

Topological recursion how to count objects on surfaces?

course 1/4

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Overview of the course

- ▶ **course 1 (march 1st)**
 - ▶ graphs, surfaces, combinatorial maps
 - ▶ generating functions
 - ▶ general idea of the topological recursion (TR)
- ▶ **course 2 (march 8th)**
 - ▶ proving TR for combinatorial maps
 - ▶ solving TR for combinatorial maps
 - ▶ consequences
- ▶ **course 3 (march 22th): asymptotics**
- ▶ **course 4 (march 29th): other TR**

References

Books

- ▶ S. K. Lando, A. K. Zvonkin *Graphs on surfaces* (2004)
- ▶ B. Eynard *Counting surfaces* (2016)

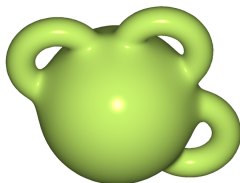
Articles

- ▶ M. Bousquet-Mélou, A. Jehanne *Polynomial equations with one catalytic variable, algebraic series and map enumeration* (2006)
- ▶ J. E. Andersen, G. Borot, L. O. Chekhov, N. Orantin *The ABCD of topological recursion* arXiv:1703.03307 [math-ph]
- ▶ M. Kontsevich, Y. Soibelman *Airy structures and symplectic geometry of topological recursion* arXiv:1701.09137 [math.AG]
- ▶ G. Borot, S. Charbonnier, N. Do, E. Garcia-Failde *Relating ordinary and fully simple maps via monotone Hurwitz numbers* (2019).

Introduction

Surfaces

closed surface: compact topological space that is "locally planar"

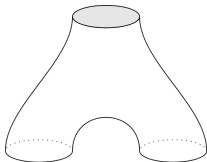


Surfaces

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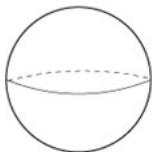


bordered surface: compact topological space that is "locally planar or half-planar"

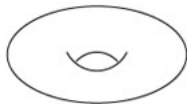


The genus

The *genus* of a surface is the maximal number of disjoint simple closed curves such that cutting along them do not disconnect the surface.



genus 0

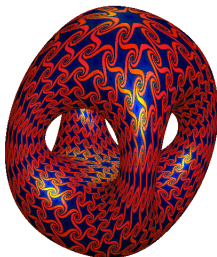


genus 1



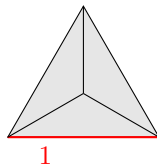
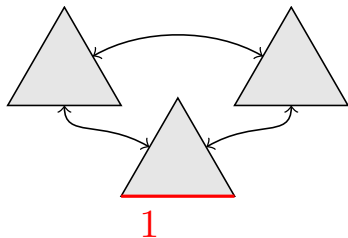
genus 2

Genus quizz



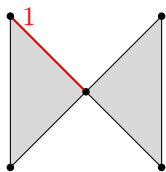
Polygon gluings or simple combinatorial maps

simple combinatorial map: finite set of polygons whose edges are glued by pair or unglued. On each boundary we mark an edge with a label from 1 to k where k is the number of boundaries.



Ordinary combinatorial maps

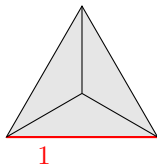
(ordinary) combinatorial map: finite set of polygons all of whose edges are glued by pair. Some of the polygons are labeled from 1 to k and called boundaries, on each of them we chose an edge and give it a number from 1 to k .



Statistics

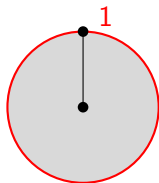
Given a combinatorial map \mathbf{m} we denote

- ▶ $n_i(\mathbf{m})$: number of i -gons, $i = 1, 2, \dots$,
- ▶ $\ell_i(\mathbf{m})$: length of the i -th boundary, $i = 1, 2, \dots, k$.



