application. Since $MPol$ preserves the decidability of membership by Corollary 4.4, we get that for all prevarieties \mathcal{C} , if membership is decidable for $LPol(\mathcal{C}) \vee RPol(\mathcal{C})$, then this is also the case for *all* levels $LP_n(\mathcal{C}) \vee RP_n(\mathcal{C})$. This can be applied for $\mathcal{C} = PT$. It is known that membership is decidable for $LPol(PT) \vee RPol(PT)$ [1, 13]. Thus, we lift this result to every level $LP_n(PT) \vee RP_n(PT)$ “for free”. This reproves a result of [12].

PROOF OF THEOREM 5.6. We fix a prevariety \mathcal{C} and $n \geq 1$. Let us start with $LP_{n+1}(\mathcal{C}) \vee RP_{n+1}(\mathcal{C}) \subseteq MPol(LP_n(\mathcal{C}) \vee RP_n(\mathcal{C}))$. By Theorem 3.15, $MPol(LP_n(\mathcal{C}) \vee RP_n(\mathcal{C}))$ is a prevariety. Hence, it suffices to prove that $LP_{n+1}(\mathcal{C})$ and $RP_{n+1}(\mathcal{C})$ are included in $MPol(LP_n(\mathcal{C}) \vee RP_n(\mathcal{C}))$. By symmetry, we only prove the former. By definition, we have $LP_{n+1}(\mathcal{C}) = LPol(RP_n(\mathcal{C}))$ which yields $LP_{n+1}(\mathcal{C}) \subseteq MPol(RP_n(\mathcal{C}))$. Finally, since it is immediate by definition that $RP_n(\mathcal{C}) \subseteq LP_n(\mathcal{C}) \vee RP_n(\mathcal{C})$, we obtain the inclusion $LP_{n+1}(\mathcal{C}) \subseteq MPol(LP_n(\mathcal{C}) \vee RP_n(\mathcal{C}))$ as desired which completes the proof for the left to right inclusion.

We now prove that the class $MPol(LP_n(\mathcal{C}) \vee RP_n(\mathcal{C}))$ is included in $LP_{n+1}(\mathcal{C}) \vee RP_{n+1}(\mathcal{C})$. We write $\mathcal{D} = LP_n(\mathcal{C}) \vee RP_n(\mathcal{C})$ for the proof. Corollary 5.2 implies that \mathcal{D} is a prevariety. Moreover, it is immediate that $\mathcal{C} \subseteq \mathcal{D} \subseteq UPol(\mathcal{C})$ ($UPol(\mathcal{C})$ is a prevariety by Theorem 3.6 and it contains both $LP_n(\mathcal{C})$ and $RP_n(\mathcal{C})$). Hence, Proposition 5.3 implies that every language in $MPol(\mathcal{D})$ is a disjoint union of mixed \mathcal{C} -deterministic marked products of languages in \mathcal{D} . It now remains to prove that for every mixed \mathcal{C} -deterministic marked product $L = L_0a_1L_1 \cdots a_nL_n$ such that $L_0, \dots, L_n \in \mathcal{D}$, we have $L \in LP_{n+1}(\mathcal{C}) \vee RP_{n+1}(\mathcal{C})$. The definition yields $H_i \in \mathcal{C}$ for each $i \leq n$ such that $L_i \subseteq H_i$ and $H_0a_1H_1 \cdots a_nH_n$ is mixed deterministic.

Consider $i \leq n$. We have $L_i \in \mathcal{D}$ and $\mathcal{D} = LP_n(\mathcal{C}) \vee RP_n(\mathcal{C})$. Hence, by definition L_i is a Boolean combination of languages in $LP_n(\mathcal{C})$ and $RP_n(\mathcal{C})$. We can put the Boolean combination in disjunctive normal form. Moreover, since $LP_n(\mathcal{C})$ and $RP_n(\mathcal{C})$ are prevarieties by Corollary 5.2, each disjunct is the intersection of a single language in $LP_n(\mathcal{C})$ with a single language in $RP_n(\mathcal{C})$. Altogether, it follows that L_i is a finite union of languages $P_i \cap Q_i$ with $P_i \in LP_n(\mathcal{C})$ and $Q_i \in RP_n(\mathcal{C})$. Moreover, since $L_i \subseteq H_i \in \mathcal{C}$, we may assume without loss of generality that all languages P_i and Q_i are included in H_i as well (otherwise we may replace them by

$P_i \cap H_i$ and $Q_i \cap H_i$). Consequently, since marked concatenation distributes over union, we obtain that $L = L_0 a_1 L_1 \cdots a_n L_n$ is a finite union of products $(P_0 \cap Q_0) a_1 (P_1 \cap Q_1) \cdots a_n (P_n \cap Q_n)$ such that $P_i \in LP_n(\mathcal{C})$ and $Q_i \in RP_n(\mathcal{C})$ are included in H_i for every $i \leq n$. It now suffices to prove that every such marked product belongs to $LP_{n+1}(\mathcal{C}) \vee RP_{n+1}(\mathcal{C})$. Since $H_0 a_1 H_1 \cdots a_n H_n$ is mixed deterministic, it is also unambiguous. Hence, since P_i and Q_i are included in H_i for every $i \leq n$, one may verify that the language $(P_0 \cap Q_0) a_1 (P_1 \cap Q_1) \cdots a_n (P_n \cap Q_n)$ is equal to the intersection,

$$(P_0 a_1 P_1 \cdots a_n P_n) \cap (Q_0 a_1 Q_1 \cdots a_n Q_n).$$

Finally, it is clear that $P_0 a_1 P_1 \cdots a_n P_n$ and $Q_0 a_1 Q_1 \cdots a_n Q_n$ are mixed deterministic marked products since this is the case for $H_0 a_1 H_1 \cdots a_n H_n$. By definition, it follows that they both belong to $MPol(LP_n(\mathcal{C}))$ and $MPol(RP_n(\mathcal{C}))$ respectively. Thus, we obtain $P_0 a_1 P_1 \cdots a_n P_n \in RP_{n+1}(\mathcal{C})$ and $Q_0 a_1 Q_1 \cdots a_n Q_n \in LP_{n+1}(\mathcal{C})$ by Lemma 5.4. Hence, the intersection of these two languages belongs to $LP_{n+1}(\mathcal{C}) \vee RP_{n+1}(\mathcal{C})$ which completes the proof. \square

6 TWO-VARIABLE FIRST-ORDER LOGIC

We look at quantifier alternation hierarchies for two-variable first-order logic over words (FO^2). We characterize several hierarchies of this kind with mixed polynomial closure.

6.1 Definitions

We first recall the definition of first-order logic over words. We view a word $w \in A^*$ as a logical structure. Its domain is the set $P(w) = \{0, \dots, |w| + 1\}$ of positions in w . A position i such that $1 \leq i \leq |w|$ carries a label in A . On the other hand, 0 and $|w| + 1$ are artificial *unlabeled* positions. We use first-order logic (FO) to express properties of words w : a formula can quantify over the positions in w and use a predetermined set of predicates to test properties of these positions. We also allow two constants “*min*” and “*max*” interpreted as the artificial unlabeled positions 0 and $|w| + 1$. Given a formula $\varphi(x_1, \dots, x_n)$ with free variables x_1, \dots, x_n , $w \in A^*$ and $i_1, \dots, i_n \in P(w)$, we write $w \models \varphi(i_1, \dots, i_n)$ to indicate that w satisfies φ when x_1, \dots, x_n are interpreted as the positions i_1, \dots, i_n . As usual, a sentence φ is a formula without free variables. It defines the language $L(\varphi) = \{w \in A^* \mid w \models \varphi\}$. We use standard predicates. For each $a \in A$, we use a unary predicate (also denoted by a) selecting all positions labeled by “ a ”. We also use three binary predicates: equality “ $=$ ”, the (strict) linear order “ $<$ ” and the successor “ $+1$ ”.

Example 6.1. The language $A^* a A^* b A^* c$ is defined by the FO sentence $(\exists x \exists y (x < y) \wedge a(x) \wedge b(y)) \wedge (\exists x c(x) \wedge (x + 1 = \text{max}))$.

A *fragment* of first-order logic consists in the specification of a (possibly finite) set V of variables and a set \mathcal{F} of FO formulas using only the variables in V which contains all quantifier-free formulas and is closed under disjunction, conjunction and quantifier-free substitution (if $\varphi \in \mathcal{F}$, a quantifier-free sub-formula can be replaced by another quantifier-free formula in \mathcal{F}). If \mathbb{S} is a set of predicates and \mathcal{F} is a fragment, we let $\mathcal{F}(\mathbb{S})$ be the class containing all languages $L(\varphi)$ where φ is a sentence of \mathcal{F} using only the predicates in \mathbb{S} , equality and the label predicates.

In this paper, we use generic sets of predicates which are built from an arbitrary input class \mathcal{C} . There are two of them. The first

one, written $\mathbb{I}_{\mathcal{C}}$, contains a binary “infix” predicate $I_L(x, y)$ for every $L \in \mathcal{C}$. Given $w \in A^*$ and two positions $i, j \in P(w)$, we have $w \models I_L(i, j)$ if and only if $i < j$ and $w(i, j) \in L$. The second set, written $\mathbb{P}_{\mathcal{C}}$, contains a unary “prefix” predicate $P_L(x)$ for every $L \in \mathcal{C}$. Given $w \in A^*$ and a position $i \in P(w)$, we have $w \models P_L(i)$ if and only if $0 < i$ and $w(0, i) \in L$. The predicates in $\mathbb{P}_{\mathcal{C}}$ are easily expressed from those in $\mathbb{I}_{\mathcal{C}}$: clearly, $P_L(x)$ is equivalent to $I_L(\text{min}, x)$. In practice, we only consider the sets $\mathbb{P}_{\mathcal{C}}$ and $\mathbb{I}_{\mathcal{C}}$ when \mathcal{C} is either a *group prevariety* \mathcal{G} or its well-suited extension \mathcal{G}^+ . This is motivated by the following lemma.

LEMMA 6.2. *If \mathcal{G} is a group prevariety and \mathcal{F} is a fragment of FO, then $\mathcal{F}(\mathbb{I}_{\mathcal{G}}) = \mathcal{F}(<, \mathbb{P}_{\mathcal{G}})$ and $\mathcal{F}(\mathbb{I}_{\mathcal{G}^+}) = \mathcal{F}(<, +1, \mathbb{P}_{\mathcal{G}})$.*

Lemma 6.2 covers many important sets of predicates. We present three important cases. If \mathcal{G} is the trivial prevariety $\text{ST} = \{\emptyset, A^*\}$, all predicates in \mathbb{P}_{ST} are trivial. Hence, we get the classes $\mathcal{F}(<)$ and $\mathcal{F}(<, +1)$. We also look at the class MOD of *modulo languages*: the Boolean combination of languages $\{w \in A^* \mid |w| \equiv k \pmod{m}\}$ with $k, m \in \mathbb{N}$ such that $k < m$. One may verify that in this case, we obtain the classes $\mathcal{F}(<, \text{MOD})$ and $\mathcal{F}(<, +1, \text{MOD})$ where “*MOD*” is the set of *modular predicates* (for all $k, m \in \mathbb{N}$ such that $k < m$, it contains a unary predicate $M_{k,m}$ selecting the positions i such that $i \equiv k \pmod{m}$). Finally, we consider the class AMT of *alphabet modulo testable languages*. If $w \in A^*$ and $a \in A$, we let $\#_a(w) \in \mathbb{N}$ be the number of occurrences of “ a ” in w . AMT contains the Boolean combinations of languages $\{w \in A^* \mid \#_a(w) \equiv k \pmod{m}\}$ where $a \in A$ and $k, m \in \mathbb{N}$ such that $k < m$ (these are the languages recognized by commutative groups). In this case, we get the classes $\mathcal{F}(<, \text{AMOD})$ and $\mathcal{F}(<, +1, \text{AMOD})$ where “*AMOD*” is the set of *alphabetic modular predicates* (for all $a \in A$ and $k, m \in \mathbb{N}$ such that $k < m$, it contains a unary predicate $M_{k,m}^a$ selecting the positions i such $\#_a(w(0, i)) \equiv k \pmod{m}$).

Quantifier alternation in FO^2 . We may now present the particular fragments that we shall consider. First, we write FO^2 for the fragment made of all first-order formulas which use at most two distinct variables (which can be reused). In the formal definition of fragments, this boils down to picking a set V of variables which has size two. We do not look at FO^2 directly. Instead, we consider its quantifier-alternation hierarchy. Let us first present the one of full first-order logic.

For every $n \in \mathbb{N}$, we associate two fragments Σ_n and $\mathcal{B}\Sigma_n$ of FO. We present the definition by induction on $n \in \mathbb{N}$. When $n = 0$, we let $\Sigma_0 = \mathcal{B}\Sigma_0$ as the fragment containing exactly the quantifier-free formulas of FO. Assume now that $n \geq 1$. We let Σ_n as the least set of expressions which contains the $\mathcal{B}\Sigma_{n-1}$ formulas and is closed under disjunction (\vee), conjunction (\wedge) and existential quantification (\exists). Moreover, we let $\mathcal{B}\Sigma_n$ as the set of all Boolean combinations of Σ_n formulas, *i.e.* the least one containing Σ_n and closed under disjunction (\vee), conjunction (\wedge) and negation (\neg).

For every $n \in \mathbb{N}$, we define Σ_n^2 (resp. $\mathcal{B}\Sigma_n^2$) as the fragment containing all formulas which belong simultaneously to FO^2 and Σ_n (resp. $\mathcal{B}\Sigma_n$). In this paper, we look at classes of the form $\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{C}})$ where \mathcal{C} is a prevariety. Our results only apply in the case when \mathcal{C} is either a group prevariety \mathcal{G} or its well-suited extension \mathcal{G}^+ (in which case Lemma 6.2 applies). However, we shall need the following general result which is specific to the classes $\mathcal{B}\Sigma_1^2(\mathbb{I}_{\mathcal{C}})$.

THEOREM 6.3. *Consider a prevariety \mathcal{C} . We have the following correspondences $\mathcal{B}\Sigma_1^2(\mathbb{I}_{\mathcal{C}}) = \mathcal{B}\Sigma_1(\mathbb{I}_{\mathcal{C}}) = BPol(\mathcal{C})$.*

PROOF. That $\mathcal{B}\Sigma_1(\mathbb{I}_{\mathcal{C}}) = BPol(\mathcal{C})$ is proved in [27]. This is a specific case of the generic correspondence between the quantifier alternation hierarchies of FO and concatenation hierarchies (which are built with *Pol* and *Bool*). The inclusion $\mathcal{B}\Sigma_1^2(\mathbb{I}_{\mathcal{C}}) \subseteq \mathcal{B}\Sigma_1(\mathbb{I}_{\mathcal{C}})$ is trivial. Hence, it suffices to show that $BPol(\mathcal{C}) \subseteq \mathcal{B}\Sigma_1^2(\mathbb{I}_{\mathcal{C}})$. By definition, $BPol(\mathcal{C})$ contains all Boolean combinations of marked products $L_0 a_1 L_1 \cdots a_n L_n$ with $L_0, \dots, L_n \in \mathcal{C}$. Since $\mathcal{B}\Sigma_1^2(\mathbb{I}_{\mathcal{C}})$ is closed under Boolean operations, it suffices to prove that all marked products of this kind belong to $\Sigma_1^2(\mathbb{I}_{\mathcal{C}})$. We use induction to build a formula $\varphi_k(x)$ of $\Sigma_1^2(\mathbb{I}_{\mathcal{C}})$ for each $k \leq n$ which has one free variable x and such that for all $w \in A^*$ and $i \in P(w)$, we have $w \models \varphi_k(i)$ if and only if $0 < i$ and $w(0, i) \in L_0 a_1 L_1 \cdots a_k L_k$. It will then follow that $L_0 a_1 L_1 \cdots a_n L_n$ is defined by the sentence $\varphi_n(\max)$ of $\Sigma_1^2(\mathbb{I}_{\mathcal{C}})$, completing the proof. If $k = 0$, it suffices to define $\varphi_0(x) := I_{L_0}(\min, x)$. Assume now that $k \geq 1$. It suffices to define $\varphi_k(x) := \exists y (\varphi_{k-1}(y) \wedge a_k(y) \wedge I_{L_k}(y, x))$ (the definition involves implicit renaming of the variables in φ_{k-1} , this is standard in FO²). By definition, $\varphi_k(x)$ is a formula of $\Sigma_1^2(\mathbb{I}_{\mathcal{C}})$ as desired. \square

6.2 Connection with mixed polynomial closure

We prove that when using the predicates $\mathbb{I}_{\mathcal{C}}$ and $\mathbb{I}_{\mathcal{C}^+}$ for a group prevariety \mathcal{C} , one may “climb” the quantifier alternation hierarchy of FO² using mixed polynomial closure.

THEOREM 6.4. *If \mathcal{C} is a group prevariety and $\mathcal{C} \in \{\mathcal{C}, \mathcal{C}^+\}$, then $\mathcal{B}\Sigma_{n+1}^2(\mathbb{I}_{\mathcal{C}}) = MPol(\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{C}}))$ for all $n \geq 1$.*

Theorem 6.3 and Theorem 6.4 imply that for every group prevariety \mathcal{C} , if $\mathcal{C} \in \{\mathcal{C}, \mathcal{C}^+\}$, then all levels $\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{C}})$ are built iteratively from $BPol(\mathcal{C})$ by applying $MPol$. By Proposition 3.1, $BPol(\mathcal{C})$ is a prevariety. Moreover, Theorem 3.15 and Corollary 4.4 imply that when $MPol$ is applied to a prevariety, it outputs a prevariety and preserves the decidability of membership. It follows that when membership is decidable for $BPol(\mathcal{C})$, this is also the case for all levels $\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{C}})$. Since $\mathcal{C} \in \{\mathcal{C}, \mathcal{C}^+\}$, it is known [31] that membership is decidable for $BPol(\mathcal{C})$ when separation is decidable for \mathcal{C} (this is based on independent techniques). Finally, we have $\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{C}}) = \mathcal{B}\Sigma_n^2(<, \mathbb{P}_{\mathcal{C}})$ and $\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{C}^+}) = \mathcal{B}\Sigma_n^2(<, +1, \mathbb{P}_{\mathcal{C}})$ by Lemma 6.2. Altogether, we obtain the following corollary.

COROLLARY 6.5. *Let \mathcal{C} be a group prevariety with decidable separation. For every $n \geq 1$, membership is decidable for $\mathcal{B}\Sigma_n^2(<, \mathbb{P}_{\mathcal{C}})$ and $\mathcal{B}\Sigma_n^2(<, +1, \mathbb{P}_{\mathcal{C}})$.*

Corollary 6.5 reproves earlier results. Separation is clearly decidable for $ST = \{\emptyset, A^*\}$. Hence, $\mathcal{B}\Sigma_n^2(<)$ and $\mathcal{B}\Sigma_n^2(<, +1)$ have decidable membership for all $n \geq 1$. For $\mathcal{B}\Sigma_n^2(<)$, this was first proved independently by Kufleitner and Weil [17] and Krebs and Straubing [11]. For $\mathcal{B}\Sigma_n^2(<, +1)$, this was first proved by Kufleitner and Lauser [14].

REMARK 6.6. *In [17], it is also shown that for every $n \geq 1$, we have $\mathcal{B}\Sigma_n^2(<) = LP_n(PT) \cap RP_n(PT)$ (with $PT = BPol(ST)$). This can be reproved using Theorem 5.5, Theorem 6.4 and the fact that $PT = LPol(PT) \cap RPol(PT)$. This is specific to $\mathcal{B}\Sigma_n^2(<)$: this fails in general. This is because the equality $PT = LPol(PT) \cap RPol(PT)$ is specific to $PT = BPol(ST)$.*

Additionally, it is known that separation is decidable for the group prevarieties MOD and AMT. This is straightforward for MOD and proved in [8] for AMT. Hence, we also obtain the decidability of membership for all levels $\mathcal{B}\Sigma_n^2(<, MOD)$ and $\mathcal{B}\Sigma_n^2(<, +1, MOD)$, as well as all levels $\mathcal{B}\Sigma_n^2(<, AMOD)$ and $\mathcal{B}\Sigma_n^2(<, +1, AMOD)$. Note that for the levels $\mathcal{B}\Sigma_n^2(<, +1, MOD)$ was already known. This was proved in [7] using a reduction to the levels $\mathcal{B}\Sigma_n^2(<, +1)$.

Theorem 6.4 also yields characterizations of FO². Indeed, one may verify from Theorem 5.1 that given a prevariety \mathcal{D} , the union of all classes built from \mathcal{D} by iteratively applying $MPol$ is $UPol(\mathcal{D})$. Hence, we obtain the following corollary.

COROLLARY 6.7. *If \mathcal{C} is a group prevariety, then $FO^2(<, \mathbb{P}_{\mathcal{C}}) = UPol(BPol(\mathcal{C}))$ and $FO^2(<, +1, \mathbb{P}_{\mathcal{C}}) = UPol(BPol(\mathcal{C}^+))$.*

Since $UPol$ preserves the decidability of membership by Theorem 3.7, the above argument also implies that for all group prevarieties \mathcal{C} with decidable separation, $FO^2(<, \mathbb{P}_{\mathcal{C}})$ and $FO^2(<, +1, \mathbb{P}_{\mathcal{C}})$ have decidable membership. This yields known results [6, 35] in the cases $\mathcal{C} = ST$ and $\mathcal{C} = MOD$.

REMARK 6.8. *Corollary 6.7 can be used to prove specialized characterizations for $FO^2(<, \mathbb{P}_{\mathcal{C}})$ and $FO^2(<, +1, \mathbb{P}_{\mathcal{C}})$. Yet, this involves the characterizations of $BPol(\mathcal{C})$ and $BPol(\mathcal{C}^+)$ presented in [31], which is outside of the scope of this paper.*

PROOF OF THEOREM 6.4. We fix a group prevariety \mathcal{C} and let $\mathcal{C} \in \{\mathcal{C}, \mathcal{C}^+\}$. For all $n \geq 1$, we write $\mathcal{D}_n = \mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{C}})$. We use induction to prove that $\mathcal{D}_{n+1} = MPol(\mathcal{D}_n)$ for all $n \geq 1$. The inclusion $\mathcal{D}_{n+1} \subseteq MPol(\mathcal{D}_n)$ is based on the algebraic characterization of $MPol$ (i.e., Theorem 4.3) and arguments analogous to Ehrenfeucht-Fraïssé games and the hypothesis $\mathcal{C} \in \{\mathcal{C}, \mathcal{C}^+\}$. It is proved in the full version of the paper.

We focus on the inclusion $MPol(\mathcal{D}_n) \subseteq \mathcal{D}_{n+1}$. It is based on a key property of $MPol$. We say that a marked product $L_0 a_1 L_1 \cdots a_m L_m$ is \mathcal{C} -pointed if for all $1 \leq i \leq m$, there are $K_i, K'_i \in BPol(\mathcal{C})$ such that $K_i a_i K'_i$ is unambiguous, $L_0 a_1 L_1 \cdots a_{i-1} L_{i-1} \subseteq K_i$ and $L_i a_{i+1} L_{i+1} \cdots a_m L_m \subseteq K'_i$. We now use Proposition 3.11 to prove the following lemma (we need the hypothesis that $\mathcal{C} \in \{\mathcal{C}, \mathcal{C}^+\}$).

LEMMA 6.9. *Every language in $MPol(\mathcal{D}_n)$ is a finite union of \mathcal{C} -pointed marked products of languages in \mathcal{D}_n .*

PROOF. We fix $L \in MPol(\mathcal{D}_n)$ for the proof. Proposition 3.13 yields a \mathcal{D}_n -morphism $\alpha : A^* \rightarrow M$ and $k \in \mathbb{N}$ such that L is a finite union of $\triangleright_{\alpha, k}$ -classes. Hence, it suffices to prove that every $\triangleright_{\alpha, k}$ -class is a finite union of \mathcal{C} -pointed marked products of languages in \mathcal{D}_n . First, we associate a language U_w to every word $w \in A^*$.

Let η be the morphism $\eta : [\cdot]_{\mathcal{C}} \circ \alpha : A^* \rightarrow M / \sim_{BPol(\mathcal{C})}$. We know that η is a $BPol(\mathcal{C})$ -morphism by Lemma 2.7. Moreover, observe that $BPol(\mathcal{C}) \subseteq \mathcal{D}_n \subseteq UPol(BPol(\mathcal{C}))$. Indeed, we know that $\mathcal{D}_1 = BPol(\mathcal{C})$ by Theorem 6.3 and induction in Theorem 6.4 implies that \mathcal{D}_n is built from \mathcal{D}_1 by applying $MPol$ iteratively. Therefore, Lemma 3.8 implies that $P_{\triangleright}(\alpha, k, w) \subseteq P_{\triangleright}(\eta, k|M, w)$. Finally, since $\mathcal{C} \in \{\mathcal{C}, \mathcal{C}^+\}$ and \mathcal{C} is a group prevariety, it follows from Proposition 3.11 that there exists another $BPol(\mathcal{C})$ -morphism, $\gamma : A^* \rightarrow Q$ such that $P_{\triangleright}(\eta, k|M, w) \subseteq P_{\triangleright}(\gamma, 1, w)$. We define,

$$\begin{aligned} (s_0, a_1, s_1, \dots, a_h, s_h) &= \sigma_{\alpha}(w, P_{\triangleright}(\alpha, k, w)). \\ (q_0, a_1, q_1, \dots, a_h, q_h) &= \sigma_{\gamma}(w, P_{\triangleright}(\alpha, k, w)). \end{aligned}$$

For all $i \leq h$, we let $V_i = \alpha^{-1}(s_i) \cap \gamma^{-1}(q_i)$. Finally, we define $U_w = V_0 a_1 V_1 \cdots a_h V_h$. By definition, $h = |\mathbb{P}_{\triangleright\alpha}(\alpha, k, w)| \leq 2|M|^k$. Thus, there are finitely many languages U_w even though there are infinitely many $w \in A^*$. Moreover, it is clear that $w \in U_w$. We now prove that U_w is included in the $\triangleright_{\alpha, k}$ -class of w and that $V_0 a_1 V_1 \cdots a_h V_h$ is a \mathcal{C} -pointed marked product of languages in \mathcal{D}_n . It will then follow that each $\triangleright_{\alpha, k}$ -class is the *finite* union of all languages U_w for the words w in the $\triangleright_{\alpha, k}$ -class, *i.e.* a finite union of \mathcal{C} -pointed marked product of languages in \mathcal{D}_n as desired.

We first show that if $u \in U_w$, then $u \triangleright_{\alpha, k} w$. By definition of U_w , there exists $P \subseteq \mathbb{P}(u)$ such that $\sigma_\alpha(u, P) = (s_0, a_1, s_1, \dots, a_h, s_h) = \sigma_\alpha(w, \mathbb{P}_{\triangleright\alpha}(\alpha, k, w))$. Hence, Lemma 3.9 yields $P = \mathbb{P}_{\triangleright\alpha}(\alpha, k, u)$ and we obtain $u \triangleright_{\alpha, k} w$ as desired.

It remains to show that $V_0 a_1 V_1 \cdots a_h V_h$ is a \mathcal{C} -pointed marked product of languages in \mathcal{D}_n . As α is a \mathcal{D}_n -morphism, γ is a $BPol(\mathcal{C})$ -morphism and $BPol(\mathcal{C}) \subseteq \mathcal{D}_n$, it is immediate by definition that $V_i \in \mathcal{D}_n$ for all $i \leq h$. We prove that $V_0 a_1 V_1 \cdots a_h V_h$ is \mathcal{C} -pointed. We fix $i \leq h$ for the proof. Let $r_i = q_0 \gamma(a_1) q_1 \cdots \gamma(a_{i-1}) q_{i-1}$ and $K_i = \gamma^{-1}(r_i)$. Moreover, we let $r'_i = q_i \gamma(a_{i+1}) q_{i+1} \cdots \gamma(a_h) q_h$ and $K'_i = \gamma^{-1}(r'_i)$. One may verify that $V_0 a_1 V_1 \cdots a_{i-1} V_{i-1} \subseteq K_i$ and $V_{i+1} a_{i+1} V_{i+1} \cdots a_h V_h \subseteq K'_i$. Hence, we have to prove that $K_i a_i K'_i$ is unambiguous. We have $\mathbb{P}_{\triangleright\alpha}(\alpha, k, w) \subseteq \mathbb{P}_{\triangleright\alpha}(\gamma, 1, w)$ by construction of γ . Therefore, all letters in the γ -snapshot $\sigma_\gamma(w, \mathbb{P}_{\triangleright\alpha}(\alpha, k, w)) = (q_0, a_1, q_1, \dots, a_h, q_h)$ correspond to positions in $\mathbb{P}_{\triangleright\alpha}(\gamma, 1, w)$. By definition, this implies that either $r_i \gamma(a_i) <_{\mathcal{R}} r_i$ or $\gamma(a_i) r'_i <_{\mathcal{L}} r'_i$. By symmetry, we assume that the former holds and prove that $K_i a_i K'_i$ is left deterministic. By contradiction, assume that there exists some word $x \in K_i \cap K_i a_i A^*$. Since $K_i = \gamma^{-1}(r_i)$, this yields $y \in A^*$ such that $r_i = r_i \gamma(a_i) \gamma(y)$, contradicting the hypothesis that $r \gamma(a_i) <_{\mathcal{R}} r$ and concluding the proof. \square

We now prove that $MPol(\mathcal{D}_n) \subseteq \mathcal{D}_{n+1}$. In view of Lemma 6.9 and since $\mathcal{D}_{n+1} = \mathcal{B}\Sigma_{n+1}^2(\mathbb{I}_{\mathcal{C}})$ is closed under union, it suffices to show that if $L_0, \dots, L_m \in \mathcal{D}_n$ and $L_0 a_1 L_1 \cdots a_m L_m$ is a \mathcal{C} -pointed marked product, then $L_0 a_1 L_1 \cdots a_m L_m \in \mathcal{D}_{n+1}$. We do so by building a $\mathcal{B}\Sigma_{n+1}^2(\mathbb{I}_{\mathcal{C}})$ sentence defining $L_0 a_1 L_1 \cdots a_m L_m$. We have $K_h, K'_h \in BPol(\mathcal{C})$ for every $h \leq m$ such that $K_h a_h K'_h$ is unambiguous, $L_0 a_1 L_1 \cdots a_{h-1} L_{h-1} \subseteq K_h$ and $L_h a_{h+1} L_{h+1} \cdots a_m L_m \subseteq K'_h$. It follows that for all $w \in A^*$, we have $w \in L_0 a_1 L_1 \cdots a_m L_m$, if and only if the two following properties are satisfied:

- There are positions $i_0, i_1, \dots, i_m, i_{m+1} \in \mathbb{P}(w)$ which satisfy $0 = i_0 < i_1 < \dots < i_m < i_{m+1} = |w| + 1$ and such that for all h such that $1 \leq h \leq m$, i_h has label a_h , $w(0, i_h) \in K_h$ and $w(i_h, |w| + 1) \in K'_h$. Observe that these positions must be *unique* since $K_h a_h K'_h$ is unambiguous.
- For $0 \leq h \leq m$, we have $w(i_h, i_{h+1}) \in L_h$.

