Topological recursion
how to count objects on surfaces?

Vincent Delecroix, CNRS, LaBRI (Bordeaux)
Recall from course 1
Three sorts of maps

Recall that a combinatorial map is encoded by a triple \((\sigma, \alpha, \phi)\). We distinguished

- **(ordinary) map**: any triple \((\sigma, \alpha, \phi)\) some of the faces are “removed” and “marked” from 1 to \(k\)
- **simple map**: at most one boundary half-edge at each vertex.
Recall from course 1

Three sorts of maps

Recall that a combinatorial map is encoded by a triple \((\sigma, \alpha, \phi)\). We distinguished

- (ordinary) map: any triple \((\sigma, \alpha, \phi)\) some of the faces are ”removed” and ”marked” from 1 to \(k\)
Three sorts of maps

Recall that a combinatorial map is encoded by a triple \((\sigma, \alpha, \phi)\). We distinguished

- *(ordinary) map*: any triple \((\sigma, \alpha, \phi)\) some of the faces are ”removed” and ”marked” from 1 to \(k\)
- *simple map*: at most one boundary half-edge at each vertex
Three sorts of maps

Recall that a combinatorial map is encoded by a triple \((\sigma, \alpha, \phi)\). We distinguished

- **(ordinary) map**: any triple \((\sigma, \alpha, \phi)\) some of the faces are "removed" and "marked" from 1 to \(k\)

- **simple map**: at most one boundary half-edge at each vertex

Which maps are simple?
Surface classification

Theorem

*Up to homeomorphism bordered surfaces are classified by their genus $g$ and their number of boundary components $k$.***
Surface classification

Theorem
Up to homeomorphism bordered surfaces are classified by their genus $g$ and their number of boundary components $k$.

Corollary (Euler characteristic)
For a polygon gluing with $k$ boundaries

$$s - a + f = 2 - 2g - k$$

where
- $s$ : number of vertices
- $a$ : number of edges
- $f$ : number of (internal) faces
Weights associated to combinatorial maps

*weight* of a map \( m \) with \( k \) boundaries

\[
\begin{align*}
  w(m) = \frac{t_1^{n_1(m)} t_2^{n_2(m)} \cdots}{x_1^{\ell_1(m)+1} x_2^{\ell_2(m)+1} \cdots x_k^{\ell_k(m)+1}}
\end{align*}
\]

- \( n_i(m) \): number of faces of degree \( i \)
- \( \ell_i(m) \): length of the \( i \)-th face
Weights associated to combinatorial maps

*weight* of a map $m$ with $k$ boundaries

$$w(m) = \frac{t_1^{n_1(m)} t_2^{n_2(m)} \ldots}{x_1^{\ell_1(m)+1} x_2^{\ell_2(m)+1} \ldots x_k^{\ell_k(m)+1}}$$

- $n_i(m)$: number of faces of degree $i$
- $\ell_i(m)$: length of the $i$-th face

*simplified weight*

$$\overline{w}(m) = \frac{t^{2e(m)+k}}{x_1^{\ell_1(m)+1} x_2^{\ell_2(m)+1} \ldots x_k^{\ell_k(m)+1}}$$

$$= w(m) \left( t_1 = t, t_2 = t^2, \ldots; x_1 = \frac{x_1}{t}, x_2 = \frac{x_2}{t}, \ldots, x_k = \frac{x_k}{t} \right)$$

where

$$2e(m) = n_1 + 2n_2 + 3n_3 + \ldots + \ell_1(m) + \ell_2(m) + \ldots \ell_k(m)$$
Generating functions

\[ W_{k}^{(g)}(t_1, t_2, \ldots; x_1, x_2, \ldots, x_k) = \sum_{m \in M_{k}^{(g)}} w(m) \]

\[ \overline{W}_{k}^{(g)}(t; x_1, x_2, \ldots, x_k) = \sum_{m \in M_{k}^{(g)}} \overline{w}(m) \]

where the sum is over the set \( M_{k}^{(g)} \) of maps with \( k \) boundaries.
Objective of course 2

1. Build/unbuild maps by induction (Tutte equations, loop equations, peeling, . . . )
2. Write associated equations on the generating functions $\mathcal{W}_k^{(g)}$
3. Solve the equations for $\mathcal{W}_0^{(1)}$
Computing $W_{k=1}^{(g=0)}$
We have

**Theorem (Tutte, 1960)**

\[
W_1^{(0)}(t; x) = \frac{t}{x} + \frac{t}{x}(W_1^{(0)}(t; x))^2 + \frac{t^2}{x}\frac{W_1^{(0)}(t; x) - W_1^{(0)}(t; 1)}{1 - x}.
\]
The Tutte equation for $W_1^{(0)}$ ("disk amplitude")

