Ergodic properties of the earthquake flow and the counting of simple closed geodesics

Modiff Seminar

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Goals

- Talk about earthquake flow,
- How to count simple closed geodesic on hyperbolic surfaces,
- Bound the rate of mixing of the earthquake flow.
Teichmuller and Moduli spaces

Let \( S \) be a surface of genus \( g \) with \( n \) punctures \((3g - 3 + n > 0)\).

- The Teichmuller space \( \mathcal{T}_{g,n} \), is the set of hyperbolic metrics marked by \( S \).
- Mapping class group \( \text{MCG}(S) = \text{Diff}^+(S)/\text{Diff}^0(S) \) acting on \( \mathcal{T}_{g,n} \).
- Moduli space \( \mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\text{MCG}(S) \).

\( \mathcal{T}_{g,n} \) is a real manifold of dimension \( 6g - 6 + 2n \).
\( \mathcal{M}_{g,n} \) is a orbifold of real dimension \( 6g - 6 + 2n \).
\( \Gamma := \) homotopic class of simple closed curves.

- For each \( X \in T_{g,n} \), there is only one geodesic in each \( \gamma \in \Gamma \),
- \( i(\alpha, \beta) \) the number of intersection,
- \( l_\alpha(X) \) is the hyperbolic length,
- \( l_{\text{sys}}(X) = \inf_{\gamma \in \Gamma} l_\gamma(X) \).
\( \Gamma := \text{homotopic class of simple closed curves.} \)

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**Lemma (Collar lemma)**

For \( \alpha, \beta \in \Gamma \) and \( X \in \mathcal{T}_{g,n} \), we always have

\[
R(l_\beta(X)) i(\alpha, \beta) \leq l_\alpha(X)
\]

with \( R(x) = 2\ln\left(\frac{e^{x/2}+1}{e^{x/2}-1}\right)^a \).

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For any \((L_1, L_2, L_3)\) there is a unique hyperbolic geometry on a surface of genus 0 with 3 boundaries components of length \(L_1, L_2, L_3\).

**Figure**: A hyperbolic pair of pants

It is called a pair of pants.
For any surface $S$ of genus $g$ with $n$ punctures, one can find $3g - 3 + n$ curves $\gamma_1, \gamma_2, \cdots, \gamma_{3g-3+n}$ such that $S - \{\gamma_1, \gamma_2, \cdots, \gamma_{3g-3+n}\}$ is a disjoint union of pair of pant.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{pair_of_pants_decomposition.png}
\caption{A pair of pant decomposition}
\end{figure}

**Theorem**

To each pair of pants decomposition the map

$$FN_\gamma : T_{g,n} \to \mathbb{R}^{6g-6+2n}, \; X \mapsto (l_{\gamma_i}(X), \tau_{\gamma_i}(X))_{i=1,\cdots,3g-3+n}$$

is a homeomorphism into its image.
A lamination is closed set foliated by simple geodesics. A measured lamination is a lamination with a transverse measure. We can see it at the limit of weighted multicurves.

We call $\mathcal{ML}$ the set of measured lamination, $P^1Tg,n$ the unit sub-bundle over the Teichmuller space and $P^1M_{g,n} = P^1Tg,n/MCG$. 
Earthquake

The earthquake flow $\hat{E}^t(\cdot, \cdot) : P^1 \mathcal{T}_{g,n} \to P^1 \mathcal{T}_{g,n}$:

- For $(X, \gamma) \in \mathcal{T}_{g,n} \times \Gamma$, $\hat{E}^t(X, \gamma)$ is a twist around $\gamma$ of length $t$.  

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- For $(X, \gamma) \in T_{g,n} \times \Gamma$, $\hat{E}^t(X, \gamma)$ is a twist around $\gamma$ of length $t$.
- For $X \in T_g$, $\sum_{i=1}^n \gamma_i \in \mathcal{ML}(\mathbb{Z})$ and $(a_i) \in \mathbb{R}^n$, then

$$\hat{E}^t \left( X, \sum a_i \gamma_i \right) = \hat{E}^{a_1} t(\cdot, \gamma_1) \circ \hat{E}^{a_2} t(\cdot, \gamma_2) \circ \cdots \circ \hat{E}^{a_n} t(X, \gamma_n).$$

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Earthquake

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$$\hat{E}^t \left( X, \sum a_i \gamma_i \right) = \hat{E}^{a_1 t}(\cdot, \gamma_1) \circ \hat{E}^{a_2 t}(\cdot, \gamma_2) \circ \cdots \circ \hat{E}^{a_n t}(X, \gamma_n).$$

- For $(X, \lambda) \in T_{g,n} \times \mathcal{ML}$, the flow is obtained by continuous extension: since the space of weighted multicurves is dense in the space of measured laminations, the earthquake along $\lambda \in \mathcal{ML}$ is the limit of earthquake along weighted multicurves converging to $\lambda^1$.