We show that both properties can be expressed in $\mathcal{B}\Sigma_{n+1}^2(\mathbb{I}_{\mathcal{C}})$. First, we build $\mathcal{B}\Sigma_1^2(\mathbb{I}_{\mathcal{C}})$ formulas with the following lemma.

LEMMA 6.10. *For $1 \leq h \leq m$, there exists a formula $\psi_h(x)$ of $\mathcal{B}\Sigma_1^2(\mathbb{I}_{\mathcal{C}})$ with one free variable x such that for every $w \in A^*$ and $i \in \mathbb{P}(w)$, we have $w \models \psi_h(i)$ if and only if i has label a_h , $w(0, i) \in K_h$ and $w(i, |w| + 1) \in K'_h$.*

Lemma 6.10 holds since $K_h, K'_h \in BPol(\mathcal{C})$ (the proof is identical to that of $BPol(\mathcal{C}) \subseteq \mathcal{B}\Sigma_1^2(\mathbb{I}_{\mathcal{C}})$ in Theorem 6.3). We fix the $\mathcal{B}\Sigma_1^2(\mathbb{I}_{\mathcal{C}})$ formulas ψ_1, \dots, ψ_m for the proof. We use them to define

new formulas $\Gamma_h(x)$ for $1 \leq h \leq m$. We let $\Gamma_1(x) := \psi_1(x)$. Additionally, for $h > 1$, we define $\Gamma_h(x) := \psi_h(x) \wedge \exists y (y < x \wedge \Gamma_{h-1}(y))$ (the definition involves implicit variable renaming, this is standard in FO²). Finally, we let $\Gamma := \exists x \Gamma_m(x)$. By definition, Γ is a sentence of $\mathcal{B}\Sigma_{n+1}^2(\mathbb{I}_{\mathcal{C}}) \subseteq \mathcal{B}\Sigma_{n+1}^2(\mathbb{I}_{\mathcal{C}})$ and it expresses Condition a).

We turn to Condition b). We define $\psi_0(x) := (x = \min)$ and $\psi_{m+1}(x) := (x = \max)$ for the construction. For every h such that $0 \leq h \leq m$, we construct a $\mathcal{B}\Sigma_{n+1}^2(\mathbb{I}_{\mathcal{C}})$ sentence φ_h which satisfies the following property: for every word $w \in A^*$ such that $w \models \Gamma$ (which yields *unique* positions $i_h, i_{h+1} \in \mathbb{P}(w)$ such that $w \models \psi_h(i_h)$ and $w \models \psi_{h+1}(i_{h+1})$), we have $w \models \varphi_h$ if and only if $w(i_h, i_{h+1}) \in L_h$. It will then be immediate that $L_0 a_1 L_1 \cdots a_m L_m$ is defined by the sentence $\varphi := \Gamma \wedge \bigwedge_{0 \leq h \leq m} \varphi_h$ of $\mathcal{B}\Sigma_{n+1}^2(\mathbb{I}_{\mathcal{C}})$, completing the proof.

We now fix h such that $0 \leq h \leq m$ and construct φ_h . By hypothesis, we have $L_h \in \mathcal{D}_n = \mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{C}})$. Hence, we get a sentence δ_h of $\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{C}})$ defining L_h . We build φ_h from δ_h by applying two kinds of modifications. First, we restrict the quantifications in δ_h to the positions that are in-between the two unique ones satisfying ψ_h and ψ_{h+1} . We recursively replace each sub-formula of the form $\exists x \zeta$ by the following (we write “ $x \leq y$ ” for the formula “ $x < y \vee x = y$ ”):

$$\exists x (\zeta \wedge (\exists y (\psi_h(y) \wedge y \leq x)) \wedge (\exists y (\psi_{h+1}(y) \wedge x \leq y))) .$$

Intuitively, we are using the unique positions satisfying ψ_h and ψ_{h+1} as substitutes for the two artificial unlabeled positions. Hence, we also need to tweak the atomic sub-formulas in δ_h . First, we replace all atomic sub-formulas $b(x)$ with $b \in A$ by,

$$b(x) \wedge (\exists y (\psi_h(y) \wedge y < x)) \wedge (\exists y (\psi_{h+1}(y) \wedge x < y)) .$$

We also need to modify the atomic sub-formulas involving the constants *min* and *max*. All sub-formulas $P(\min, x)$ with $P(\min, x) := (\min = x)$ or $P(\min, x) := I_L(\min, x)$ where $L \in \mathcal{C}$ are replaced by $\exists y (\psi_h(y) \wedge P(y, x))$. Symmetrically, all sub-formulas $P(x, \max)$ with $P(x, \max) := (x = \max)$ or $P(x, \max) := I_L(x, \max)$ where $L \in \mathcal{C}$ are replaced by $\exists y (\psi_{h+1}(y) \wedge P(x, y))$. Finally, all sub-formulas $I_L(\min, \max)$ for $L \in \mathcal{C}$ are replaced by the formula $\exists x \exists y (\psi_h(x) \wedge \psi_{h+1}(y) \wedge I_L(x, y))$. There can be other atomic sub-formulas involving *min* and *max* such as $b(\min)$, $(\min = \max)$ or $I_L(\max, x)$. We do not modify them since they are equivalent to \perp (*i.e.*, false).

By definition, φ_h is built by nesting the $\mathcal{B}\Sigma_1^2(\mathbb{I}_{\mathcal{C}})$ formulas ψ_h and ψ_{h+1} under the sentence δ_h of $\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{C}})$. Thus, one may verify that φ_h is a sentence of $\mathcal{B}\Sigma_{n+1}^2(\mathbb{I}_{\mathcal{C}})$ as desired. One may also verify that φ_h satisfies the desired property: for every word $w \in A^*$ such that $w \models \Gamma$ (we get *unique* positions $i_h, i_{h+1} \in \mathbb{P}(w)$ such that $w \models \psi_h(i_h)$ and $w \models \psi_{h+1}(i_{h+1})$), we know that $w \models \varphi_h$ if and only if $w(i_h, i_{h+1}) \models \delta_h$ (*i.e.*, $w(i_h, i_{h+1}) \in L_h$). This concludes the proof. \square

7 SEPARATION

We prove that if \mathcal{C} is a *finite* prevariety with decidable separation and \mathcal{D} is a prevariety which satisfy $\mathcal{C} \subseteq \mathcal{D} \subseteq UPol(\mathcal{C})$, separation is decidable for the classes $LPol(\mathcal{D})$, $RPol(\mathcal{D})$ and $MPol(\mathcal{D})$. This result is designed to handle the hierarchies of Section 5 associated to an input class \mathcal{C} which is a *finite* prevariety. Using Theorem 6.4, we also get the decidability of separation for all levels $\mathcal{B}\Sigma_n^2(<)$ in the quantifier-alternation hierarchy of FO²(<).

REMARK 7.1. *Separation is often considered alongside a more general problem called covering (see [25]). We do not discuss it since this involves too much machinery. However, the results of this section can be generalized to this problem.*

7.1 Preliminaries

We present notions that we use in our separation algorithms. First, we explain how *finite* prevarieties are handled.

\mathcal{C} -compatible morphisms. Let \mathcal{C} be a finite prevariety. A morphism $\alpha : A^* \rightarrow M$ is \mathcal{C} -compatible if and only if it is surjective and recognizes all languages in \mathcal{C} .

FACT 7.2. *Let \mathcal{C} be a finite prevariety. Given as input two regular languages L_0 and L_1 , one may compute a \mathcal{C} -compatible morphism recognizing both L_0 and L_1 .*

Non-separable pairs. Let $\alpha : A^* \rightarrow M$ be a morphism and \mathcal{D} a prevariety. We let $\mathcal{N}_{\mathcal{D}}[\alpha] \subseteq M^2$ as the set of all \mathcal{D} -pairs for α , i.e. $(s, s') \in \mathcal{N}_{\mathcal{D}}[\alpha]$ if and only if $\alpha^{-1}(s)$ is not \mathcal{D} -separable from $\alpha^{-1}(s')$. Clearly, $\mathcal{N}_{\mathcal{D}}[\alpha]$ can be computed from α if \mathcal{D} -separation is decidable. We have the converse.

LEMMA 7.3. *Let \mathcal{D} be a prevariety, $\alpha : A^* \rightarrow M$ a morphism and $F_0, F_1 \subseteq M$. In this case, $\alpha^{-1}(F_0)$ is \mathcal{D} -separable from $\alpha^{-1}(F_1)$ if and only if $(F_0 \times F_1) \cap \mathcal{N}_{\mathcal{D}}[\alpha] = \emptyset$.*

For a fixed finite prevariety \mathcal{C} , Fact 7.2 implies that given as input two regular languages L_0 and L_1 , one may compute a single \mathcal{C} -compatible morphism $\alpha : A^* \rightarrow M$ recognizing both L_0 and L_1 . Thus, by Lemma 7.3, finding a \mathcal{D} -separation algorithm boils down to exhibiting an algorithm for computing $\mathcal{N}_{\mathcal{D}}[\alpha]$ from an input \mathcal{C} -compatible morphism $\alpha : A^* \rightarrow M$. We use this approach below.

We complete the definition with a useful property of the sets $\mathcal{N}_{\mathcal{D}}[\alpha]$. A set $S \subseteq M^2$ is *saturated* (for α) if $(\alpha(w), \alpha(w)) \in S$ for every $w \in A^*$ and S is closed under multiplication: if $(s_i, s'_i) \in S$ for $i = 1, 2$, then $(s_1s_2, s'_1s'_2) \in S$.

LEMMA 7.4. *Let \mathcal{D} be a prevariety and $\alpha : A^* \rightarrow M$ a morphism. Then, $\mathcal{N}_{\mathcal{D}}[\alpha]$ is saturated for α .*

Alphabet testable languages. The key applications of our separation results consider a particular finite prevariety \mathcal{C} . Let AT be the class containing the Boolean combinations of languages A^*aA^* where $a \in A$. One may verify that AT is a finite prevariety. This class is connected to the class $\text{PT} = \text{BPol}(\text{ST})$ of piecewise testable languages.

LEMMA 7.5. *The following equalities hold: $\text{UPol}(\text{AT}) = \text{UPol}(\text{PT})$, $\text{LPol}(\text{AT}) = \text{LPol}(\text{PT})$ and $\text{RPol}(\text{AT}) = \text{RPol}(\text{PT})$.*

REMARK 7.6. *This fails for MPol : $\text{MPol}(\text{AT}) \subsetneq \text{MPol}(\text{PT})$.*

7.2 Left/right polynomial closure

Given a *finite* prevariety \mathcal{C} and a prevariety \mathcal{D} which satisfy the inclusions $\mathcal{C} \subseteq \mathcal{D} \subseteq \text{UPol}(\mathcal{C})$, we characterize $\mathcal{N}_{\text{LPol}(\mathcal{D})}[\alpha]$ and $\mathcal{N}_{\text{RPol}(\mathcal{D})}[\alpha]$ for an arbitrary \mathcal{C} -compatible morphism α . If \mathcal{D} -separation is decidable, this yields procedures for computing both sets. By Lemma 7.3, it follows that separation is decidable for $\text{LPol}(\mathcal{D})$ and $\text{RPol}(\mathcal{D})$.

We first present the characterization. Let $\alpha : A^* \rightarrow M$ be a surjective morphism. Recall that by Lemma 2.7, the quotient $M/\sim_{\mathcal{C}}$ is a

monoid. We use the Green relations $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ defined on $M/\sim_{\mathcal{C}}$. Let $P \subseteq M^2$. We say that another set $S \subseteq M^2$ is (LPol, P) -saturated (for α) when it is saturated (for α), and satisfies the following additional property:

$$\begin{aligned} &\text{if } (e, e') \in S \text{ is a pair of idempotents and } (s, s') \in P \\ &\text{satisfies } [e]_{\mathcal{C}} \leq_{\mathcal{R}} [s]_{\mathcal{C}}, \text{ then } (es, e's') \in S. \end{aligned} \quad (3)$$

Symmetrically, S is (RPol, P) -saturated when it is saturated (for α), and satisfies the following additional property:

$$\begin{aligned} &\text{if } (e, e') \in S \text{ is a pair of idempotents and } (s, s') \in P \\ &\text{satisfies } [e]_{\mathcal{C}} \leq_{\mathcal{L}} [s]_{\mathcal{C}}, \text{ then } (se, s'e') \in S. \end{aligned} \quad (4)$$

We are ready to state the characterization.

THEOREM 7.7. *Let \mathcal{C} be a finite prevariety and \mathcal{D} a prevariety such that $\mathcal{C} \subseteq \mathcal{D} \subseteq \text{UPol}(\mathcal{C})$. Let $\alpha : A^* \rightarrow M$ be a \mathcal{C} -compatible morphism and $P = \mathcal{N}_{\mathcal{D}}[\alpha]$. Then, $\mathcal{N}_{\text{LPol}(\mathcal{D})}[\alpha]$ is the least (LPol, P) -saturated subset of M^2 and $\mathcal{N}_{\text{RPol}(\mathcal{D})}[\alpha]$ is the least (RPol, P) -saturated subset of M^2 for α .*

In view of Theorem 7.7, once we have $P = \mathcal{N}_{\mathcal{D}}[\alpha]$ in hand (this boils down to \mathcal{D} -separation by definition), it is possible to compute $\mathcal{N}_{\text{LPol}(\mathcal{D})}[\alpha]$ and $\mathcal{N}_{\text{RPol}(\mathcal{D})}[\alpha]$ from α using least fixpoint procedures. Therefore, we obtain the following corollary from Fact 7.2 and Lemma 7.3.

COROLLARY 7.8. *Let \mathcal{C} be a finite prevariety and \mathcal{D} a prevariety with decidable separation such that $\mathcal{C} \subseteq \mathcal{D} \subseteq \text{UPol}(\mathcal{C})$. Then separation is decidable for $\text{LPol}(\mathcal{D})$ and $\text{RPol}(\mathcal{D})$.*

A key application is the case $\mathcal{C} = \text{AT}$. Since AT is finite, AT -separation is decidable (there are finitely many separator candidates and we may test them all). Therefore, we may apply Corollary 7.8 recursively to obtain that for every $n \in \mathbb{N}$, $\text{LP}_n(\text{AT})$ and $\text{RP}_n(\text{AT})$ are both decidable. By Lemma 7.5, these are also the classes $\text{LP}_n(\text{PT})$ and $\text{RP}_n(\text{PT})$.

COROLLARY 7.9. *For every $n \in \mathbb{N}$, separation is decidable for $\text{LP}_n(\text{PT})$ and $\text{RP}_n(\text{PT})$.*

7.3 Mixed polynomial closure

Given a *finite* prevariety \mathcal{C} and a prevariety \mathcal{D} which satisfy the inclusions $\mathcal{C} \subseteq \mathcal{D} \subseteq \text{UPol}(\mathcal{C})$, we characterize $\mathcal{N}_{\text{MPol}(\mathcal{D})}[\alpha]$ for an arbitrary \mathcal{C} -compatible morphism α . If \mathcal{D} -separation is decidable, this yields an algorithm for computing this set. By Lemma 7.3, we get that separation is decidable for $\text{MPol}(\mathcal{D})$.

We first present the characterization. Let $\alpha : A^* \rightarrow M$ be a surjective morphism. By Lemma 2.7, the quotient $M/\sim_{\mathcal{C}}$ is a monoid. We use the Green relation \mathcal{J} on $M/\sim_{\mathcal{C}}$. Let $P, P_1, P_2 \subseteq M^2$. We start with a preliminary notion. A (P_1, P, P_2) -block is a pair $(s_1e_1s_3e_2s_2, s'_1e'_1s'_3e'_2s'_2) \in M^2$ which satisfies $(s_1, s'_1), (e_1, e'_1) \in P_1$, $(s_2, s'_2), (e_2, e'_2) \in P_2$ and $(s_3, s'_3) \in P$. Moreover, e_1, e'_1, e_2, e'_2 must be idempotents of M such that $[e_1]_{\mathcal{C}} \mathcal{J} [e_2]_{\mathcal{C}} \mathcal{J} [s_1e_1s_3e_2s_2]_{\mathcal{C}}$. We consider a set $S \subseteq M^2$. We say that S is $(\text{MPol}, P_1, P, P_2)$ -saturated (for α) when it is saturated (for α), and satisfies the following additional property for every $n \in \mathbb{N}$:

$$\begin{aligned} &\text{if } (s_0, s'_0), \dots, (s_n, s'_n) \in M^2 \text{ are } (P_1, P, P_2)\text{-blocks and} \\ &(t_1, t'_1), \dots, (t_n, t'_n) \in P \text{ satisfy } [s_{i-1}t_i]_{\mathcal{C}} \mathcal{J} [s_{i-1}]_{\mathcal{C}} \\ &\text{and } [t_i s_i]_{\mathcal{C}} \mathcal{J} [s_i]_{\mathcal{C}} \text{ for } 1 \leq i \leq n, \\ &\text{then } (s_0 t_1 s_1 \cdots t_n s_n, s'_0 t'_1 s'_1 \cdots t'_n s'_n) \in S. \end{aligned} \quad (5)$$

When $n = 0$, (5) states that if (s_0, s'_0) is a (P_1, P, P_2) -block, then $(s_0, s'_0) \in S$. We may now present the characterization.

THEOREM 7.10. *Let \mathcal{C} be a finite prevariety and \mathcal{D} a prevariety such that $\mathcal{C} \subseteq \mathcal{D} \subseteq \text{UPol}(\mathcal{C})$. Moreover, let $\alpha : A^* \rightarrow M$ be a \mathcal{C} -compatible morphism and let $P = \mathcal{N}_{\mathcal{D}}[\alpha]$, $P_1 = \mathcal{N}_{\text{LPol}(\mathcal{D})}[\alpha]$ and $P_2 = \mathcal{N}_{\text{RPol}(\mathcal{D})}[\alpha]$. Then, the set $\mathcal{N}_{\text{MPol}(\mathcal{D})}[\alpha]$ is the least $(\text{MPol}, P_1, P, P_2)$ -saturated subset of M^2 for α .*

By Theorem 7.10, once we have the three sets $P = \mathcal{N}_{\mathcal{D}}[\alpha]$, $P_1 = \mathcal{N}_{\text{LPol}(\mathcal{D})}[\alpha]$ and $P_2 = \mathcal{N}_{\text{RPol}(\mathcal{D})}[\alpha]$ in hand, we may compute $\mathcal{N}_{\text{MPol}(\mathcal{D})}[\alpha]$ from α using a least fixpoint algorithm (note that while (5) must hold for every $n \in \mathbb{N}$, one may verify using a pumping argument that this is equivalent to (5) holding for $n \leq |M|^3$). Computing P boils down to \mathcal{D} -separation by definition. It is also possible to compute P_1 and P_2 when \mathcal{D} -separation is decidable by Theorem 7.7. Thus, we get the following corollary from Lemma 7.3.

COROLLARY 7.11. *Let \mathcal{C} be a finite prevariety and \mathcal{D} a prevariety with decidable separation such that $\mathcal{C} \subseteq \mathcal{D} \subseteq \text{UPol}(\mathcal{C})$. Then separation is decidable for $\text{MPol}(\mathcal{D})$.*

Corollary 7.11 applies to the quantifier alternation hierarchy of $\text{FO}^2(<)$. Indeed, it follows from Theorem 6.3 and Lemma 6.2 that $\mathcal{B}\Sigma_1^2(<) = \text{BPol}(\text{ST}) = \text{PT}$. Moreover, the class PT of piecewise testable languages is known to have decidable separation (see [5, 24]). Additionally, we have $\text{AT} \subseteq \text{PT} \subseteq \text{UPol}(\text{AT})$ by Lemma 7.5 and AT is a finite prevariety. Finally, Theorem 6.4 implies that the levels $\mathcal{B}\Sigma_n^2(<)$ are built from PT by applying MPol iteratively. We obtain the following result from Corollary 7.11 and a simple induction.

COROLLARY 7.12. *For all $n \in \mathbb{N}$, separation is decidable for the level $\mathcal{B}\Sigma_n^2(<)$.*

Let us point out that Corollary 7.12 was proved independently in [9] using distinct techniques.

REMARK 7.13. *Using independent techniques, one may lift Corollary 7.12 to the levels $\mathcal{B}\Sigma_n^2(<, +1)$ and $\mathcal{B}\Sigma_n^2(<, +1, \text{MOD})$ in the hierarchies of $\text{FO}^2(<, +1)$ and $\text{FO}^2(<, +1, \text{MOD})$. It turns out that $\mathcal{B}\Sigma_n^2(<)$, $\mathcal{B}\Sigma_n^2(<, +1)$ and $\mathcal{B}\Sigma_n^2(<, +1, \text{MOD})$ are connected by another operator called “enrichment” or “wreath product” which is used to combine two classes into a larger one. First, we have $\mathcal{B}\Sigma_n^2(<, +1) = \mathcal{B}\Sigma_n^2(<) \circ \text{SU}$ with SU as the class of “suffix languages” (the Boolean combinations of languages A^*w with $w \in A^*$). A proof is available in [18]. Moreover, $\mathcal{B}\Sigma_n^2(<, +1, \text{MOD}) = \mathcal{B}\Sigma_n^2(<, +1) \circ \text{MOD}$ (this is a standard property which holds for many fragments of first-order logic, see [23] for example). Finally, it is known that the operators $\mathcal{C} \mapsto \mathcal{C} \circ \text{SU}$ and $\mathcal{C} \mapsto \mathcal{C} \circ \text{SU} \circ \text{MOD}$ preserve the decidability of separation [23, 30]. Therefore, Corollary 7.12 also implies that for every $n \in \mathbb{N}$, separation is decidable for both $\mathcal{B}\Sigma_n^2(<, +1)$ and $\mathcal{B}\Sigma_n^2(<, +1, \text{MOD})$.*

8 CONCLUSION

We investigated the operators LPol , RPol and MPol , and the associated deterministic hierarchies. We proved that these three operators preserve the decidability of membership. Moreover, we used MPol to characterize the quantifier alternation hierarchies of the variants

$\text{FO}^2(<, \mathbb{P}_{\mathcal{G}})$ and $\text{FO}^2(<, +1, \mathbb{P}_{\mathcal{G}})$ of FO^2 for a group prevariety \mathcal{G} . They imply the decidability of membership for all levels when separation is decidable for \mathcal{G} . Finally, we looked at separation for our operators and used the results to show that all levels in the quantifier alternation hierarchy of $\text{FO}^2(<)$ have decidable separation. In particular, MPol is the linchpin upon which most of our results are based.

There are several follow-up questions. A first point concerns membership for the levels $\text{LP}_n(\mathcal{C}) \vee \text{RP}_n(\mathcal{C})$ of the hierarchies introduced in Section 5. These are the only levels which we are not able to handle in a generic manner. Indeed, it follows from Theorems 4.3 and 5.6 that membership is decidable for all these levels as soon as this is the case for the first one: $\text{LPol}(\mathcal{C}) \vee \text{RPol}(\mathcal{C})$. Yet, we do not have a generic result for handling this initial level. Another question is whether our separation results for the levels $\mathcal{B}\Sigma_n^2(<)$ can be generalized to the variants $\mathcal{B}\Sigma_n^2(<, \mathbb{P}_{\mathcal{G}})$ and $\mathcal{B}\Sigma_n^2(<, +1, \mathbb{P}_{\mathcal{G}})$ for arbitrary group prevarieties \mathcal{G} . Such a result is proved in [29] for the first level: if \mathcal{G} has decidable separation, then so does $\mathcal{B}\Sigma_1^2(<, \mathbb{P}_{\mathcal{G}})$ (the proof considers $\text{BPol}(\mathcal{G})$ which characterizes $\mathcal{B}\Sigma_1^2(<, \mathbb{P}_{\mathcal{G}})$ by Theorem 6.3) Finally, one may also look at the other variants of FO^2 : the classes $\text{FO}^2(\mathbb{I}_{\mathcal{G}})$ for an arbitrary prevariety \mathcal{G} . Unfortunately, our results fail in the general case. An example is considered in [10]: FO^2 with “between relations”. It is simple to verify from the definition that this class corresponds to $\text{FO}^2(\mathbb{I}_{\text{AT}})$. The results of [10] imply that $\text{FO}^2(\mathbb{I}_{\text{AT}})$ is distinct from $\text{UPol}(\text{BPol}(\text{AT}))$ which means that Corollary 6.7 fails in this case.

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A APPENDIX TO SECTION 2

We provide proof for the statements presented in Section 2. We start with those concerning \mathcal{C} -morphisms. First, we consider the following proposition.

PROPOSITION 2.2. *Let \mathcal{C} be a prevariety. A regular language belongs to \mathcal{C} iff its syntactic morphism is a \mathcal{C} -morphism.*

PROOF. We fix a regular language L for the proof. Moreover, we let $\alpha_L : A^* \rightarrow M_L$ as its syntactic morphism. Since L is recognized

by α_L , it is immediate that if α_L is a \mathcal{C} -morphism, then $L \in \mathcal{C}$. We prove the converse implication. Assume that $L \in \mathcal{C}$. We show that $\alpha^{-1}(F) \in \mathcal{C}$ for every $F \subseteq M$. Since \mathcal{C} is closed under union, it suffices to consider the case when $F = \{s\}$ for some $s \in M$. By definition of the syntactic morphism, $\alpha^{-1}(s)$ is an equivalence class of \equiv_L . Moreover, since \equiv_L has finite index, it is immediate by definition that it is a finite Boolean combination of languages of the form $\{w \mid xwy \in L\}$ for $w, y \in A^*$. Since $L \in \mathcal{C}$ and \mathcal{C} is closed under quotients, all languages $\{w \mid xwy \in L\}$ belong to \mathcal{C} . Hence, since \mathcal{C} is a Boolean algebra, we get $\alpha^{-1}(s) \in \mathcal{C}$ which completes the proof. \square

We turn to the proposition used to construct \mathcal{C} -morphisms.

PROPOSITION 2.3. *Let \mathcal{C} be a prevariety and consider finitely many languages $L_1, \dots, L_k \in \mathcal{C}$. There exists a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ such that L_1, \dots, L_k are recognized by η .*

PROOF. For every $i \leq k$, we let $\alpha_i : A^* \rightarrow M_i$ as the syntactic morphism of L_i . We know from Proposition 2.2 that α_i is a \mathcal{C} -morphism. Let $M = M_1 \times \dots \times M_n$ be the monoid equipped with the componentwise multiplication and $\alpha : A^* \rightarrow M$ the morphism defined by $\alpha(w) = (\alpha_1(w), \dots, \alpha_n(w))$ for every $w \in A^*$. Clearly, the languages L_1, \dots, L_n are all recognized by α . Moreover, for every $\bar{s} = (s_1, \dots, s_n) \in M$, it is immediate that $\alpha^{-1}(\bar{s}) = \alpha_1^{-1}(s_1) \cap \dots \cap \alpha_n^{-1}(s_n)$. Hence, $\alpha^{-1}(\bar{s}) \in \mathcal{C}$ by closure under intersection. It follows that every language recognized by α belongs to \mathcal{C} . Hence, it suffices to define η as the surjective restriction of α to get the desired \mathcal{C} -morphism. \square

We turn to the statements concerning the canonical equivalence $\sim_{\mathcal{C}}$ associated to every morphism.

LEMMA 2.5. *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ be a morphism. The equivalence $\sim_{\mathcal{C}, \alpha}$ on M is the reflexive transitive closure of the \mathcal{C} -pair relation associated to α .*

PROOF. We write \cong for the reflexive transitive closure of the \mathcal{C} -pair relation. We show that $\cong = \sim_{\mathcal{C}, \alpha}$. We start with the left to right inclusion. Since $\sim_{\mathcal{C}, \alpha}$ is an equivalence by definition, it suffices that for every \mathcal{C} -pair $(s, t) \in M^2$, we have $s \sim_{\mathcal{C}, \alpha} t$. Given $F \subseteq M$ such that $\alpha^{-1}(F) \in \mathcal{C}$, we have to show that $s \in F \Leftrightarrow t \in F$. We first prove that $s \in F \Rightarrow t \in F$. If $s \in F$, then $\alpha^{-1}(s) \subseteq \alpha^{-1}(F)$. Hence, since $\alpha^{-1}(F) \in \mathcal{C}$ and (s, t) is a \mathcal{C} -pair (which means that $\alpha^{-1}(s)$ is not \mathcal{C} -separable from $\alpha^{-1}(t)$), we have $\alpha^{-1}(F) \cap \alpha^{-1}(t) \neq \emptyset$. Thus, we get $t \in F$ as desired. The implication $t \in F \Rightarrow s \in F$ is proved symmetrically since (t, s) is also a \mathcal{C} -pair (this is immediate from the definition as \mathcal{C} is closed under complement).

We now prove that $\sim_{\mathcal{C}, \alpha} \subseteq \cong$. Let $s, t \in M$ such that $s \sim_{\mathcal{C}, \alpha} t$. We show that $s \cong t$. Let $F = \{q \in M \mid s \cong q\}$. We prove that $\alpha^{-1}(F) \in \mathcal{C}$. Since $s \sim_{\mathcal{C}, \alpha} t$, it will follow that $t \in F$ by definition, i.e. that $s \cong t$ as desired. Let $q \in F$ and $r \notin F$. By definition of \cong , it is immediate that (q, r) is not a \mathcal{C} -pair. Hence, there exists $H_{q,r} \in \mathcal{C}$ such that $\alpha^{-1}(q) \subseteq H_{q,r}$ and $\alpha^{-1}(r) \cap H_{q,r} = \emptyset$. For every $q \in F$, we define,

$$H_q = \bigcap_{r \notin F} H_{q,r}.$$

Clearly $H_q \in \mathcal{C}$ since \mathcal{C} is a lattice. Moreover, $\alpha^{-1}(q) \subseteq H_q$ and $\alpha^{-1}(r) \cap H_r = \emptyset$ for every $r \notin F$. This yields,

$$\alpha^{-1}(F) = \bigcup_{q \in F} H_q.$$

Therefore, $\alpha^{-1}(F) \in \mathcal{C}$ since \mathcal{C} is a lattice. \square

We turn to Lemma 2.6.

LEMMA 2.6. *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ be a surjective morphism. Then, $\sim_{\mathcal{C}, \alpha}$ is a congruence of M .*

PROOF. We fix $s_1, t_1, s_2, t_2 \in M$ such that $s_1 \sim_{\mathcal{C}} t_1$ and $s_2 \sim_{\mathcal{C}} t_2$. We prove that $s_1 s_2 \sim_{\mathcal{C}} t_1 t_2$. Let $F \subseteq M$ such that $\alpha^{-1}(F) \in \mathcal{C}$. We show that $s_1 s_2 \in F \Leftrightarrow t_1 t_2 \in F$. By symmetry, we only prove the left to right implication. Hence, we assume that $s_1 s_2 \in F$. Let $u, v \in A^*$ such that $\alpha(u) = s_1$ and $\alpha(v) = t_2$ (this is where we use the hypothesis that α is surjective). Let $X = \{q \in M \mid s_1 q \in F\}$. We have $s_2 \in X$ and $\alpha^{-1}(X) = \{w \in A^* \mid uw \in \alpha^{-1}(F)\}$ which belongs to \mathcal{C} by closure under quotients. Hence, since $s_2 \sim_{\mathcal{C}} t_2$, we get $t_2 \in X$ which yields $s_1 t_2 \in F$. Let $Y = \{r \in M \mid r t_2 \in F\}$. We know that $s_1 \in Y$ since $s_1 t_2 \in F$. Moreover, we know that $\alpha^{-1}(Y) = \{w \in A^* \mid w v \in \alpha^{-1}(F)\}$ which belongs to \mathcal{C} by closure under quotients. Hence, since $s_1 \sim_{\mathcal{C}} t_1$, we get $t_1 \in Y$ which yields $t_1 t_2 \in F$ as desired. \square

It remains to prove Lemma 2.7.

