

# Topological recursion how to count objects on surfaces?

course 3/4

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## Recall from course 2

## Tutte equation

The generating series of maps of genus 0 with 1 boundary satisfy the Tutte equation

$$(tx - t)F^2 - (t^2 + x^2 - x)F + (t^2f + tx - t) = 0.$$

where  $F = W_1^{(0)}(t; x) \in \mathbb{Q}[x^{-1}][[t]]$  and  $f = W_1^{(0)}(t; 1) \in \mathbb{Q}[[t]]$ .

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$$(27t^3)f^2 + (1 - 18t^2)f + (16t^3 - t) = 0$$

and concluded that

$$f = \frac{18t^2 - 1 + (1 - 12t^2)^{3/2}}{54t^3} = 1 + 2t^3 + 9t^5 + 54t^7 + 378t^9 + \dots$$

## Loop equation (generalized Tutte formula)

# Loop equation (generalized Tutte formula)

$$\begin{aligned}
 W_k^{(g)}(t; x_1, x_2, \dots, x_k) = & \\
 & \frac{t}{x_1} \sum_{\substack{l_1 \sqcup l_2 = \{2, \dots, k\} \\ g_1 + g_2 = g}} W_{k_1+1}^{(g_1)}(t; x_1, x_{l_1}) \cdot W_{k_2+1}^{(g_2)}(t; x_1, x_{l_2}) \\
 & + \frac{t}{x_1} W_{k+1}^{(g-1)}(t; x_1, x_1, x_2, \dots, x_k) \\
 & + \frac{t^2}{x_1} \frac{W_k^{(g)}(t; x_1, x_2, \dots, x_k) - W_k^{(g)}(t; 1, x_2, \dots, x_k)}{1 - x_1} \\
 & + \frac{t^2}{x_1} \sum_{j=2}^k \frac{\partial}{\partial x_j} \frac{W_{k-1}^{(g)}(t; x_1, \dots, \widehat{x_j}, \dots, x_k) - W_{k-1}^{(g)}(t; x_2, \dots, x_k)}{x_1 - x_j}
 \end{aligned}$$

## Objective of course 3

- ▶ rewrite  $W_1^{(0)}(t; x_1)$  in terms of Zhukovsky's variable
- ▶ compute  $W_k^{(g)}(t; x_1, \dots, x_k)$  for any  $(g, k)$
- ▶ ribbon graphs and Kontsevich topological recursion



Disk amplitude  $W_1^{(0)}(t; x)$  and Zhukovsky's  
variable

# 1-cut for disk amplitude (Brown's lemma)

## Theorem

Let  $T = \frac{-1 + \sqrt{1 - 12t^2}}{6}$ . Then the discriminant  $\Delta = (t^2 + x^2 - x)^2 - 4(tx - t)(t^2 + tx - t)$  of the Tutte equation factorizes as  $\Delta = (x - (T + 1))^2(x^2 + 2Tx + T - 3t^2)$ .

## Corollary

We have

$$W_1^{(0)}(x; t) = \frac{t^2 + x^2 - x - (x - (T + 1))\sqrt{(x^2 + 2Tx + T - 3t^2)}}{2t(x - 1)}$$

## Zhukovsky's variable

Let  $a, b \in \mathbb{Q}[[t]](T)$  be such that  
 $x^2 + 2Tx + T - 3t^2 = (x - a)(x - b)$  and define

$$\alpha := \frac{a + b}{2} = -T \quad \text{and} \quad \gamma := \frac{a - b}{4} = \frac{\sqrt{T^2 - T + 3t^2}}{2}$$

*Zhukovsky's variable:*

$$x(z) = \frac{a + b}{2} + \frac{a - b}{4} \left( z + \frac{1}{z} \right)$$

$$z(x) = \frac{1}{2\gamma} \left( x - \alpha + \sqrt{(x - \alpha)^2 - 4\gamma^2} \right)$$

## Zhukovsky's variable

The Zhukovsky's variable is such that

$$\sqrt{(x(z) - a)(x(z) - b)} = \gamma \left( z - \frac{1}{z} \right).$$

And one has

$$W_1^{(0)}(t; x(z)) = \sum_{k \geq 1} u_k z^{-k}$$

where  $u_k \in \mathbb{Q}[[t]]$ .