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Properties of the earthquake

The earthquake has nice properties.

- It is commuting with $MCG$, so we have a flow $E^t(\cdot, \cdot)$ on $P^1\mathcal{M}_{g,n}$
- Two points on $\mathcal{T}_{g,n}$ are link by a unique earthquake .
- It is ergodic with respect to some measure $\nu^2$.

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- Two points on \( T_{g,n} \) are linked by a unique earthquake.
- It is ergodic with respect to some measure \( \nu^2 \).

But remain difficult to grasp

- We don’t know often it visits the cusp of \( \mathcal{M}_{g,n} \).
- We don’t know its rate of mixing.
- The normalizer of the earthquake flow inside the group of orbifold automorphisms of \( P^1\mathcal{M}_g \) is the flow itself\(^3\).


\(^3\)Francisco Arana-Herrera and Alex Wright. The asymmetry of Thurston’s earthquake flow. 2022.
Definition (Thurston measure on $\mathcal{ML}$)

$$\mu'_{Th}(A) = \lim_{M \to \infty} \frac{\#\{\delta \in \Gamma, 1/M\delta \in A\}}{M^{6g-6+2n}}$$
Measure

Definition (Thurston measure on $\mathcal{M}_g$)

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Definition (Weil-Petterson measure on $\mathcal{T}_{g,n}$)

Given a pants decomposition $\gamma_1, \ldots, \gamma_{3g-3}$, we consider:

$$d\omega_{WP} = \sum_{i=1}^{3g-3} dl_{\gamma_i} \wedge d\tau_{\gamma_i}.$$ 

It does not depend on the choice of pants decomposition\(^a\), and it induces a volume form $\mu_{WP}$ called the Weil-Petersson measure.

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We have a measure on $P^1\mathcal{M}\mathcal{L}$ by setting for any $X \in \mathcal{T}_{g,n}$,
$$\mu_{Th}(X)(A) = \mu'_{Th}\{t\lambda, \lambda \in A, t \in [0, 1]\}$$

The measure $\mu_{Th} \times \mu_{WP}$ on $P^1\mathcal{T}_{g,n}$ is mapping class group invariant and goes down to measure on $P^1\mathcal{M}_{g,n}$. 
$\alpha$ and $\beta$ have same topological type $\equiv$ there is $\phi \in MCG$ such that $\phi(\alpha) = \beta$. 
Mirzakhani result

Theorem

Let $X \in T_{g,n}$, if $c(L, X, \gamma) = \#\{\delta \in \text{MCG} \cdot \gamma, l_\delta(X) \leq L\}$ alors

$$c(L, X, \gamma) \sim \frac{B(X)}{b_{g,n}} r(\gamma)L^{6g-6+2n}$$

and $r(\gamma)$ can be explicitly compute.\(^a\)

• \( c(X, \lambda) = \sum_{\gamma \text{ type}} c(X, \gamma, L) \) count all the simple closed curves.

• We can look at the frequency of a certain type:

\[
p(\gamma) = \frac{r(\gamma)}{\sum_{\delta \text{ type}} r(\delta)}
\]

Example

For a genus 2 surfaces \( \frac{r(\text{cutting})}{r(\text{non cutting}) + r(\text{cutting})} = \frac{48}{49} \)
Unfolding

\[ c(X, \gamma, L) = \#\{\alpha \in MCG \cdot \gamma, l_\alpha(X) \leq L\} \]

\[ c(X, \gamma, L) = \sum_{\alpha \in MCG \cdot \gamma} 1_{[0,1]}\left(\frac{1}{L} l_\alpha(X)\right) \]

\[ = \sum_{\phi \in MCG/\text{Stab}(\gamma)} 1_{[0,1]}\left(\frac{1}{L} l_{\phi \cdot \gamma}(X)\right) \]

\[ = \sum_{\phi \in MCG/\text{Stab}(\gamma)} 1_{[0,1]}\left(\frac{1}{L} l_{\gamma}(\phi \cdot X)\right) \]