LEMMA 2.7. *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ be a surjective morphism. The languages recognized by $[\cdot]_{\mathcal{C}} \circ \alpha : A^* \rightarrow M/\sim_{\mathcal{C}}$ are exactly those which are simultaneously in \mathcal{C} and recognized by α .*

PROOF. By definition, the languages recognized by $[\cdot]_{\mathcal{C}} \circ \alpha$ are those of the form $\alpha^{-1}(F)$ where F is a $\sim_{\mathcal{C}}$ -class. Hence, it suffices to prove that for every $F \subseteq M$, we have $\alpha^{-1}(F) \in \mathcal{C}$ if and only if F is a $\sim_{\mathcal{C}}$ -class. We fix $F \subseteq M$ for the proof. Assume first that $\alpha^{-1}(F) \in \mathcal{C}$ we prove that F is a $\sim_{\mathcal{C}}$ -class. Let $s \in F$ and $t \in M$ such that $s \sim_{\mathcal{C}} t$. By definition of $\sim_{\mathcal{C}}$ and since $\alpha^{-1}(F) \in \mathcal{C}$, we have $t \in F$. Hence F is a $\sim_{\mathcal{C}}$ -class as desired. Conversely, assume that F is a $\sim_{\mathcal{C}}$ -class. We prove that $\alpha^{-1}(F) \in \mathcal{C}$. Consider $s \in F$. By definition, we know that for every element $r \notin F$, we have $s \not\sim_{\mathcal{C}} r$. Hence, there exists a set $F_{s,r} \subseteq M$ such that $\alpha^{-1}(F_{s,r}) \in \mathcal{C}$, $s \in F_{s,r}$ and $r \notin F_{s,r}$. It is now immediate that,

$$F = \bigcup_{s \in F} \left(\bigcap_{r \notin F} F_{s,r} \right).$$

Hence, since inverse image commutes with Boolean operation, we obtain,

$$\alpha^{-1}(F) = \bigcup_{s \in F} \left(\bigcap_{r \notin F} \alpha^{-1}(F_{s,r}) \right).$$

This yields $\alpha^{-1}(F) \in \mathcal{C}$ since \mathcal{C} is a Boolean algebra. \square

B APPENDIX TO SECTION 3

In this appendix, we prove the statements involved in the framework that we use for handling *LPol*, *RPol* and *MPol*. We separate Proposition 3.11 from the rest as it requires quite a bit of work.

B.1 Statements involved in the definition

We start with Lemma 3.8.

LEMMA 3.8. *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ a $UPol(\mathcal{C})$ -morphism. For every $h \in \mathbb{N}$ and every word $w \in A^*$, $P_{\triangleright}(\alpha, h, w) \subseteq P_{\triangleright}([\cdot]_{\mathcal{C}} \circ \alpha, h|M|, w)$ and $P_{\triangleleft}(\alpha, h, w) \subseteq P_{\triangleleft}([\cdot]_{\mathcal{C}} \circ \alpha, h|M|, w)$.*

PROOF. We write $N = M/\sim_{\mathcal{C}}$ and $\eta = [\cdot]_{\mathcal{C}} \circ \alpha : A^* \rightarrow N$ for the proof. We show that $P_{\triangleright}(\alpha, h, w) \subseteq P_{\triangleright}(\eta, h|M|, w)$ for all $w \in A^*$ and $h \in \mathbb{N}$. The other inclusion is symmetrical and left to the reader. Let $a_1, \dots, a_\ell \in A$ be the letters such that $w = a_1 \cdots a_\ell$. We use induction on h . If $h = 0$, then $P_{\triangleright}(\alpha, 0, w) = P_{\triangleright}(\eta, 0, w) = \emptyset$. Assume now that $h \geq 1$ and let $i \in P_{\triangleright}(\alpha, h, w)$. We show that $i \in P_{\triangleright}(\eta, h|M|, w)$. By definition, there is $j \in P_{\triangleright}(\alpha, h-1, w) \cup \{0\}$ such that $j < i$ and $\alpha(w(j, i)a_i) <_{\mathcal{R}} \alpha(w(j, i))$. By induction, we get $j \in P_{\triangleright}(\eta, (h-1)|M|, w) \cup \{0\}$. Let $i_1, \dots, i_n \in P_c(w)$ be all the positions in w which satisfy $j < i_1 < \dots < i_n$ and $\alpha(w(j, i_\ell)a_{i_\ell}) <_{\mathcal{R}} \alpha(w(j, i_\ell))$ for $1 \leq \ell \leq n$. Note that $n \leq |M|$ by definition. Since $i \in \{i_1, \dots, i_n\}$ by hypothesis, it now suffices to prove that $i_1, \dots, i_n \in P_{\triangleright}(\eta, h|M|, w)$. We write $i_0 = j$. For every ℓ such that $1 \leq \ell \leq n$, we prove that $\eta(w(i_{\ell-1}, i_\ell)a_{i_\ell}) <_{\mathcal{R}} \eta(w(i_{\ell-1}, i_\ell))$. Since we have $i_0 = j \in P_{\triangleright}(\eta, h|M| - |M|, w) \cup \{0\}$ and $n \leq |M|$, this implies that $i_1, \dots, i_n \in P_{\triangleright}(\eta, h|M|, w)$ by definition.

We proceed by contradiction. Assume that there exists an index $1 \leq \ell \leq n$ such that $\eta(w(i_{\ell-1}, i_\ell)a_{i_\ell}) \not\mathcal{R} \eta(w(i_{\ell-1}, i_\ell))$. We write $u = w(j, i_{\ell-1})a_{i_{\ell-1}}$, $v = w(i_{\ell-1}, i_\ell)$. Our contradiction hypothesis states that $\eta(va_{i_\ell}) \not\mathcal{R} \eta(v)$. This yields $y \in A^*$ such that $\eta(va_{i_\ell}y) = \eta(v)$. Moreover, $\alpha(uva_{i_\ell}) <_{\mathcal{R}} \alpha(uv) \mathcal{R} \alpha(u)$ by definition of i_1, \dots, i_n . Hence, we get a word $z \in A^*$ such that $\alpha(uz) = \alpha(u)$. Since $\eta(va_{i_\ell}y) = \eta(v)$, we have $\eta(va_{i_\ell}yz) = \eta(vz)$, i.e. $\alpha(va_{i_\ell}yz) \sim_{\mathcal{C}} \alpha(vz)$ by definition of η . Therefore, since α is a $UPol(\mathcal{C})$ -morphism, it follows from Theorem 3.7 that $(\alpha(vz))^{\omega+1} = (\alpha(vz))^\omega \alpha(va_{i_\ell}yz) (\alpha(vz))^\omega$. We multiply on the left by $\alpha(u)$. Since $\alpha(uz) = \alpha(u)$, we get $\alpha(u) = \alpha(u) \alpha(va_{i_\ell}yz) (\alpha(vz))^\omega$. Hence, we obtain $\alpha(uv) \leq_{\mathcal{R}} \alpha(uva_{i_\ell})$, contradicting the hypothesis that $\alpha(uva_{i_\ell}) <_{\mathcal{R}} \alpha(uv)$. This concludes the proof. \square

We now prove Lemma 3.9. We first present a preliminary result that we shall reuse to prove other statements.

FACT B.1. *Let $\eta : A^* \rightarrow N$ be a surjective morphism and consider $w, w' \in A^*$, $P \subseteq P_c(w)$ and $P' \subseteq P_c(w')$. Assume that $\sigma_\eta(w, P) = \sigma_\eta(w', P')$ and let $P_1, P_2 \subseteq P$ such that $P_1 \cup P_2 = P$. There exist $P'_1, P'_2 \subseteq P'$ such that $P'_1 \cup P'_2 = P'$, $\sigma_\eta(w, P_1) = \sigma_\eta(w', P'_1)$ and $\sigma_\eta(w, P_2) = \sigma_\eta(w', P'_2)$.*

PROOF. Since $\sigma_\eta(w, P) = \sigma_\eta(w', P')$, we have $|P| = |P'|$. Hence, there exists a unique increasing bijection $f : P \rightarrow P'$ (by “increasing”, we mean that $i < j \Rightarrow f(i) < f(j)$ for every $i, j \in P$). We let $P'_1 = f(P_1)$ and $P'_2 = f(P_2)$. Clearly, we have $P'_1 \cup P'_2 = P'$ since $P_1 \cup P_2 = P$. One may then verify using our hypothesis on (w, P) and (w', P') that $\sigma_\eta(w, P_1) = \sigma_\eta(w', P'_1)$ and $\sigma_\eta(w, P_2) = \sigma_\eta(w', P'_2)$. \square

We now turn to Lemma 3.9 itself.

LEMMA 3.9. *Let $\eta : A^* \rightarrow N$ be a surjective morphism, $k \in \mathbb{N}$ and $x \in \{\triangleright, \triangleleft, \bowtie\}$. Consider two words $w, w' \in A^*$ and $P' \subseteq P_c(w')$. If $\sigma_\eta(w, P_x(\eta, k, w)) = \sigma_\eta(w', P')$, then $P' = P_x(\eta, k, w')$.*

PROOF. First, note that the case $x = \bowtie$ is a corollary of the other two. Indeed, assume for now that they hold and that we have $\sigma_\eta(w, P_{\bowtie}(\eta, k, w)) = \sigma_\eta(w', P')$. By definition, we know that $P_{\bowtie}(\eta, k, w) = P_{\triangleright}(\eta, k, w) \cup P_{\triangleleft}(\eta, k, w)$. Consequently, Fact B.1 yields $P'_1, P'_2 \subseteq P'$ which satisfy $P' = P'_1 \cup P'_2$, $\sigma_\eta(w, P_{\triangleright}(\eta, k, w)) = \sigma_\eta(w', P'_1)$ and $\sigma_\eta(w, P_{\triangleleft}(\eta, k, w)) = \sigma_\eta(w', P'_2)$. Hence, the cases when $x \in \{\triangleright, \triangleleft\}$ yield $P'_1 = P_{\triangleright}(\eta, k, w')$ and $P'_2 = P_{\triangleleft}(\eta, k, w')$. We get $P' = P_{\triangleright}(\eta, k, w') \cup P_{\triangleleft}(\eta, k, w') = P_{\bowtie}(\eta, k, w')$ as desired.

We turn to the case $x = \triangleright$ (the case $x = \triangleleft$ is symmetrical and left to the reader). Let $w = a_1 \cdots a_m \in A^*$ and $w' = b_1 \cdots b_n \in A^*$. Additionally, we assume that $\sigma_\eta(w, P_{\triangleright}(\eta, k, w)) = \sigma_\eta(w', P')$. We prove $P' = P_{\triangleright}(\eta, k, w')$. We have $|P_{\triangleright}(\eta, k, w)| = |P'|$ by hypothesis. Consider the unique increasing bijection $f : P_{\triangleright}(\eta, k, w) \rightarrow P'$ (by “increasing”, we mean that $i < j \Rightarrow f(i) < f(j)$ for all i, j). We extend it to the unlabeled positions 0 and $|w| + 1$ by defining $f(0) = 0$ and $f(|w| + 1) = |w'| + 1$. One may now verify that our hypothesis implies the following two properties:

- (1) for all $i \in P_{\triangleright}(\eta, k, w)$, we have $a_i = b_{f(i)}$ (i and $f(i)$ have the same label), and,
- (2) for all $i, j \in P_{\triangleright}(\eta, k, w) \cup \{0, |w| + 1\}$, if $i < j$, then we have $\eta(w(i, j)) = \eta(w'(f(i), f(j)))$.

First, we show that $P' \subseteq P_{\triangleright}(\eta, k, w')$. Let $h \leq k$. Using induction on h , we prove that $f(i) \in P_{\triangleright}(\eta, h, w')$ for all $i \in P_{\triangleright}(\eta, h, w)$. Since f is surjective, the case $h = k$ yields $P' \subseteq P_{\triangleright}(\eta, k, w')$. We consider $i \in P_{\triangleright}(\eta, h, w)$. By definition, $h \geq 1$ and there exists $j \in P_{\triangleright}(\eta, h-1, w) \cup \{0\}$ such that $j < i$ and we have the inequality $\eta(w(j, i)a_i) <_{\mathcal{R}} \eta(w(j, i))$. We have $f(j) < f(i)$ since f is increasing. Moreover we know that $f(j) \in P_{\triangleright}(\eta, h-1, w') \cup \{0\}$ by induction. We know that $a_i = b_{f(i)}$ and $\eta(w(j, i)) = \eta(w'(f(j), f(i)))$. Thus, we get $\eta(w'(f(j), f(i))b_{f(i)}) <_{\mathcal{R}} \eta(w'(f(j), f(i)))$ and we conclude that $f(i) \in P_{\triangleright}(\eta, h, w')$ as desired.

We now prove that $P_{\triangleright}(\eta, k, w') \subseteq P'$. Let $h \leq k$. Using induction on h , we prove that for all $i' \in P_{\triangleright}(\eta, h, w')$, there is $i \in P_{\triangleright}(\eta, h, w)$ such that $i' = f(i)$. This implies $P_{\triangleright}(\eta, k, w') \subseteq P'$ as desired. We fix $i' \in P_{\triangleright}(\eta, h, w')$. By definition, $h \geq 1$, and there exists $j' \in P_{\triangleright}(\eta, h-1, w') \cup \{0\}$ such that $j' < i'$ and $\eta(w'(j', i')b_{i'}) <_{\mathcal{R}} \eta(w'(j', i'))$. Induction yields a position $j \in P_{\triangleright}(\eta, h-1, w) \cup \{0\}$ such that $j' = f(j)$. Let i_1, \dots, i_p be all positions of w such that $j < i_1 < \dots < i_p$ and $\eta(w(j, i_\ell)a_{i_\ell}) <_{\mathcal{R}} \eta(w(j, i_\ell))$ for $1 \leq \ell \leq n$. Since we have $j \in P_{\triangleright}(\eta, h-1, w) \cup \{0\}$, we get $i_1, \dots, i_p \in P_{\triangleright}(\eta, h, w)$. Thus, it suffices to prove that $i' = f(i_\ell)$ for some $\ell \leq p$. We proceed by contradiction. Assume that $i' \neq f(i_\ell)$ for $1 \leq \ell \leq p$. For the proof, we write $i_0 = j$ and $i_{p+1} = |w| + 1$. Clearly, we have $i_0 < i_1 < \dots < i_{p+1}$ which implies that $f(i_0) < f(i_1) < \dots < f(i_{p+1})$. Hence, by hypothesis on i' and since $f(i_0) = j' < i'$, there exists ℓ such that $0 \leq \ell \leq n$ and $f(i_\ell) < i' < f(i_{\ell+1})$. By definition of i_1, \dots, i_p , we have $\eta(w(j, i_\ell)a_{i_\ell}) \mathcal{R} \eta(w(j, i_{\ell+1}))$. Since $j' = f(j)$, we get $\eta(w'(j', f(i_\ell))b_{f(i_\ell)}) \mathcal{R} \eta(w'(j', f(i_{\ell+1})))$. Therefore, since $f(i_\ell) < i' < f(i_{\ell+1})$, we get $\eta(w'(j', i')) \mathcal{R} \eta(w'(j', i')b_{i'})$. This is a contradiction since $\eta(w'(j', i')b_{i'}) <_{\mathcal{R}} \eta(w'(j', i'))$ by hypothesis. \square

We turn to Lemma 3.10.

LEMMA 3.10. *Let $\eta : A^* \rightarrow N$ be a morphism, $w \in A^*$ and $k \in \mathbb{N}$. Let P be the set $P_{\triangleright}(\eta, k, w)$ (resp. $P_{\triangleleft}(\eta, k, w)$, $P_{\bowtie}(\eta, k, w)$) and $(s_0, a_1, s_1, \dots, a_n, s_n) = \sigma_\eta(w, P)$. Then, the marked product*

$\eta^{-1}(s_0)a_1\eta^{-1}(s_1) \cdots a_n\eta^{-1}(s_n)$ is left (resp. right, mixed) deterministic.

PROOF. We treat the case when $P = P_{\bowtie}(\eta, k, w)$ (the other cases are similar and left to the reader). For all h such that $1 \leq h \leq n$, we define $U_h = \eta^{-1}(s_0)a_1\eta^{-1}(s_1) \cdots a_{h-1}\eta^{-1}(s_{h-1})$ and we define $V_h = \eta^{-1}(s_h)a_{h+1} \cdots \eta^{-1}(s_{n-1})a_n\eta^{-1}(s_n)$. We show that either $U_h \cap U_h a_h A^* = \emptyset$ or $V_h \cap A^* a_h V_h = \emptyset$. Let $i_1 < \dots < i_n$ such that $P_{\bowtie}(\eta, k, w) = \{i_1, \dots, i_n\}$ (i_h has label a_h). By definition of $P_{\bowtie}(\eta, k, w)$, we know that either $i_h \in P_{\triangleright}(\eta, k, w)$ or $i_h \in P_{\triangleleft}(\eta, k, w)$ for $1 \leq h \leq n$. In the former case, one may prove that $U_h \cap U_h a_h A^* = \emptyset$ and in the latter case, one may prove that $V_h \cap A^* a_h V_h = \emptyset$. By symmetry, we only prove the former property. Let h such that $1 \leq h \leq n$ and assume that $i_h \in P_{\triangleright}(\eta, k, w)$. We use induction on the least number m such that $i_h \in P_{\triangleright}(\eta, m, w)$ to show that $U_h \cap U_h a_h A^* = \emptyset$.

By definition, we get a position $j \in P_{\triangleright}(\eta, m-1, w) \cup \{0\}$ such that $\eta(w(j, i_h)a_h) <_{\mathcal{R}} \eta(w(j, i_h))$. Let $q = \eta(w(j, i_h))$. Observe that $\eta^{-1}(q)a_h A^* \cap \eta^{-1}(q) = \emptyset$. Indeed, otherwise we get $x \in A^*$ such that $q = q\eta(a_h)\eta(x)$ which contradicts $q\eta(a_h) <_{\mathcal{R}} q$. This concludes the proof when $j = 0$. Since $q = \eta(w(0, i_h))$ in this case, one may verify that $U_h \subseteq \eta^{-1}(q)$. Hence, we get $U_h \cap U_h a_h A^* = \emptyset$. Assume now that $j \neq 0$. Hence, $j \in P_{\triangleright}(\eta, m-1, w)$ which implies that $j = i_g$ for some $g \leq h$. By induction, $U_g \cap U_g a_g A^* = \emptyset$. We use contradiction to prove that $U_h \cap U_h a_h A^* = \emptyset$. Assume that there exists $u \in U_h \cap U_h a_h A^*$. Since $q = \eta(w(i_g, i_h))$, one may verify that $U_h \subseteq U_g a_g \eta^{-1}(q)$. Hence, we get $x, x' \in U_g, y, y' \in \eta^{-1}(q)$ and $z \in A^*$ such that $u = x a_g y a_h z = x' a_g y'$. Since we have $U_g \cap U_g a_g A^* = \emptyset$, this yields $x = x'$. Thus, $y a_h z = y'$. This is a contradiction since $\eta^{-1}(q)a_h A^* \cap \eta^{-1}(q) = \emptyset$. \square

We now prove Lemma 3.12.

LEMMA 3.12. *If $\eta : A^* \rightarrow N$ is a surjective morphism and $k \in \mathbb{N}$, then $\triangleright_{\eta, k}$, $\triangleleft_{\eta, k}$ and $\bowtie_{\eta, k}$ are congruences of finite index.*

PROOF. We present a proof for $\bowtie_{\eta, k}$. It is immediate from the definition that $\bowtie_{\eta, k}$ is an equivalence. Moreover, it has finite index. Indeed, one may verify using induction on k that for every $w \in A^*$, we have $|P_{\bowtie}(\eta, k, w)| \leq 2|N|^k$ (the key point being that this bound does not depend on w). Hence, there are only finite possible η -snapshots $\sigma_\eta(w, P_{\bowtie}(\eta, k, w))$ for $w \in A^*$. It follows that $\bowtie_{\eta, k}$ has finite index. It remains to prove that $\bowtie_{\eta, k}$ is a congruence. Let $u_1, u_2, v_1, v_2 \in A^*$ such that $u_h \bowtie_{\eta, k} v_h$ for $h = 1, 2$. We prove that $u_1 u_2 \bowtie_{\eta, k} v_1 v_2$. We let P as the set of all positions $i \in P_c(u_1 u_2)$ such that either $i \in P_{\bowtie}(\eta, k, u_1)$ or $i - |u_1| \in P_{\bowtie}(\eta, k, u_2)$. Symmetrically, we let Q as the set of all positions $i \in P_c(v_1 v_2)$ such that either $i \in P_{\bowtie}(\eta, k, v_1)$ or $i - |v_1| \in P_{\bowtie}(\eta, k, v_2)$.

By hypothesis, $\sigma_\eta(u_h, P_{\bowtie}(\eta, k, u_h)) = \sigma_\eta(v_h, P_{\bowtie}(\eta, k, v_h))$ for $h = 1, 2$. This yields $\sigma_\eta(u_1 u_2, P) = \sigma_\eta(v_1 v_2, Q)$ by definition. Moreover, one may verify from the definitions that $P_{\bowtie}(\eta, k, u_1 u_2) \subseteq P$. Hence, Fact B.1 yields $Q' \subseteq Q$ such that $\sigma_\eta(u_1 u_2, P_{\bowtie}(\eta, k, u_1 u_2)) = \sigma_\eta(v_1 v_2, Q')$. Therefore, $Q' = P_{\bowtie}(\eta, k, v_1 v_2)$ by Lemma 3.9. Altogether, this yields $u_1 u_2 \bowtie_{\eta, k} v_1 v_2$, completing the proof. \square

We turn to the proof of Proposition 3.13.

PROPOSITION 3.13. *Let \mathcal{C} be a prevariety and $L \subseteq A^*$. Then, $L \in \text{LPol}(\mathcal{C})$ (resp. $L \in \text{RPol}(\mathcal{C})$, $L \in \text{MPol}(\mathcal{C})$) if and only if there*

exist a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$ such that L is a union of $\triangleright_{\eta,k}$ -classes (resp. $\triangleleft_{\eta,k}$ -classes, $\bowtie_{\eta,k}$ -classes).

PROOF. We present a proof argument for $MPol(\mathcal{C})$ (the other cases are similar and left to the reader). Assume first that $L \in MPol(\mathcal{C})$. We exhibit a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$ such that L is a union of $\bowtie_{\eta,k}$ -classes. By definition of $MPol(\mathcal{C})$, there exists a finite set \mathbf{H} of languages in \mathcal{C} and $m \geq 1$ such that L is a finite disjoint union of mixed deterministic marked products of at most m languages in \mathbf{H} . By definition, every unambiguous product of languages in \mathbf{H} belongs to $UPol(\mathcal{C})$. Hence, since $UPol(\mathcal{C})$ is a prevariety by Theorem 3.6, Proposition 2.3 yields a $UPol(\mathcal{C})$ -morphism $\alpha : A^* \rightarrow M$ recognizing every unambiguous marked product of at most m languages in \mathbf{H} . Consider the congruence $\sim_{\mathcal{C}}$ on M . We let $N = M/\sim_{\mathcal{C}}$ and $\eta = [\cdot]_{\mathcal{C}} \circ \alpha : A^* \rightarrow N$ and $k = |M|$. Lemma 2.7 implies that η is a \mathcal{C} -morphism. Moreover, since all $H \in \mathbf{H}$ belong to \mathcal{C} and are recognized by α (by definition), the lemma also implies that η recognizes every $H \in \mathbf{H}$. It remains to prove that L is a union of $\bowtie_{\eta,k}$ -classes. For all $w, w' \in A^*$ such that $w \bowtie_{\eta,k} w'$, we prove that $w \in L \Leftrightarrow w' \in L$. By symmetry, we only prove the left to right implication. Thus, we assume that $w \in L$ and prove that $w' \in L$.

Since $w \in L$, it follows from the definition of \mathbf{H} and m that there exist $H_0, \dots, H_n \in \mathbf{H}$ and $a_1, \dots, a_n \in A$ such that $n+1 \leq m$, $w \in H_0 a_1 H_1 \dots a_n H_n \subseteq L$ and $H_0 a_1 H_1 \dots a_n H_n$ is mixed deterministic. Consequently, it suffices to prove $w' \in H_0 a_1 H_1 \dots a_n H_n$. Since $w \in H_0 a_1 H_1 \dots a_n H_n$, we get $w_j \in H_j$ for $0 \leq j \leq n$ such that $w = w_0 a_1 w_1 \dots a_n w_n$. Let $P \subseteq P_c(w)$ be the set of all positions carrying the letters a_1, \dots, a_n . We prove that $P \subseteq P_{\bowtie}(\eta, k, w)$. Let us first explain why this implies $w' \in H_0 a_1 H_1 \dots a_n H_n$. Assume for now that $P \subseteq P_{\bowtie}(\eta, k, w)$. Since we have $w \bowtie_{\eta,k} w'$, Fact B.1 yields a set $P' \subseteq P_{\bowtie}(\eta, k, w')$ such that $\sigma_{\eta}(w, P) = \sigma_{\eta}(w', P')$. By definition of P , this exactly says that w' admits a decomposition $w' = w'_0 a_1 w'_1 \dots a_n w'_n$ such that $\eta(w'_j) = \eta(w_j)$ for every $j \leq n$. Since $H_0, \dots, H_n \in \mathbf{H}$ are recognized by η and $w_j \in H_j$ for every $j \leq n$, this yields $w'_j \in H_j$ for every $j \leq n$. Therefore, we get $w' \in H_0 a_1 H_1 \dots a_n H_n \subseteq L$ as desired.

It remains to prove that $P \subseteq P_{\bowtie}(\eta, k, w)$. Since $\alpha : A^* \rightarrow M$ is a $UPol(\mathcal{C})$ -morphism and $k = |M|$, Lemma 3.8 yields $P_{\bowtie}(\alpha, 1, w) \subseteq P_{\bowtie}(\eta, k, w)$. We prove that $P \subseteq P_{\bowtie}(\alpha, 1, w)$. We fix a position $i \in P$ for the proof. By definition of P , there exists $j \leq n$ such that the position i is the one labeled by the highlighted letter a_j in $w = w_0 a_1 w_1 \dots a_n w_n$. We let $u = w_0 a_1 w_1 \dots w_{j-1} \in H_0 a_1 H_1 \dots H_{j-1}$. Moreover, we let $v = w_j \dots a_n w_n \in H_j \dots a_n H_n$. Clearly, we have $w = ua_j v$. Since $H_0 a_1 H_1 \dots a_n H_n$ is mixed deterministic, we know that the marked concatenation $(H_0 a_1 H_1 \dots H_{j-1}) a_j (H_j \dots a_n H_n)$ is either left deterministic or right deterministic. By symmetry, we only treat the former case and prove that $i \in P_{\triangleright}(\alpha, 1, w) \subseteq P_{\bowtie}(\alpha, 1, w)$ (in the latter case, one proves that $i \in P_{\triangleleft}(\alpha, 1, w)$). Thus, we assume that $(H_0 a_1 H_1 \dots H_{j-1}) a_j (H_j \dots a_n H_n)$ is left deterministic. Recall that i is the position carrying the highlighted letter a_j in the decomposition $w = ua_j v$ of w . Hence, we have to prove that $\alpha(ua_j) <_{\mathcal{R}} \alpha(u)$. This will imply $i \in P_{\triangleright}(\alpha, 1, w)$ as desired. By contradiction, assume that $\alpha(ua_j) \mathcal{R} \alpha(u)$. This yields $x \in A^*$ such that $\alpha(ua_j x) = \alpha(u)$. By definition of u , we have $u \in H_0 a_1 H_1 \dots H_{j-1}$. Moreover, since the whole product $H_0 a_1 H_1 \dots a_n H_n$ is mixed deterministic, one may verify that $H_0 a_1 H_1 \dots H_{j-1}$ is unambiguous

which means that it is recognized by α (it is a unambiguous marked product of $j \leq n \leq m$ languages in \mathbf{H}). Hence, since $\alpha(ua_j x) = \alpha(u)$, we obtain that $ua_j x \in H_0 a_1 H_1 \dots H_{j-1}$. Since it is also clear that $ua_j x \in H_0 a_1 H_1 \dots H_{j-1} a_j A^*$, we obtain a contradiction to the hypothesis that $(H_0 a_1 H_1 \dots H_{j-1}) a_j (H_j \dots a_n H_n)$ is left deterministic. This concludes the proof for the left to right implication in Proposition 3.13.

We turn to the converse implication. We fix a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$. We prove that every $\bowtie_{\eta,k}$ -class is defined by a mixed deterministic marked product of languages in \mathcal{C} . Since equivalence classes are pairwise disjoint and $\bowtie_{\eta,k}$ has finite index, this implies that every union of $\bowtie_{\eta,k}$ -classes belongs to $MPol(\mathcal{C})$ as desired. We fix $w \in A^*$ and consider its $\bowtie_{\eta,k}$ -class. We define $\sigma_{\eta}(w, P_{\bowtie}(\eta, k, w)) = (s_0, a_1, s_1, \dots, a_n, s_n)$. Moreover, we let $L_h = \eta^{-1}(s_h)$ for every $h \leq n$. We have $L_h \in \mathcal{C}$ since η is a \mathcal{C} -morphism. Let $L = L_0 a_1 L_1 \dots a_n L_n$. We know from Lemma 3.10 that $L_0 a_1 L_1 \dots a_n L_n$ is mixed deterministic. Hence, $L \in MPol(\mathcal{C})$. We show that L is the $\bowtie_{\eta,k}$ -class of w , completing the proof. Let $w' \in A^*$. We prove that $w \bowtie_{\eta,k} w'$ if and only if $w' \in L$. If $w' \bowtie_{\eta,k} w$, then $\sigma_{\eta}(w', P_{\bowtie}(\eta, k, w')) = \sigma_{\eta}(w, P_{\bowtie}(\eta, k, w))$. Hence, $\sigma_{\eta}(w', P_{\bowtie}(\eta, k, w')) = (s_0, a_1, s_1, \dots, a_n, s_n)$ which yields $w' \in L$ by definition of η -snapshots. Assume now that $w' \in L$. By definition of L , we have $w' = w'_0 a_1 w'_1 \dots a_n w'_n$ with $\alpha(w'_h) = s_h$ for every $h \leq n$. Let $P' \subseteq P_c(w')$ be the set containing all positions carrying the highlighted letters a_1, \dots, a_n . Clearly, $\sigma_{\eta}(w', P') = (s_0, a_1, s_1, \dots, a_n, s_n)$. Therefore, $\sigma_{\eta}(w, P_{\bowtie}(\eta, k, w)) = \sigma_{\eta}(w', P')$. It then follows from Lemma 3.9 that $P' = P_{\bowtie}(\eta, k, w')$. Altogether, we get $w \bowtie_{\eta,k} w'$ as desired. \square

Finally, we prove Corollary 3.14.

