Computing Dirichlet fundamental domains

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Models of hyperbolic geometry

In $\mathbb{R}^{n+1}$ we consider

- **hyperboloid model**
  
  $\mathbb{H}^n = \{(x_1, \ldots, x_n, y) : x_1^2 + \ldots + x_n^2 - y^2 = -1 \text{ and } y > 0\}$

- **disk model**
  
  $\mathbb{D}^n = \{(x_1, \ldots, x_n, 0) : x_1^2 + \ldots + x_n^2 < 1\}$

- **Klein model**
  
  $\mathbb{K}^n = \{(x_1, \ldots, x_n, 1) : x_1^2 + \ldots + x_n^2 < 1\}$
Geodesics

The quadratic form $x_1^2 + \ldots + x_n^2 - y^2$ induces on the hyperboloid $\mathbb{H}^n$ a Riemannian metric whose geodesics are the intersection of linear subspaces in $\mathbb{R}^{n+1}$ with $\mathbb{H}^n$. 

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Geodesics project as circles orthogonal to the boundary on $\mathbb{D}^n$. 
Geodesics (Klein projection)

picture by Martin von Gagern
Geodesics (disk projection)

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Isometries

The group $\text{SO}^+(n, 1)$ of transformations of $\mathbb{R}^{n+1}$ preserving the quadratic form $x_1^2 + \ldots + x_n^2 - y^2$ is the group of isometries of $\mathbb{H}^n (\approx \mathbb{D}^n \approx \mathbb{K}^n)$. 
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We are interested in discrete subgroups of $\text{SO}^+(n, 1)$ that are relevant in hyperbolic geometry: if $M$ is a hyperbolic $n$-manifold then $M$ is isometric to a quotient $\mathbb{H}^n/\Gamma$,
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2. number theory: automorphic forms of arithmetic groups (e.g., congruence groups ($n = 2$) and Bianchi groups ($n = 3$)),

3. combinatorial group theory: Coxeter groups.
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Fix $\mathcal{O} \subset \mathbb{Q}[\sqrt{-d}]$ an order (e.g., $\mathcal{O} = \mathbb{Z}[i]$). Then $\text{PSL}_2(\mathcal{O}) \subset \text{PSL}_2(\mathbb{C})$ is a discrete subgroup called a Bianchi group.
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Geometry $\leftrightarrow$ number theory

1. covolume of $\text{PSL}_2(\mathcal{O})$ $\leftrightarrow$ Dedekind zeta function of $K$,
2. cusps of $\text{PSL}_2(\mathcal{O})$ $\leftrightarrow$ ideal classes of $\mathcal{O}$. 

We will describe an algorithm that computes a presentation ($i.e.$. generators and relations). Example (Swan) $\text{SL}_2(\mathbb{Z}[i]) = \langle (1 \ 1 \ 0 \ 1), (1 \ i \ 0 \ 1), (-i \ 0 \ 0 \ i), (0 \ -1 \ 1 \ 0) \rangle$ 

Study of Bianchi group is not new: Bianchi (1892), Swan (1971), Riley (1983), ..., Yasaki (Gunnell's algorithm, 2010), Aranes (SageMath), Page (Magma), ...
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Poincaré’s polyhedron theorem (compact case)

Let $P$ be a polyhedral cone in $\mathbb{R}^{n+1}$ which intersects the hyperboloid $\mathbb{H}^n$ compactly.

**Theorem (Poincaré)**

Let $P$ be a Poincaré polyhedron with facets $\{F_s\}_{s \in S}$. Then the group $\Gamma_P$ generated by $S$ is a discrete subgroup of $\text{SO}^+(n, 1)$. Moreover we have a presentation $\Gamma_P = \langle S | R \rangle$ where $R$ are the relations $(s_k s_{k-1} \ldots s_1)^m$ obtained along the ridges.
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A labeling of the facets $F_s$ of $P$ with elements $s \in \text{SO}^+(n,1)$ is called a face pairing if $S$ is symmetric, $s(F_{s^{-1}}) = F_s$ and $s$ maps the inward normal of $F_{s^{-1}}$ to the outward normal of $F_s$.
It is a Poincaré polyhedron if furthermore following identifications along the ridges, we see total dihedral angles of the form $2\pi/m$.