With \( \text{Stab}(\gamma) = \{\phi \in MCG, \phi \cdot \gamma = \gamma\} \)
Definition

\[ \tilde{\mu}_{\gamma,L} := 1_{[0,1]}(\frac{1}{L} l_{\gamma}(X))\tilde{\mu}_{WP}(X) \]

is a measure on \( \mathcal{T}_{g,n}/\text{Stab}(\gamma) \) with total mass

\[ m_{\gamma,L} := \tilde{\mu}_{\gamma,L}(\mathcal{T}_{g,n}/\text{Stab}(\gamma)) \]
Let $\eta_\epsilon$ be a positive function with support in $B(X, \epsilon)$ and of measure 1 for $\hat{\mu}_{WP}$.

$$c(X, \gamma, L) \sim \int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y)c(Y, \gamma, L)d\hat{\mu}_{WP}$$

$$= \int_{T_{g,n}/\text{Stab}(\gamma)} \eta_\epsilon(Y)1_{[0,1]}\left(\frac{1}{L}l_\gamma(\phi \cdot Y)\right) \frac{d\tilde{\mu}_{WP}}{m_{\gamma,L}} m_{\gamma,L}$$

$$= \int_{T_{g,n}/\text{Stab}(\gamma)} \eta_\epsilon(Y) \frac{d\tilde{\mu}_{\gamma,L}}{m_{\gamma,L}} m_{\gamma,L}$$
Weak convergence of measure

Let’s consider the family of measure \( \tilde{\nu}_{\gamma,L} = \delta_{\gamma/l_{\gamma}(Y)} \times \frac{\tilde{\mu}_{\gamma,L}(Y)}{m_{\gamma,L}} \)

All of this measured are

- Earthquake invariant
- With total mass 1

We also have

- There is no escape of mass
- Every open set is charged eventually

By ergodicity of the earthquake flow we have,

\[
\frac{\tilde{\nu}_{\gamma,L}}{m_{\gamma,L}} \to \frac{\tilde{\nu}}{b_{g,n}}
\]

And

\[
\frac{\tilde{\mu}_{\gamma,L}}{m_{\gamma,L}} \to \frac{\tilde{\mu}B(Y)}{b_{g,n}}
\]
Background
Counting simple closed curves
Rate of mixing

Integration à la Mirzakhani

Theorem

Let \( g : \mathbb{R}_+ \to \mathbb{R}_+ \), \( g_\gamma(X) = \sum_{\alpha \in \text{MCG}_\gamma} g(l_\alpha(X)) \) then

\[
\int_{\mathcal{M}_{g,n}} g_\gamma(X) d\hat{\mu}_{WP}(X) = \int_{X} g(|x|) V_{g,n}^\gamma(x) dx
\]

With \( V_{g,n} \) an explicit polynomial of degree \( L^{6g-7+2n} \)

\footnote{Maryam Mirzakhani. “Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces”. In: Inventiones mathematicae 167.1 (2007), pp. 179–222.}

We can conclude that \( \frac{m_\gamma L}{L^{6g-6+2n}} \to r(\gamma) \).
End of the proof

\[
c(X, \gamma, L) \sim \int_{T_{g,n}/\text{Stab}(\gamma)} \eta_\epsilon(Y) \frac{d\tilde{\mu}_{\gamma,L}}{m_{\gamma,L}} \frac{m_{\gamma,L}}{L^{6g-6+2n}} L^{6g-6+2n}
\]

\[
\sim \int_{T_{g,n}/\text{Stab}(\gamma)} \eta_\epsilon(Y) \frac{B(Y)}{b_{g,n}} d\tilde{\mu}_{WP} r(\gamma) L^{6g-6+2n}
\]

\[
\sim \frac{B(X)}{b_{g,n}} r(\gamma) L^{6g-6+2n}
\]
The Earthquake flow is no more than polynomially mixing for Lipschitz functions.