Maps with few edges

\[
\begin{align*}
\text{Maps with few edges} & \\
\begin{array}{c}
\begin{array}{c}
\bullet \\
\frac{t}{x}
\end{array} & \\
\begin{array}{c}
\bullet
\end{array} & \\
\frac{t^3}{x^2} & \\
\frac{t^3}{x^3}
\end{array}
\end{align*}
\]
Maps with few edges

- The coefficient of $t^{2n+1}$ is in $\mathbb{Z}[x^{-1}]$,
- The coefficient of $(t/x)^{2n+1}$ is the number of rooted planar trees with $n$ edges.
Peeling for $g = 0$ and $k = 1$

Separating case

\[
\frac{t^{2e_1+2e_2+3}}{x^{\ell_1+\ell_2+3}}
\]

\[
\frac{t^{2e_1+1}}{x^{\ell_1+1}}
\]

\[
\frac{t^{2e_2+1}}{x^{\ell_2+1}}
\]

\[
\frac{t}{x} \cdot (W_1^{(0)})^2
\]
Peeling for $g = 0$ and $k = 1$

Non-separating case

\[
\begin{align*}
t_{2e+\ell+3} & \cdot x^{\ell+2} \\
t_{2e+\ell+3} & \cdot x^{\ell+1} \\
t_{2e+\ell+3} & \cdot x^{\ell-1} \\
\cdots & \\
t_{2e+\ell+3} & \cdot x^2 \\
\cdots & \\
t_{2e+\ell+1} & \cdot x^{\ell+1}
\end{align*}
\]
Peeling for $g = 0$ and $k = 1$

Non-separating case

\[ t^{2e+\ell+3} \frac{x^{\ell+2}}{x^{\ell+1}} + t^{2e+\ell+3} \frac{x^{\ell+1}}{x^{\ell}} + \ldots + t^{2e+\ell+3} \frac{x^{\ell-1}}{x^2} \]

Rewriting \( \frac{1}{x} + 1 + \ldots + x^{\ell-1} = \frac{1}{x} \frac{1-x^{\ell+1}}{1-x} \) we get

\[ \frac{t^2}{x} \frac{W_1^{(0)}(t; x) - W_1^{(0)}(t; 1)}{1 - x}. \]
Solving the Tutte equation via Bousquet-Mélou–Jehanne

Tutte equation determines both $W_1^{(0)}(t; x)$ and $W_1^{(0)}(1; x)$

Tutte equation is a polynomial equation of the form

$$E(W_1^{(0)}(t; x), W_1^{(0)}(t; 1), t, x) = 0$$

where $E(F, f, t, x) \in \mathbb{Q}[F, f, t, x]$. 
Solving the Tutte equation via Bousquet-Mélou–Jehanne

Tutte equation determines both \( W_1^{(0)}(t; x) \) and \( W_1^{(0)}(1; x) \)

Tutte equation is a polynomial equation of the form

\[
E(W_1^{(0)}(t; x), W_1^{(0)}(t; 1), t, x) = 0
\]

where \( E(F, f, t, x) \in \mathbb{Q}[F, f, t, x] \).

Lemma

The equation \( E(F, f, t, x) = 0 \) with unknowns \( (F, f) \) has a unique solution in \( \mathbb{Q}[x][[t]] \times \mathbb{Q}[[t]] \).
Solving the Tutte equation via Bousquet-Mélou–Jehanne BMJ system ("variation of parameters")

Differentiate \( E(F(t, x), f(t), t, x) = 0 \) with respect to \( x \)
Solving the Tutte equation via Bousquet-Mélou–Jehanne
BMJ system ("variation of parameters")

Differentiate $E(F(t, x), f(t), t, x) = 0$ with respect to $x$

$$\frac{\partial}{\partial t} F(t, x)E'_1(F(t, x), f(t), t, x) + E'_4(F(t, x), F(t), t, x) = 0$$
Solving the Tutte equation via Bousquet-Mélou–Jehanne
BMJ system ("variation of parameters")

Differentiate $E(F(t, x), f(t), t, x) = 0$ with respect to $x$

$$\frac{\partial}{\partial t} F(t, x) E_1'(F(t, x), f(t), t, x) + E_4'(F(t, x), F(t), t, x) = 0$$

Idea: find a power series $X(t) = O(t)$ such that
$E(F(t; X(t)), f(t), t, X(t)) = 0$ and
$E_1'(F(t, X(t)), f(t), t, X(t)) = 0$. 

Solving the Tutte equation via Bousquet-Mélou–Jehanne
BMJ system ("variation of parameters")

Differentiate $E(F(t, x), f(t), t, x) = 0$ with respect to $x$

$$\frac{\partial}{\partial t} F(t, x) E'_1(F(t, x), f(t), t, x) + E'_4(F(t, x), F(t), t, x) = 0$$

Idea: find a power series $X(t) = O(t)$ such that

$E(F(t; X(t)), f(t), t, X(t)) = 0$ and

$E'_1(F(t, X(t)), f(t), t, X(t)) = 0$.

We automatically get $E'_4(F(t, X(t)), F(t), t, X(t))$. 