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The \((6,6,6)\)-triangle group

\[
\langle r, s, t | r^2, s^2, t^2, (rs)^6, (st)^6, (tr)^6 \rangle
\]

picture by Anton Sherwood
A genus 2 surface group

\[ \langle a, b, c, d \mid b^{-1}aba^{-1}d^{-1}cdc^{-1} \rangle \simeq \langle a, b, c, d \mid [a, b][c, d] \rangle \]
Dirichlet domains

**Definition**

Let $\Gamma \subseteq \text{SO}^+(n, 1)$ be discrete and $p_0 \in \mathbb{H}^n$. The *Dirichlet domain* of $\Gamma$ at $p_0$ is

$$\text{Dir}(\Gamma, p_0) := \{ q \in \mathbb{H}^n : \forall \gamma \in \Gamma, d(p_0, q) \leq d(p_0, \gamma q) \}.$$
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Theorem

The Dirichlet domain is a (generalized) Poincaré polyhedron.

Warning: when $n \geq 3$, even if $\Gamma$ is finitely generated, $P_{\Gamma, p_0}$ can have infinitely many sides.
Dirichlet domain is a Poincaré polyhedron

To each $\gamma \in \text{SO}^+(n, 1)$ we associate a half-space

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$H_\gamma$ can be seen as a linear half-space in $\mathbb{R}^{n+1}$ and the associated hyperplanes $F_\gamma$ satisfy the conditions of the face pairing $\gamma(F_{\gamma^{-1}}) = F_\gamma$. 
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By definition, the Dirichlet domain is the (infinite) intersection

$$P_{\Gamma, p_0} = \bigcap_{\gamma \in \Gamma \setminus \{1\}} H_\gamma.$$
The algorithm

We are given generators $S_0$ of a discrete subgroup $\Gamma$ of $\text{SO}^+(n, 1)$ and a point $p_0$ and the aim is to construct the associated Dirichlet domain $\text{Dir}(\Gamma, p_0)$.

$\text{Dirichlet}(S_0, p_0)$:
Input: $S_0 \subset \text{SO}^+(n, 1), p_0 \in \mathbb{H}^n$
Output: $\text{Dir}(S_0, p_0)$
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**Dirichlet**\((S_0, p_0)\):

Input: \( S_0 \subset \text{SO}^+(n, 1), \ p_0 \in \mathbb{H}^n \)

Output: \( \text{Dir}(S_0, p_0) \)

We will need the following procedure

```plaintext
procedure reduce(x, S):
Input: \( x \in \mathbb{H}^n, S \subset \text{SO}^+(n, 1) \) finite
Output: an element \( \gamma \in \langle S \rangle \) so that \( \gamma x \in P_S \) where \( P_S := \bigcap_{s \in S} H_s \) and

\[
H_s = \{ q \in \mathbb{H}^n : d(p_0, q) \leq d(p_0, s^{-1} q) \}
\]
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\textbf{Dirichlet}(S_0, p_0):

\textbf{Input:} \( S_0 \subset \text{SO}^+(n, 1), \ p_0 \in \mathbb{H}^n \)

\textbf{Output:} \( \text{Dir}(S_0, p_0) \)

- \textbf{initialization:} set \( P_0 = P_{S_0} = \bigcap_{s \in S_0 \cup S_0^{-1}} H_s \)
- \textbf{inductive step:} we construct \( P_{n+1} \) from \( P_n \) by adding the following half-spaces and removing the redundant ones
  - for each equivalence class of ridge, add partial monodromies along the ridge and define \( S'_{n+1} \),
  - for each \( s \in S_0 \), add \( H_{s^{-1}s'} \) where \( s' = \text{reduce}(s p_0, S'_{n+1}) \) to define \( S_{n+1} \),
- \textbf{Stop} when \( P_n \) is a Poincaré polyhedron.
It works

**Theorem (Voigt, Page)**

If $\Gamma = \langle S \rangle \subset \text{SO}^+(n, 1)$ is geometrically finite, then the procedure described above terminates and constructs the Dirichlet domain.
An implementation detail

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- a non-redundant system of hyperplanes,
- the ridges as pairs of supporting hyperplanes.
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The most efficient, is (probably) through linear programming (for $n \geq 4$)?
Concrete implementation: SageMath, e-antic and Normaliz

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...demo...
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