**Theorem**

Suppose that there are constants $d$ and $C$ so that

$$\left| \int f g \circ E^t d\nu - \int f d\nu \int g d\nu \right| < C \frac{1}{t^d} \| f \|_{\text{Lip}} \| g \|_{\text{Lip}}$$

for all Lipschitz functions $f, g : P^1 \mathcal{M}_g \to \mathbb{R}$ and all times $t \geq 1$. Then $d \leq 6g - 5$. 
The asymmetric Thurston distance on $\mathcal{T}_g$ between $Y$ and $Y'$ is

$$d_{Th}^{asy} (Y, Y') := \log \left( \sup_{\lambda \in \mathcal{ML}} \frac{l_\lambda(Y)}{l_\lambda(Y')} \right)$$

By symmetrizing $d_{Th}^{asy}$, we get the Thurston distance:

$$d_{Th}(Y, Y') := \max(d_{Th}^{asy}(Y, Y'), d_{Th}^{asy}(Y', Y))$$
Distance on $\mathcal{ML}$

Fix a pants decomposition $\gamma_1, \ldots, \gamma_{3g-3}$ $i(\cdot, \cdot)$ the intersection number and $t(\cdot)$ the twisting number

**Theorem (Dehn-Thurston coordinates)**

The map

$$
\mathcal{ML} \to (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3} \cup (\{0\} \times \mathbb{R}_{\geq 0})^{3g-3}
$$

$$
\lambda \mapsto (i(\gamma_1, \lambda), t_{\gamma_1}(\lambda), \ldots, i(\gamma_{3g-3}, \lambda), t_{\gamma_{3g-3}}(\lambda))
$$

is a bijection.

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Pull back the $L^\infty$ norm from $\mathbb{R}^{6g-6}$ to $\mathcal{ML}$ to get a distance $d_{lam}$. 
Distance on $\mathcal{T}_g \times \mathcal{ML}$:

$$d_{Th} \times d_{lam}((X, \lambda), (X', \lambda')) = \max(d_{Th}(X, X'), d_{lam}(\lambda, \lambda')).$$

Distance on $P^1\mathcal{M}_g$:

$$d_{P^1\mathcal{M}_g}([Y, \lambda], [Y', \lambda']) = \inf_{h \in \text{Mod}(S)} d_{Th} \times d_{lam}((Y, \lambda), (h.Y', h.\lambda')).$$

We will consider Lipschitz function for this distance.
Strategy

- Find to a sequence of test function with support on $l_{sys}^{-1}(]0, \epsilon[)$,
- Control the amount of time it need to reach region with large systole,
- Control their integral and Lipschitz norms,
- Plug it into the previous inequality.

Same strategy as in\(^4\).

Given $\rho$ a positive real number, $\lambda \in \mathcal{ML}$, $X \in \mathcal{T}_g$ and $\gamma \in \Gamma$, let

$$J_{\gamma,X}^\lambda(\rho) := \{ t \in \mathbb{R}, l_{\gamma}(\hat{E}_t(X, \lambda)) \leq \rho \}$$

and

$$\epsilon_{\gamma,X}^\lambda = \min_{t \in \mathbb{R}} l_{\gamma}(\hat{E}_t(X, \lambda)).$$

There are constants $\rho$ and $C_{\text{lem}}$, depending only on $S$, such that for any $(X, \lambda) \in P^1 \mathcal{T}_g, \gamma \in \Gamma$ and all $t \in J_{\gamma,X}^\lambda(\rho)$,

$$i(\lambda, \gamma)|t - t_\gamma| - C_{\text{lem}}\epsilon_{\gamma,X}^\lambda \leq l_{\gamma}(\hat{E}_t(X, \lambda)) \leq i(\lambda, \gamma)|t - t_\gamma| + \epsilon_{\gamma,X}^\lambda$$

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Figure: The length of a curve is controlled alongside the earthquake flow during a small time near its minimum.
Time without intersection

- Fix $\epsilon_0(2 + C_{lem}) < \frac{\rho}{2}$
- We consider a parameter $\mu$

$$\Omega_{\epsilon_0,\mu} = \{[X, \lambda] \in P^1\mathcal{M}_g, l_{sys}(X) \in \left[\frac{\epsilon_0}{2}; \epsilon_0\right]$$

and $i(\lambda, \gamma) \in [0, \mu]$ when $l_\gamma(X) \in \left[\frac{\epsilon_0}{2}; \epsilon_0\right]$} 

- $D = \{[X, \lambda] \in \mathcal{P}\mathcal{M}^1, l_{sys}(X) > \rho\}$ (non-empty with positive measure).
Time without intersection 2