Cylinder amplitude:  $W_2^{(0)}(t; x_1, x_2)$

## The loop equation

Theorem (loop equation (special case  $g = 0, k = 2$ ))

$$\begin{aligned} W_2^{(0)}(t; x_1, x_2) = & 2 \frac{t}{x_1} W_2^{(0)}(t; x_1, x_2) \cdot W_1^{(0)}(t; x_1) \\ & + \frac{t^2}{x_1} \cdot \frac{W_2^{(0)}(t; x_1, x_2) - W_2^{(0)}(t; 1, x_2)}{1 - x_1} \\ & + \frac{t^2}{x_1} \cdot \frac{\partial}{\partial x_2} \frac{W_1^{(0)}(t; x_1) - W_1^{(0)}(t; x_2)}{x_1 - x_2} \end{aligned}$$

- ▶ no more quadratic term
- ▶ allows to compute term by term

Computation of  $W_2^{(0)}(t; x_1, x_2)$ 

$$W_2^{(0)}(t; x_1, x_2) = \left( \frac{1}{x_1^2 x_2^2} \right) t^3 + \\ \left( \frac{3}{x_1^4 x_2^2} + \frac{2}{x_1^3 x_2^3} + \frac{3}{x_1^2 x_2^4} + \frac{2}{x_1^3 x_2^2} + \frac{2}{x_1^2 x_2^3} + \frac{1}{x_1^2 x_2^2} \right) t^5 + \\ O(t^7)$$

We can rewrite the loop equation as

$$\begin{aligned} & \left( x_1 - 2tW_1^{(0)}(t; x_1) + \frac{t^2}{x_1 - 1} \right) W_2^{(0)}(t; x_1, x_2) = \\ & + t^2 \cdot \frac{W_2^{(0)}(t; 1, x_2)}{x_1 - 1} \\ & + t^2 \cdot \frac{\partial}{\partial x_2} \frac{W_1^{(0)}(t; x_1) - W_1^{(0)}(t; x_2)}{x_1 - x_2} \end{aligned}$$



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Recall

$$2tW_1^{(0)}(t; x) = \frac{t^2 + x^2 - x - (x - (T + 1))\sqrt{(x^2 + 2Tx + T - 3t^2)}}{x - 1}$$

We can rewrite the loop equation as

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Recall

$$2tW_1^{(0)}(t; x) = \frac{t^2 + x^2 - x - (x - (T + 1))\sqrt{(x^2 + 2Tx + T - 3t^2)}}{x - 1}$$

rewrites

$$\frac{t^2}{x - 1} + x - 2tW_1^{(0)}(t; x) = \frac{(x - (T + 1))\sqrt{(x^2 + 2Tx + T - 3t^2)}}{x - 1}$$

Hence

$$W_2^{(0)}(t; x_1, x_2) = \frac{t^2 \frac{W_2^{(0)}(t; 1, x_2)}{x_1 - 1} + t^2 \cdot \frac{\partial}{\partial x_2} \frac{W_1^{(0)}(t; x_1) - W_1^{(0)}(t; x_2)}{x_1 - x_2}}{M(x_1) \sqrt{(x_1 - a)(x_1 - b(t))}}$$

# 1-cut for cylinder amplitude (Brown's lemma)

## Theorem

$W_2^{(0)}(x(z_1), x(z_2))x'(z_1)x'(z_2)$  is a rational function of  $z_1$  and  $z_2$  which behaves as  $O(z_1^{-2})$  as large  $z_1$  and has a pole only at  $z_1 = 1/z_2$  and this pole is a double pole with coefficients  $-z_2^{-2}$  and no residue.

## Corollary

$$\begin{aligned} W_2^{(0)}(x(z_1), x(z_2))x'(z_1)x'(z_2) &= \frac{-1}{(z_1 z_2 - 1)^2} \\ &= \frac{1}{(z_1 - z_2)^2} - \frac{x'(z_1)x'(z_2)}{(x(z_1) - x(z_2))^2}. \end{aligned}$$

## Proof

$W_2^{(0)}(x(z_1), x(z_2))$  is a rational function of  $z_2$  with possible poles at  $+1, -1, z_2, 1/z_2$ .

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$W_2^{(0)}(x(z_1), x(z_2))$  is a rational function of  $z_2$  with possible poles at  $+1, -1, z_2, 1/z_2$ . One shows that there is no poles at the zeros  $\{+1, -1\}$  of  $M(x(z_1))$ .