COROLLARY 3.14. *Let \mathcal{C} be a prevariety and L_1, \dots, L_m finitely many languages in $LPol(\mathcal{C})$ (resp. $RPol(\mathcal{C})$, $MPol(\mathcal{C})$). There is a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$ such that L_1, \dots, L_m are unions of $\triangleright_{\eta,k}$ -classes (resp. $\triangleleft_{\eta,k}$ -classes, $\bowtie_{\eta,k}$ -classes).*

PROOF. In this case as well, we only consider $MPol(\mathcal{C})$ (the other cases are similar and left to the reader). Hence, we assume that $L_1, \dots, L_m \in MPol(\mathcal{C})$. For every $i \leq m$, it follows from Proposition 3.13 that there exist a \mathcal{C} -morphism $\eta_i : A^* \rightarrow N_i$ and $k_i \in \mathbb{N}$ such that L_i is a union of \bowtie_{η_i, k_i} -classes. Let $M = N_1 \times \dots \times N_m$ be the monoid equipped with the componentwise multiplication and $\alpha : A^* \rightarrow M$ be the morphism defined by $\alpha(w) = (\eta_1(w), \dots, \eta_m(w))$ for every $w \in A^*$. We let $\eta : A^* \rightarrow N$ as the surjection induced by α . One may verify that η is a \mathcal{C} -morphism since \mathcal{C} is closed under intersection and $\eta_i : A^* \rightarrow N_i$ was a \mathcal{C} -morphism for every $i \leq m$. Finally, let $k = \max(k_1, \dots, k_m)$. One may now verify from the definitions that $\bowtie_{\eta,k}$ is finer than \bowtie_{η_i, k_i} for every $i \leq m$. Hence, L_1, \dots, L_m are unions of $\bowtie_{\eta,k}$ -classes as desired. \square

B.2 Proof of Proposition 3.11

We start with preliminary results that we require to prove the proposition. Let $\alpha : A^* \rightarrow M$ be a morphism. An α -monomial is a marked product of the form $\alpha^{-1}(s_0) a_1 \alpha^{-1}(s_1) \dots a_d \alpha^{-1}(s_d)$ where $s_1, \dots, s_d \in M$. The number d is called the degree of this α -monomial. Moreover, an α -polynomial is a finite union of α -monomials. Its degree is the maximum among the degrees of all α -monomials in the finite union. We have the following simple lemma.

LEMMA B.2. *Let α be a morphism and K, L which are defined by α -polynomials of degrees $m, n \in \mathbb{N}$. Then $K \cap L$ is defined by an α -polynomial of degree at most $m + n$.*

PROOF. Since intersection distributes over union, we may assume without loss of generality that K, L are defined by α -monomials of degrees $m, n \in \mathbb{N}$. Moreover, since there are finitely many α -monomials of degree at most $m + n$, it suffices to prove that for every $w \in K \cap L$, there exists $H \subseteq A^*$ which is defined by an α -monomial of degree at most $m + n$ and such that $w \in H \subseteq K \cap L$. The finite union of all these languages H will then define $K \cap L$. We fix $w \in K \cap L$.

By hypothesis, $K = \alpha^{-1}(s_0)a_1\alpha^{-1}(s_1) \cdots a_m\alpha^{-1}(s_m)$ and $L = \alpha^{-1}(t_0)b_1\alpha^{-1}(t_1) \cdots b_n\alpha^{-1}(t_n)$. Hence, since we have $w \in K \cap L$, there are $P, Q \subseteq P(w)$ such that $\sigma_\alpha(w, P) = (s_0, a_1, s_1, \dots, a_m, s_m)$ and $\sigma_\alpha(w, Q) = (t_0, b_1, t_1, \dots, b_n, t_n)$. We define $R = P \cup Q$. Clearly, $\ell = |R| \leq |P| + |Q| = m + n$. Let $(q_0, c_1, q_1, \dots, c_\ell, q_\ell) = \sigma_\alpha(w, R)$. We let H as the language defined by $\alpha^{-1}(q_0)c_1\alpha^{-1}(q_1) \cdots c_\ell\alpha^{-1}(q_\ell)$ of degree $\ell \leq m + n$. One may now verify that $w \in H \subseteq K \cap L$. \square

We complete the definition with two lemmas for α -polynomials. They are designed to exploit our hypothesis on the class \mathcal{C} in Proposition 3.11 (i.e. $\mathcal{C} \in \{\mathcal{S}, \mathcal{S}^+\}$ for a group prevariety \mathcal{S}).

LEMMA B.3. *Let $\alpha : A^* \rightarrow G$ be a morphism into a group and $x, y, w \in A^*$ such that $\alpha(xw) = \alpha(w)$ and $\alpha(yw) = \alpha(w)$. For every α -polynomial $H \subseteq A^*$, we have $w \in H \Rightarrow xwy \in H$.*

PROOF. Assume that $w \in H$. Since G is a group our hypotheses on x and y imply that $\alpha(x) = \alpha(y) = 1_G$. Moreover, if $w \in H$, there exists an α -monomial K in the union defining H such that $w \in K$. One may now verify that $K = \alpha^{-1}(1_G)K\alpha^{-1}(1_G)$. Hence, $xwy \in K \subseteq H$ as desired. \square

We now consider the morphisms $\alpha : A^* \rightarrow M$ such that $\alpha(A^+)$ is a group.

LEMMA B.4. *Let $\alpha : A^* \rightarrow M$ be a morphism, $u, v \in A^*$ and $n \in \mathbb{N}$ such that $G = \alpha(A^+)$ is a group and $|u| = |v| = n$. Let $x, y, w \in A^*$ such that $\alpha(xw) = \alpha(w)$, $\alpha(yw) = \alpha(w)$, $w \in uA^*v$ and $xwy \in uA^*v$. For every α -polynomial $H \subseteq A^*$ of degree at most n , we have $w \in H \Rightarrow xwy \in H$.*

PROOF. When $n = 0$, the lemma is trivial. The α -polynomials of degree 0 are exactly the languages recognized by α . Thus, since our hypotheses yields $\alpha(xwy) = \alpha(w)$, we get that $w \in H \Rightarrow xwy \in H$ for every α -polynomial H of degree 0.

Assume that $n \geq 1$ and $w \in H$. We get an α -monomial K in the union defining H such that $w \in K$. We write $d \leq n$ for the degree of K . By definition, we know that K is of the form $K = \alpha^{-1}(s_0)a_1\alpha^{-1}(s_1) \cdots a_d\alpha^{-1}(s_d)$. Consequently, we have $w = w_0a_1w_1 \cdots a_dw_d$ where $\alpha(w_i) = s_i$ for every $i \leq d$. Since $w \in uA^*v$ and $|u| = |v| = n$, we know that $|w| \geq 2n$. Thus, since $d \leq n$, there exists $i \leq d$ such that $w_i \neq \varepsilon$. We let $h \leq d$ and $\ell \leq d$ as the least and the greatest such i respectively, $u' = w_0a_1 \cdots w_{h-1}a_h = a_1 \cdots a_h$ (if $h = 0$, then $u' = \varepsilon$) and $v' = a_{\ell+1}w_{\ell+1} \cdots a_dw_d = a_{\ell+1} \cdots a_d$ (if $\ell = d$, then $v' = 0$). By definition, we have $y = u'w_ha_{h+1}w_{h+1} \cdots a_\ell w_\ell v'$ and $w_h, w_\ell \in A^+$. By definition, $|u'| \leq d \leq n$ and $|v'| \leq d \leq n$. Thus, since $y \in uA^*v$ and $|u| = |v| = n$, it follows that u' is a prefix of u and v' is a suffix of v . Since we also know that $xwz \in uA^*v$,

this yields $z \in A^*$ such that $xwy = u'zv'$. By hypothesis on w , we also know that $xwy = xu'w_ha_{h+1}w_{h+1} \cdots a_\ell w_\ell v'y$. Thus, we get $x', y' \in A^*$ such that $u'x' = xu'$ and $y'v' = v'y$. Altogether, it follows that $xwy = u'x'w_ha_{h+1}w_{h+1} \cdots a_\ell w_\ell y'v'$. We now prove that $\alpha(x'w_h) = s_h$ and $\alpha(w_\ell y') = s_\ell$. By symmetry, we only detail the former. This is trivial if $x' = \varepsilon$. Thus, we assume that $x' \in A^+$. Since $u'x' = xu'$, we have $x \in A^+$ as well. Hence, since $\alpha(xw) = \alpha(w)$, $w \in A^+$ and $G = \alpha(A^+)$ is a group, we get $\alpha(x) = 1_G$. Thus, since $u'x' = xu'$ and $u' \in A^+$, we get $\alpha(u'x') = \alpha(u')$. It follows that $\alpha(x') = 1_G$. Finally, since $w_h \in A^+$, we have $\alpha(w_h) \in G$ and it follows that $\alpha(x'w_h) = \alpha(w_h) = s_h$. We may now complete the proof that $xwy \in H$. We obtain,

$$x'w_ha_{h+1} \cdots a_\ell w_\ell y' \in \alpha^{-1}(s_h)a_{h+1}\alpha^{-1}(s_{h+1}) \cdots a_\ell\alpha^{-1}(s_\ell).$$

By definition, we know that $u' \in \alpha^{-1}(s_0)a_1 \cdots \alpha^{-1}(s_{h-1})a_h$ and $v' \in a_\ell\alpha^{-1}(a_\ell) \cdots a_d\alpha^{-1}(s_d)$. Consequently, we obtain that $xwy = u'x'w_ha_{h+1}w_{h+1} \cdots a_\ell w_\ell y'v' \in K \subseteq H$. \square

We may now prove Proposition 3.11. We first recall the statement.

PROPOSITION 3.11. *Let \mathcal{S} be a group prevariety and $\mathcal{C} \in \{\mathcal{S}, \mathcal{S}^+\}$. If $\eta : A^* \rightarrow N$ is a $BPol(\mathcal{C})$ -morphism and $k \in \mathbb{N}$, there exists a $BPol(\mathcal{C})$ -morphism, $\gamma : A^* \rightarrow Q$ such that $P_{\triangleright}(\eta, k, w) \subseteq P_{\triangleright}(\gamma, 1, w)$ and $P_{\triangleleft}(\eta, k, w) \subseteq P_{\triangleleft}(\gamma, 1, w)$.*

PROOF. Let us first define γ . By hypothesis, η is a $BPol(\mathcal{C})$ -morphism. Hence, there exists a finite set L of languages in \mathcal{C} such that all languages recognized by η are Boolean combinations of marked products of languages in L . Proposition 2.3 yields a \mathcal{C} -morphism $\alpha : A^* \rightarrow M$ recognizing every $L \in L$. Therefore, since union distributes over marked concatenation, every language recognized by η is a Boolean combination of α -monomials. These Boolean combinations can be put into disjunctive normal form. Moreover, intersection of α -monomials are finite unions of \mathcal{C} -monomials by Lemma B.2. Consequently, there exists a number $n \in \mathbb{N}$ such that every language recognized by η is a finite union of languages of the form $L \setminus H$ where L is an α -monomial of degree at most n and H is a finite union of α -monomials of degree at most n (i.e., an α -polynomial of degree at most n). Clearly, there are finitely many α -polynomials of degree at most $(3n + 1) \times k$ and since α is a \mathcal{C} -morphism, they all belong to $Pol(\mathcal{C}) \subseteq BPol(\mathcal{C})$. Hence, Proposition 2.3 yields a $BPol(\mathcal{C})$ -morphism $\gamma : A^* \rightarrow Q$ recognizing every α -polynomial of degree at most $(3n + 1) \times k$.

It remains to prove the inclusions $P_{\triangleright}(\eta, k, w) \subseteq P_{\triangleright}(\gamma, 1, w)$ and $P_{\triangleleft}(\eta, k, w) \subseteq P_{\triangleleft}(\gamma, 1, w)$ for every $w \in A^*$. By symmetry, we only prove the former. We fix $w \in A^*$ for the proof. The hypothesis that $\mathcal{C} \in \{\mathcal{S}, \mathcal{S}^+\}$ implies the following lemma.

LEMMA B.5. *Let h such that $1 \leq h \leq k$, $i \in P_{\triangleright}(\eta, h, w)$ and $a \in A$ the label of i . There exists an α -monomial K of degree at most $(3n + 1)h - 1$ such that $w(0, i) \in K$ and $w(0, i) \notin KaA^*$.*

Let us first apply Lemma B.5 to complete the main argument. Let $i \in P_{\triangleright}(\eta, k, w)$. We show that $i \in P_{\triangleright}(\gamma, 1, w)$. Let a be the label of i . By definition, we have to prove that $\gamma(w(0, i)a) \not\prec_{\mathcal{C}} \gamma(w(0, i))$. Since γ is surjective (recall that it is a $BPol(\mathcal{C})$ -morphism), this boils down to proving that $\gamma(w(0, i)) \neq \gamma(w(0, i)au)$ for every $u \in A^*$. We fix u for the proof. Lemma B.5 yields an α -monomial K of degree at most $(3n + 1)k - 1$ such that $w(0, i) \in K$ and $w(0, i) \notin KaA^*$.

Clearly, KaA^* is defined by an α -polynomial of degree at most $(3n+1)k$. Hence, KaA^* is recognized by γ . Since we have $w(0, i)au \in KaA^*$ and $w(0, i) \notin KaA^*$, we obtain $\gamma(w(0, i)) \neq \gamma(w(0, i)au)$ which completes the proof.

It remains to prove Lemma B.5. We consider a number h such that $1 \leq h \leq k$, $i \in P_{\triangleright}(\eta, h, w)$ and $a \in A$ the label of i . We have to construct an α -monomial K of degree at most $(3n+1)h-1$ such that $w(0, i) \in K$ and $w(0, i) \notin KaA^*$. We proceed by induction on h . By definition, there exists $j \in P_{\triangleright}(\eta, h-1, w) \cup \{0\}$ such that $\eta(w(j, i)a) <_{\mathcal{R}} \eta(w(j, i))$. We first prove an important result about the word $w(j, i)$. Recall that by hypothesis, we have $\mathcal{C} \in \{\mathcal{G}, \mathcal{G}^+\}$. Hence, $\alpha : A^* \rightarrow M$ is a \mathcal{G}^+ morphism and since \mathcal{G} is a group prevariety, Lemma E.1 implies that $G = \alpha(A^+)$ is a group. We prove that there exists an α -monomial V of degree at most $3n$ which satisfies the following property:

$$w(j, i) \in V \text{ and } w(j, i) \notin (\{\varepsilon\} \cup \alpha^{-1}(1_G))VaA^*. \quad (6)$$

Let $t = \eta(w(i, j))$. By construction, since $w(i, j) \in \eta^{-1}(t)$, there exist an α -monomial L and an α -polynomial H , both of degree at most n and such that $w(i, j) \in L \setminus H \subseteq \eta^{-1}(t)$. There are two cases depending on α .

Construction of V , first case. We assume that $1_M \in G$, i.e. $\alpha(A^*) = G$ and $1_M = 1_G$. We let $V = L$ which is an α -monomial of degree at most $n \leq 3n$. We already know that $w(i, j) \in L$. We show that $w(j, i) \notin (\{\varepsilon\} \cup \alpha^{-1}(1_G))LaA^*$. We proceed by contradiction. Assume that $w(j, i) = xyz$ with $\alpha(x) = 1_G$ (since $1_M = 1_G$, this covers the case when $x = \varepsilon$), $y \in L$ and $z \in A^*$. We show that $\eta(xy) = \eta(w(j, i)) = t$. Since $w(j, i) = xyz$, this yields $\eta(w(j, i)) = \eta(w(j, i)az)$, contradicting the hypothesis that $\eta(w(j, i)a) <_{\mathcal{R}} \eta(w(j, i))$. Since $L \setminus H \subseteq \eta^{-1}(t)$, it suffices to prove that $xy \in L \setminus H$. Since $\alpha(x) = 1_G = 1_M$, we have $\alpha(xy) = \alpha(y)$. We also have $y \in L$ which is an α -monomial. Thus, since $\alpha(A^*) = G$ is a group, Lemma B.3 yields $xy \in L$. It remains to prove $xy \notin H$. By contradiction, we assume that $xy \in H$. Since $xy \in L$ and $w(j, i) \in L$, one may verify from the definition of α -monomials that $\alpha(xy) = \alpha(w(j, i))$. Since $w(j, i) = xyz$, we obtain $\alpha(xy) = \alpha(xyz)$. Moreover, H is an α -polynomial by definition. Thus, since $\alpha(A^*) = G$ is a group, Lemma B.3 yields $w(j, i) = xyz \in H$. This is a contradiction since $w(j, i) \in L \setminus H$ by hypothesis.

Construction of V , second case. We assume that $1_M \notin G$. Since $\alpha(A^+) = G$, it follows that $\alpha^{-1}(1_N) = \{\varepsilon\}$. We consider two sub-cases. First, assume that $|w(j, i)| \leq 3n$. In this case, we let $V = \{w(j, i)\}$. Since $\alpha^{-1}(1_N) = \{\varepsilon\}$, this is an α -monomial of degree $|w(j, i)| \leq 3n$. Since $w(j, i) \in V$ and $w(j, i) \notin (\{\varepsilon\} \cup \alpha^{-1}(1_G))VaA^*$, (6) is proved.

We now consider the sub-case when $|w(j, i)| > 3n$. This hypothesis yields $u, v \in A^+$ such that $|u| = |v| = n$ and $w(j, i) \in uA^*v$. Since $\alpha^{-1}(1_N) = \{\varepsilon\}$, it is immediate that uA^*v is defined by an α -polynomial of degree $2n$. Since L is an α -monomial of degree at most n , Lemma B.2 yields that $L \cap uA^*v$ is defined by an α -polynomial of degree at most $3n$. Since $w(j, i) \in L \cap uA^*v$, we get an α -monomial V of degree at most $3n$ such that $w(j, i) \in V \subseteq L \cap uA^*v$. It remains to prove that $w(j, i) \notin (\{\varepsilon\} \cup \alpha^{-1}(1_G))VaA^*$. By contradiction, we assume that $w(j, i) = xyz$ with $x = \varepsilon$ or $\alpha(x) = 1_G$, $y \in V$ and $z \in A^*$. We prove that $\eta(xy) = \eta(w(j, i)) = t$. Since

$w(j, i) = xyz$, this implies that $\eta(w(j, i)) = \eta(w(j, i)az)$, contradicting the hypothesis that $\eta(w(j, i)a) <_{\mathcal{R}} \eta(w(j, i))$. Since $L \setminus H \subseteq \eta^{-1}(t)$, it suffices to prove that $xy \in L \setminus H$. By hypothesis on V , we have $y \in L \cap uA^*v$. Thus, $xy \in A^*uA^*v$ and since $w(j, i) = xyz \in uA^*v$, it follows that $xy \in uA^*v$. Since $y \in A^+$ (which means that $\alpha(y) \in G$) and either $x = \varepsilon$ or $\alpha(x) = 1_G$, we also have $\alpha(xy) = \alpha(y)$. Hence, since L is an α -monomial of degree at most n and $\alpha(A^+) = G$ is a group, it follows from Lemma B.4 that $xy \in L$. It remains to show that $xy \notin H$. By contradiction, we assume that $xy \in H$. Since $w(j, i) = xyz$ and xy both belong to L which is an α -monomial, we have $\alpha(xy) = \alpha(xyz)$. Moreover, $xy \in uA^*v$ and $xyz = w(j, i) \in uA^*v$. Hence, since H is an α -polynomial of degree at most n by definition and $\alpha(A^+) = G$ is a group, Lemma B.4 yields $w(j, i) = xyz \in H$. This is a contradiction since $w(j, i) \in L \setminus H$ by hypothesis. This completes the construction of V .

Construction of K . With our α -monomial V of degree at most $3n$ in hand, we may build K . There are two cases depending on whether $j = 0$ or $j \geq 1$. Assume first that $j = 0$. In that case, we choose $K = V$ which has degree $3n \leq (3n+1)h-1$. By (6), we have $w(0, i) \in K$ and $w(0, i) \notin KaA^*$ as desired.

Assume now that $1 \leq j < i$. Since $j \in P_{\triangleright}(\eta, h-1, w)$, it follows that $h-1 \geq 1$. Let b be the label of j . Induction on h in Lemma B.5 yields an α -monomial U with degree at most $(3n+1)(h-1)-1$ such that $w(0, j) \in U$ and $w(0, j) \notin UbA^*$. We define $K = UbV$. By hypothesis on U and V , we know that K is an α -monomial of degree at most $(3n+1)(h-1)-1+1+3n = (3n+1)h-1$. Moreover, we have $w(0, i) = w(0, j)bw(j, i) \in UbV = K$. It remains to prove that $w(0, i) \notin KaA^*$. We use contradiction. We assume that $w(0, i) \in KaA^* = UbVaA^*$. We get $x \in U$, $y \in V$ and $z \in A^*$ such that $w(0, i) = xbyaz$. Moreover, we know that $w(0, i) = w(0, j)bw(j, i)$ and since $w(0, j) \notin UbA^*$, the word $xb \in Ub$ cannot be a prefix $w(0, j)$. Hence, we have $x' \in A^*$ such that $xb = w(0, j)bx'$ and $x'yaz = w(j, i)$. Since U is an α -monomial and $x, w(0, j) \in U$, we have $\alpha(x) = \alpha(w(0, j))$. Hence, $\alpha(xb) = \alpha(w(0, j)b)$ and since $xb = w(0, j)bx'$, it follows that either $x' = \varepsilon$ or $\alpha(x') = 1_G$. We conclude that $w(j, i) = x'yaz \in (\{\varepsilon\} \cup \alpha^{-1}(1_G))VaA^*$. This contradicts (6) in the definition of V which concludes the proof. \square

C APPENDIX TO SECTION 3

In this appendix, we prove Theorem 4.1 and the present the missing proof for Lemma 4.5 (which is used to prove Theorem 4.3 in the main text). We omit the proof of Theorem 4.2 since the argument is symmetrical to the one for Theorem 4.1. Additionally, we present a simple lemma which reformulates the characterizations of $LPol$ and $RPol$. We shall use it multiple times in the sequel.

C.1 Characterization proofs

We start with the proof of Theorem 4.1.

THEOREM 4.1. *Let \mathcal{C} be a prevariety and $\alpha : A^* \rightarrow M$ a surjective morphism. The following properties are equivalent:*

- α is an $LPol(\mathcal{C})$ -morphism.
- $s^{\omega+1} = s^{\omega}t$ for all \mathcal{C} -pairs $(s, t) \in M^2$.
- $s^{\omega+1} = s^{\omega}t$ for all $s, t \in M$ such that $s \sim_{\mathcal{C}} t$.

PROOF. We first prove that $a) \Rightarrow b)$. We assume α is an $LPol(\mathcal{C})$ -morphism and prove that b) holds. Consider a \mathcal{C} -pair $(s, t) \in M^2$. We show that $s^{\omega+1} = s^\omega t$. Corollary 3.14 yields a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$ such that for every language recognized by α is a union of $\triangleright_{\eta,k}$ -classes. Since (s, t) is a \mathcal{C} -pair and η is a \mathcal{C} -morphism, one may verify that there exist $u, v \in A^*$ such that $\eta(u) = \eta(v)$, $\alpha(u) = s$ and $\alpha(v) = t$. Let $p = \omega(M) \times \omega(N)$, $w = u^{pk}u$ and $w' = u^{pk}v$. We have the following lemma.

LEMMA C.1. *For every $i \in P_{\triangleright}(\eta, k, w)$, we have $i \leq |u^{pk}|$.*

PROOF. We use induction on h to show that for every $h \leq k$ and every $i \in P_{\triangleright}(\eta, h, w)$, we have $i \leq |u^{ph}|$. The lemma follows from the case $h = k$. We write $w = a_1 \cdots a_\ell$ for the proof. Let $h \leq k$. By contradiction, assume that there exists $i \in P_{\triangleright}(\eta, h, w)$ such that $i > |u^{ph}|$. By definition, there exists $j \in P_{\triangleright}(\eta, h-1, w) \cup \{0\}$ such that $j < i$ and the strict inequality $\eta(w(j, i)a_i) <_{\mathcal{R}} \eta(w(j, i))$ holds. By induction, we have $j \leq |u^{p(h-1)}|$. Hence, since $i > |u^{ph}|$ and $w = u^{pk}u$, the infix $w(j, i)$ must contain an infix u^p : we have $x, y \in A^*$ and $n \in \mathbb{N}$ such that $w(j, i) = xu^p y$ and $w(j, |w|+1) = xu^n$. Let $h \in \mathbb{N}$ such that $n+h$ is a multiple of p . By definition of p , $\eta(u^p)$ is an idempotent of N . Hence, $\eta(w(j, |w|+1)u^h y) = \eta(xu^p y) = \eta(w(j, i))$. Since $w(j, i)a_i$ is a prefix of $w(j, |w|+1)$, it follows that $\eta(w(j, i)) \leq_{\mathcal{R}} \eta(w(j, i)a_i)$. This is a contradiction since $\eta(w(j, i)a_i) <_{\mathcal{R}} \eta(w(j, i))$ by hypothesis. \square

We may now prove that $s^{\omega+1} = s^\omega t$. By Lemma C.1, every position in $P_{\triangleright}(\eta, k, w)$ belong to the prefix u^{pk} of $w = u^{pk}u$. Therefore, since u^{pk} is also a prefix of $w' = u^{pk}v$, $P_{\triangleright}(\eta, k, w) \subseteq P_c(w')$. Since $\eta(u) = \eta(v)$, we get $\sigma_\eta(w, P_{\triangleright}(\eta, k, w)) = \sigma_\eta(w', P_{\triangleright}(\eta, k, w))$. Hence, Lemma 3.9 yields $P_{\triangleright}(\eta, k, w) = P_{\triangleright}(\eta, k, w')$. Altogether, we get $w \triangleright_{\eta,k} w'$ and it follows that $\alpha(w) = \alpha(w')$ since the languages recognized by α are unions of $\triangleright_{\eta,k}$ -classes. By definition of w, w' and since p is a multiple of $\omega(M)$, this yields $s^{\omega+1} = s^\omega t$ as desired.

We turn to the implication $b) \Rightarrow c)$. We assume that b) holds and consider $s, t \in M$ such that $s \sim_{\mathcal{C}} t$. We show that $s^{\omega+1} = s^\omega t$. By Lemma 2.5, there exist $r_0, \dots, r_n \in M$ such that $r_0 = e$, $r_n = t$ and (r_i, r_{i+1}) is a \mathcal{C} -pair for all $i < n$. We use induction on i to show that $s^{\omega+1} = s^\omega r_i$ for every $i \leq n$. The case $i = n$ yields the desired result as $t = r_n$. When $i = 0$, the result is immediate as $r_0 = s$. Assume now that $i \geq 1$. Since (r_{i-1}, r_i) is a \mathcal{C} -pair, one may verify that $(s^\omega r_{i-1}, s^\omega r_i)$ is a \mathcal{C} -pair as well. Therefore, we get from b) that $(s^\omega r_{i-1})^{\omega+1} = (s^\omega r_{i-1})^\omega s^\omega r_i$. Finally, induction yields $s^{\omega+1} = s^\omega r_{i-1}$. Combined with the previous equality, this yields $s^{\omega+1} = (s^\omega r_{i-1})^{\omega+1} = (s^\omega r_{i-1})^\omega s^\omega r_i = s^\omega r_i$ as desired.

It remains to prove $c) \Rightarrow a)$. We assume that c) holds and show that α is an $LPol(\mathcal{C})$ -morphism. Let $N = M/\sim_{\mathcal{C}}$ and recall that N is a monoid since $\sim_{\mathcal{C}}$ is a congruence by Lemma 2.6. We write $\eta = [\cdot]_{\mathcal{C}} \circ \alpha : A^* \rightarrow N$ which is a \mathcal{C} -morphism by Lemma 2.7. We let $k = |M|$ and consider the equivalence $\triangleright_{\eta,k}$ on A^* . We prove the following property:

$$\text{for every } w, w' \in A^*, \quad w \triangleright_{\eta,k} w' \Rightarrow \alpha(w) = \alpha(w'). \quad (7)$$

This implies that every language recognized by α is a union of $\triangleright_{\eta,k}$ -classes. Together with Proposition 3.13 this yields that every language recognized by α belongs to $LPol(\mathcal{C})$ since η is a \mathcal{C} -morphism. We now concentrate on 7. Let $w, w' \in A^*$ such that

$w \triangleright_{\eta,k} w'$. We show that $\alpha(w) = \alpha(w')$. For the proof, we write $P = P_{\triangleright}(\alpha, 1, w)$. We use the hypothesis that $w \triangleright_{\eta,k} w'$ to prove the following lemma.

LEMMA C.2. *There is $P' \subseteq P_c(w')$ s.t. $\sigma_\eta(w, P) = \sigma_\eta(w', P')$.*

PROOF. Since c) holds, we know that for all $s, t \in M$ such that $s \sim_{\mathcal{C}} t$, we have $s^{\omega+1} = s^\omega t$. We may multiply by s^ω on the right to get $s^{\omega+1} = s^\omega t s^\omega$. Hence, it follows from Theorem 3.7 that α is a $UPol(\mathcal{C})$ -morphism. Since $k = |M|$, it follows from Lemma 3.8 that $P = P_{\triangleright}(\alpha, 1, w) \subseteq P_{\triangleright}(\eta, k, w)$. Finally, since $w \triangleright_{\eta,k} w'$, we have $\sigma_\eta(w, P_{\triangleright}(\eta, k, w)) = \sigma_\eta(w', P_{\triangleright}(\eta, k, w'))$. Thus, Fact B.1 yields a set $P' \subseteq \sigma_\eta(w', P_{\triangleright}(\eta, k, w'))$ such that $\sigma_\eta(w, P) = \sigma_\eta(w', P')$ as desired. \square

Consider the α -snapshots $(s_0, a_1, s_1, \dots, a_n, s_n) = \sigma_\alpha(w, P)$ and $(t_0, b_1, t_1, \dots, b_m, t_m) = \sigma_\alpha(w', P')$. Lemma C.2 yields $\sigma_\eta(w, P) = \sigma_\eta(w', P')$. We get $n = m$ and $a_i = b_i$ for $1 \leq i \leq n$ and $s_i \sim_{\mathcal{C}} t_i$ for $0 \leq i \leq n$ by definition of η . Thus $\alpha(w) = s_0 a_1 s_1 \cdots a_n s_n$ and $\alpha(w') = t_0 a_1 t_1 \cdots a_n t_n$ by definition of α -snapshots. It now remains to prove that $s_0 a_1 s_1 \cdots a_n s_n = t_0 a_1 t_1 \cdots a_n t_n$. We let $q_h = s_0 a_1 s_1 \cdots a_h$ and $r_h = t_0 a_1 t_1 \cdots a_h$ for $0 \leq h \leq n$ (in particular, $q_0 = r_0 = 1_M$). We use induction on h to show that $q_h s_h = r_h t_h$ for $0 \leq h \leq n$. Clearly, the case $h = n$ yields the desired result.