- Take \([X, \lambda] \in \Omega_{\epsilon_0, \mu}\) and \(\gamma \in \Gamma\) such that \(\tilde{\epsilon} := l_\gamma(X) \in \left[\frac{\epsilon_0}{2}; \epsilon_0\right]\),
- \(\epsilon^{\lambda,X}_\gamma = \epsilon_\gamma\) reach at time \(t_\gamma\),
- The lemma at \(t = 0\) yield

\[
\tilde{\epsilon} - \epsilon_\gamma \leq i(\lambda, \gamma)|t_\gamma| \leq \tilde{\epsilon} + C_{\text{lem}}\epsilon_\gamma.
\]

- Because \(0 < \epsilon_\gamma \leq \tilde{\epsilon}\), we have \(i(\lambda, \gamma)|t_\gamma| \in [0; \tilde{\epsilon}(1 + C_{\text{lem}})]\).
- Then for any \(t \in J^{\lambda,X}_\gamma(\rho)\), one has

\[
l_\gamma(\hat{E}_t(X, \lambda)) \leq \tilde{\epsilon}(2 + C_{\text{lem}}) + i(\lambda, \gamma)|t| \leq \epsilon_0(2 + C_{\text{lem}}) + \mu |t| \leq \frac{\rho}{2} + \mu |t|.
\]

- Hence, for all \(|t| \leq \frac{\rho}{2\mu} := t_{\text{lim}}\), we have that \(l_\gamma(\hat{E}_t(X, \lambda)) \leq \rho\).
  In particular, \(E_t([X, \lambda]) \notin D\) for all \(|t| \leq t_{\text{lim}}\).
Take any Lipschitz function $g$ with support on $D$ and positive integral. We define $\hat{f}_{\epsilon_0, \mu}$ on $P^1 T_g$ as follows: $\hat{f}_{\epsilon_0, \mu} = h_{\epsilon_0} j_\mu$ with

$$h_{\epsilon_0}(X) = \begin{cases} 
0 & \text{if } l_{\mathrm{sys}}(X) \notin [\epsilon_0/2; \epsilon_0] \\
1 & \text{if } l_{\mathrm{sys}}(X) \in [\frac{4\epsilon_0}{6}; \frac{5\epsilon_0}{6}] \\
\frac{6}{\epsilon_0} (l_{\mathrm{sys}}(X) - \frac{\epsilon_0}{2}) & \text{if } l_{\mathrm{sys}} \in [\frac{\epsilon_0}{2}; \frac{4\epsilon_0}{6}] \\
-\frac{6}{\epsilon_0} (l_{\mathrm{sys}}(X) - \epsilon_0) & \text{if } l_{\mathrm{sys}} \in [\frac{5\epsilon_0}{6}; \epsilon_0]
\end{cases}$$

and

$$j_\mu(X, \lambda) = \prod_{\gamma \in \Gamma, \frac{\epsilon_0}{2} < l_\gamma(X) \leq \epsilon_0} j_{\mu, \gamma}(\lambda)$$

where

$$j_{\mu, \gamma}(\lambda) = \begin{cases} 
0 & \text{if } i(\lambda, \gamma) > \mu \\
1 & \text{if } i(\lambda, \gamma) \leq \mu/2 \\
\frac{2}{\mu} (\mu - i(\lambda, \gamma)) & \text{if } i(\lambda, \gamma) \in [\mu/2; \mu].
\end{cases}$$
Compute Lipschitz norm

Lemma

For each $\epsilon > 0$, the function $l_{sys}$ is Lipschitz on $\mathcal{M}^\epsilon_g = \{ l_{sys}(X) \geq \epsilon, X \in \mathcal{M}_g \}$ (and we denote this Lipshitz constant $C_{sys,\epsilon}^{\text{Lip}}$).

Lemma

For a a simple closed curve $\gamma$ and two measured laminations $\lambda$ and $\lambda'$ we have

$$|i(\lambda, \gamma) - i(\lambda', \gamma)| \leq C_{\text{int}, \gamma}^{\text{Lip}} d_{\text{lam}}(\lambda, \lambda')$$

With the previous lemma we can derive

\[ \| \hat{f}_{\epsilon_0,\mu} \|_{\text{Lip}} \leq \frac{C_{f_{\text{Lip}}}}{\mu}. \]

The function \( \hat{f}_{\epsilon_0,\mu} \) is \( \text{Mod}(S) \)-invariant and descends to \( f_{\epsilon_0,\mu} \) on \( P^1 M_g \) with the same Lipschitz norm.
Now we want to give a lower bound on $\int f_{\epsilon_0,\mu}$.

