

Polygonal billiards

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1 Introduction

Given P a polygon in the Euclidean plane \mathbb{R}^2 we consider the *billiard dynamics* in P : a frictionless point mass evolve in straight lines and bounces off the edges according to the law of geometric optics ¹.

The billiard flow is not well-defined when a trajectory hits a corner.

The dynamics happen on the unit tangent bundle to P which is 3-dimensional. The Liouville measure on the unit tangent bundle is preserved by the dynamics.

Exercise 1. *Let P be a triangle.*

1. *Write down in coordinates the first return map of the billiard flow on the edges of P .*
2. *Deduce that the first return map preserves the Liouville measure.*

It will be useful to consider polygonal billiards as a family of dynamical systems.

- How chaotic a polygonal billiard can be ? eg does there exists a mixing billiard?
- What dynamical properties hold for all billiards ? eg does all polygonal billiard admit a periodic orbit?
- What dynamical properties hold for a generic (in the topological sense or measurable sense) billiard ?

Let P be a polygon with angles $\alpha_1, \dots, \alpha_n$. The *double* of P consists in taking P and a reflection P' and glue the corresponding sides together. We obtain a *flat sphere with conical singularities* of angles $2\alpha_1, \dots, 2\alpha_n$. The billiard trajectory in P lifts as trajectories of the geodesic flow in $S^2(P)$.

All questions on polygonal billiards can legitimately be asked in the more general setting of flat sphere with conical singularities
...and why not in higher genera?

¹the angle of incidence equals the angle of reflection

TODO

Figure 1: A billiard in a triangle and the associated geodesic in its double.

2 Geometry of Euclidean structures

Theorem 2 ("Uniformisation" [Tro86]). *Let X be a compact conformal surface and x_1, \dots, x_n distinct points on X . Let $\alpha_1, \dots, \alpha_n$ be positive real numbers such that*

$$\sum_{i=1}^n (2\pi - \alpha_i) = 2\pi\chi(X).$$

Then there exists a (conformal) euclidean structure on X with conical metrics of angle α_i at x_i . It is furthermore unique if we normalize the total area to be one.

Both the proof of uniqueness and existence uses harmonic functions on the surface X . We refer to the article of [Tro86].

Corollary 3. *For any genus and any $\alpha_1, \dots, \alpha_n$ with $\sum_{i=1}^n (2\pi - \alpha_i) = 2\pi(2 - 2g)$ the moduli space of euclidean structures on X with angles $\mathcal{M}_{g,n}(\alpha_1, \dots, \alpha_n)$ is canonically isomorphic to the moduli space of conformal structures $\mathcal{M}_{g,n}$.*

It is in particular connected of complex dimension $3g - 3 + n$.

Theorem 4. *All conical structures on compact surfaces admit a geodesic triangulation whose vertices are the conical points and each edge bounds two distinct faces.*

Exercise 5. 1. *Show that for $g = 0, n = 3$ all conical metrics are the double of a triangle (hint: what is the dimension of the moduli space?)*

2. *Does there exist a conical metric on the sphere which is not the double of a Euclidean disk? What is the minimal n needed for this construction?*

2.1 Holonomy, foliation and unfolding

Let (X, x_1, \dots, x_n) be a compact surface endowed with a conical euclidean structure with singularities at the vertices x_1, \dots, x_n .

The equivalence classes of unit tangent vectors under parallel transport form a foliation endowed with a parallel vector field. Two such leaves are isomorphic (as Riemann surfaces) and the different vector fields differ by rotation. Such a family of vector field is called a *translation surface*.

Theorem 6. *If the polygon is rational then the leaves are compact. If the polygon is irrational then the leaves are dense.*

The unfolding can be identified to the Abelian covering determined by the kernel of the holonomy mapping.

Exercise 7. *What is the topology of the unfolding? hint: apply Riemann-Hurwitz*

2.2 Periodic trajectories

Remark zero : if we have a closed orbit we have a metric cylinder!

The objective of this section is to discuss the following conjecture and partial result.