We fix $h \leq n$. Since $P = P_{\triangleright}(\alpha, 1, w)$, one may verify from the definitions that $q_h s_h \mathcal{R} q_h$ for $0 \leq h \leq n$. We get $x \in M$ such that $q_h = q_h s_h x$. Since $s_h \sim_{\mathcal{C}} t_h$ and $\sim_{\mathcal{C}}$ is a congruence, we have $x s_h \sim_{\mathcal{C}} x t_h$. Hence, it follows from c) that $(x s_h)^{\omega+1} = (x s_h)^\omega x t_h$. We may now multiply on the left by s_h to obtain $(s_h x)^{\omega+1} s_h = (s_h x)^\omega t_h$. We combine this with $q_h = q_h s_h x$ to obtain $q_h s_h = q_h t_h$. This concludes the proof when $h = 0$: this merely states that $s_0 = t_0$. Finally, if $h \geq 1$, induction yields $q_{h-1} s_{h-1} = r_{h-1} t_{h-1}$. Since $q_h = q_{h-1} a_h$ and $r_h = r_{h-1} a_h$ by definition, it follows that $q_h = t_h$. Altogether, we get $q_h s_h = r_h t_h$ which completes the proof. \square

We turn to the proof of Lemma 4.5 which is part of the larger proof argument for Theorem 4.3. Recall that a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$ are fixed. Moreover, we have $p \geq 1$ which is a multiple of $\omega(N)$ and four words $u, v, x, y \in A^*$ such that $\eta(u) = \eta(v)$. Finally, we defined $w = (ux)^{pk}u(yu)^{pk}$ and $w' = (ux)^{pk}v(yu)^{pk}$.

LEMMA 4.5. *For every $i \in P_{\triangleright}(\eta, k, w)$, either $i \leq |(ux)^{pk}|$ or $i > |(ux)^{pk}u|$.*

PROOF. Since $P_{\triangleright}(\eta, k, w) = P_{\triangleright}(\eta, k, w) \cup P_{\triangleleft}(\eta, k, w)$, there are two cases depending on whether $i \in P_{\triangleright}(\eta, k, w)$ or $i \in P_{\triangleleft}(\eta, k, w)$. By symmetry, we only treat the former case. Given a position $i \in P_{\triangleright}(\eta, k, w)$, we show that either $i \leq |(ux)^{pk}|$ or $i > |(ux)^{pk}u|$. We write $w = a_1 \cdots a_\ell$ for the proof. We consider a slightly stronger property. Let $h \leq k$. Using induction on h , we show that for every $i \in P_{\triangleright}(\eta, h, w)$, either $i \leq |(ux)^{ph}|$ or $i > |(ux)^{pk}u|$. By contradiction, assume that there exists some position $i \in P_{\triangleright}(\eta, h, w)$ such that $|(ux)^{ph}| < i \leq |(ux)^{pk}u|$. This yields $j \in P_{\triangleright}(\eta, h-1, w) \cup \{0\}$ such that $j < i$ and $\eta(w(j, i)a_i) <_{\mathcal{R}} \eta(w(j, i))$. By induction, we have $j \leq |(ux)^{p(h-1)}|$. Therefore, since we have $|(ux)^{ph}| < i \leq |(ux)^{pk}u|$ and $w = (ux)^{pk}u(yu)^{pk}$, the infix $w(j, i)$ must contain an infix $(ux)^p$: we have $z, z' \in A^*$ and $n \in \mathbb{N}$ such that $w(j, i) = z(ux)^p z'$ and $w(j, |(ux)^{pk}u|+1) = z(ux)^n u$. Let $m \in \mathbb{N}$ be a number such that $n+1+m$ is a multiple of p . By

definition of p , $\eta(u^p)$ is an idempotent of N . Therefore, we have $\eta(w(j, |(ux)^{p^k}u| + 1)x(ux)^{h_z}z') = \eta(z(ux)^p z') = \eta(w(j, i))$. By definition, $w(j, i)a_i$ is a prefix of $\eta(w(j, |(ux)^{p^k}u| + 1))$. Consequently, it follows that $\eta(w(j, i)) \leq_{\mathcal{R}} \eta(w(j, i)a_i)$. This is a contradiction since $\eta(w(j, i)a_i) <_{\mathcal{R}} \eta(w(j, i))$ by hypothesis. \square

C.2 Additional lemma

We present a corollary of Theorems 4.1 and 4.2. We shall use it multiple times in the sequel. Given two prevarieties \mathcal{C} and \mathcal{D} such that $\mathcal{C} \subseteq \mathcal{D} \subseteq UPol(\mathcal{C})$, we present a property of the $LPol(\mathcal{D})$ - and $RPol(\mathcal{D})$ -morphisms. Recall that when \mathcal{C} is a prevariety and $\alpha : A^* \rightarrow M$ is a surjective morphism, the equivalence $\sim_{\mathcal{C}}$ on M is a congruence by Lemma 2.6. We consider the Green relations of the quotient $M/\sim_{\mathcal{C}}$.

LEMMA C.3. *Let \mathcal{C}, \mathcal{D} be prevarieties such that $\mathcal{C} \subseteq \mathcal{D}$ and $\mathcal{D} \subseteq UPol(\mathcal{C})$ and $\alpha : A^* \rightarrow M$ a morphism. Let $e, q, r \in M$ such that e is an idempotent and $q \sim_{\mathcal{D}} r$. If α is an $LPol(\mathcal{D})$ -morphism and $[e]_{\mathcal{C}} \leq_{\mathcal{R}} [q]_{\mathcal{C}}$, then $eq = er$. Moreover, if α is a $RPol(\mathcal{D})$ -morphism and $[e]_{\mathcal{C}} \leq_{\mathcal{L}} [q]_{\mathcal{C}}$, then $qe = re$.*

PROOF. By symmetry, we only consider the case when α is an $LPol(\mathcal{D})$ -morphism. Thus, we assume that $[e]_{\mathcal{C}} \leq_{\mathcal{R}} [q]_{\mathcal{C}}$ and prove $eq = er$. By hypothesis, we get $s \in M$ such that $[e]_{\mathcal{C}} = [qs]_{\mathcal{C}}$, i.e. $e \sim_{\mathcal{C}} qs$. Since $\mathcal{C} \subseteq \mathcal{D} \subseteq UPol(\mathcal{C})$, we have $LPol(\mathcal{D}) \subseteq UPol(\mathcal{C})$ and α is a $UPol(\mathcal{C})$ -morphism. Therefore, since $e \sim_{\mathcal{C}} qs$ and e is an idempotent, Theorem 3.7 yields $e = eqse$. Hence, $eq = eqseq = eq(seq)^{\omega+1}$. Moreover, since $q \sim_{\mathcal{D}} r$ and $\sim_{\mathcal{D}}$ is a congruence we have $seq \sim_{\mathcal{D}} ser$. Hence, since α is an $LPol(\mathcal{D})$ -morphism, it follows from Theorem 4.1 that $(seq)^{\omega+1} = (seq)^{\omega} ser$. Since we already know that $eq = eq(seq)^{\omega+1}$, we obtain $eq = eq(seq)^{\omega} ser$. This exactly says that $eq = (eqse)^{\omega+1} r$ and since $e = eqse$ is an idempotent, we obtain $eq = er$ as desired. \square

D APPENDIX TO SECTION 5

We prove the statements presented in Section 5. Let us start with Lemma 5.4.

LEMMA 5.4. *Let \mathcal{C} be a prevariety. For every $n \geq 1$, we have $LP_{n+1}(\mathcal{C}) = MPol(RP_n(\mathcal{C}))$ and $RP_{n+1}(\mathcal{C}) = MPol(LP_n(\mathcal{C}))$.*

PROOF. We prove that $LP_{n+1}(\mathcal{C}) = MPol(RP_n(\mathcal{C}))$ (the other property is symmetrical). Since $LP_{n+1}(\mathcal{C}) = LPol(RP_n(\mathcal{C}))$ by definition, the left to right inclusion is immediate. We concentrate on the converse one. We write $\mathcal{D} = LP_{n-1}(\mathcal{C})$ for the proof. By definition, we need to prove that,

$$MPol(RPol(\mathcal{D})) \subseteq LPol(RPol(\mathcal{D})).$$

Every language in $MPol(RPol(\mathcal{D}))$ is a finite disjoint union of mixed deterministic marked products of languages in $RPol(\mathcal{D})$. Hence, since $LPol(RPol(\mathcal{D}))$ is closed under union, it suffices to prove that if $L = L_0 a_1 L_1 \cdots a_k L_k$ is a mixed deterministic marked product such that $L_1, \dots, L_k \in RPol(\mathcal{D})$, then $L \in LPol(RPol(\mathcal{D}))$. We proceed by induction on k . If $k = 0$, then $L = L_0 \in RPol(\mathcal{D}) \subseteq LPol(RPol(\mathcal{D}))$ and we are finished. Assume now that $k \geq 1$. Since $L_0 a_1 L_1 \cdots a_k L_k$ is mixed deterministic, we know that the

marked concatenation $(L_0 a_1 L_1 \cdots L_{k-1}) a_k (L_k)$ is either left deterministic or right deterministic. We handle these two cases separately. Assume first that $(L_0 a_1 L_1 \cdots a_{k-1} L_{k-1}) a_k (L_k)$ is left deterministic. One may verify that the product of $k-1$ languages $L_0 a_1 L_1 \cdots a_{k-1} L_{k-1}$ remains a mixed deterministic product. Hence, $L_0 a_1 L_1 \cdots a_{k-1} L_{k-1} \in LPol(RPol(\mathcal{D}))$ by induction. Moreover, since $L_0 \in RPol(\mathcal{D}) \subseteq LPol(RPol(\mathcal{D}))$ and the marked concatenation $(L_0 a_1 L_1 \cdots a_{k-1} L_{k-1}) a_k (L_k)$ is left deterministic, we obtain $L_0 a_1 L_1 \cdots a_k L_k \in LPol(RPol(\mathcal{D}))$ from Lemma 3.4. Assume now that $(L_0 a_1 L_1 \cdots a_{k-1} L_{k-1}) a_k (L_k)$ is right deterministic. Hence, $L_{k-1} a_k L_k$ is right deterministic. Thus, since $L_{k-1}, L_k \in RPol(\mathcal{D})$, we obtain from Lemma 3.4 that $L_{k-1} a_k L_k \in RPol(\mathcal{D})$. One may now verify that the product of $k-1$ languages $L_0 a_1 L_1 \cdots a_{k-1} (L_{k-1} a_k L_k)$ is mixed deterministic. Thus, we obtain from induction on k that $L = L_0 a_1 L_1 \cdots a_k L_k \in LPol(RPol(\mathcal{D}))$. This completes the proof. \square

We turn to Theorem 5.5. We start with a preliminary lemma concerning classes of the form $\mathcal{C} \cap \mathcal{D}$ that we shall need for the proof.

LEMMA D.1. *Let \mathcal{C}, \mathcal{D} be a prevarieties and $\mathcal{E} = \mathcal{C} \cap \mathcal{D}$. Let $\alpha : A^* \rightarrow M$ be a surjective morphism. The equivalence $\sim_{\mathcal{E}}$ on M is the least one containing both $\sim_{\mathcal{C}}$ and $\sim_{\mathcal{D}}$.*

PROOF. We write \equiv for the least equivalence of M containing both $\sim_{\mathcal{C}}$ and $\sim_{\mathcal{D}}$. We have to prove that $\equiv = \sim_{\mathcal{E}}$. It is clear that $\equiv \subseteq \sim_{\mathcal{E}}$ since $\sim_{\mathcal{E}}$ contains both $\sim_{\mathcal{C}}$ and $\sim_{\mathcal{D}}$ (this is immediate by definition as \mathcal{C} and \mathcal{D} both contain \mathcal{E}).

Conversely, consider $s, t \in M$ such that $s \sim_{\mathcal{E}} t$. We show that $s \equiv t$. Let $F \subseteq M$ be the \equiv -class of s . We have to show that $t \in F$. By definition of \equiv , F is simultaneously a union of $\sim_{\mathcal{C}}$ -classes and $\sim_{\mathcal{D}}$ -classes. Thus, Lemma 2.7 yields that $\alpha^{-1}(F)$ belongs to \mathcal{C} and \mathcal{D} . In other words, we have $\alpha^{-1}(F) \in \mathcal{E}$. Since $s \in F$ and $s \sim_{\mathcal{E}} t$, we get $t \in F$ by definition of $\sim_{\mathcal{E}}$. This concludes the proof. \square

We are ready to prove Theorem 5.5.

THEOREM 5.5. *Let \mathcal{C} be a prevariety. For every $n \geq 1$, we have $LP_{n+1}(\mathcal{C}) \cap RP_{n+1}(\mathcal{C}) = MPol(LP_n(\mathcal{C}) \cap RP_n(\mathcal{C}))$.*

PROOF. We start with right to left inclusion. It is immediate that $MPol(LP_n(\mathcal{C}) \cap RP_n(\mathcal{C}))$ is included in both $MPol(LP_n(\mathcal{C}))$ and $MPol(RP_n(\mathcal{C}))$. Moreover, these two classes are equal to $RP_{n+1}(\mathcal{C})$ and $LP_{n+1}(\mathcal{C})$ respectively as shown in Lemma 5.4. Altogether, we obtain $MPol(LP_n(\mathcal{C}) \cap RP_n(\mathcal{C})) \subseteq LP_{n+1}(\mathcal{C}) \cap RP_{n+1}(\mathcal{C})$.

We turn to the converse inclusion. For the sake of avoiding clutter, we write \mathcal{D} for the class $LP_n(\mathcal{C}) \cap RP_n(\mathcal{C})$. Consider $L \in LP_{n+1}(\mathcal{C}) \cap RP_{n+1}(\mathcal{C})$. We show that $L \in MPol(\mathcal{D})$. By Theorem 3.15, \mathcal{D} and $MPol(\mathcal{D})$ are prevarieties. Hence, by Proposition 2.2, it suffices to verify that the syntactic morphism $\alpha : A^* \rightarrow M$ of L satisfies the characterization of $MPol(\mathcal{D})$ presented in Theorem 4.3. Let $q, r, s, t \in M$ such that $s \sim_{\mathcal{D}} t$. We prove that $(sq)^{\omega} s (rs)^{\omega} = (sq)^{\omega} t (rs)^{\omega}$. Since $\mathcal{D} = LP_n(\mathcal{C}) \cap RP_n(\mathcal{C})$, Lemma D.1 yields $p_0, \dots, p_{\ell} \in M$ such that $p_0 = s, p_{\ell} = t$ and for $i < \ell$, either $p_i \sim_{LP_n(\mathcal{C})} p_{i+1}$ or $p_i \sim_{RP_n(\mathcal{C})} p_{i+1}$. We prove that for every $i < \ell$, we have $(sq)^{\omega} p_i (rs)^{\omega} = (sq)^{\omega} p_{i-1} (rs)^{\omega}$. By transitivity, this implies that $(sq)^{\omega} s (rs)^{\omega} = (sq)^{\omega} t (rs)^{\omega}$ as desired. We fix $i < \ell$ for the proof. We only treat the case when $p_{i-1} \sim_{LP_n(\mathcal{C})} p_i$ (the case when $p_{i-1} \sim_{RP_n(\mathcal{C})} p_i$ is symmetrical

and left to the reader). With this hypothesis in hand, we prove that $p_i(rs)^\omega = p_{i-1}(rs)^\omega$ which implies the desired result.

We have $L \in RPol(LP_n(\mathcal{G}))$ by hypothesis. Consequently, its syntactic morphism α is a $RPol(LP_n(\mathcal{G}))$ -morphism by Proposition 2.2. It is also clear that $\mathcal{G} \subseteq LP_n(\mathcal{G}) \subseteq UPol(\mathcal{G})$. Moreover, by hypothesis, we have $p_{i-1} \sim_{LP_n(\mathcal{G})} p_i$ and $(rs)^\omega$ is an idempotent. Finally, since \mathcal{G} is included in both $LP_n(\mathcal{G})$ and $RP_n(\mathcal{G})$, the equivalences $\sim_{LP_n(\mathcal{G})}$ and $\sim_{RP_n(\mathcal{G})}$ are included in $\sim_{\mathcal{G}}$. Hence, we have $s \sim_{\mathcal{G}} p_i$ by definition which implies that $[(rs)^\omega]_{\mathcal{G}} \leq_{\mathcal{G}} [p_i]_{\mathcal{G}}$. Altogether, it follows from Lemma C.3 that $p_i(rs)^\omega = p_{i-1}(rs)^\omega$ as desired. \square

E APPENDIX TO SECTION 6

In this appendix we present the missing proofs of Section 6. First, we prove Lemma 6.2. Then, we introduce definitions and results concerning the quantifier-alternation hierarchy of two-variable first-order logic that we shall need in the proof of Theorem 6.4. The last part of the appendix is devoted to Theorem 6.4 itself.

E.1 Proof of Lemma 6.2

We first present a useful preliminary statement about \mathcal{G} - and \mathcal{G}^+ -morphisms when \mathcal{G} is a group prevariety.

LEMMA E.1. *Let \mathcal{G} be a group prevariety and $\eta : A^* \rightarrow N$ a morphism. If η is a \mathcal{G} -morphism, then $\eta(A^*)$ is a group. Moreover, if η is \mathcal{G}^+ -morphism, then $\eta(A^+)$ is a group.*

PROOF. We treat the case when η is \mathcal{G}^+ -morphism. The other one is handled with a similar argument which is left to the reader. Let $G = \alpha(A^+)$. We show that G is a group. By definition of groups, it suffices to prove that there is only one idempotent in G . Hence, we consider two idempotents $e, f \in G$ and show that $e = f$. Let $u, v \in A^+$ such that $\eta(u) = e$ and $\eta(v) = f$. By hypothesis we have $\eta^{-1}(e) \in \mathcal{G}^+$. This yields $L \in \mathcal{G}$ such that either $\eta^{-1}(e) = L \cup \{\varepsilon\}$ or $\eta^{-1}(e) = L \cap A^+$. Since L is group language, we have a morphism $\beta : A^* \rightarrow H$ into a group H recognizing L . Let $p = \omega(H)$. Since e is idempotent, we have $\alpha(u^p) = e$ and since $u^p \in A^+$, this yields $u^p \in L$. Moreover, since H is a group, we have $\beta(u^p) = 1_H = \beta(v^p)$. Hence, $v^p \in L$ since β recognizes L . Since $v^p \in A^+$, it follows that $\eta(v^p) = e$. Finally, since $\eta(v) = f$ which is an idempotent, we also have $\eta(v^p) = f$ and we get $e = f$ as desired. \square

We turn to Lemma 6.2.

LEMMA 6.2. *If \mathcal{G} is a group prevariety and \mathcal{F} is a fragment of FO, then $\mathcal{F}(\mathbb{I}_{\mathcal{G}}) = \mathcal{F}(<, \mathbb{P}_{\mathcal{G}})$ and $\mathcal{F}(\mathbb{I}_{\mathcal{G}^+}) = \mathcal{F}(<, +1, \mathbb{P}_{\mathcal{G}})$.*

PROOF. We first handle the inclusions $\mathcal{F}(<, \mathbb{P}_{\mathcal{G}}) \subseteq \mathcal{F}(\mathbb{I}_{\mathcal{G}})$ and $\mathcal{F}(<, +1, \mathbb{P}_{\mathcal{G}}) \subseteq \mathcal{F}(\mathbb{I}_{\mathcal{G}^+})$. It suffices to prove that we may express all atomic formulas of $\mathcal{F}(<, \mathbb{P}_{\mathcal{G}})$ and $\mathcal{F}(<, +1, \mathbb{P}_{\mathcal{G}})$ using atomic formulas of $\mathcal{F}(\mathbb{I}_{\mathcal{G}})$ and $\mathcal{F}(\mathbb{I}_{\mathcal{G}^+})$ respectively. The linear order $x < y$ is expressed by $I_{A^+}(x, y)$. For every $L \in \mathcal{G}$, $P_L(x)$ is expressed by $I_L(\min, x)$. Finally, $x + 1 = y$ is expressed $I_{\{\varepsilon\}}(x, y)$ (note that $I_{\{\varepsilon\}}$ is a predicate of $\mathbb{I}_{\mathcal{G}^+}$ but not of $\mathbb{I}_{\mathcal{G}}$). We get the desired inclusions.

We now prove that $\mathcal{F}(\mathbb{I}_{\mathcal{G}}) \subseteq \mathcal{F}(<, \mathbb{P}_{\mathcal{G}})$. By definition of fragments, it suffices to prove that for every $L \in \mathcal{G}$, the atomic formula $I_L(x, y)$ is equivalent to a quantifier-free formula of $\mathcal{F}(<, \mathbb{P}_{\mathcal{G}})$. Proposition 2.3 yields a \mathcal{G} -morphism $\eta : A^* \rightarrow G$ recognizing L .

Since \mathcal{G} is a group prevariety, G is a group by Lemma E.1. For every $g \in G$, the language $\alpha^{-1}(g)$ belongs to \mathcal{G} , whence $P_{\alpha^{-1}(g)}$ is a predicate in $\mathbb{P}_{\mathcal{G}}$. Let $F \subseteq G$ be the set such that $\alpha^{-1}(F) = L$. Since G is a group, we have $\alpha(v) = (\alpha(ua))^{-1}\alpha(uav)$ for all $u, v \in A^*$ and $a \in A$. We define $T = \{(g, a, h) \in G \times A \times G \mid (g\alpha(a))^{-1}h \in F\}$. Consider the following quantifier-free formula of $\mathcal{F}(<, \mathbb{P}_{\mathcal{G}})$:

$$\varphi(x, y) = (x < y) \wedge \left(\bigvee_{(g,a,h) \in T} (P_{\alpha^{-1}(g)}(x) \wedge a(x) \wedge P_{\alpha^{-1}(h)}(y)) \right).$$

One may verify that $I_L(x, y)$ is equivalent to the following quantifier-free formula of $\mathcal{F}(<, \mathbb{P}_{\mathcal{G}})$:

$$(x = \min \wedge P_L(y)) \vee \varphi(x, y).$$

This concludes the proof for $\mathcal{F}(\mathbb{I}_{\mathcal{G}}) \subseteq \mathcal{F}(<, \mathbb{P}_{\mathcal{G}})$.

Finally, we prove that $\mathcal{F}(\mathbb{I}_{\mathcal{G}^+}) \subseteq \mathcal{F}(<, +1, \mathbb{P}_{\mathcal{G}})$. By definition, it suffices to show that for every language $K \in \mathcal{G}^+$, the atomic formula $I_K(x, y)$ is equivalent to a quantifier-free formula of $\mathcal{F}(<, +1, \mathbb{P}_{\mathcal{G}})$. By definition of \mathcal{G}^+ , there exists $L \in \mathcal{G}$ such that either $L = \{\varepsilon\} \cup K$ or $L = A^+ \cap K$. Consequently, $I_K(x, y)$ is equivalent to either $I_{\{\varepsilon\}}(x, y) \vee I_L(x, y)$ or $I_{A^+}(x, y) \wedge I_L(x, y)$. Since, $L \in \mathcal{G}$, we already proved above that $I_L(x, y)$ is equivalent to a quantifier-free formula of $\mathcal{F}(<, \mathbb{P}_{\mathcal{G}}) \subseteq \mathcal{F}(<, +1, \mathbb{P}_{\mathcal{G}})$. Moreover, $I_{\{\varepsilon\}}(x, y)$ is equivalent to $x + 1 = y$ and I_{A^+} is equivalent to $x < y \wedge \neg(x + 1 = y)$. This concludes the proof. \square

E.2 Preorders associated to FO²

We define preorders that we use to characterize the quantifier alternation hierarchy of FO²($\mathbb{I}_{\mathcal{G}}$) for some prevariety \mathcal{G} . The definitions are based on standard constructions in finite model theory. We then prove properties of these preorders that we shall need in the proof of Theorem 6.4.

Relations. We start with two preliminary definitions. We use the standard notion of *quantifier rank*. The quantifier rank (or simply rank) of an FO² formula φ is the maximal nesting depth of quantifiers in φ . Moreover, for every morphism $\eta : A^* \rightarrow N$, we associate a set \mathbb{I}_{η} of predicates. For every language $L \subseteq A^*$ which is recognized by η , the set \mathbb{I}_{η} contains the binary predicate I_L . Recall that for $w \in A^*$ and $i, j \in P(w)$, we have $w \models I_L(i, j)$ if and only if $i < j$ and $w(i, j) \in L$. Note that \mathbb{I}_{η} is a finite set of predicates.

Let $\eta : A^* \rightarrow N$ be a morphism, $k \in \mathbb{N}$ and $n \geq 1$. We associate a preorder $\leq_{\eta, k, n}$ which compares pairs (w, i) where $w \in A^*$ and $i \in P(w)$. Consider $w, w' \in A^*$, $i \in P(w)$ and $i' \in P(w')$. We let $w, i \leq_{\eta, k, n} w', i'$ if and only if for every formula $\varphi(x)$ of $\Sigma_n^2(\mathbb{I}_{\eta})$ with quantifier rank at most k and at most one free variable “ x ” the following implication holds:

$$w \models \varphi(i) \Rightarrow w' \models \varphi(i').$$

It is immediate by definition that $\leq_{\eta, k, n}$ is a preorder and it has *finitely many upper sets* (there are finitely many non-equivalent formulas $\varphi(x)$ of $\Sigma_n^2(\mathbb{I}_{\eta})$ with quantifier-rank at most k since \mathbb{I}_{η} is finite). One may verify the following fact.

FACT E.2. *Let $\eta : A^* \rightarrow N$ be a morphism, $k \in \mathbb{N}$, $n \geq 1$, $w \in A^*$ and $i \in P(w)$. There exists a formula $\varphi(x)$ of $\Sigma_n^2(\mathbb{I}_{\eta})$ with quantifier rank at most k such that for all $w' \in A^*$ and $i' \in P(w')$, we have $w' \models \varphi(i')$ if and only if $w, i \leq_{\eta, k, n} w', i'$.*

We restrict the preorders $\leq_{\eta,k,n}$ to single words in A^* . Let $w, w' \in A^*$. We let $w \leq_{\eta,k,n} w'$ if and only if $w, 0 \leq_{\eta,k,n} w', 0$. This is a pre-order on A^* . Finally, we write $\cong_{\eta,k,n}$ for the equivalence associated to $\leq_{\eta,k,n}$: $w \cong_{\eta,k,n} w'$ if and only if $w \leq_{\eta,k,n} w'$ and $w' \leq_{\eta,k,n} w$. Clearly, this equivalence has finite index. We use it to characterize the classes $\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{G}})$.

LEMMA E.3. *Let \mathcal{C} be a prevariety, $n \geq 1$ and $L \subseteq A^*$. We have $L \in \mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{G}})$ if and only if there exists a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$ such that L is a union of $\cong_{\eta,k,n}$ -classes.*

PROOF. For the “only if” direction, assume that $L \in \mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{G}})$ and let φ be the sentence of $\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{G}})$ which defines L . Let $k \in \mathbb{N}$ be the rank of φ . Proposition 2.3 yields a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ such that $\varphi \in \mathcal{B}\Sigma_n^2(\mathbb{I}_{\eta})$. One may verify that L is a union of $\cong_{\eta,k,n}$ -classes. For the “if” direction, consider a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$. We prove that every union of $\cong_{\eta,k,n}$ -classes belongs to $\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{G}})$. As $\cong_{\eta,k,n}$ has finite index, it suffices to show that all $\cong_{\eta,k,n}$ -classes belong to $\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{G}})$. For every $u \in A^*$, Fact E.2 yields a formula $\psi_u(x)$ of $\Sigma_n^2(\mathbb{I}_{\eta})$ of rank at most k such that for every $v \in A^*$ and $j \in P(v)$, we have $v \models \psi_u(j)$ if and only if $u, 0 \leq_{\eta,k,n} v, j$. Let $w \in A^*$. We define,

$$\varphi_w = \psi_w(\min) \wedge \left(\bigwedge_{w \leq_{\eta,k,n} u \text{ and } u \not\cong_{\eta,k,n} w} \neg \psi_u(\min) \right).$$

Note the conjunction boils down to a finite one since there finitely many non-equivalent $\Sigma_n^2(\mathbb{I}_{\mathcal{G}})$ of rank at most k . One may now verify that φ_w defines the $\cong_{\eta,k,n}$ -class of w which concludes the proof since it a $\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{G}})$ sentence. \square

We complete the definitions with a useful proposition. It provides an alternate definition of the preorders $\leq_{\eta,k,n}$. Intuitively, it boils down Ehrenfeucht-Fraïssé games. Yet, formulating it as an inductive definition rather than a game is more convenient. We start with a preliminary notion. Let $\eta : A^* \rightarrow N$ be a morphism, $w, w' \in A^*$, $i \in P(w)$ and $i' \in P(w')$. We say that (w, i) and (w', i') are η -equivalent if and only if one of the three following conditions holds:

- $i = i' = 0$, and $\eta(w) = \eta(w')$ or,
- $i = |w| + 1$, $i' = |w'| + 1$ and $\eta(w) = \eta(w')$ or,
- $i \in P_c(w)$, $i' \in P_c(w')$, the positions i and i' have the same label, $\eta(w(i, |w| + 1)) = \eta(w'(i', |w'| + 1))$ and $\eta(w(0, i)) = \eta(w'(0, i'))$.

We may now present the proposition.

PROPOSITION E.4. *Let $\eta : A^* \rightarrow N$ be a morphism, $k \in \mathbb{N}$, $n \geq 1$, $w, w' \in A^*$, $i \in P(w)$ and $i' \in P(w')$. Then, we have $w, i \leq_{\eta,k,n} w', i'$ if and only if the following properties hold:*

- (1) (w, i) and (w', i') are η -equivalent.
- (2) If $n \geq 2$, then $w', i' \leq_{\eta,k,n-1} w, i$.
- (3) If $k \geq 1$, then for all $j \in P(w)$ such that $i < j$, we have $j' \in P(w')$ such that $i' < j'$, $\eta(w(i, j)) = \eta(w'(i', j'))$ and $w, j \leq_{\eta,k-1,n} w', j'$.
- (4) If $k \geq 1$, then for all $j \in P(w)$ such that $j < i$, we have $j' \in P(w')$ such that $j' < i'$, $\eta(w(j, i)) = \eta(w'(j', i'))$ and $w, j \leq_{\eta,k-1,n} w', j'$.