If $\gamma_1$ and $\gamma_2$ are simple closed curves the collar lemma states that:

$$i(\gamma_1, \gamma_2) \leq \frac{l_{\gamma_1}(X)}{R(l_{\gamma_2}(X))}.$$

Now in our case, if $\gamma$ is such that $\frac{\epsilon_0}{2} < l_{\gamma}(X) \leq \epsilon_0$ we have for every curve $\delta$:

$$i(\gamma, \delta) \leq \frac{l_{\delta}(X)}{R(\epsilon_0)}.$$
If $\mu R(\epsilon_0) \leq 1$, we get

$$\{\delta, l_\delta(X) \leq \mu R(\epsilon_0)L\} \subset \{\delta, i(\delta, \gamma) \leq \mu L, l_\delta(X) \leq L\}$$

Indeed $i(\delta, \gamma) \leq \frac{1}{R(\epsilon_0)}l_\delta(X) \leq \mu L$. for any $L > 0$.

if $X \in T_g$ and $\delta_1, \ldots, \delta_k$ are the curves such that $\frac{\epsilon_0}{2} < l_{\delta_j}(X) \leq \epsilon_0$, we get that:

$$(\mu R(\epsilon_0))^{6g-6} \frac{\#\{\delta, l_\delta(X) \leq \mu R(\epsilon_0)L\}}{(\mu R(\epsilon_0)L)^{6g-6}} = \frac{\#\{\delta, l_\delta(X) \leq \mu R(\epsilon_0)L\}}{L^{6g-6}} \leq \frac{\#\{\delta, i(\delta, \delta_j) \leq \mu L, j \in [1, k], l_\delta(X) \leq L\}}{L^{6g-6}}$$

Taking the limit $L \rightarrow \infty$, we find that:

$$(\mu R(\epsilon_0))^{6g-6} B(X) \leq \mu_{Th}(X)\{\lambda \in \mathcal{ML}, i(\lambda, \delta_j) \leq \mu, j \in [1, k]\},$$

where $B(X)$ is the Thurston volume of the unit ball in the space of lamination.
In this way, we derive that

\[
\int f_{\epsilon_0, \mu} d\nu \geq \int 1_{\text{sys}}(X) \in \left[\frac{4\epsilon_0}{6}, \frac{5\epsilon_0}{6}\right] 1 j_{\mu}(X, \lambda) = 1 d\nu \\
= \int_{\Theta} 1_{\text{sys}}(X) \in \left[\frac{4\epsilon_0}{6}, \frac{5\epsilon_0}{6}\right] \int_{\mathcal{ML}} 1 i(\lambda, \delta_j) \leq \mu/2, j \in [1, k] d\mu Th(X)(\lambda) d\mu_{WP} \\
\geq K_{vol}/\mu^{6g-6}
\]

where $\Theta$ is a fundamental domain of the $\text{Mod}(S)$-action on $T_g$ and

\[
K_{vol} = \left(\frac{R(\epsilon_0)}{2}\right)^{6g-6} \int_{\Theta} 1_{\text{sys}}(X) \in \left[\frac{4\epsilon_0}{6}, \frac{5\epsilon_0}{6}\right] B(X) d\mu_{WP}(X). \quad (2)
\]
Conclusion

Suppose we have

$$\left| \int f_{\epsilon_0, \mu} g \circ E^t d\nu - \int f d\nu \int g d\nu \right| < C \frac{1}{t_d} \| f_{\epsilon_0, \mu} \|_{\text{Lip}} \| g \|_{\text{Lip}}$$

(3)

So at time $t_{lim}$ we have

$$\left| \int f_{\epsilon_0, \mu} d\nu \int g d\nu \right| < C \frac{1}{t_d} \| f_{\epsilon_0, \mu} \|_{\text{Lip}} \| g \|_{\text{Lip}}$$

(4)

- $t_{lim} = \frac{\rho}{2\mu}$
- $\int f_{\epsilon_0, \mu} d\nu \geq K_{\text{Vol}} \mu^{6g-6}$
- $\| f_{\epsilon_0, \mu} \|_{\text{Lip}} \leq \frac{C_f^{\text{Lip}}}{\mu}$

So we get

$$\mu^{6g-5} < K \mu^d.$$

And then $d \leq 6g - 5$. 