Topological recursion
how to count objects on surfaces?

course 1/4

Vincent Delecroix, CNRS, LaBRI (Bordeaux)
Overview of the course
Overview of the course

- **course 1 (march 1st)**
  - graphs, surfaces, combinatorial maps
  - generating functions
  - general idea of the topological recursion (TR)

- **course 2 (march 8th)**
  - proving TR for combinatorial maps
  - solving TR for combinatorial maps
  - consequences

- **course 3 (march 22th)**: asymptotics

- **course 4 (march 29th)**: other TR
References

Books
▶ B. Eynard *Counting surfaces* (2016)

Articles
▶ M. Bousquet-Mélou, A. Jehanne *Polynomial equations with one catalytic variable, algebraic series and map enumeration* (2006)
▶ M. Kontsevich, Y. Soibelman *Airy structures and symplectic geometry of topological recursion* arXiv:1701.09137 [math.AG]
Introduction
Surfaces

*closed surface*: compact topological space that is "locally planar"
Surfaces

*closed surface*: compact topological space that is "locally planar"

*bordered surface*: compact topological space that is "locally planar or half-planar"
The genus

The genus of a surface is the maximal number of disjoint simple closed curves such that cutting along them do not disconnect the surface.
Genus quizz
**Simple combinatorial map**: finite set of polygons whose edges are glued by pair or unglued. On each boundary we mark an edge with a label from 1 to \( k \) where \( k \) is the number of boundaries.
(ordinary) combinatorial map: finite set of polygons all of whose edges are glued by pair. Some of the polygons are labeled from 1 to $k$ and called boundaries, on each of them we chose an edge and give it a number from 1 to $k$. 
Statistics

Given a combinatorial map $m$ we denote

- $n_i(m)$: number of $i$-gons, $i = 1, 2, \ldots$,
- $\ell_i(m)$: length of the $i$-th boundary, $i = 1, 2, \ldots, k$.

$n_3(m) = 3$, $\ell_1(m) = 3$. 
Combinatorial map examples

1. $n_3 = 1$
   $\ell_1 = 1$

2. $n_4 = 4$
   $\ell_1 = \ell_2 = 4$

3. $n_1 = 1$
   $n_2 = 2$
   $\ell_1 = 1$
Combinatorial map examples

$n_3 = 1$
ℓ₁ = 1

$n_4 = 4$
ℓ₁ = ℓ₂ = 4

$n_1 = 1$
n₂ = 2
ℓ₁ = 1

Which maps are simple?
Why using generating function?

*generating function*: formal power series whose coefficients count something

1. Inductive construction of objects translate into equations on generating functions,
2. Analysis can be used to study asymptotics.
The generating series of combinatorial maps

For $g \geq 0$, $k \geq 0$ define

$$W_k^{(g)}(x_1, x_2, \ldots, x_k; t_1, t_2, \ldots) =$$

$$\sum_{m \in \text{Maps}_k^{(g)}} \frac{1}{\# \text{Aut}(m)} \frac{t_1^{n_1(m)} t_2^{n_2(m)} \ldots}{x_1^{1+\ell_1(m)} x_2^{1+\ell_2(m)} \ldots x_k^{1+\ell_k(m)}}$$

$$WFS_k^{(g)}(x_1, x_2, \ldots, x_k; t_1, t_2, \ldots) =$$

$$\sum_{m \in \text{FSMaps}_k^{(g)}} \frac{1}{\# \text{Aut}(m)} \frac{t_1^{n_1(m)} t_2^{n_2(m)} \ldots}{x_1^{1+\ell_1(m)} x_2^{1+\ell_2(m)} \ldots x_k^{1+\ell_k(m)}}$$

where $\text{Maps}_k^{(g)}$ and $\text{FSMaps}_k^{(g)}$ are respectively the ordinary and simple combinatorial maps of genus $g$ with $k$ boundary components.
The main theorem (informal version)

\[ W_k^{(g)}(x_1, x_2, \ldots, x_k; t_3, t_4, \ldots; t) = \sum_{m \in M_k^{(g)}} \frac{1}{\# \text{Aut}(m)} \frac{t_1^{n_1(m)} t_2^{n_2(m)} \ldots}{x_1^{1+\ell_1(m)} x_2^{1+\ell_2(m)} \ldots x_k^{1+\ell_k(m)}} \]