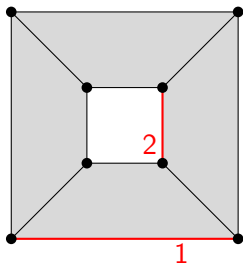
$$n_3(\mathbf{m}) = 3, \ell_1(\mathbf{m}) = 3.$$

Combinatorial map examples



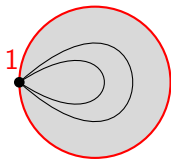
$$n_3 = 1$$

$$l_1 = 1$$



$$n_4 = 4$$

$$l_1 = l_2 = 4$$

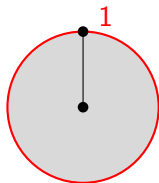


$$n_1 = 1$$

$$n_2 = 2$$

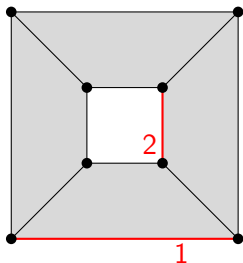
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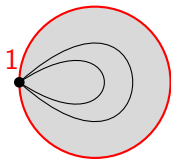
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Which maps are simple?

Why using generating function?

generating function: formal power series whose coefficients count something

1. Inductive construction of objects translate into equations on generating functions,
2. Analysis can be used to study asymptotics.

The generating series of combinatorial maps

For $g \geq 0$, $k \geq 0$ define

$$\begin{aligned}
 W_k^{(g)}(x_1, x_2, \dots, x_k; t_1, t_2, \dots) &= \\
 \sum_{\mathbf{m} \in \text{Maps}_k^{(g)}} \frac{1}{\#\text{Aut}(\mathbf{m})} \frac{t_1^{n_1(\mathbf{m})} t_2^{n_2(\mathbf{m})} \dots}{x_1^{1+\ell_1(\mathbf{m})} x_2^{1+\ell_2(\mathbf{m})} \dots x_k^{1+\ell_k(\mathbf{m})}} \\
 WFS_k^{(g)}(x_1, x_2, \dots, x_k; t_1, t_2, \dots) &= \\
 \sum_{\mathbf{m} \in \text{FSMaps}_k^{(g)}} \frac{1}{\#\text{Aut}(\mathbf{m})} \frac{t_1^{n_1(\mathbf{m})} t_2^{n_2(\mathbf{m})} \dots}{x_1^{1+\ell_1(\mathbf{m})} x_2^{1+\ell_2(\mathbf{m})} \dots x_k^{1+\ell_k(\mathbf{m})}}
 \end{aligned}$$

where $\text{Maps}_k^{(g)}$ and $\text{FSMaps}_k^{(g)}$ are respectively the ordinary and simple combinatorial maps of genus g with k boundary components.

The main theorem (informal version)

$$W_k^{(g)}(x_1, x_2, \dots, x_k; t_3, t_4, \dots; t) = \sum_{\mathbf{m} \in M_k^{(g)}} \frac{1}{\#\text{Aut}(\mathbf{m})} \frac{t_1^{n_1(\mathbf{m})} t_2^{n_2(\mathbf{m})} \dots}{x_1^{1+\ell_1(\mathbf{m})} x_2^{1+\ell_2(\mathbf{m})} \dots x_k^{1+\ell_k(\mathbf{m})}}$$

Theorem

topological recursion: inductive formulas on the $W_k^{(g)}$

initial conditions: explicit solutions for $W_{k=1}^{(g=0)}$ and $W_{k=2}^{(g=0)}$

References: Bender–Canfield (1994), Tutte (1960), Brown (1965), Bousquet-Mélou–Jehanne (2006), Eynard (2006), ...

Ordinary versus simple maps

Theorem

There is a formula relating $W_k^{(g)}$ to $WFS_k^{(g)}$.

initial conditions: explicit solutions for $WFS_{k=1}^{(g=0)}$ and $WFS_{k=2}^{(g=0)}$

Conjecturally, $WFS_k^{(g)}$ also satisfies a topological recursion.

