Topological recursion how to count objects on surfaces?

course 2/4

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Recall from course 1

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Which maps are simple?

Surface classification

Theorem

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Corollary (Euler characteristic)

For a polygon gluing with k boundaries

$$s-a+f=2-2g-k$$

where

- s : number of vertices
- a : number of edges
- ▶ f : number of (internal) faces

Weights associated to combinatorial maps weight of a map m with k boundaries

$$w(\boldsymbol{m}) = \frac{t_1^{n_1(\boldsymbol{m})} t_2^{n_2(\boldsymbol{m})} \cdots}{x_1^{\ell_1(\boldsymbol{m})+1} x_2^{\ell_2(\boldsymbol{m})+1} \cdots x_k^{\ell_k(\boldsymbol{m})+1}}$$

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n_i(m) : number of faces of degree *i ℓ_i(m)* : length of the *i*-th face
 simplified weight

$$\overline{w}(\boldsymbol{m}) = \frac{t^{2e(\boldsymbol{m})+k}}{x_1^{\ell_1(\boldsymbol{m})+1} x_2^{\ell_2(\boldsymbol{m})+1} \cdots x_k^{\ell_k(\boldsymbol{m})+1}} \\ = w(\boldsymbol{m}) \left(t_1 = t, t_2 = t^2, \dots; x_1 = \frac{x_1}{t}, x_2 = \frac{x_2}{t}, \dots, x_k = \frac{x_k}{t} \right)$$

where

$$2e(\boldsymbol{m}) = n_1 + 2n_2 + 3n_3 + \ldots + \ell_1(\boldsymbol{m}) + \ell_2(\boldsymbol{m}) + \ldots \ell_k(\boldsymbol{m})$$

Generating functions

$$W_k^{(g)}(t_1, t_2, \dots; x_1, x_2, \dots, x_k) = \sum_{\boldsymbol{m} \in M_k^{(g)}} w(\boldsymbol{m})$$
$$\overline{W}_k^{(g)}(t; x_1, x_2, \dots, x_k) = \sum_{\boldsymbol{m} \in M_k^{(g)}} \overline{w}(\boldsymbol{m})$$

where the sum is over the set $M_k^{(g)}$ of maps with k boundaries.

Objective of course 2

- 1. Build/unbuild maps by induction (Tutte equations, loop equations, peeling, ...)
- 2. Write associated equations on the generating functions $W_k^{(g)}$
- 3. Solve the equations for $W_0^{(1)}$

Topological recursion (course 2/4) \Box Computing $W_{k=1}^{(g=0)}$

Computing
$$W_{k=1}^{(g=0)}$$

We have Theorem (Tutte, 1960)

$$W_1^{(0)}(t;x) = rac{t}{x} + rac{t}{x}(W_1^{(0)}(t;x))^2 + rac{t^2}{x} \; rac{W_1^{(0)}(t;x) - W_1^{(0)}(t;1)}{1-x}.$$

Maps with few edges



 $\begin{array}{l} \mbox{Topological recursion (course 2/4)} \\ \mbox{-} \mbox{Computing $W_{k=1}^{(g=0)}$} \\ \mbox{-} \mbox{The Tutte equation for $W_1^{(0)}$ ("disk amplitude")} \end{array}$

Maps with few edges



- The coefficient of t^{2n+1} is in $\mathbb{Z}[x^{-1}]$,
- The coefficient of (t/x)²ⁿ⁺¹ is the number of rooted planar trees with n edges

Peeling for g = 0 and k = 1Separating case



Peeling for g = 0 and k = 1Non-separating case



Peeling for g = 0 and k = 1Non-separating case



Rewriting $\frac{1}{x}+1+\ldots+x^{\ell-1}=\frac{1}{x}\frac{1-x^{\ell+1}}{1-x}$ we get

$$\frac{t^2}{x} \frac{W_1^{(0)}(t;x) - W_1^{(0)}(t;1)}{1-x}$$

Solving the Tutte equation via Bousquet-Mélou–Jehanne Tutte equation determines both $W_1^{(0)}(t; x)$ and $W_1^{(0)}(1; x)$

Tutte equation is a polynomial equation of the form

$$E(W_1^{(0)}(t;x), W_1^{(0)}(t;1), t, x) = 0$$

where $E(F, f, t, x) \in \mathbb{Q}[F, f, t, x]$.

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Lemma

The equation E(F, f, t, x) = 0 with unknowns (F, f) has a unique solution in $\mathbb{Q}[x][[t]] \times \mathbb{Q}[[t]]$.