Theorem
topological recursion: inductive formulas on the \( W_k^{(g)} \)
initial conditions: explicit solutions for \( W_k^{(g=0)} \) and \( W_k^{(g=0)} \)

Topological recursion (course 1/4)

Introduction

Topological recursion for combinatorial maps

Ordinary versus simple maps

Theorem

There is a formula relating \( W_k^{(g)} \) to \( WFS_k^{(g)} \).

Initial conditions: explicit solutions for \( WFS_{k=1}^{(g=0)} \) and \( WFS_{k=2}^{(g=0)} \).

Conjecturally, \( WFS_k^{(g)} \) also satisfies a topological recursion.

Combinatorial maps
Combinatorial maps

*labeled combinatorial map with $k$ boundaries:* triple $(\sigma, \alpha, \phi)$ of permutations on $\{1, 2, \ldots, 2N\}$ where $\alpha$ is an involution without fixed points, $\sigma \alpha \phi = \text{Id}$ and $1, 2, \ldots, k$ belong to distinct orbits of $\phi$. 

![Diagram of a combinatorial map](attachment:image.png)
Combinatorial maps

*labeled combinatorial map with k boundaries:* triple \((\sigma, \alpha, \phi)\) of permutations on \(\{1, 2, \ldots, 2N\}\) where \(\alpha\) is an involution without fixed points, \(\sigma\alpha\phi = \text{Id}\) and 1, 2, \ldots, \(k\) belong to distinct orbits of \(\phi\).

1, \ldots, 2N are called *half-edges* or *oriented edges*
Combinatorial maps

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1, \ldots, 2N are called *half-edges or oriented edges*.

The *vertices* and *edges* of the map are respectively the cycles of \(\sigma\) and \(\alpha\).
Combinatorial maps

*labeled combinatorial map with $k$ boundaries:* triple $(\sigma, \alpha, \phi)$ of permutations on $\{1, 2, \ldots, 2N\}$ where $\alpha$ is an involution without fixed points, $\sigma \alpha \phi = \text{Id}$ and $1, 2, \ldots, k$ belong to distinct orbits of $\phi$.

1, $\ldots$, $2N$ are called *half-edges* or *oriented edges*

The *vertices* and *edges* of the map are respectively the cycles of $\sigma$ and $\alpha$.

The *faces* and *boundaries* of the map are respectively the cycles of $\phi$ that does not contain and contain 1, $\ldots$, $k$. 
Labeled combinatorial map examples

\[ \sigma = (1, 2, 4)(3) \]
\[ \alpha = (1, 2)(3, 4) \]
\[ \phi = (1)(2, 4, 3) \]
Isomorphism

Two labeled combinatorial maps with \( k \) boundaries \((\sigma, \alpha, \phi)\) are \textit{isomorphic} if there exists a permutation \( \tau \) such that 
\[
\forall i \in \{1, \ldots, k\}, \tau(i) = i \quad \text{and} \quad \tau \sigma \tau^{-1} = \sigma', \quad \tau \alpha \tau^{-1} = \beta', \quad \tau \phi \tau^{-1} = \phi'.
\]

\[
\sigma = (1, 2, 4)(3) \quad \text{and} \quad \sigma = (1, 3, 4)(2)
\]
\[
\alpha = (1, 2)(3, 4) \quad \text{and} \quad \epsilon = (1, 4)(2, 3)
\]
\[
\phi = (1)(2, 4, 3) \quad \text{and} \quad \phi = (1)(3, 4, 2)
\]
Simple maps

A map with \( k \) boundaries \((\sigma, \alpha, \phi)\) is simple if for each cycle of \( \sigma \) (corresponding to a vertex) at most one half-edge belongs to a boundary.
Surfaces
Polygon gluings are surfaces

To a map \((\sigma, \alpha, \phi)\) with \(k\) boundaries one can associate a topological space which consists of gluing the polygons associated to the faces according to the permutation \(\alpha\).