Question 8. *Let (X, x_1, \dots, x_n) be a Euclidean structure on a compact surface. Does it admit infinitely many closed regular geodesics?*

Theorem 9 (R. E. Schwartz [Sch09]). *Let T be a triangle all of whose angles smaller than $\frac{11}{18}\pi$ then T admits a periodic orbit.*

2.3 R. Schwartz strategy

To prove the existence of periodic trajectory, one strategy is to build them ! More precisely, we want to tile the moduli space of Euclidean structures on $\mathbb{CP}^1 \setminus \{0, 1, +\infty\}$ with open sets that contains a fixed combinatorial trajectory.

The main difficulty in R. Schwartz proof is that such a tiling must be infinite because there are points that can not be tiled.

2.4 Stable and unstable geodesics

Definition 10. *Let X be an Euclidean structure and γ a periodic trajectory. We say that γ is stable if it persists under small deformations in the space of all conical metrics with n conical singularities.*

Exercise 11. *When angles are fixed, all periodic trajectories are stable.*

Proposition 12. *Let X be an Euclidean structure and γ a periodic trajectory. Then the γ is stable if and only if its image is zero in $H_1(X \setminus \{x_1, \dots, x_n\}; \mathbb{Z})$.*

Proof. The only thing that could go wrong in a neighbouring surface is that we get some non-trivial holonomy at the junction. But this is exactly measured in $H_1(X)$. \square

Combinatorial version for billiard in polygons.

Exercise 13. *$\#odd - \#even$ is the same for all letters,*

Theorem 14 (W. P. Hooper [Hoo07]). *All periodic trajectories in a right triangle are unstable.*

Question 15. *Can we characterize Euclidean structures all of whose trajectories are unstable?*

Proof. We can assume that α is irrational.

Let X be the double of P and $\tilde{X} \rightarrow X$ its translation cover (it is a \mathbb{Z} covering). \tilde{X} admits an involution ι which acts as a rotation by π in each rhombus. (the quotient by this involution is actually a sphere)

We claim that for any periodic trajectory γ , the associated maximal cylinder is stabilized by ι . In particular it has two fixed points on its circumference.

Now γ and $\iota(\gamma)$ projects to the same geodesic on the double! Its homology class is hence $2\gamma_3$ where γ_3 is the closed loop around p_3 (that supports the conical singularity of angle $\pi/2$). \square

Theorem 16 (H. Masur[Mas86, Mas88]). *Let T be a rational polygon. Then there are periodic trajectories in a dense set of directions. Furthermore the growth rate is $\Theta(L^2)$.*

2.5 Entropy

Topological entropy is zero (equivalently, word complexity grows slower than any exponential).

Theorem 17. *For rational polygons it is $\Theta(L^3)$ (and conjecturally $\sim cL^3$).*

Question 18. *Is the word complexity of any polygonal billiard $\Theta(L^3)$? $\sim c(T)L^3$?*

interesting approach: compute the constant c for all rational triangle and see whether $c(T)$ admits some continuation point.

Question 19. *Let $\mathcal{L} \subset \{1, 2, 3\}^*$ be the subset of words on the alphabet on $\{1, 2, 3\}$ that encode some piece of orbit in some triangular billiard. Does there exists k such that*

$$\#\{w \in \mathcal{L} : |w| \leq R\} = O(R^k).$$

2.6 Ergodicity

Theorem 20 (KMS). *For rational polygons, the unit tangent bundle is foliated by invariant compact surfaces. This gives rise to a one parameter decomposition of the Liouville measure. Almost every of them are ergodic.*

The proof consists of two steps - (elegant) Masur's criterion - (very technical) almost every direction satisfies Masur's criterion (-¿ Could the second point be deduced from Minsky-Weiss?)

Vorobets quantitative version

2.7 Weak-mixing

Theorem 21 (Katok). *Translation flows are never mixing*

Theorem 22 (AF). *Almsot sure in strata + rank one.*

3 Questions

- What does look like a generic geodesic flow on the 2-sphere? (C^1 topology? C^2 topology?)

References

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