Topological recursion (course 2/4)
- Computing $W_{g=0}^{(g=0)}$
- Solving Tutte equation
Solving the Tutte equation via Bousquet-Mélou–Jehanne

BMJ system ("variation of parameters")

Differentiate $E(F(t, x), f(t), t, x) = 0$ with respect to $x$

$$\frac{\partial}{\partial t} F(t, x) E'_1(F(t, x), f(t), t, x) + E'_4(F(t, x), F(t), t, x) = 0$$

Idea: find a power series $X(t) = O(t)$ such that

$E(F(t; X(t)), f(t), t, X(t)) = 0$ and $E'_1(F(t, X(t)), f(t), t, X(t)) = 0$.

We automatically get $E'_4(F(t, X(t)), F(t), t, X(t))$.

Three equations and three unknowns!
Solving the Tutte equation via Bousquet-Mélou–Jehanne

The Tutte system

MBJ system For Tutte equation

\[
\begin{align*}
E & = t x F^2 + t^2 f - t^2 F - x^2 F - t F^2 + t x + x F - t \\
E'_1 & = 2 t x F - t^2 - x^2 - 2 t F + x \\
E'_4 & = t F^2 - 2 x F + t + F
\end{align*}
\]
Topological recursion (course 2/4)

Computing $W_{k=1}^{(g=0)}$

Solving Tutte equation

Solving the Tutte equation via Bousquet-Mélou–Jehanne

Solving in SageMath

```
sage: R = PolynomialRing(QQ, 't,x,f,F')
sage: t, x, f, F = R.gens()
sage: E = x*(1-x)*F - (1-x)*t*(1+F^2) - t^2*(F-f)
sage: EF = E.derivative(F)
sage: Ex = E.derivative(x)
sage: I = R.ideal([E, EF, Ex])
sage: J = I.elimination_ideal([F, x])
sage: len(J.gens())
1
sage: J.gens()[0].factor()
t^2 * (t*f - 1) * (27*t^3*f^2 + 16*t^3 - 18*t^2*f - t + f)
```
Solving the Tutte equation via Bousquet-Mélou–Jehanne Formula

**Theorem**

\[ f(t) = W_1^{(0)}(t; 1) \text{ satisfies} \]

\[ (27t^3)f^2 + (1 - 18t^2)f + (16t^3 - t) = 0 \]

**In other words**

\[ W_1^{(0)}(t; 1) = \frac{18t^2 - 1 + (1 - 12t^2)^{3/2}}{54t^3} \]

Taylor expansion: \( W_1^{(0)}(t; 1) = t + 2t^3 + 9t^5 + 54t^7 + 378t^9 + \ldots \)
Solving the Tutte equation via Bousquet-Mélou–Jehanne Formula bis

Theorem

\[ [t^{2n+1}] W_1^{(0)}(t; 1) = \frac{2 \cdot 3^n}{(n+2)(n+1)} \frac{(2n)!}{(n!)^2}. \]
General topological recursion
Recall that for genus 0 with 1 boundary we proved

**Theorem (Tutte, 1960)**

\[
W_1^{(0)}(t; x) = \frac{t}{x} + \frac{t}{x} (W_1^{(0)}(t; x))^2 + \frac{t^2}{x} \frac{W_1^{(0)}(t; x) - W_1^{(0)}(1; x)}{1 - x}.
\]

We will now generalize to any \((g, k)\).
Generalized Tutte formula

Theorem

\[
W_{k}^{(g)}(t; x_{1}, x_{2}, \ldots, x_{k}) = \frac{t}{x_{1}x_{2} \ldots x_{k}} \\
+ \frac{t}{x_{1}} \sum_{l_{1} \sqcup l_{2} = \{2, \ldots, k\}} W_{k_{1}+1}^{(g_{1})}(t; x_{1}, x_{l_{1}}) \cdot W_{k_{2}+1}^{(g_{2})}(t; x_{1}, x_{l_{2}}) \\
+ \frac{t}{x_{1}} W_{k+1}^{(g-1)}(t; x_{1}, x_{1}, x_{2}, \ldots, x_{k}) \\
+ \frac{t^{2}}{x_{1}} \frac{W_{k}^{(g)}(t; x_{1}, x_{2}, \ldots, x_{k}) - W_{k}^{(g)}(t; 1, x_{2}, \ldots, x_{k})}{1 - x_{1}} \\
+ \frac{t^{2}}{x_{1}} \sum_{j=2}^{k} \frac{\partial}{\partial x_{j}} \frac{W_{k-1}^{(g)}(t; x_{1}, \ldots, \tilde{x}_{j}, \ldots, x_{k}) - W_{k-1}^{(g)}(t; x_{2}, \ldots, x_{k})}{x_{1} - x_{j}}
\]
Proof of the formula

same face, disconnected
Proof of the formula
same face, connected
Proof of the formula

different faces, unrooted neighbor
Proof of the formula

different faces, rooted neighbor