**Lemma**

*The topological space obtained from \((\sigma, \alpha, \phi)\) is a surface with boundaries if and only if \((\sigma, \alpha, \phi)\) is a simple map.*

**exercise:** describe the list of possible neighborhoods of a point in an ordinary map.
Surfaces

Theorem (Jordan curve theorem)

A simple closed curve on the sphere separates it into two regions homeomorphic to an open disk.

Corollary

The sphere has genus zero.
Surface classification theorem

Theorem

*Up to homeomorphism bordered surfaces are classified by their genus $g$ and their number of boundary components $k$.***
Proof of the classification theorem

step 1: remove boundaries
Proof of the classification theorem

step 2: reduction to combinatorial maps
Proof of the classification theorem

step 3: connectedness of the flip graph

Consider two moves on the space of combinatorial maps (without boundary)

1. contract edges whose endpoints are distinct
2. remove edges whose adjacent faces are distinct

The moves do not modify the topology of the surface.
Proof of the classification theorem

step 3: connectedness of the flip graph

Consider two moves on the space of combinatorial maps (without boundary)

1. contract edges whose endpoints are distinct
2. remove edges whose adjacent faces are distinct

The moves do not modify the topology of the surface. By applying these moves we are reduced to the problem of connecting maps with a single face and a single vertex. We encode such a map by a circular word in which each letters appear twice.
Proof of the classification theorem

step 3: connectedness of the flip graph

Any map is connected to the canonical map \( a_1 b_1 a_1 b_1 \cdots a_g b_g a_g b_g \) by applying at most 4g moves of type 2 (2g forward and 2g backward)

\[
\begin{array}{c}
\text{\textbf{X}} & \text{\textbf{Y}} & \text{\textbf{Z}} \\
a & b & a
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{\textbf{T}} & \text{\textbf{b}} & \text{\textbf{Z}} \\
T & b & Z
\end{array}
\]

\[
\begin{array}{c}
\text{\textbf{X}} & \text{\textbf{Y}} & \text{\textbf{Z}} \\
a & b & a
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{\textbf{X}} & \text{\textbf{b}} & \text{\textbf{Y}} \\
a' & b' & a'
\end{array}
\]

\[
a X b Y a Z b T \quad \quad \quad a' b' a' b' T Z Y X
\]
step 4: show that the canonical map indeed has genus $g$

In order to show that the map with one face and one vertex with circular code $a_1b_1a_1b_1 \cdots a_gb_ga_gb_g$ has genus $g$ one has to apply arguments from algebraic topology involving singular homology. It is admitted for this course that this map has genus $g$. 
Euler characteristic

**Theorem**

Let $m$ be a combinatorial map (without boundary) with $v(m)$ vertices, $e(m)$ edges and $f(m)$ faces. Then

$$v(m) - e(m) + f(m) = 2 - 2g.$$
Proof of the Euler characteristic

The map considered in the previous proof $a_1 b_1 a_1 b_1 \cdots a_g b_g a_g b_g$ has $v = 1$, $e = 2g$ and $f = 1$ and hence $v - e + f = 2 - 2g$. 
Proof of the Euler characteristic

The map considered in the previous proof $a_1 b_1 a_1 b_1 \cdots a_g b_g a_g b_g$ has $v = 1$, $e = 2g$ and $f = 1$ and hence $v - e + f = 2 - 2g$.

The operations of adding/removing edges from the previous proof preserves Euler characteristic.