PROOF. We start with the “only if” implication. Assume that $w, i \leq_{\eta,k,n} w', i'$. We show that the four conditions in the lemma are satisfied. The first one is immediate as one may check η -equivalence using quantifier-free formulas in $\Sigma_n^2(\mathbb{I}_{\eta})$. We turn to Condition 2. Assume that $n \geq 2$. We prove $w', i' \leq_{\eta,k,n-1} w, i$. Given a formula $\varphi(x)$ of $\Sigma_{n-1}^2(\mathbb{I}_{\eta})$ with rank at most k , we show that $w' \models \varphi(i') \Rightarrow w \models \varphi(i)$. By definition, $\neg\varphi(x) \in \Sigma_n^2(\mathbb{I}_{\eta})$ and it has rank at most k . Hence, since $w, i \leq_{\eta,k,n} w', i'$, we have $w \models \neg\varphi(i) \Rightarrow w' \models \neg\varphi(i')$. The contrapositive is exactly the desired implication. It remains to handle Conditions 3 and 4. By symmetry, we only detail the former. Assume that $k \geq 1$ and let $j \in P(w)$ such that $i < j$. We have to exhibit $j' \in P(w')$ such that $i' < j'$, $\eta(w(i, j)) = \eta(w'(i', j'))$ and $w, j \leq_{\eta,k-1,n} w', j'$. Fact E.2 yields a formula $\varphi(x)$ of $\Sigma_n^2(\mathbb{I}_{\eta})$ with rank at most $k-1$ such that for all $u \in A^*$ and $h \in P(u)$, $u \models \varphi(h)$ if and only if $w, j \leq_{\eta,k-1,n} u, h$. Moreover, we let $s = \eta(w(i, j)) \in N$ (recall that $i < j$) and $L = \eta^{-1}(s)$. Consider the following formula:

$$\psi(x) := \exists y (I_L(x, y) \wedge \varphi(y)).$$

Clearly, $\psi(x) \in \Sigma_n^2(\mathbb{I}_{\eta})$ and it has rank at most k . Moreover, it is clear that $w \models \psi(i)$ (one may use j as the position quantified by y). Hence, since $w, i \leq_{\eta,k,n} w', i'$, it follows that $w' \models \psi(i')$. This yields $j' \in P(w')$ such that $i' < j'$, $w'(i', j') \in L$ and $w' \models \varphi(j')$. By definition of L , the fact that $w'(i', j') \in L$ yields $\eta(w'(i', j')) = s = \eta(w(i, j))$. Finally, since $w' \models \varphi(j')$, we obtain $w, j \leq_{\eta,k-1,n} w', j'$ by definition of φ . This concludes the proof for the “only if” direction.

We turn to the “if” implication. Assume that the four conditions are satisfied. We show that $w, i \leq_{\eta,k,n} w', i'$. We have to prove that given a $\Sigma_n^2(\mathbb{I}_{\eta})$ formula $\varphi(x)$ with rank at most k , the implication $w \models \varphi(i) \Rightarrow w' \models \varphi(i')$ holds. First, we put $\varphi(x)$ into normal form. The following lemma can be verified from the definition of Σ_n^2 and DeMorgan’s laws.

LEMMA E.5. *The formula $\varphi(x)$ is equivalent to another formula of rank at most k which belongs to the least set of expressions closed under disjunction, conjunction and existential quantification, and containing atomic formulas as well as their negations and, if $n \geq 2$, the negations of $\Sigma_{n-1}^2(\mathbb{I}_{\eta})$ formulas.*

We assume that $\varphi(x)$ is of the form described in Lemma E.5 and prove that $w \models \varphi(i) \Rightarrow w' \models \varphi(i')$ by structural induction on φ . If $\varphi(x)$ is an atomic formula or its negation, the implication can be verified from Condition 1. We turn to the case when $\varphi(x) := \neg\psi(x)$ where $\psi(x)$ is a $\Sigma_{n-1}^2(\mathbb{I}_{\eta})$ formula (this may only happen when $n \geq 2$). Clearly, $\psi(x)$ has rank at most k by hypothesis on $\varphi(x)$. Since $w', i' \leq_{\eta,k,n-1} w, i$ by Condition 2, $w' \models \psi(i') \Rightarrow w \models \psi(i)$. The contrapositive yields $w \models \varphi(i) \Rightarrow w' \models \varphi(i')$. We turn to conjunction and disjunction. If $\varphi = \psi_1 X \psi_2$ for $X \in \{\vee, \wedge\}$, we get $w \models \psi_h(i) \Rightarrow w' \models \psi_h(i')$ for $h = 1, 2$ by structural induction. Hence, $w \models \varphi(i) \Rightarrow w' \models \varphi(i')$ as desired.

It remains to handle existential quantification. Assume that $\varphi(x) = \exists y \psi(x, y)$ (since variables can be renamed, we may assume that $y \neq x$). By hypothesis on φ , we know that ψ has rank at most $k-1$. Assume that $w \models \varphi(i)$. We show that $w \models \varphi(i')$. By hypothesis on φ , we get $j \in P(w)$ such that $w \models \psi(i, j)$. We use it to define $j' \in P(w')$. There are several cases depending on whether $j = i$, $i < j$ or $j < i$. By symmetry, we only treat the case when

$i < j$. In this case, Condition 3 yields $j' \in P(w')$ such that $i' < j'$, $w, j \preceq_{\eta, k-1, n} w', j'$ and $\eta(w(i, j)) = \eta(w(i', j'))$. We use a sub-induction on the structure of $\psi(x, y)$ to show that $w' \models \psi(i', j')$ which implies that $w', i' \models \varphi(i')$ as desired. If x is the *only* free variable in ψ , then our hypothesis states that $w \models \psi(i)$ and the main induction yields $w' \models \psi(i')$ as desired. If y is the *only* free variable in ψ , then our hypothesis states that $w \models \psi(j)$. Hence, since $w, j \preceq_{\eta, k-1, n} w', j'$ and ψ has rank at most $k-1$, we obtain $w' \models \psi(j')$ as desired. If $\psi(x, y)$ is an atomic formula or its negation involving both x and y (i.e. $x = y$, $\neg(x = y)$, $I_L(x, y)$ or $\neg I_L(x, y)$ with L recognized by η), since $w \models \psi(i, j)$, $i < j$, $i' < j'$ and $\eta(w(i, j)) = \eta(w(i', j'))$, one may verify that $w \models \psi(i', j')$. Finally, disjunction and conjunction are handled by sub-induction as in the main induction. This concludes the proof. \square

Properties. We now present important properties of the preorders $\preceq_{\eta, k, n}$. We start with a simple preliminary lemma which can be verified from Proposition E.4.

LEMMA E.6. *Let $\eta : A^* \rightarrow N$ be a morphism, $k \in \mathbb{N}$ and $n \geq 1$. Let $x_1, x_2, y_1, y_2 \in A^*$ and $a \in A$ such that $x_1 \preceq_{\eta, k, n} y_1$ and $x_2 \preceq_{\eta, k, n} y_2$. Moreover, let $i = |x_1| + 1$ and $j = |y_1| + 1$. Then, $x_1 x_2 \preceq_{\eta, k, n} y_1 y_2$ and $x_1 a x_2, i \preceq_{\eta, k, n} y_1 a y_2, i'$.*

We turn to the property that we shall use in the proof of Theorem 6.4. They are specific to morphisms $\eta : A^* \rightarrow N$ such that the set $\eta(A^+)$ is a *finite group*. This reflects the fact that Theorem 6.4 only applies to group prevarieties \mathcal{G} and their well-suited extensions \mathcal{G}^+ . We first present two preliminary results for the preorders $\preceq_{\eta, k, 1}$. The first one considers the case when η is a morphism into a group.

LEMMA E.7. *Consider a morphism $\eta : A^* \rightarrow G$ into a group and p a multiple of $\omega(G)$. Let $u, v, x, y \in A^*$ and $\ell \in \mathbb{N}$ such that $\eta(u) = \eta(v)$. Then, $v \preceq_{\eta, \ell, 1} u(yv)^p$ and $v \preceq_{\eta, \ell, 1} (vx)^p u$.*

PROOF. By symmetry, we only prove that $v \preceq_{\eta, \ell, 1} u(yv)^p$. Since G is a group, we have $\eta((vy)^p) = 1_G$. Since $\eta(u) = \eta(v)$, this yields $\eta(uy(yv)^{p-1}) = 1_G$. Thus, one may verify from Proposition E.4 that $\varepsilon \preceq_{\eta, \ell, 1} uy(yv)^{p-1}$. Hence, Lemma E.6 yields $v \preceq_{\eta, \ell, 1} u(yv)^p$ as desired. \square

We now consider the case of morphisms $\eta : A^* \rightarrow N$ such that $\eta(A^+)$ is a group. We prove a slightly weaker result.

LEMMA E.8. *Consider a morphism $\eta : A^* \rightarrow N$ such that $G = \alpha(A^+)$ is group, $\ell \in \mathbb{N}$ and p a multiple of $\omega(G)$. We consider $u, v, w, x \in A^*$ such that $|w| \geq \ell$ and $\eta(u) = \eta(v)$. We have $wv \preceq_{\eta, \ell, 1} wu(xwv)^p$ and $wv \preceq_{\eta, \ell, 1} (vwx)^p u$.*

PROOF. By symmetry, we only prove that $wv \preceq_{\eta, \ell, 1} wu(xwv)^p$. We consider a slightly more general property that we prove by induction. We let $z = wv$ and $z' = wu(xwv)^p$. Let $m = |wu(xwv)^p x|$. Clearly, if $i \in P(z)$, then $m+i$ is the corresponding position in the suffix $z = wv$ of $z' = wu(xwv)^p$. We prove that the two following properties are satisfied for every $h \leq \ell$:

- if $i \leq \ell - h$, then $z, i \preceq_{\eta, h, 1} z', i$.
- if $i > \ell - h$, then $z, i \preceq_{\eta, h, 1} z', m+i$.

In the case $h = \ell$, we may apply the first assertion for $i = 0$ which yields $wv \preceq_{\eta, \ell, 1} wu(xwv)^p$ as desired.

We now prove that the two above properties hold for every $i \in P(wv)$ and $h \leq \ell$. We proceed by induction on h . By symmetry, we only consider the first property and leave the other to the reader. Thus, we assume that $i \leq \ell - h$ and show that $z, i \preceq_{\eta, h, 1} z', i$. We use Proposition E.4. There are only three conditions to verify: Condition 2 is trivial since we are in the case $n = 1$. Moreover, it is straightforward to verify Condition 1 from our hypotheses. We turn to Conditions 3 and 4. By symmetry, we only detail the former. Assume that $h \geq 1$ and let $j \in P(v)$ such that $i < j$, we show that there exists $j' \in P(w)$ such that $i < j'$, $\eta(z(i, j)) = \eta(z'(i, j'))$ and $z, j \preceq_{\eta, h-1, 1} z, j'$. There are two sub-cases depending on j . First, assume that $j \leq \ell - (h-1)$. In this case, we let $j' = j$. Clearly, we have $\eta(z(i, j)) = \eta(z'(i, j))$ since $z(i, j) = z'(i, j)$ (this is because w is a common prefix of z and z' , and $|w| \geq \ell$). Since $j \leq \ell - (h-1)$, we get $v, j \preceq_{\eta, h-1, 1} w, j$ by induction on h . We turn to the second sub-case. Assume that $\ell - (h-1) < j$. We define $j' = m + j$. Clearly, $i < j'$ since we have $i < j$. Moreover, since $j > \ell - (h-1)$ and $j' = m + j$, induction on h yields $z, j \preceq_{\eta, h-1, 1} z', j'$. We show that $\eta(z(i, j)) = \eta(z'(i, j'))$. By definition j' is the position corresponding to $j \in P(z)$ in the suffix $z = wv$ of z' . Hence, there exists $y \in A^*$ such that $z(i, |z| + 1) = z(i, j)y$ and $z'(i, |z'| + 1) = z'(i, j')y$. Moreover, by definition of z' , we have $z'(i, |z'| + 1) = z(i, |z| + 1)(xwv)^p$. Since p is a multiple of $\omega(G)$ and $xwv \in A^+$ (we have $|w| \geq \ell$), we get $\eta(xwv) = 1_G$. Moreover, $z(i, |z| + 1) \in A^+$ since we have $i \leq \ell - h$ and $h \geq 1$. Altogether, it follows that $\eta(z(i, j)y) = \eta(z'(i, j')y)$. If $y = \varepsilon$, this concludes the proof. Otherwise, $y \in A^+$ and since we have $i \leq \ell - h$ and $\ell - (h-1) < j$, we also know that $z(i, j), z'(i, j') \in A^+$. Since $G = \alpha(A^+)$ is a group, we get $\eta(z(i, j)) = \eta(z'(i, j'))$ as desired. \square

We are ready to present the main property. We state it in the following proposition.

PROPOSITION E.9. *Consider a morphism $\eta : A^* \rightarrow N$ such that $G = \alpha(A^+)$ is a group. For all $k \in \mathbb{N}$, we have $p \geq 1$ such that if $n \geq 1$ and $u, v, x, y, z \in A^*$ satisfy $u \preceq_{\eta, k, n} v \preceq_{\eta, k, 1} z$,*

$$(zx)^p u (yz)^p \preceq_{\eta, k, n+1} (zx)^p v (yz)^p.$$

PROOF. We fix $k \in \mathbb{N}$. Let us first define $p \geq 1$. By Lemma E.6 the equivalence $\cong_{\eta, k, 1}$ is a congruence of finite index. Hence, the quotient set $A^*/\cong_{\eta, k, 1}$ is a *finite monoid*. We now define $p = \omega(G) \times \omega(A^*/\cong_{\eta, k, 1})$. By definition, we have the following key property of p :

$$\text{for every } \ell \leq k \text{ and } w \in A^*, w^{2p} \cong_{\eta, \ell, 1} w^p. \quad (8)$$

Let $n \geq 1$ and $x, y, z \in A^*$. Moreover we write $w_1 = (zx)^p$ and $w_2 = (yz)^p$. We prove a more general property.

LEMMA E.10. *Let $\ell \leq k$, $1 \leq m \leq n$ and $u, v \in A^*$ such that $u \preceq_{\eta, \ell, 1} z$ and $v \preceq_{\eta, \ell, 1} z$. Let $w = w_1 u w_2$ and $w' = w_1 v w_2$. The three following properties hold:*

- (1) if $0 \leq i \leq |w_1|$ and $u \preceq_{\eta, \ell, m} v$, then $w, i \preceq_{\eta, \ell, m+1} w', i$.
- (2) if $1 \leq i \leq |w_2| + 1$ and $u \preceq_{\eta, \ell, m} v$, then $w, |w_1 u| + i \preceq_{\eta, \ell, m+1} w', |w_1 v| + i$.
- (3) if $i \in P_c(u)$ and $i' \in P_c(v)$ satisfy $u, i \preceq_{\eta, \ell, m} v, i'$, then $w, |w_1| + i \preceq_{\eta, \ell, m+1} w', |w_1| + i'$.

Let us first apply the lemma to compete the main argument. Consider $u, v \in A^*$ such that $u \preceq_{\eta, k, n} v \preceq_{\eta, k, 1} z$. The first assertion

in Lemma E.10 yields $w_1 u w_2, 0 \preceq_{\eta, k, n+1} w_1 v w_2, 0$. This exactly says that $w_1 u w_2 \preceq_{\eta, k, n+1} w_1 v w_2$ by definition and Proposition E.9 is proved. It remains to prove Lemma E.10.

We fix $\ell \leq k, 1 \leq m \leq n$ and $u, v \in A^*$ such that $u \preceq_{\eta, \ell, 1} z$ and $v \preceq_{\eta, \ell, 1} z$. We write $w = w_1 u w_2$ and $w' = w_1 v w_2$. We use induction on ℓ and m (in any order) to prove that the three properties in the lemma hold. Since the three of them are handled using similar arguments, we only detail the third one and leave the other two to the reader. Hence, we consider $i \in P_c(u)$ and $i' \in P_c(v)$ such that $u, i \preceq_{\eta, \ell, m} v, i'$. We show that $w, |w_1| + i \preceq_{\eta, \ell, m+1} w', |w_1| + i'$. The argument is based on Proposition E.4. There are four conditions to verify. For Condition 1, that $(w, |w_1| + i)$ and $(w', |w_1| + i')$ are η -equivalent can be verified from $u, i \preceq_{\eta, \ell, m} v, i'$ which implies that (u, i) and (v, i') are η -equivalent. We turn to Condition 2. we have to prove that $w', |w_1| + i' \preceq_{\eta, \ell, m} w, |w_1| + i$. There are two sub-cases depending on m . First, assume that $m \geq 2$. Since $u, i \preceq_{\eta, \ell, m} v, i'$, Proposition E.4 implies that $v, i' \preceq_{\eta, \ell, m-1} u, i$. Hence, by induction on m , the third assertion in Lemma E.10 yields $w', |w_1| + i' \preceq_{\eta, \ell, m} w, |w_1| + i$ as desired. We now assume that $m = 1$: we prove that $w', |w_1| + i' \preceq_{\eta, \ell, 1} w, |w_1| + i$. Consider the decompositions $u = u_1 a u_2$ and $v = v_1 a v_2$ where the positions carrying the highlighted letters “a” are i and i' . We prove that $w_1 v_1 \preceq_{\eta, \ell, 1} w_1 u_1$ and $v_2 w_2 \preceq_{\eta, \ell, 1} u_2 w_2$. Since $w = w_1 u_1 a u_2 w_2$ and $w' = w_1 v_1 a v_2 w_2$, it will then follow from Lemma E.6 that $w', |w_1| + i' \preceq_{\eta, \ell, 1} w, |w_1| + i$ as desired. By symmetry, we only prove that $v_2 w_2 \preceq_{\eta, \ell, 1} u_2 w_2$. If $u_2 = v_2$, this is trivial. Hence, we assume that $u_2 \neq v_2$. Since $u, i \preceq_{\eta, \ell, 1} v, i'$, one may verify from Proposition E.4 that $\eta(u_2) = \eta(v_2)$. We prove that $v_2 \preceq_{\eta, \ell, 1} u_2 (y v)^p$. Let us first explain why this implies the desired result. By (8), we have $(y v)^{2p} \preceq_{\eta, \ell, 1} (y v)^p$. Together, with $v_2 \preceq_{\eta, \ell, 1} u_2 (y v)^p$ and Lemma E.6, this implies $v_2 (y v)^p \preceq_{\eta, \ell, 1} u_2 (y v)^{2p} \preceq_{\eta, \ell, 1} u_2 (y v)^p$ as desired. It remains to prove that $v_2 \preceq_{\eta, \ell, 1} u_2 (y v)^p$. Let $y' = y v_1 a$. Clearly, we have $y v = y' v_2$. Thus, we have to show that $v_2 \preceq_{\eta, \ell, 1} u_2 (y' v_2)^p$. There are two cases depending on η . If $\eta(A^*) = G$, the result is immediate from Lemma E.7 since $\eta(u_2) = \eta(v_2)$ and p is a multiple of $\omega(G)$. Assume now that $\eta(A^*) \neq G$. Since $\eta(A^+) = G$, it follows that $\eta^{-1}(1_N) = \{\varepsilon\}$. Hence, since $u, i \preceq_{\eta, \ell, 1} v, i'$ and $u_2 \neq v_2$, one may verify from Proposition E.4 that $|u_2| \geq \ell, |v_2| \geq \ell$ and $u_2(0, \ell+1) = v_2(0, \ell+1)$. Hence, we may apply Lemma E.8 to obtain $v_2 \preceq_{\eta, \ell, 1} u_2 (y' v_2)^p$ since p is a multiple of $\omega(G)$. This completes the proof for Condition 2.

It remains to handle Conditions 3 and 4. Since those are symmetrical, we only present an argument for the former. Let $j \in P(w)$ such that $|w_1| + i < j$. We have to exhibit $j' \in P(w')$ such that $|w_1| + i' < j', \eta(w'(|w_1| + i', j')) = \eta(w(|w_1| + i, j))$ and $w, j \preceq_{\eta, \ell-1, m+1} w', j'$. We distinguish two sub-cases depending on j . First, assume that $|w_1| + i < j \leq |w_1 u|$. In this case, there exists a position $h \in P_c(u)$ such that $j = |w_1| + h$. In particular, we have $i \leq h$. Hence, since $u, i \preceq_{\eta, \ell, m} v, i'$, Proposition E.4 yields $h' \in P_c(v)$ such that $\eta(u(i, h)) = \eta(v(i', h'))$ and $u, h \preceq_{\eta, \ell-1, m} v, h'$. We now define $j' = |w_1| + h'$. Clearly, $w'(|w_1| + i', j') = v(i', h')$ and $w(|w_1| + i, j) = u(i, h)$. Hence, it is immediate that $\eta(w'(|w_1| + i', j')) = \eta(w(|w_1| + i, j))$. Moreover, since $u, h \preceq_{\eta, \ell-1, m} v, h'$, it follows from induction on ℓ that we may apply the third assertion in Lemma E.10 to get $w, j \preceq_{\eta, \ell-1, m+1} w', j'$. We turn to

the second sub-case: $j > |w_1 u|$. In this case, there exists a position $1 \leq h \leq |w_2| + 1$ of w_2 such that $j = |w_1 u| + h$. We let $j' = |w_1 v| + h$. Clearly, we have $|w_1| + i' < j'$. It is also immediate that $w'(|w_1| + i', j') = v(i', |v| + 1) w_2(0, h)$ and $w(|w_1| + i, j) = u(i, |u| + 1) w_2(0, h)$. Additionally, since $u, i \preceq_{\eta, \ell, m} v, i'$, one may verify from Proposition E.4 that $\eta(u(i, |u| + 1)) = \eta(v(i', |v| + 1))$. Hence, we get $\eta(w'(|w_1| + i', j')) = \eta(w(|w_1| + i, j))$. Finally, it follows from induction on ℓ that we may apply the second assertion in Lemma E.10 to get $w, j \preceq_{\eta, \ell-1, m+1} w', j'$. This completes the proof of Lemma E.10. \square

Finally, we complete Proposition E.9 with a useful corollary. In fact, this is the result that we shall actually need.

COROLLARY E.11. *Consider a morphism $\eta : A^* \rightarrow N$ such that $G = \alpha(A^+)$ is a group. For all $k \in \mathbb{N}$, we have $p \geq 1$ such that for $n \geq 1$ and $u, v, x, y \in A^*$ satisfying $u \cong_{\eta, k, n} v$, we have,*

$$(vx)^p u (yv)^p \cong_{\eta, k, n+1} (vx)^p v (yv)^p.$$

PROOF. We fix $k \in \mathbb{N}$ and define $p \geq 1$ as the number given by Proposition E.9. Since $u \preceq_{\eta, k, n} v \preceq_{\eta, k, 1} v$, the case $z = v$ in the proposition yields,

$$(vx)^p u (yv)^p \preceq_{\eta, k, n+1} (vx)^p v (yv)^p.$$

Moreover, we also have $v \preceq_{\eta, k, n} u \preceq_{\eta, k, 1} v$. Therefore, we may apply Proposition E.9 in the case when u and v have been swapped and $z = v$. This yields,

$$(vx)^p v (yv)^p \preceq_{\eta, k, n+1} (vx)^p u (yv)^p.$$

We get $(vx)^p u (yv)^p \cong_{\eta, k, n+1} (vx)^p v (yv)^p$ as desired. \square

E.3 Proof of Theorem 6.4

We now concentrate on proving Theorem 6.4. Let us first recall the statement.

THEOREM 6.4. *If \mathcal{G} is a group prevariety and $\mathcal{C} \in \{\mathcal{G}, \mathcal{G}^+\}$, then $\mathcal{B}\Sigma_{n+1}^2(\mathbb{I}_{\mathcal{G}}) = MPol(\mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{G}}))$ for all $n \geq 1$.*

PROOF. For all $n \geq 1$, we write $\mathcal{D}_n = \mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{G}})$. We use induction on n to prove that $\mathcal{D}_{n+1} = MPol(\mathcal{D}_n)$ for all $n \geq 1$. Fix $n \geq 1$ for the proof. we already proved the inclusion $MPol(\mathcal{D}_n) \subseteq \mathcal{D}_{n+1}$ in the main text. Here, we concentrate on the converse one: $\mathcal{D}_{n+1} \subseteq MPol(\mathcal{D}_n)$. The argument is based on Corollary E.11. Let $L \in \mathcal{D}_{n+1}$. Since \mathcal{D}_n is a prevariety, it follows from Proposition 2.2 that it suffices to prove that the syntactic morphism $\alpha : A^* \rightarrow M$ of L is an $MPol(\mathcal{D}_n)$ -morphism. We use Theorem 4.3: for every $q, r, s, t \in M$ such that $(s, t) \in M^2$ is a \mathcal{D}_n -pair, we prove that $(sq)^\omega s (rs)^\omega = (sq)^\omega t (rs)^\omega$. By definition of \mathcal{D}_{n+1} , we have $L \in \mathcal{B}\Sigma_{n+1}^2(\mathbb{I}_{\mathcal{G}})$. Therefore, Lemma E.3 yields a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ and $k \in \mathbb{N}$ such that L is a union of $\cong_{\eta, k, n+1}$ -classes. Let K be the union of all $\cong_{\eta, k, n}$ -classes which intersect $\alpha^{-1}(s)$. Lemma E.3 yields $K \in \mathcal{B}\Sigma_n^2(\mathbb{I}_{\mathcal{G}}) = \mathcal{D}_n$. Moreover, $\alpha^{-1}(s) \subseteq K$ by hypothesis. Since $(s, t) \in M^2$ is a \mathcal{D}_n -pair, it follows that $K \cap \alpha^{-1}(t) \neq \emptyset$. Hence, we get $u, v \in A^*$ such that $\alpha(u) = s, \alpha(v) = t$ and $u \cong_{\eta, k, n} v$. We also let $x, y \in A^*$ such that $\alpha(x) = q$ and $\alpha(y) = r$. Since $\mathcal{C} \in \{\mathcal{G}, \mathcal{G}^+\}$ and $\eta : A^* \rightarrow N$ is a \mathcal{C} -morphism, Lemma E.1 implies that $G = \eta(A^+)$ is a group. Hence,

since $u \cong_{\eta,k,n} v$, Corollary E.11 and Lemma E.6 yields $p \geq 1$ such that,

$$w(vx)^p u(yv)^p w' \cong_{\eta,k,n} w(vx)^p v(yv)^p w' \text{ for all } w, w' \in A^*.$$

By definition of the syntactic morphism, it follows that the words $(vx)^p v(yv)^p$ and $(vx)^p u(yv)^p$ have the same image under α . We get $(sq)^p s(rs)^p = (sq)^p t(rs)^p$. It now suffices to multiply by the right amount of copies of tq on the left and of rt on the right to obtain $(tq)^\omega s(rt)^\omega = (st)^\omega t(rt)^\omega$. This completes the proof of $\mathcal{D}_{n+1} \subseteq MPol(\mathcal{D}_n)$. \square

F APPENDIX TO SECTION 7

This appendix contains the proofs of all statements in Section 7. Its organization mimics the one of Section 7.

F.1 Preliminary statements

We start with the proof of Fact 7.2

FACT 7.2. *Let \mathcal{C} be a finite prevariety. Given as input two regular languages L_0 and L_1 , one may compute a \mathcal{C} -compatible morphism recognizing both L_0 and L_1 .*

PROOF. Proposition 2.3 yields a \mathcal{C} -morphism $\eta : A^* \rightarrow N$ recognizing all languages in \mathcal{C} since \mathcal{C} is finite. Moreover, since L_0 and L_1 are regular, one may compute morphism $\alpha_0 : A^* \rightarrow M_0$ and $\alpha_1 : A^* \rightarrow M_1$ which recognize L_0 and L_1 respectively. We consider the monoid $M_0 \times M_1 \times N$ equipped with the componentwise multiplication. Moreover, we let $\alpha : A^* \rightarrow M_0 \times M_1 \times N$ as the morphism defined by $\alpha(w) = (\alpha_0(w), \alpha_1(w), \eta(w))$ for every $w \in A^*$. One may now verify that the surjective restriction of α is \mathcal{C} -compatible and recognizes both L_0 and L_1 . \square

We complete Fact 7.2 with a useful result on \mathcal{C} -compatible morphisms.

FACT F.1. *Let \mathcal{C} be a finite prevariety, $\alpha : A^* \rightarrow M$ a \mathcal{C} -compatible morphism and $\eta : A^* \rightarrow N$ an arbitrary morphism. For every $u, v \in A^*$, if $\alpha(u) \sim_{\mathcal{C}} \alpha(v)$, then $\eta(u) \sim_{\mathcal{C}} \eta(v)$.*

PROOF. Let $u, v \in A^*$ such that $\alpha(u) \sim_{\mathcal{C}} \alpha(v)$. We show that $\eta(u) \sim_{\mathcal{C}} \eta(v)$. Given $F \subseteq N$ such that $\eta^{-1}(F) \in \mathcal{C}$, we have to prove that $\eta(u) \in F \Leftrightarrow \eta(v) \in F$. By hypothesis, $\eta^{-1}(F)$ is recognized by α . We get $P \subseteq M$ such that $\eta^{-1}(F) = \alpha^{-1}(P)$. In particular, $\alpha^{-1}(P) \in \mathcal{C}$ and since $\alpha(u) \sim_{\mathcal{C}} \alpha(v)$, we get $\alpha(u) \in P \Leftrightarrow \alpha(v) \in P$. Therefore, $u \in \alpha^{-1}(P) \Leftrightarrow v \in \alpha^{-1}(P)$ and since $\eta^{-1}(F) = \alpha^{-1}(P)$, we get $\eta(u) \in F \Leftrightarrow \eta(v) \in F$, completing the proof. \square

We now prove Lemma 7.3.

LEMMA 7.3. *Let \mathcal{D} be a prevariety, $\alpha : A^* \rightarrow M$ a morphism and $F_0, F_1 \subseteq M$. In this case, $\alpha^{-1}(F_0)$ is \mathcal{D} -separable from $\alpha^{-1}(F_1)$ if and only if $(F_0 \times F_1) \cap \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha] = \emptyset$.*

PROOF. Assume first that $\alpha^{-1}(F_0)$ is \mathcal{D} -separable from $\alpha^{-1}(F_1)$ and let $L \in \mathcal{D}$ be a separator. Clearly, for every $H_0 \subseteq \alpha^{-1}(F_0)$ and every $H_1 \subseteq \alpha^{-1}(F_1)$, L separates H_0 from H_1 . Hence, for every $(s_0, s_1) \in F_0 \times F_1$, the language $L \in \mathcal{D}$ separates $\alpha^{-1}(s_0)$ from $\alpha^{-1}(s_1)$. It follows that $(s_0, s_1) \notin \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$. Therefore, $(F_0 \times F_1) \cap \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha] = \emptyset$ as desired.

We now assume that $(F_0 \times F_1) \cap \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha] = \emptyset$. Consider a pair $(s_0, s_1) \in F_0 \times F_1$. By hypothesis $\alpha^{-1}(s_0)$ is \mathcal{D} -separable from $\alpha^{-1}(s_1)$. We let $L_{s_0, s_1} \in \mathcal{D}$ as a separator. We now define,

$$L = \bigcup_{s_0 \in F_0} \left(\bigcap_{s_1 \in F_1} L_{s_0, s_1} \right).$$

Clearly, $L \in \mathcal{D}$ since \mathcal{D} is a prevariety. One may verify that L separates $\alpha^{-1}(F_0)$ from $\alpha^{-1}(F_1)$, completing the proof. \square

We complete the presentation with a lemma which connects the set $\mathcal{N}\mathcal{S}_{\mathcal{D}}[\eta, \alpha]$ to \mathcal{D} -morphisms. It will be useful in proofs arguments.

LEMMA F.2. *Let \mathcal{D} be a prevariety, and $\alpha : A^* \rightarrow M$ be a morphism. The following properties hold:*

- (1) *For all $(s, s') \in \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$, if $\eta : A^* \rightarrow N$ is a \mathcal{D} -morphism, there exist $u, u' \in A^*$ such that $\alpha(u) = s$, $\alpha(u') = s'$ and $\eta(u) = \eta(u')$.*
- (2) *There exists a \mathcal{D} -morphism $\eta : A^* \rightarrow N$ such that for all $u, u' \in A^*$, if $\eta(u) = \eta(u')$, then $(\alpha(u), \alpha(u')) \in \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$.*

PROOF. For the first assertion, consider $(s, s') \in \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$ and some \mathcal{D} -morphism $\eta : A^* \rightarrow N$. By hypothesis $\alpha^{-1}(s)$ is not \mathcal{D} -separable from $\alpha^{-1}(s')$. In particular, these two languages cannot be separated by a language recognized by η . Hence, there exists some $q \in N$ such that $\eta^{-1}(q)$ intersects both $\alpha^{-1}(s)$ and $\alpha^{-1}(s')$. This yields $u, u' \in A^*$ such that $\alpha(u) = s$, $\alpha(u') = s'$ and $\eta(u) = \eta(u') = q$.