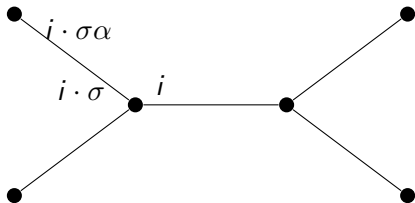
References: Krikun (2007), Borot–Garcia-Failde (2020),

Borot–Charbonnier–Do–Garcia-Failde (2019), . . .

Combinatorial maps

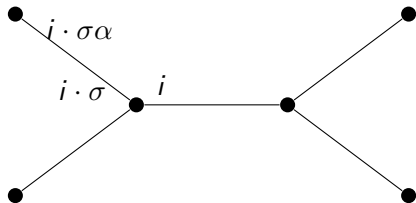
Combinatorial maps

labeled combinatorial map with k boundaries: triple (σ, α, ϕ) of permutations on $\{1, 2, \dots, 2N\}$ where α is an involution without fixed points, $\sigma\alpha\phi = \text{Id}$ and $1, 2, \dots, k$ belong to distinct orbits of ϕ .



Combinatorial maps

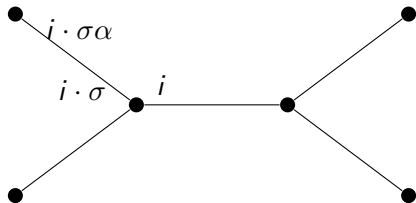
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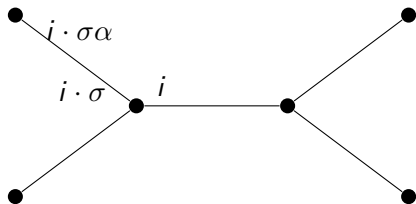


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The *vertices* and *edges* of the map are respectively the cycles of σ and α .

Combinatorial maps

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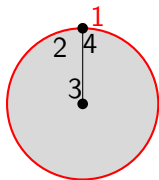


$1, \dots, 2N$ are called *half-edges* or *oriented edges*

The *vertices* and *edges* of the map are respectively the cycles of σ and α .

The *faces* and *boundaries* of the map are respectively the cycles of ϕ that does not contain and contain $1, \dots, k$.

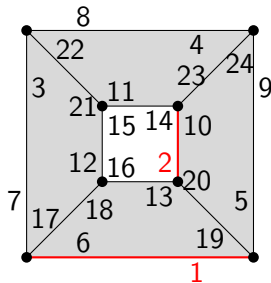
Labeled combinatorial map examples



$$\sigma = (1, 2, 4)(3)$$

$$\alpha = (1, 2)(3, 4)$$

$$\phi = (1)(2, 4, 3)$$



$$\sigma = ?$$

$$\alpha = ?$$

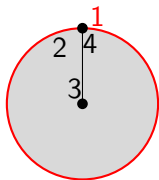
$$\phi = ?$$

Isomorphism

Two labeled combinatorial maps with k boundaries (σ, α, ϕ) are *isomorphic* if there exists a permutation τ such that

$\forall i \in \{1, \dots, k\}, \tau(i) = i$ and

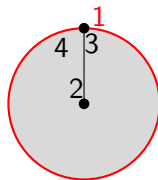
$$\tau\sigma\tau^{-1} = \sigma' \quad \tau\alpha\tau^{-1} = \beta' \quad \tau\phi\tau^{-1} = \phi'.$$



$$\sigma = (1, 2, 4)(3)$$

$$\alpha = (1, 2)(3, 4)$$

$$\phi = (1)(2, 4, 3)$$



$$\sigma = (1, 3, 4)(2)$$

$$\epsilon = (1, 4)(2, 3)$$

$$\phi = (1)(3, 4, 2)$$

Simple maps

A map with k boundaries (σ, α, ϕ) is *simple* if for each cycle of σ (corresponding to a vertex) at most one half-edge belongs to a boundary.