Solving the Tutte equation via Bousquet-Mélou–Jehanne BMJ system ("varation of parameters")

Differentiate E(F(t, x), f(t), t, x) = 0 with respect to x



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$$\frac{\partial}{\partial t}F(t,x)E_1'(F(t,x),f(t),t,x)+E_4'(F(t,x),F(t),t,x)=0$$

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Idea: find a power series X(t) = O(t) such that E(F(t; X(t)), f(t), t, X(t)) = 0 and $E'_1(F(t, X(t)), f(t), t, X(t)) = 0$.

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Solving the Tutte equation via Bousquet-Mélou–Jehanne The Tutte system

MBJ system For Tutte equation

$$\begin{cases} E = t \times F^{2} + t^{2} f - t^{2} F - x^{2} F - t F^{2} + t \times x + x F - t \\ E'_{1} = 2 t \times F - t^{2} - x^{2} - 2 t F + x \\ E'_{4} = t F^{2} - 2 \times F + t + F \end{cases}$$

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Solving the Tutte equation via Bousquet-Mélou–Jehanne Solving in SageMath

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sage: R = PolynomialRing(QQ, 't,x,f,F')
sage: t, x, f, F = R.gens()
sage: E = x*(1-x)*F - (1-x)*t*(1+F^2) - t^2*(F-f)
sage: EF = E.derivative(F)
sage: Ex = E.derivative(x)
sage: I = R.ideal([E, EF, Ex])
sage: J = I.elimination_ideal([F, x])
sage: len(J.gens())
1
sage: J.gens()[0].factor()
t^2 * (t*f - 1) * (27*t^3*f^2 + 16*t^3 - 18*t^2*f - t + f)
```

Solving the Tutte equation via Bousquet-Mélou–Jehanne Formula

Theorem

$$f(t) = W_1^{(0)}(t; 1)$$
 satisfies
 $(27t^3)f^2 + (1 - 18t^2)f + (16t^3 - t) = 0$

In other words

$$W_1^{(0)}(t;1) = rac{18t^2 - 1 + (1 - 12t^2)^{3/2}}{54t^3}$$

Taylor expansion: $W_1^{(0)}(t; 1) = t + 2t^3 + 9t^5 + 54t^7 + 378t^9 + \dots$



Solving the Tutte equation via Bousquet-Mélou–Jehanne Formula bis

Theorem

$$[t^{2n+1}]W_1^{(0)}(t;1) = \frac{2 \cdot 3^n}{(n+2)(n+1)} \frac{(2n)!}{(n!)^2}.$$

General topological recursion

General topological recursion

Recall that for genus 0 with 1 boundary we proved Theorem (Tutte, 1960)

$$W_1^{(0)}(t;x) = rac{t}{x} + rac{t}{x}(W_1^{(0)}(t;x))^2 + rac{t^2}{x} \; rac{W_1^{(0)}(t;x) - W_1^{(0)}(1;x)}{1-x}.$$

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We will now generalize to any (g, k).

Generalized Tutte formula

Theorem

$$\begin{split} W_k^{(g)}(t; x_1, x_2, \dots, x_k) &= \frac{t}{x_1 x_2 \dots x_k} \\ &+ \frac{t}{x_1} \sum_{\substack{l_1 \sqcup l_2 = \{2, \dots, k\} \\ g_1 + g_2 = g}} W_{k_1 + 1}^{(g_1)}(t; x_1, x_{l_1}) \cdot W_{k_2 + 1}^{(g_2)}(t; x_1, x_{l_2}) \\ &+ \frac{t}{x_1} W_{k+1}^{(g-1)}(t; x_1, x_1, x_2, \dots, x_k) \\ &+ \frac{t^2}{x_1} \frac{W_k^{(g)}(t; x_1, x_2, \dots, x_k) - W_k^{(g)}(t; 1, x_2, \dots, x_k)}{1 - x_1} \\ &+ \frac{t^2}{x_1} \sum_{j=2}^k \frac{\partial}{\partial x_j} \frac{W_{k-1}^{(g)}(t; x_1, \dots, \widehat{x_j}, \dots, x_k) - W_{k-1}^g(t; x_2, \dots, x_k)}{x_1 - x_j} \end{split}$$

Proof of the formula same face, disconnected

Proof of the formula same face, connected

Proof of the formula different faces, unrooted neighbor

Proof of the formula different faces, rooted neighbor