We turn to the second assertion. Let $S = M^2 \setminus \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$. By definition, for every pair $(s, s') \in S$, there exists a language $L_{s, s'} \in \mathcal{D}$ which separates $\alpha^{-1}(s)$ from $\alpha^{-1}(s')$. Proposition 2.3 yields a \mathcal{D} -morphism $\eta : A^* \rightarrow N$ recognizing all languages $L_{s, s'} \in \mathcal{D}$ for $(s, s') \in S$. We now consider $u, u' \in A^*$ such that $\eta(u) = \eta(u')$ and show that $(\alpha(u), \alpha(u')) \in \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$. Since $\eta(u) = \eta(u')$, no language recognized by η can separate $\alpha^{-1}(\alpha(u))$ from $\alpha^{-1}(\alpha(u'))$. Hence, $(\alpha(u), \alpha(u')) \notin S$ by definition of η . This exactly says that $(\alpha(u), \alpha(u')) \in \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$ as desired. \square

We now turn to the proof of Lemma 7.4.

LEMMA 7.4. *Let \mathcal{D} be a prevariety and $\alpha : A^* \rightarrow M$ a morphism. Then, $\mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$ is saturated for α .*

PROOF. First, it is clear that $(\alpha(w), \alpha(w)) \in \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$ for every $w \in A^*$. Indeed, $\alpha^{-1}(\alpha(w))$ is not \mathcal{D} -separable from $\alpha^{-1}(\alpha(w))$ since these two languages intersect (w is in the intersection). We prove closure under multiplication. For $i = 1, 2$, we let $(s_i, s'_i) \in \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$ and prove that $(s_1 s_2, s'_1 s'_2) \in \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$. Lemma 7.4 yields a \mathcal{D} -morphism $\eta : A^* \rightarrow N$ such that for every $w, w' \in A^*$, if $\eta(w) = \eta(w')$, then $(\alpha(w), \alpha(w')) \in \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$. Since $(s_i, s'_i) \in \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$ for $i = 1, 2$, Lemma 7.4 yields $u_i, u'_i \in A^*$ such that $\alpha(u_i) = s_i$, $\alpha(u'_i) = s'_i$ and $\eta(u_i) = \eta(u'_i)$. Therefore, $\alpha(u_1 u_2) = s_1 s_2$, $\alpha(u'_1 u'_2) = s'_1 s'_2$, $\eta(u_1 u_2) = \eta(u'_1 u'_2)$. By definition of η , it follows that $(s_1 s_2, s'_1 s'_2) \in \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$ as desired. \square

We now consider Lemma 7.5

LEMMA 7.5. *The following equalities hold: $UPol(AT) = UPol(PT)$, $LPol(AT) = LPol(PT)$ and $RPol(AT) = RPol(PT)$.*

PROOF. Since it is clear that $AT \subseteq PT$ the left to right inclusions are immediate. We prove that $LPol(PT) \subseteq LPol(AT)$ and $UPol(PT) \subseteq UPol(AT)$ (the case of $RPol$ is symmetrical and left to the reader). We prove that $PT \subseteq LPol(AT)$. This will imply that $LPol(PT) \subseteq LPol(LPol(AT)) = LPol(AT)$ and $UPol(PT) \subseteq UPol(LPol(AT)) = UPol(AT)$ as desired. Every language in PT is a Boolean combination of marked products $A^*a_1A^* \cdots a_nA^*$. Therefore, since $LPol(AT)$ is a prevariety by Theorem 3.15, it suffices to prove that every such marked product belongs to $LPol(AT)$. Observe that $A^*a_1A^* \cdots a_nA^*$ is also defined by the marked product $(A \setminus \{a_1\})^*a_1(A \setminus \{a_2\})^*a_2 \cdots (A \setminus \{a_n\})^*a_nA^*$. One may verify that this is a left deterministic marked product of languages in AT . Thus, $A^*a_1A^* \cdots a_nA^* \in LPol(AT)$ which concludes the proof. \square

Finally, we shall need the following standard lemma about the Green relations of finite monoids.

LEMMA F.3. *Let M be a finite monoid and $s, t \in M$. If $s \leq_{\mathcal{R}} t$ and $t \leq_{\mathcal{J}} s$, then $s \mathcal{R} t$. Symmetrically, if $s \leq_{\mathcal{L}} t$ and $t \leq_{\mathcal{J}} s$, then $s \mathcal{L} t$.*

PROOF. By symmetry, we only prove the first property. Assume that $s \leq_{\mathcal{R}} t$ and $t \leq_{\mathcal{J}} s$. We show that $s \mathcal{R} t$. Since we already know that $t \leq_{\mathcal{R}} s$, this amounts to proving that $s \leq_{\mathcal{R}} t$. Since $t \leq_{\mathcal{R}} s$, we have $x \in M$ such that $sx = t$. Since $s \leq_{\mathcal{J}} t$, we have $y, z \in M$ such that $ytz = s$. This yields,

$$s = ystz = y^{\omega} s(xz)^{\omega} = y^{\omega} s(xz)^{\omega} (xz)^{\omega} = s(xz)^{\omega}.$$

Therefore, $s = sx(xz)^{\omega-1}z = t(xz)^{\omega-1}z$ and we get $s \leq_{\mathcal{R}} t$, completing the proof. \square

F.2 Left/right polynomial closure

This part of the appendix is devoted to the proof of Theorem 7.7. Let us first recall the statement.

THEOREM 7.7. *Let \mathcal{C} be a finite prevariety and \mathcal{D} a prevariety such that $\mathcal{C} \subseteq \mathcal{D} \subseteq UPol(\mathcal{C})$. Let $\alpha : A^* \rightarrow M$ be a \mathcal{C} -compatible morphism and $P = \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$. Then, $\mathcal{N}\mathcal{S}_{LPol(\mathcal{D})}[\alpha]$ is the least $(LPol, P)$ -saturated subset of M^2 and $\mathcal{N}\mathcal{S}_{RPol(\mathcal{D})}[\alpha]$ is the least $(RPol, P)$ -saturated subset of M^2 for α .*

By symmetry, we only prove that $\mathcal{N}\mathcal{S}_{LPol(\mathcal{D})}[\alpha]$ is the least $(LPol, P)$ -saturated subset of M^2 . The proof involves two independent arguments. First, we show that $\mathcal{N}\mathcal{S}_{LPol(\mathcal{D})}[\alpha]$ is $(LPol, P)$ -saturated. Then, we show it includes all $(LPol, P)$ -saturated subsets. We start with the former.

Soundness. We prove that $\mathcal{N}\mathcal{S}_{LPol(\mathcal{D})}[\alpha]$ is $(LPol, P)$ -saturated. We write $S = \mathcal{N}\mathcal{S}_{LPol(\mathcal{D})}[\alpha]$ for the proof. We already know from Lemma 7.4 that it is saturated since $LPol(\mathcal{D})$ is a prevariety by Theorem 3.15. Hence, we focus on (3). Let $(e, e') \in S$ be a pair of idempotents and $(s, s') \in P$ such that $[e]_{\mathcal{C}} \leq_{\mathcal{R}} [s]_{\mathcal{C}}$. We show that $(es, e's') \in S$. By Lemma F.2, there exists an $LPol(\mathcal{D})$ -morphism $\eta : A^* \rightarrow N$ such that for every pair $u, u' \in A^*$, such that $\eta(u) = \eta(u')$, we have $(\alpha(u), \alpha(u')) \in S$.

Since $(e, e') \in S$, $S = \mathcal{N}\mathcal{S}_{LPol(\mathcal{D})}[\alpha]$ and η is an $LPol(\mathcal{D})$ -morphism, Lemma F.2 yields two words $x, x' \in A^*$ such that $\eta(x) = \eta(x')$, $\alpha(x) = e$ and $\alpha(x') = e'$. We shall write $t = \eta(x) = \eta(x')$. Moreover, since $(s, s') \in P$, $P = \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$ and $[\cdot]_{\mathcal{D}} \circ \eta : A^* \rightarrow N/\sim_{\mathcal{D}}$ is a \mathcal{D} -morphism by Lemma 2.7, we get from Lemma F.2 that there exist $y, y' \in A^*$ such that $\eta(y) \sim_{\mathcal{D}}$

$\eta(y')$, $\alpha(y) = s$ and $\alpha(y') = s'$. We write $p = \eta(y)$ and $p' = \eta(y')$: we have $p \sim_{\mathcal{D}} p'$. Finally, since $[e]_{\mathcal{C}} \leq_{\mathcal{R}} [s]_{\mathcal{C}}$, we have $[\alpha(x)]_{\mathcal{C}} \leq_{\mathcal{R}} [\alpha(y)]_{\mathcal{C}}$. Since α is \mathcal{C} -compatible, Fact F.1 yields $[\eta(x)]_{\mathcal{C}} \leq_{\mathcal{R}} [\eta(y)]_{\mathcal{C}}$, i.e. $[t]_{\mathcal{C}} \leq_{\mathcal{R}} [p]_{\mathcal{C}}$. Clearly, this implies that $[t^{\omega}]_{\mathcal{C}} \leq_{\mathcal{R}} [p]_{\mathcal{C}}$. By hypothesis, $\mathcal{C} \subseteq \mathcal{D} \subseteq UPol(\mathcal{C})$ and η is an $LPol(\mathcal{D})$ -morphism. Hence, since t^{ω} is an idempotent of N and we have $p \sim_{\mathcal{D}} p'$, Lemma C.3 yields $t^{\omega}p = t^{\omega}p'$. Let $n = \omega(N)$. Since we have $t = \eta(x) = \eta(x')$, $p = \eta(y)$ and $p' = \eta(y')$, we just proved that $\eta(x^n y) = \eta((x')^n y')$. As e and e' are idempotents, we have $\alpha(x^n y) = es$ and $\alpha((x')^n y') = e's'$. We obtain $(es, e's') \in \mathcal{N}\mathcal{S}_{LPol(\mathcal{D})}[\alpha] = S$ by definition of η .

Completeness. Consider an arbitrary $(LPol, P)$ -saturated set $S \subseteq M^2$. We prove that $\mathcal{N}\mathcal{S}_{LPol(\mathcal{D})}[\alpha] \subseteq S$. Since we have $P = \mathcal{N}\mathcal{S}_{\mathcal{D}}[\alpha]$, it follows from Lemma F.2 that there exists a \mathcal{D} -morphism $\eta : A^* \rightarrow N$ such that for every $u, u' \in A^*$ which satisfy $\eta(u) = \eta(u')$, we have $(\alpha(u), \alpha(u')) \in P$. Moreover, we define two notions that we shall use as induction parameters. In the definition we consider the Green relations $\leq_{\mathcal{J}}$ of $M/\sim_{\mathcal{C}}$ and $\leq_{\mathcal{R}}$ of S (note that $S \subseteq M^2$ is a monoid since it is saturated for α):

- (1) The \mathcal{J} -rank $r(w) \in \mathbb{N}$ of a word $w \in A^*$ is the number of elements $q \in M/\sim_{\mathcal{C}}$ such that $[\alpha(w)]_{\mathcal{C}} \leq_{\mathcal{J}} q$.
- (2) The \mathcal{R} -index $d(t, t') \in \mathbb{N}$ of a pair $(t, t') \in S$ is the number of pairs $(p, p') \in S$ such that $(p, p') \leq_{\mathcal{R}} (t, t')$.

Let $m = |M^2|$. We prove the following lemma by induction.

LEMMA F.4. *Let $d, r \in \mathbb{N}$ and $k \geq mr + d$. For all $(t, t') \in S$ such that $d(t, t') \leq d$ and all $w, w' \in A^*$ such that $r(w) \leq r$ and $w \triangleright_{\eta, k} w'$, we have $(t\alpha(w), t'\alpha(w')) \in S$.*

We first use Lemma F.4 to prove that $\mathcal{N}\mathcal{S}_{LPol(\mathcal{D})}[\alpha] \subseteq S$. Let $(s, s') \in \mathcal{N}\mathcal{S}_{LPol(\mathcal{D})}[\alpha]$. We show that $(s, s') \in S$. We let $k = m \times |M/\sim_{\mathcal{C}}| + m$ and consider the equivalence $\triangleright_{\eta, k}$. By Proposition 3.13 every union of $\triangleright_{\eta, k}$ -class belongs to $LPol(\mathcal{D})$ since η is a \mathcal{D} -morphism. Therefore, we obtain $w, w' \in A^*$ such that $\alpha(w) = s$, $\alpha(w') = s'$ and $w \triangleright_{\eta, k} w'$ (otherwise, $\alpha^{-1}(s)$ would be separated from $\alpha^{-1}(s')$ by a union of $\triangleright_{\eta, k}$ -classes, which contradicts the hypothesis that $(s, s') \in \mathcal{N}\mathcal{S}_{LPol(\mathcal{D})}[\alpha]$). Clearly, we have $r(w) \leq |M/\sim_{\mathcal{C}}|$. Moreover, $(1_M, 1_M) \in S$ since S is saturated and it is clear that $d(1_M, 1_M) \leq |M^2| = m$. Hence, since $k = m \times |M/\sim_{\mathcal{C}}| + m$ and $w \triangleright_{\eta, k} w'$, Lemma F.4 yields $(\alpha(w), \alpha(w')) \in S$, i.e., $(s, s') \in S$ as desired.

We now prove Lemma F.4. We fix $d, r \in \mathbb{N}$ and $k \geq mr + d$ for the proof. We let $(t, t') \in S$ such that $d(t, t') \leq d$ and $w, w' \in A^*$ such that $r(w) \leq r$ and $w \triangleright_{\eta, k} w'$. We need to prove that $(t\alpha(w), t'\alpha(w')) \in S$. We proceed by induction on r and d (in that order of importance). There are two cases.

Base case: we have $(s, s') \in S$ such that $[s]_{\mathcal{C}} \mathcal{R} [\alpha(w)]_{\mathcal{C}}$ and $(ts, ts') \mathcal{R} (t, t')$. By hypothesis, we get $(q, q') \in S$ such that $(tsq, t's'q') = (t, t')$. Let $(e, e') = ((sq)^{\omega}, (s'q')^{\omega})$. Clearly, this is a pair of idempotents in M , $(te, t'e') = (t, t')$ and $(e, e') \in S$ since S is saturated. Since $[s]_{\mathcal{C}} \mathcal{R} [\alpha(w)]_{\mathcal{C}}$, it is clear that $[e]_{\mathcal{C}} \leq_{\mathcal{R}} [\alpha(w)]_{\mathcal{C}}$. Finally, since $w \triangleright_{\eta, k} w'$, we have $\eta(w) = \eta(w')$ which yields $(\alpha(w), \alpha(w')) \in P$ by definition of η . Altogether, since S is $(LPol, P)$ -saturated, it follows from (3) that $(e\alpha(w), e'\alpha(w')) \in S$. By closure under multiplication, we get $(te\alpha(w), t'e'\alpha(w')) \in S$ which exactly says that $(t\alpha(w), t'\alpha(w')) \in S$ as desired.

Inductive case: for every $(s, s') \in S$ such that $[s]_{\mathcal{C}} \mathcal{R} [\alpha(w)]_{\mathcal{C}}$, we have $(ts, t's') <_{\mathcal{R}} (t, t')$. Let $x \in A^*$ be the least prefix of w such that $[\alpha(x)]_{\mathcal{C}} \mathcal{R} [\alpha(w)]_{\mathcal{C}}$. Observe that x is nonempty. Indeed, if $x = \varepsilon$, then $(1_M, 1_M) = (\alpha(\varepsilon), \alpha(\varepsilon)) \in S$ since S is saturated and we have $[1_M]_{\mathcal{C}} \mathcal{R} [\alpha(w)]_{\mathcal{C}}$. This contradicts our hypothesis as $(t1_M, t'1_M) = (t, t') \mathcal{R} (t, t')$.

Since $x \in A^*$ we get $u \in A^*$ and $a \in A$ such that $x = ua$. We let $v \in A^*$ as the suffix such $w = uav$. By definition of x as the least prefix of w such that $[\alpha(x)]_{\mathcal{C}} \mathcal{R} [\alpha(w)]_{\mathcal{C}}$, we have $[\alpha(w)]_{\mathcal{C}} \mathcal{R} [\alpha(ua)]_{\mathcal{C}} <_{\mathcal{R}} [\alpha(u)]_{\mathcal{C}}$. Using this property and the hypothesis that $w \triangleright_{\eta, k} w'$, we prove the following fact.

FACT F.5. *We have $u', v' \in A^*$ such that $w' = u'av', u \triangleright_{\eta, k} u'$ and $v \triangleright_{\eta, k-1} v'$.*

PROOF. We first show that $\eta(ua) <_{\mathcal{R}} \eta(u)$. By contradiction assume that there exists $y \in A^*$ such that $\eta(uy) = \eta(u)$. By definition of η , it follows that $(\alpha(uy), \alpha(ua)) \in \mathcal{N}_{\mathcal{D}}[\alpha]$ and since $\mathcal{C} \subseteq \mathcal{D}$, we get $(\alpha(uy), \alpha(ua)) \in \mathcal{N}_{\mathcal{C}}[\alpha]$. One may verify that this implies $\alpha(uy) \sim_{\mathcal{C}} \alpha(ua)$, contradicting the hypothesis that $[\alpha(ua)]_{\mathcal{C}} <_{\mathcal{R}} [\alpha(u)]_{\mathcal{C}}$.

We may now prove the fact. Let $i \in P_c(w)$ be the position carrying the highlighted letter “a” in $w = uav$. Since $\eta(ua) <_{\mathcal{R}} \eta(u)$, it follows that $i \in P_{\triangleright}(\eta, 1, w)$. We define $X = P_{\triangleright}(\eta, k, u)$ and $Y = P_{\triangleright}(\eta, k-1, v)$. Since $i \in P_{\triangleright}(\eta, 1, w)$, we have $X \cup \{i\} \cup \{i+j \mid j \in Y\} \subseteq P_{\triangleright}(\eta, k, w)$ by definition. Moreover, since we have $w \triangleright_{\eta, k} w'$, we know that $\sigma_{\eta}(w, P_{\triangleright}(\eta, k, w)) = \sigma_{\eta}(w', P_{\triangleright}(\eta, k, w'))$. Thus, one may verify that there exist $i' \in P_c(w')$, $X' \subseteq P_c(u')$ for $u' = w'(0, i')$ and $Y' \subseteq P_c(v')$ for $v' = w'(i', |w'|+1)$ such that i' has label a , $\sigma_{\eta}(u, X) = \sigma_{\eta}(u', X')$ and $\sigma_{\eta}(v, Y) = \sigma_{\eta}(v', Y')$. By construction, we have $w' = u'av'$. Moreover, since we have $\sigma_{\eta}(u, X) = \sigma_{\eta}(u', X')$ and $X = P_{\triangleright}(\eta, k, u)$, Lemma 3.9 yields $X' = P_{\triangleright}(\eta, k, u')$. Hence, we get $u \triangleright_{\eta, k} u'$. Finally, since $\sigma_{\eta}(v, Y) = \sigma_{\eta}(v', Y')$ and $Y = P_{\triangleright}(\eta, k-1, v)$, Lemma 3.9 yields $Y' = P_{\triangleright}(\eta, k-1, v')$. Hence, $v \triangleright_{\eta, k-1} v'$. \square

Since $k \geq mr + d$, we have $k \geq m(r-1) + m$. By definition, $[\alpha(w)]_{\mathcal{C}} <_{\mathcal{R}} [\alpha(u)]_{\mathcal{C}}$ which yields $[\alpha(w)]_{\mathcal{C}} <_{\mathcal{F}} [\alpha(u)]_{\mathcal{C}}$ by Lemma F.3. Hence, $r(u) < r(w)$ and since $r(w) \leq r$, we get $r(u) \leq r-1$. Recall that $d(1_M, 1_M) \leq m = |M^2|$. Hence, since $u \triangleright_{\eta, k} u'$, it follows by induction on r in Lemma F.4 (our most important parameter) that $(\alpha(u), \alpha(u')) \in S$. Moreover, we know that $(\alpha(a), \alpha(a)) \in S$ since S is saturated. Thus closure under multiplication yields that $(\alpha(ua), \alpha(u'a)) \in S$. We let $(s, s') = (\alpha(ua), \alpha(u'a))$. Since $s = \alpha(ua)$, we have $[\alpha(w)]_{\mathcal{C}} \mathcal{R} [s]_{\mathcal{C}}$ by definition of u and a . Hence, our hypothesis in the inductive case yields $(ts, t's') <_{\mathcal{R}} (t, t')$. It follows that $d(ts, t's') < d(t, t')$ which yields $d(ts, t's') \leq d-1$. Moreover, since $w = uav$, we have $[\alpha(w)]_{\mathcal{C}} \leq_{\mathcal{F}} [\alpha(v)]_{\mathcal{C}}$. Thus, $r(v) \leq r(w) \leq r$. Finally, since $k \geq mr + d$, we have $k-1 \geq mr + (d-1)$. Hence, since $v \triangleright_{\eta, k-1} v'$, induction on d in Lemma F.4 yields $(t\alpha(v), t'\alpha(v')) \in S$. By definition of (s, s') , this exactly says that $(t\alpha(w), t'\alpha(w')) \in S$ as desired.

F.3 Mixed polynomial closure

This final part of the appendix is devoted to the proof of Theorem 7.10. Let us first recall the statement.

THEOREM 7.10. *Let \mathcal{C} be a finite prevariety and \mathcal{D} a prevariety such that $\mathcal{C} \subseteq \mathcal{D} \subseteq \text{UPol}(\mathcal{C})$. Moreover, let $\alpha : A^* \rightarrow M$ be a \mathcal{C} -compatible morphism and let $P = \mathcal{N}_{\mathcal{D}}[\alpha]$, $P_1 = \mathcal{N}_{\text{LPol}(\mathcal{D})}[\alpha]$ and $P_2 = \mathcal{N}_{\text{RPol}(\mathcal{D})}[\alpha]$. Then, the set $\mathcal{N}_{\text{MPol}(\mathcal{D})}[\alpha]$ is the least $(\text{MPol}, P_1, P, P_2)$ -saturated subset of M^2 for α .*

The proof involves two independent arguments. First, we show that $\mathcal{N}_{\text{MPol}(\mathcal{D})}[\alpha]$ is $(\text{MPol}, P_1, P, P_2)$ -saturated. Then, we show it includes all $(\text{MPol}, P_1, P, P_2)$ -saturated subsets.

Soundness. We write $S = \mathcal{N}_{\text{MPol}(\mathcal{D})}[\alpha]$. We have to prove that S is $(\text{MPol}, P_1, P, P_2)$ -saturated. We already know from Lemma 7.4 that it is saturated since $\text{MPol}(\mathcal{D})$ is a prevariety by Theorem 3.15. Hence, we focus on (5). By Lemma F.2, there exists an $\text{MPol}(\mathcal{D})$ -morphism $\eta : A^* \rightarrow N$ such that for every pair $u, u' \in A^*$ satisfying $\eta(u) = \eta(u')$, we have $(\alpha(u), \alpha(u')) \in S$. We start with a preliminary lemma.

LEMMA F.6. *Let $(s, s') \in M^2$ be a (P_1, P, P_2) -block. We have $u, u' \in A^*$ such that $\alpha(u) = s$, $\alpha(u') = s'$ and $\eta(u) = \eta(u')$. Moreover, there exists an infix v of u such that $\eta(v)$ is an idempotent of N and $[\alpha(v)]_{\mathcal{C}} \mathcal{F} [s]_{\mathcal{C}}$.*

PROOF. We have $(s_1, s'_1), (e_1, e'_1) \in P_1$, $(s_2, s'_2), (e_2, e'_2) \in P_2$ and $(s_3, s'_3) \in P$ such that $e_1, e'_1, e_2, e'_2 \in M$ are idempotents satisfying $[e_1]_{\mathcal{C}} \mathcal{F} [e_2]_{\mathcal{C}} \mathcal{F} [s_1e_1s_3e_2s_2]_{\mathcal{C}}$ and,

$$(s, s') = (s_1e_1s_3e_2s_2, s'_1e'_1s'_3e'_2s'_2).$$

We have $P = \mathcal{N}_{\mathcal{D}}[\alpha]$. Hence since $[\cdot]_{\mathcal{D}} \circ \eta$ is a \mathcal{D} -morphism by Lemma 2.7 and $(s_3, s'_3) \in P$, Lemma F.2 yields $x_3, x'_3 \in A^*$ such that $\alpha(x_3) = s_3$, $\alpha(x'_3) = s'_3$ and $\eta(x_3) \sim_{\mathcal{D}} \eta(x'_3)$. Since $P_1 = \mathcal{N}_{\text{LPol}(\mathcal{D})}[\alpha]$ and $(s_1, s'_1), (e_1, e'_1) \in P_1$, a similar argument yields $x_1, x'_1, y_1, y'_1 \in A^*$ such that $\alpha(x_1) = s_1$, $\alpha(x'_1) = s'_1$, $\alpha(y_1) = e_1$, $\alpha(y'_1) = e'_1$, $\eta(x_1) \sim_{\text{LPol}(\mathcal{D})} \eta(x'_1)$ and $\eta(y_1) \sim_{\text{LPol}(\mathcal{D})} \eta(y'_1)$. Moreover, since $P_2 = \mathcal{N}_{\text{RPol}(\mathcal{D})}[\alpha]$ and $(s_2, s'_2), (e_2, e'_2) \in P_2$, we also get $x_2, x'_2, y_2, y'_2 \in A^*$ which satisfy $\alpha(x_2) = s_2$, $\alpha(x'_2) = s'_2$, $\alpha(y_2) = e_2$, $\alpha(y'_2) = e'_2$, $\eta(x_2) \sim_{\text{RPol}(\mathcal{D})} \eta(x'_2)$ and $\eta(y_2) \sim_{\text{RPol}(\mathcal{D})} \eta(y'_2)$. Finally, we let $n = \omega(M) \times \omega(N)$. We now define $u = x_1y_1^n x_3y_2^n x_2$, $u' = x'_1(y'_1)^n x'_3(y'_2)^n x'_2$ and $v = y_1^n$. Since e_1, e_2, e'_1, e'_2 are idempotents, it is immediate by definition that $\alpha(u) = s$ and $\alpha(u') = s'$. Moreover, v is an infix of u and $\alpha(v) = e_1$. Hence, we have $[\alpha(v)]_{\mathcal{C}} \mathcal{F} [s]_{\mathcal{C}}$ by hypothesis on e_1 .

It remains to prove that $\eta(u) = \eta(u')$. We write $q_1 = \eta(x_1)$, $f_1 = \eta(y_1^n)$, $q_3 = \eta(x_3)$, $f_2 = \eta(y_2^n)$ and $q_2 = \eta(x_2)$. Clearly, $\eta(u) = q_1f_1q_3f_2q_2$. Moreover, we define $q'_1, f'_1, q'_3, f'_2, q'_2$ analogously such that $\eta(u') = q'_1f'_1q'_3f'_2q'_2$. We have to prove that $q_1f_1q_3f_2q_2 = q'_1f'_1q'_3f'_2q'_2$. As $[e_1]_{\mathcal{C}} \mathcal{F} [e_2]_{\mathcal{C}} \mathcal{F} [s_1e_1s_3e_2s_2]_{\mathcal{C}}$, we get from Lemma F.3 that $[e_1]_{\mathcal{C}} \leq_{\mathcal{R}} [e_1s_3e_2s_2]_{\mathcal{C}}$ and $[e_2]_{\mathcal{C}} \leq_{\mathcal{F}} [s_1e_1s_3e_2]_{\mathcal{C}}$. Since α is \mathcal{C} -compatible, one may verify from the definitions and Fact F.1 that this implies $[f_1]_{\mathcal{C}} \leq_{\mathcal{R}} [f_1q_3f_2q_2]_{\mathcal{C}}$ and $[f_2]_{\mathcal{C}} \leq_{\mathcal{F}} [q_1f_1q_3f_2]_{\mathcal{C}}$.

FACT F.7. *We have $q_3f_2 \sim_{\text{RPol}(\mathcal{D})} q'_3f'_2$.*

PROOF. We first define $\beta : N \rightarrow N/\sim_{\text{RPol}(\mathcal{D})}$ as the morphism $q \mapsto [q]_{\text{RPol}(\mathcal{D})}$ and $\gamma = \beta \circ \eta : A^* \rightarrow N/\sim_{\text{RPol}(\mathcal{D})}$. It follows from Lemma 2.7 that γ is an $\text{RPol}(\mathcal{D})$ -morphism. We know that $\mathcal{C} \subseteq \mathcal{D} \subseteq \text{UPol}(\mathcal{C})$ by hypothesis. Moreover, $\beta(f_2)$ is an idempotent of $N/\sim_{\text{RPol}(\mathcal{D})}$ and since $q_3 \sim_{\mathcal{D}} q'_3$ which yields $q_3f_2 \sim_{\mathcal{D}} q'_3f_2$, one may verify $\beta(q_3f_2) \sim_{\mathcal{D}} \beta(q'_3f_2)$. Moreover, since $[f_2]_{\mathcal{C}} \leq_{\mathcal{F}}$

$[q_1f_1q_3f_2]_{\mathcal{C}}$, one may verify that $[\beta(f_2)]_{\mathcal{C}} \leq_{\mathcal{D}} [\beta(q_3f_2)]_{\mathcal{C}}$. Altogether, Lemma C.3 yields $\beta(q_3f_2)\beta(f_2) = \beta(q'_3f_2)\beta(f_2)$. Since we have $f_2 \sim_{RPol(\mathcal{D})} f'_2$ by definition, $\beta(f_2) = \beta(f'_2)$. We get $\beta(q_3f_2) = \beta(q'_3f'_2)$ which exactly says that $q_3f_2 \sim_{RPol(\mathcal{D})} q'_3f'_2$ as desired. \square

We now prove that $q_1f_1q_3f_2q_2 = q'_1f'_1q'_3f'_2q'_2$. There are two steps: one proves independently that $q_1f_1q_3f_2q_2 = q_1f_1q'_3f'_2q'_2$ and that $q_1f_1q'_3f'_2q'_2 = q'_1f'_1q'_3f'_2q'_2$. As the arguments are symmetrical, we only prove the former. Since α is an $MPol(\mathcal{D})$ -morphism, Lemma 5.4 implies that it is an $LPol(RPol(\mathcal{D}))$ -morphism. Moreover, we have $\mathcal{C} \subseteq RPol(\mathcal{D}) \subseteq UPol(\mathcal{C})$ by hypothesis on \mathcal{D} . By Fact F.7, we have $q_3f_2 \sim_{RPol(\mathcal{D})} q'_3f'_2$ and since $q_2 \sim_{RPol(\mathcal{D})} q'_2$ by definition of q_2, q'_2 , we have $f_1q_3f_2q_2 \sim_{RPol(\mathcal{D})} f_1q'_3f'_2q'_2$. We have $[f_1]_{\mathcal{C}} \leq_{\mathcal{D}} [f_1q_3f_2q_2]_{\mathcal{C}}$. Altogether, Lemma C.3 yields $f_1f_1q_3f_2q_2 = f_1f_1q'_3f'_2q'_2$. We get $q_1f_1q_3f_2q_2 = q_1f_1q'_3f'_2q'_2$ since f_1 is idempotent. \square

We prove that S satisfies (5). Let $n \in \mathbb{N}$ and $(s_0, s'_0), \dots, (s_n, s'_n) \in M^2$ be (P_1, P, P_2) -blocks. Moreover, let $(t_1, t'_1), \dots, (t_n, t'_n) \in P$ such that $[s_{i-1}t_i]_{\mathcal{C}} \mathcal{F} [s_{i-1}]_{\mathcal{C}}$ and $[t_i s_i]_{\mathcal{C}} \mathcal{F} [s_i]_{\mathcal{C}}$ for $1 \leq i \leq n$. We have to prove that,

$$(s_0 t_1 s_1 \cdots t_n s_n, s'_0 t'_1 s'_1 \cdots t'_n s'_n) \in S. \quad (9)$$

For $0 \leq i \leq n$, Lemma F.6 yields $u_i, u'_i, v_i \in A^*$ such that we have $\alpha(u_i) = s_i$, $\alpha(u'_i) = s'_i$, $\eta(u_i) = \eta(u'_i)$, v_i is an infix of u_i , $\eta(v_i)$ is an idempotent of N and $[\alpha(v_i)]_{\mathcal{C}} \mathcal{F} [s_i]_{\mathcal{C}}$. Moreover, recall that $[\cdot]_{\mathcal{D}} \circ \eta$ is a \mathcal{D} -morphism by Lemma 2.7. Hence, as $(t_i, t'_i) \in P$ and $P = \mathcal{N}_{\mathcal{D}}[\alpha]$, Lemma F.2 yields $w_i, w'_i \in A^*$ such that $\alpha(w_i) = t_i$, $\alpha(w'_i) = t'_i$ and $\eta(w_i) \sim_{\mathcal{D}} \eta(w'_i)$ for $1 \leq i \leq n$. We may now define $w = u_0 w_1 u_1 \cdots w_n u_n$ and $w' = u'_0 w'_1 u'_1 \cdots w'_n u'_n$. We show that $\eta(w) = \eta(w')$. By definition η , this will imply that $(\alpha(w), \alpha(w')) \in S$. Moreover, by definition of w and w' , this exactly says that (9) holds, concluding the proof.