Surfaces

Polygon gluings are surfaces

To a map (σ, α, ϕ) with k boundaries one can associate a topological space which consists of gluing the polygons associated to the faces according to the permutation α .

Lemma

The topological space obtained from (σ, α, ϕ) is a surface with boundaries if and only if (σ, α, ϕ) is a simple map.

exercise: describe the list of possible neighborhoods of a point in an ordinary map.

Surfaces

Theorem (Jordan curve theorem)

A simple closed curve on the sphere separates it into two regions homeomorphic to an open disk.

Corollary

The sphere has genus zero.

Surface classification theorem

Theorem

Up to homeomorphism bordered surfaces are classified by their genus g and their number of boundary components k .

Proof of the classification theorem

step 1: remove boundaries

Proof of the classification theorem

step 2: reduction to combinatorial maps

Proof of the classification theorem

step 3: connectedness of the flip graph

Consider two moves on the space of combinatorial maps (without boundary)

1. contract edges whose endpoints are distinct
2. remove edges whose adjacent faces are distinct

The moves do not modify the topology of the surface.

Proof of the classification theorem

step 3: connectedness of the flip graph

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1. contract edges whose endpoints are distinct
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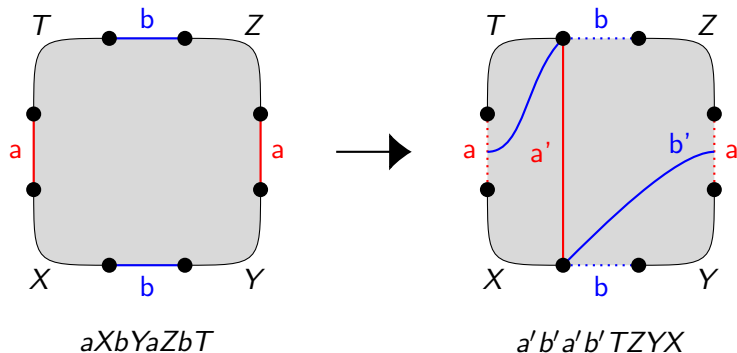
The moves do not modify the topology of the surface.

By applying these moves we are reduced to the problem of connecting maps with a single face and a single vertex. We encode such a map by a circular word in which each letters appear twice.

Proof of the classification theorem

step 3: connectedness of the flip graph

Any map is connected to the canonical map $a_1 b_1 a_1 b_1 \cdots a_g b_g a_g b_g$ by applying at most $4g$ moves of type 2 ($2g$ forward and $2g$ backward)



step 4: show that the canonical map indeed has genus g

In order to show that the map with one face and one vertex with circular code $a_1 b_1 a_1 b_1 \cdots a_g b_g a_g b_g$ has genus g one has to apply arguments from algebraic topology involving singular homology. It is admitted for this course that this map has genus g .

Euler characteristic

Theorem

Let \mathbf{m} be a combinatorial map (without boundary) with $v(\mathbf{m})$ vertices, $e(\mathbf{m})$ edges and $f(\mathbf{m})$ faces. Then

$$v(\mathbf{m}) - e(\mathbf{m}) + f(\mathbf{m}) = 2 - 2g.$$

Proof of the Euler characteristic

The map considered in the previous proof $a_1 b_1 a_1 b_1 \cdots a_g b_g a_g b_g$ has $v = 1$, $e = 2g$ and $f = 1$ and hence $v - e + f = 2 - 2g$.

Proof of the Euler characteristic

The map considered in the previous proof $a_1 b_1 a_1 b_1 \cdots a_g b_g a_g b_g$ has $v = 1$, $e = 2g$ and $f = 1$ and hence $v - e + f = 2 - 2g$.

The operations of adding/removing edges from the previous proof preserves Euler characteristic.