## Higher topological recursion

## Higher topological recursion

For  $(g, k)$  with  $2g + k - 2 > 0$  (ie  $\neq \{(0, 1), (0, 2)\}$ ) let

$$\omega_k^{(g)}(z_1, \dots, z_k) := W_k^{(g)}(x(z_1), x(z_2), \dots, x(z_k))x'(z_1) \dots x'(z_k)$$

### Theorem

*For  $(g, k)$  with  $2g + k - 2 > 0$ ,  $\omega_k^{(g)}$  is a rational function of its Zhukovsky variables  $z_1, \dots, z_k$  with poles only at the branch points  $z_i = \pm 1$  and which behaves as  $O(z_i^{-2})$  at large  $z_i$ . They satisfy the topological recursion (next slide)*



# Topological recursion

$$\omega_{k+1}^{(g)}(z_0, L) = \frac{1}{2} \operatorname{Res}_{z \rightarrow \pm 1} \left( \frac{1}{z_0 - z} - \frac{1}{z_0 - 1/z} \right) \frac{z}{2y(z)x'(1/z)} \left( \sum'_{\substack{g_1 + g_2 = g \\ l_1 \sqcup l_2 = l}} \omega_{1+|l_1|}^{(g_1)}(z, z_{l_1}) \omega_{1+|l_2|}^{(g_2)}(z^{-1}, z_{l_2}) + \omega_{k+2}^{(g-1)}(z, z^{-1}, z_l) \right)$$

where  $y(z) = \frac{1}{2} M(x) \sqrt{(x-a)(x-b)}$ ,  $l = \{2, 3, \dots, k\}$  and  $\sum'$  means that we exclude the terms in  $\omega_0^{(1)}$ .

## Eynard's general statement

- ▶ What we have sketched holds in a very general setting (including bipartite maps, triangulations, quadrangulations, etc).

## Kontsevich recursion

## Motivation

- ▶ *Kontsevich TR* is a "simpler" topological recursion (introduced by Kontsevich for proving Witten's conjecture)
- ▶ (next course) We will see that Kontsevich TR is the "double scaling limit" of the combinatorial maps TR.

## Ribbon graphs and combinatorial moduli space

For  $2g - 2 + k > 0$ , let  $\mathcal{R}_{g,k}$  be the set of trivalent ribbon graphs of genus  $g$  and  $k$  boundaries.

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Set

$$\mathcal{M}_{g,k}^{comb} := \bigcup_{G \in \mathcal{R}_{g,k}} (G \times (\mathbb{R}_+)^{E(G)}) / \text{Aut}(G)$$

and  $\mathcal{M}_{g,k}^{comb}(L_1, L_2, \dots, L_k)$  be the graphs where the boundary lengths  $L_i$  are fixed.

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Example of  $(g, k) = (0, 3)$

Example of  $(g, k) = (1, 1)$



## The Kontsevich measure

$$\text{measure } \mu_{g,k} \text{ on } \mathcal{M}_{g,k}: d\mu_{g,k} \prod_{i=1}^k dL_i = 2^{2g-2+k} \prod_{e \in E(G)} \ell_e.$$

## The Kontsevich measure

measure  $\mu_{g,k}$  on  $\mathcal{M}_{g,k}$ : 
$$d\mu_{g,k} \prod_{i=1}^k dL_i = 2^{2g-2+k} \prod_{e \in E(G)} \ell_e.$$

### Theorem (Kontsevich)

Let

$$V_{g,k}(L_1, \dots, L_k) := \mu_{g,k}(\mathcal{M}_{g,k}^{comb}(L_1, \dots, L_k)).$$

Then  $V_{g,k}$  is an homogeneous polynomial of degree  $6g - 6 + 2k$  with only even powers of  $L_i$ . Its Laplace transform

$$\omega_{g,k}^K(z_1, \dots, z_k) = \left( \int_{\mathbb{R}_+^k} L_1 \cdots L_k e^{-z_i L_i} V_{g,k}(L_1, \dots, L_k) dL_1 \cdots dL_k \right)$$

satisfies a topological recursion (next slide).

# Topological recursion for (Laplace transform's of) Kontsevich volumes

$$\omega_{g,k}^K(z_1, \dots, z_k) = \text{Res}_{z \rightarrow 0} K(z_1, z) \left( \omega_{g-1,k+1}(z, -z, z_I) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = I}} \omega_{g_1,1+|I_1|}(z, z_{I_1}) \omega_{g_2,1+|I_2|}(z, z_{I_2}) \right)$$

where  $I = \{2, \dots, k\}$  and

$$K(z_1, z) = \frac{1}{2z(z - z_1)^2} \quad \text{and} \quad \omega_{0,2}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2}$$

# Tutte's equation for Kontsevich volumes

## Lemma

$$\omega_{g,k}^{Kon} = \sum_{G \in \mathcal{R}_{g,k}} \frac{1}{\# \text{Aut}(G)} \prod_{e \text{ between face } i \text{ and } j} \frac{1}{z_i + z_j}$$