We prove that $\eta(w) = \eta(w')$. Let $w'' = u_0 w'_1 u_1 \cdots w'_n u_n$. Since $\eta(u_i) = \eta(u'_i)$ for $0 \leq i \leq n$, we have $\eta(w') = \eta(w'')$. Hence, we have to prove that $\eta(w) = \eta(w'')$. By definition of w , it suffices to prove that $\eta(u_{i-1} w_i u_i) = \eta(u_{i-1} w'_i u_i)$ for $1 \leq i \leq n$. By transitivity, it will follow that $\eta(w) = \eta(w'')$. We fix i such that $1 \leq i \leq n$ for the proof. Let $q_i = \eta(w_i)$ and $q'_i = \eta(w'_i)$. We have the following lemma

LEMMA F.8. *There exist two elements $r_{i-1}, r_i \in N$ such that we have $\eta(u_{i-1}) = \eta(u_{i-1})(q_i r_{i-1})^\omega$ and $\eta(u_i) = (r_i q_i)^\omega \eta(u_i)$.*

PROOF. By symmetry, we only prove the existence of r_i . Recall that by hypothesis, we have an infix v_i of u_i such that $\eta(u_i)$ is idempotent and $[\alpha(v_i)]_{\mathcal{C}} \mathcal{F} [s_i]_{\mathcal{C}}$. We get $x, y \in A^*$ such that $u_i = x v_i y$. Moreover, $[t_i s_i]_{\mathcal{C}} \mathcal{F} [s_i]_{\mathcal{C}}$. Since $s_i = \alpha(u_i)$ and $t_i = \alpha(w_i)$. We get $[\alpha(w_i x v_i y)]_{\mathcal{C}} \mathcal{F} [s_i]_{\mathcal{C}} \mathcal{F} [\alpha(v_i)]_{\mathcal{C}}$. It follows that $[\alpha(v_i)]_{\mathcal{C}} \leq_{\mathcal{D}} [\alpha(w_i x v_i y)]_{\mathcal{C}}$. Moreover, since it is clear that we have $[\alpha(w_i x v_i)]_{\mathcal{C}} \leq_{\mathcal{D}} [\alpha(v_i)]_{\mathcal{C}}$, we get $[\alpha(w_i x v_i)]_{\mathcal{C}} \mathcal{L} [\alpha(v_i)]_{\mathcal{C}}$ from Lemma F.3. We get $z \in A^*$ such that $\alpha(z w_i x v_i) \sim_{\mathcal{C}} \alpha(v_i)$. Since α is \mathcal{C} -compatible, it then follows from Lemma F.1 that $\eta(z w_i x v_i) \sim_{\mathcal{C}} \eta(v_i)$. Since η is an $MPol(\mathcal{D})$ -morphism and $\mathcal{D} \subseteq UPol(\mathcal{C})$, we know that η is $UPol(\mathcal{C})$ -morphism. Hence, since $\eta(v_i)$ is an idempotent of N , Theorem 3.7 yields $\eta(v_i) = \eta(v_i z w_i x v_i)$. Hence, $\eta(x v_i) = \eta(x v_i z w_i x v_i) = (\eta(x v_i z w_i))^\omega \eta(x v_i)$.

Let $r_i = \eta(x v_i z)$. Since $u_i = x v_i y$ and $q_i = \eta(w_i)$, we obtain that $\eta(u_i) = (r_i q_i)^\omega \eta(u_i)$ as desired. \square

Recall that by definition, $\eta(w_i) \sim_{\mathcal{D}} \eta(w'_i)$, i.e. $q_i \sim_{\mathcal{D}} q'_i$. Hence, since η is an $MPol(\mathcal{D})$ -morphism by definition, it follows from Theorem 4.3 that,

$$(q_i r_{i-1})^\omega q_i (r_i q_i)^\omega = (q_i r_{i-1})^\omega q'_i (r_i q_i)^\omega.$$

We may now multiply by $\eta(u_{i-1})$ on the left and by $\eta(u_i)$ on the right. Since $q_i = \eta(w_i)$ and $q'_i = \eta(w'_i)$, in view of Lemma F.8, this yields $\eta(u_{i-1} w_i u_i) = \eta(u_{i-1} w'_i u_i)$ which completes the soundness proof.

Completeness. We now prove that for every set $S \subseteq M^2$ which is $(MPol, P_1, P, P_2)$ -saturated for α , we have $\mathcal{N}_{\mathcal{D}MPol(\mathcal{D})}[\alpha] \subseteq S$. We fix S for the proof. First, we use the sets P, P_1 and P_2 to construct a \mathcal{D} -morphism $\eta : A^* \rightarrow N$. Since \mathcal{D} is a prevariety, Theorem 3.15 implies that $LPol(\mathcal{D})$ and $RPol(\mathcal{D})$ are prevarieties as well. Since $P = \mathcal{N}_{\mathcal{D}}[\alpha]$, Lemma F.2 yields a \mathcal{D} -morphism $\eta_3 : A^* \rightarrow N_3$ such that for every $u, u' \in A^*$, if $\eta_3(u) = \eta_3(u')$, then $(\alpha(u), \alpha(u')) \in P$. Similarly, since $P_1 = \mathcal{N}_{\mathcal{D}LPol(\mathcal{D})}[\alpha]$ and $P_2 = \mathcal{N}_{\mathcal{D}RPol(\mathcal{D})}[\alpha]$, we get an $LPol(\mathcal{D})$ -morphism $\gamma_1 : A^* \rightarrow Q_1$ and an $RPol(\mathcal{D})$ -morphism $\gamma_2 : A^* \rightarrow Q_2$ such that for every $u, u' \in A^*$ and $i \in \{1, 2\}$, if $\gamma_i(u) = \gamma_i(u')$, then $(\alpha(u), \alpha(u')) \in P_i$. Moreover, Proposition 3.13 yields two \mathcal{D} -morphisms $\eta_1 : A^* \rightarrow N_1$ and $\eta_2 : A^* \rightarrow N_2$ and $k_1, k_2 \in \mathbb{N}$ such that all languages recognized by γ_1 are unions of $\triangleright_{\eta_1, k_1}$ -classes and all languages recognized by γ_2 are unions of $\triangleleft_{\eta_2, k_2}$ -classes. Since $\mathcal{C} \subseteq \mathcal{D}$ is finite, Proposition 2.3 yields a \mathcal{D} -morphism $\eta : A^* \rightarrow N$ recognizing all languages which are in \mathcal{C} , or which are recognized by η_1, η_2 or η_3 . In particular, note that η is \mathcal{C} -compatible. We have the following fact.

FACT F.9. *Let $u, u' \in A^*$ and $k \in \mathbb{N}$. The following hold:*

- If $\eta(u) = \eta(u')$, then $(\alpha(u), \alpha(u')) \in P$.
- If $k \geq k_1$ and $u \triangleright_{\eta, k} u'$, then $(\alpha(u), \alpha(u')) \in P_1$.
- If $k \geq k_2$ and $u \triangleleft_{\eta, k} u'$, then $(\alpha(u), \alpha(u')) \in P_2$.

PROOF. By definition, if $\eta(u) = \eta(u')$, then $\eta_3(u) = \eta_3(u')$. This yields $(\alpha(u), \alpha(u')) \in P$ by definition of η_3 . We prove the second assertion (the third one is symmetrical and left to the reader). Assume that $k \geq k_1$ and $u \triangleright_{\eta, k} u'$. By definition of η and since $k \geq k_1$, one may verify that $u \triangleright_{\eta_1, k_1} u'$. Hence, since the languages recognized by γ_1 are unions of $\triangleright_{\eta_1, k_1}$ -classes, we have $\gamma_1(u) = \gamma_1(u')$. By definition of γ_1 , this yields $(\alpha(u), \alpha(u')) \in P_1$ as desired. \square

We let $m = \max(k_1, k_2)$ and $p = |M|^2$. Moreover, we define $k = (2p + 2)^{|N|} + p + m$. The argument is based on the following key proposition.

PROPOSITION F.10. *For all $w, w' \in A^*$ such that $w \triangleright_{\eta, k} w'$, we have $(\alpha(w), \alpha(w')) \in S$.*

We first use Proposition F.10 to prove $\mathcal{N}_{\mathcal{D}MPol(\mathcal{D})}[\alpha] \subseteq S$. Let $(s, s') \in \mathcal{N}_{\mathcal{D}MPol(\mathcal{D})}[\alpha]$. By Proposition 3.13 every union of $\triangleright_{\eta, k}$ -classes belongs to $MPol(\mathcal{D})$ since η is a \mathcal{D} -morphism. Hence, we obtain $w, w' \in A^*$ such that $\alpha(w) = s$, $\alpha(w') = s'$ and $w \triangleright_{\eta, k} w'$ (otherwise, $\alpha^{-1}(s)$ would be separated from $\alpha^{-1}(s')$ by a union of $\triangleright_{\eta, k}$ -classes, which contradicts the hypothesis that $(s, s') \in \mathcal{N}_{\mathcal{D}MPol(\mathcal{D})}[\alpha]$). Thus, Proposition F.10 yields $(s, s') \in S$ as desired.

We turn to the proof of Proposition F.10. We start with a preliminary statement. Consider a word $w \in A^*$ and $n \in \mathbb{N}$. We say that w is an n -iteration if it admits a decomposition $w = xw_1 \cdots w_n y$ with $x, y, w_1, \dots, w_n \in A^*$ such that $\eta(w_i) \mathcal{F} \eta(w)$ for every $i \leq n$. We have the following lemma.

LEMMA F.11. *Let $w, w' \in A^*$ such that $w \vDash_{\eta, p+m} w'$ and w is a $2p$ -iteration. Then, $(\alpha(w), \alpha(w'))$ is a (P_1, P, P_2) -block.*

PROOF. We first consider the case when $\eta(w) \mathcal{F} \eta(\varepsilon)$. Since $w \vDash_{\eta, p+m} w'$, we have $\eta(w) = \eta(w')$. In view of Fact F.9, this yields $(\alpha(w), \alpha(w')) \in P$. Moreover, it is clear that $(1_M, 1_M) \in P_1$ and $(1_M, 1_M) \in P_2$ by Lemma 7.4. Finally, since $\eta(w) \mathcal{F} \eta(\varepsilon)$, we have $[\eta(w)]_{\mathcal{G}} \mathcal{F} [\eta(\varepsilon)]_{\mathcal{G}}$ and since η is \mathcal{G} -compatible, we get $[\alpha(w)]_{\mathcal{G}} \mathcal{F} [\alpha(\varepsilon)]_{\mathcal{G}}$ by Fact F.1. In other words $[\alpha(w)]_{\mathcal{G}} \mathcal{F} [1_M]_{\mathcal{G}}$. By definition, it follows that the pair $(\alpha(w), \alpha(w')) = (1_M 1_M \alpha(w) 1_M 1_M, 1_M 1_M \alpha(w') 1_M 1_M)$ is a (P_1, P, P_2) -block as desired.

We now assume that $\eta(w) <_{\mathcal{F}} \eta(\varepsilon)$ (which implies that if $\eta(u) \mathcal{F} \eta(w)$, then u is nonempty). One may verify from the definition of $2p$ -iterations that there exist $2p$ positions $i_1, \dots, i_p \in P_{\triangleright}(\eta, p, w)$ and $j_1, \dots, j_p \in P_{\triangleleft}(\eta, p, w)$ such that we have $0 = i_0 < i_1 < \dots < i_p < j_p < \dots < j_1 < j_0 = |w| + 1$ and, for $1 \leq h \leq p$, if we define $u_h = w(i_{h-1}, i_h + 1)$ and $v_h = w(j_h - 1, j_{h-1})$, then $\eta(w) \mathcal{F} \eta(u_h) \mathcal{F} \eta(v_h)$. Finally, we let $x = w(i_p, j_p)$. By definition, $w = u_1 \cdots u_p x v_p \cdots v_1$.

Consequently, since $w \vDash_{\eta, p+m} w'$, one may verify that w' admits a decomposition $w' = u'_1 \cdots u'_p x' v'_p \cdots v'_1$ such that $\eta(x) = \eta(x')$, $u_h \triangleright_{\eta, m} u'_h$ and $v_h \triangleleft_{\eta, m} v'_h$ for $1 \leq h \leq p$. We define $(s, s') = (\alpha(x), \alpha(x'))$. We have $(s, s') \in P$ by Fact F.9. Also, for $1 \leq h \leq p$, we let $(q_h, q'_h) = (\alpha(u_h), \alpha(u'_h))$ and $(r_h, r'_h) = (\alpha(v_h), \alpha(v'_h))$. Since $m = \max(k_1, k_2)$, Fact F.9 yields $(q_h, q'_h) \in P_1$ and $(r_h, r'_h) \in P_2$. By definition,

$$(\alpha(w), \alpha(w')) = (q_1 \cdots q_p s r_p \cdots r_1, q'_1 \cdots q'_p s' r'_p \cdots r'_1).$$

Recall that $p = |M|^2$. Thus, the pigeon-hole principle yields h, h' such that $0 \leq h < h' \leq p$, $q_1 \cdots q_h = q_1 \cdots q_{h'}$ and $q'_1 \cdots q'_h = q'_1 \cdots q'_{h'}$. We also get ℓ, ℓ' such that $\ell < \ell' \leq p$, $r_{\ell} \cdots r_1 = r_{\ell'} \cdots r_1$ and $r_{\ell} \cdots r_1 = r_{\ell'} \cdots r_1$. We let,

$$\begin{aligned} (s_1, s'_1) &= (q_1 \cdots q_h, q'_1 \cdots q'_h) \\ (e_1, e'_1) &= ((q_{h+1} \cdots q_h)^{\omega}, (q'_{h+1} \cdots q'_{h'})^{\omega}) \\ (s_3, s'_3) &= (q_{h'+1} \cdots q_p s r_p \cdots r_{\ell'+1}, q'_{h'+1} \cdots q'_p s' r'_p \cdots r'_{\ell'+1}) \\ (e_2, e'_2) &= ((r_{\ell'} \cdots r_{\ell+1})^{\omega}, (r'_{\ell'} \cdots r'_{\ell'+1})^{\omega}) \\ (s_2, s'_2) &= (r_{\ell} \cdots r_1, r'_{\ell} \cdots r'_1) \end{aligned}$$

By Lemma 7.4, P, P_1 and P_2 are closed under multiplication. Hence, we get $(s_1, s'_1), (e_1, e'_1) \in P_1$ and $(s_2, s'_2), (e_2, e'_2) \in P_1$. Moreover, since \mathcal{D} is included in $L\text{Pol}(\mathcal{D})$ and $R\text{Pol}(\mathcal{D})$, we have $P_1 \subseteq P$ and $P_2 \subseteq P$. Hence, $(s_3, s'_3) \in P$. It is also immediate by definition that e_1, e'_1, e_2, e'_2 are idempotents of M . Thus, as $(\alpha(w), \alpha(w')) = (s_1 e_1 s_3 e_2 s_2, s'_1 e'_1 s'_3 e'_2 s'_2)$ by construction, it remains to prove that $[e_1]_{\mathcal{G}} \mathcal{F} [e_2]_{\mathcal{G}} \mathcal{F} [\alpha(w)]_{\mathcal{G}}$ to conclude that $(\alpha(w), \alpha(w'))$ is a (P_1, P, P_2) -block. By symmetry, we only show that $[e_1]_{\mathcal{G}} \mathcal{F} [\alpha(w)]_{\mathcal{G}}$. Clearly, we have $[\alpha(w)]_{\mathcal{G}} \leq_{\mathcal{F}} [e_1]_{\mathcal{G}}$. Moreover, by definition of e_1 , we have $[e_1]_{\mathcal{G}} \leq_{\mathcal{F}} [q_h]_{\mathcal{G}} = [\alpha(u_h)]_{\mathcal{G}}$. By definition, we know that $\eta(w) \mathcal{F} \eta(u_h)$ which yields $[\eta(w)]_{\mathcal{G}} \mathcal{F} [\eta(u_h)]_{\mathcal{G}}$. As η is \mathcal{G} -compatible, it follows that $[\alpha(w)]_{\mathcal{G}} \mathcal{F} [\alpha(u_h)]_{\mathcal{G}}$ by

Fact F.1. Altogether, we get $[e_1]_{\mathcal{G}} \leq_{\mathcal{F}} [\alpha(w)]_{\mathcal{G}}$, completing the proof. \square

We now introduce a technique for decomposing an arbitrary word into factors that are n -iterations. Consider a $n \in \mathbb{N}$. Let $w \in A^*$. An n -template for w consists in a sequence of positions $i_0, j_0, \dots, i_{\ell}, j_{\ell} \in P(w)$ within w which satisfy the three following conditions:

- $i_0 = 0$ and $j_{\ell} = |w| + 1$,
- for $0 \leq h \leq \ell$, we have $i_h < j_h$ and the factor $w(i_h, j_h)$ is an n -iteration.
- for $1 \leq h \leq \ell$, we have $j_{h-1} \leq i_h$. Moreover, we have the relations $\alpha(w(i_{h-1}, j_{h-1})) \mathcal{R} \alpha(w(i_{h-1}, i_h))$ and $\alpha(w(i_h, j_h)) \mathcal{L} \alpha(w(j_{h-1}, j_h))$.

We prove for all $w \in A^*$ there exists a particular n -template.

LEMMA F.12. *Let $n \in \mathbb{N}$ and $k' = (n+2)^{|N|}$. For every word $w \in A^*$, there exists an n -template $i_0, j_0, \dots, i_{\ell}, j_{\ell}$ for w such that $i_1, \dots, i_n \in P_{\triangleright}(\eta, k', w)$ and $j_0, \dots, j_{n-1} \in P_{\triangleleft}(\eta, k', w)$.*

PROOF. We proceed in three steps. The first two steps consist in proving that every word $w \in A^*$ admits a decomposition satisfying specific properties. Let $w \in A^*$ and $\ell \geq 1$. A good decomposition of length ℓ for w is a decomposition $w = w_1 a_1 \cdots w_{\ell-1} a_{\ell-1} w_{\ell}$ such that every factor $w_i \in A^*$ is a $(n+2)$ -iteration and $a_1, \dots, a_{\ell-1} \in A$. We first prove that every word $w \in A^*$ admits a good decomposition of length at most $k' = (n+2)^{|N|}$. Given $w \in A^*$, we define the \mathcal{F} -depth of w denoted by $d(w) \in \mathbb{N}$ as number of elements $q \in N$ satisfying $\eta(w) <_{\mathcal{F}} q$. Clearly, $d(w) \leq |N| - 1$ for every $w \in A^*$. Hence, it suffices to prove that every $w \in A^*$ admits a good decomposition of length at most $(n+2)^{d(w)}$. We proceed by induction on $d(w)$. If $d(w) = 0$, then $\eta(w) \mathcal{F} 1_N$ which implies that $w = \varepsilon^{n+2} w$ is an $(n+2)$ -iteration. In particular, w admits a good decomposition whose length is $1 = (n+2)^0$. Assume now that $d(w) \geq 1$. In that case, we have $\alpha(w) <_{\mathcal{F}} 1_M$. Hence, there exists $\ell > 1$, $w_1, \dots, w_{\ell} \in A^*$ and $a_1, \dots, a_{\ell} \in A$ such that $w = w_1 a_1 \cdots w_{\ell} a_{\ell} w_{\ell+1}$ and for every $i \leq n$, we have $\alpha(w) \mathcal{F} \alpha(w_i a_i) <_{\mathcal{F}} \alpha(w_i)$ and $\alpha(w) <_{\mathcal{F}} \alpha(w_{\ell+1})$. We consider two independent cases depending on ℓ . First, assume that $\ell \geq n+2$. In that case, since $\alpha(w_i a_i) \mathcal{F} \alpha(w)$ for every $i \leq \ell$, it is immediate that w is an $(n+2)$ -iteration. In particular, w admits a good decomposition of length $1 \leq (n+2)^{d(w)}$ and we are finished. Conversely, assume that $\ell < n+2$. Since $\alpha(w) <_{\mathcal{F}} \alpha(w_i)$ for every $i \leq \ell$, we have $d(w_i) \leq d(w) - 1$ by definition. Hence, induction yields that each word w_i admits a good decomposition of length at most $(p+2)^{d(w)-1}$. Consequently, since $\ell+1 \leq n+2$, the word $w = w_1 a_1 \cdots w_{\ell} a_{\ell} w_{\ell+1}$ admits a good decomposition of length at most $(n+2) \times (n+2)^{d(w)-1} = (n+2)^{d(w)}$. This concludes the first step.

We turn to the second step. Let $w \in A^*$ and a good decomposition $w = w_1 a_1 \cdots w_{\ell-1} a_{\ell-1} w_{\ell}$ of w . We say that this good decomposition is irreducible to indicate that for every i such that $1 \leq i \leq \ell-1$, we have $\alpha(w_i a_i w_{i+1}) <_{\mathcal{R}} \alpha(w_i)$ and $\alpha(w_i a_i w_{i+1}) <_{\mathcal{L}} \alpha(w_{i+1})$. In view of the first step, it is simple to verify that every word $w \in A^*$ admits a irreducible good decomposition of length at most $k' = (n+2)^{|N|}$. Indeed, we already know that it admits an arbitrary good decomposition of length at most k . If it is not irreducible, it is possible to make it smaller by merging the consecutive factors

which do not satisfy the condition in the definition (clearly, the merging remains an $n + 2$ -iteration. One may then iterate the process to get a irreducible good decomposition of w whose length is at most k' .

We complete the proof. Let $w \in A^*$. The second step yields a irreducible good decomposition $w = w_1 a_1 \cdots w_\ell a_\ell w_{\ell+1}$ where $\ell + 1 \leq k'$. Let us define our n -template $i_0, j_0, \dots, i_\ell, j_\ell$. We start with i_0, \dots, i_ℓ . We let $i_0 = 0$. Then, for $1 \leq h \leq \ell$, we define i_ℓ as a particular position of the factor $a_\ell w_{\ell+1}$ of w . By definition of irreducible good decompositions, we know that $\alpha(w_h a_h w_{h+1}) <_{\mathcal{R}} \alpha(w_h)$. We let i_h as the least position within $a_h w_{h+1}$ where this change in \mathcal{R} -class happens. One may verify from the definition that for $1 \leq h \leq \ell$, we have $\alpha(w(i_{h-1}, i_h + 1)) <_{\mathcal{R}} \alpha(w(i_{h-1}, i_h))$. Hence, we have $i_1, \dots, i_\ell \in P_{\triangleright}(\eta, k', w)$. Let us now define j_0, \dots, j_ℓ . First, we let $j_\ell = |w| + 1$. Then, for $0 \leq h \leq \ell - 1$, we define j_h as a position of the factor $w_h u_h$ of w . By definition of irreducible good decompositions, we know that $\alpha(w_h a_h w_{h+1}) <_{\mathcal{L}} \alpha(w_{h+1})$. We let j_h as the greatest position in $w_h a_h$ at which this change in \mathcal{L} -class happens. One may verify from the definition that for $0 \leq h \leq \ell - 1$, $\alpha(w(j_h - 1, j_{h+1})) <_{\mathcal{L}} \alpha(w(j_h, j_{h+1}))$. Hence, $j_0, \dots, j_{\ell-1} \in P_{\triangleleft}(\eta, k', w)$. One may now verify from the definitions that $i_0, j_0, \dots, i_\ell, j_\ell$ is an n -template for w . \square

We are now ready to prove Proposition F.10. Recall that we have $k = (2p + 2)^{|N|} + p + m$ and consider $w, w' \in A^*$ such that $w \preceq_{\eta, k} w'$. We need to prove that $(\alpha(w), \alpha(w')) \in S$. Let $k' = (2p + 2)^{|N|}$: we have $k = k' + p + m$. Lemma F.12 yields a $2p$ -template $i_0, j_0, \dots, i_\ell, j_\ell$ for w such that $i_1, \dots, i_\ell \in P_{\triangleright}(\eta, k', w)$ and $j_0, \dots, j_{\ell-1} \in P_{\triangleleft}(\eta, k', w)$. By definition, we have $0 = i_0 < j_0 \leq i_1 < j_1 \leq \dots \leq i_\ell < j_\ell = |w| + 1$. Since $w \preceq_{\eta, k} w'$ and $k = k' + m + p$, one may verify that there exist positions $i'_0, j'_0, \dots, i'_\ell, j'_\ell \in P(w)$ such that:

- $0 = i'_0 < j'_0 \leq i'_1 < j'_1 \leq \dots \leq i'_n < j'_n = |w'| + 1$.
- j_h and j'_h have the same label " b_h " for $0 \leq h \leq n - 1$.
- i_h and i'_h have the same label " c_h " for $1 \leq h \leq n$.
- $w(i_h, j_h) \preceq_{\eta, p+m} w'(i'_h, j'_h)$ for $0 \leq h \leq n$.
- $j_{h-1} < i_h \Leftrightarrow j_{h-1} < i'_h$ for $1 \leq i \leq h$ and in that case, we have $\eta(w(j_{h-1}, i_h)) = \eta(w'(j'_{h-1}, i'_h))$.

For h, g such that $0 \leq h \leq g \leq \ell$, we write $w_{h,g} = w(i_h, j_g)$ and $w'_{h,g} = w'(i'_h, j'_g)$. In particular, we write w_h and w'_h for $w_{h,h}$ and $w'_{h,h}$. We prove that $(\alpha(w_{h,g}), \alpha(w'_{h,g})) \in S$ for all h, g such that $0 \leq h \leq g \leq \ell$. By definition, the case $h = 0$ and $g = \ell$ yields $(\alpha(w), \alpha(w')) \in S$ as desired. We fix h, g for the proof and proceed by induction on the number $g - h \in \mathbb{N}$. There are two cases.

First, assume that there exists n such that $h < n \leq g$ and $j_{n-1} = i_n$. We handle this case by induction. Since $j_{n-1} = i_n$, we also have $j'_{n-1} = i'_n$ and i_n, i'_n both have label b_n . Hence, it is immediate by definition that $w_{h,g} = w_{h,n-1} b_n w_{n,g}$ and $w'_{h,g} = w'_{h,n-1} b_n w'_{n,g}$. It is immediate from induction that $(\alpha(w_{h,n-1}), \alpha(w'_{h,n-1})) \in S$ and $(\alpha(w_{n,g}), \alpha(w'_{n,g})) \in S$. We have $(\alpha(b_n), \alpha(b_n)) \in S$ since S is saturated. By closure under multiplication, we get $(\alpha(w_{h,g}), \alpha(w'_{h,g})) \in S$.

We now assume that for every n such that $h < n \leq g$, we have $j_{n-1} < i_n$. By definition, we get $j'_{n-1} < i'_n$ as well. For all n such that $h < n \leq g$, we may define $v_n = w(j_{n-1}, i_n)$ and $v'_n = w'(j'_{n-1}, i'_n)$.

By hypothesis, we have $\eta(v_n) = \eta(v'_n)$ and Fact F.9 implies that $(\alpha(v_n), \alpha(v'_n)) \in P$. We also define $u_h = w_h b_{h+1}$ and $u'_h = w'_h b_{h+1}$, $u_n = c_n w_n b_{n+1}$ and $u'_n = c_n w'_n b_{n+1}$ for $h + 1 \leq n \leq g - 1$ and $u_g = c_g w_g$ and $u'_g = c_g w'_g$. By hypothesis and since \preceq_{p+m} is a congruence, we have $u_n \preceq_{p+m} u'_n$ for $h \leq n \leq g$. Moreover, one may verify from the properties of $2p$ -templates that u_h, \dots, u_g are $2p$ -iterations. Hence, it follows from Lemma F.11 that $(\alpha(u_n), \alpha(u'_n))$ is a (P_1, P, P_2) -block for $h \leq n \leq g$. Finally, by definition of $2p$ -templates, we have $\eta(u_{n-1} v_n) \mathcal{F} \eta(u_{n-1})$ and $\eta(v_n u_n) \mathcal{F} \eta(u_n)$ for $h + 1 \leq n \leq g$. Since η is \mathcal{C} -compatible, we get $[\alpha(u_{n-1} v_n)]_{\mathcal{C}} \mathcal{F} [\alpha(u_{n-1})]_{\mathcal{C}}$ and $[\alpha(v_n u_n)]_{\mathcal{C}} \mathcal{F} [\alpha(u_n)]_{\mathcal{C}}$. It now follows from (5) in the definition of $(MPol, P_1, P, P_2)$ -saturated sets that,

$$(\alpha(u_h v_{h+1} u_{h+1} \cdots v_g u_g), \alpha(u'_h v'_{h+1} u'_{h+1} \cdots v'_g u'_g)) \in S.$$

This exactly says that $(\alpha(w_{h,g}), \alpha(w'_{h,g})) \in S$ by definition which completes the proof.