

# Topological recursion how to count graphs on surfaces?

course 4/4

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## Recall from course 3

## From Tutte equation to TR

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- ▶ they simplify into an Eynard-Orantin topological recursion

## Objective of course 4

- ▶ a bit more about Kontsevich TR
- ▶ "double scaling" limit of  $W_k^{(g)}(t; x_1, \dots, x_k)$  and  $\Omega_k^{(g)}(x_1, \dots, x_k)$

## Kontsevich TR

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Laplace transform of polynomials

$$\mathcal{L} : L^d \mapsto \frac{d!}{z^{d+1}}.$$

## Laplace transform

### Lemma

*Given a ribbon graph  $G$  the volume of the associated (orbifold) cell of  $\mathcal{M}_{g,k}^{comb}(L_1, \dots, L_k)$  has Laplace transform*

$$\frac{1}{\# \operatorname{Aut}(G)} \prod_{\substack{e \text{ between} \\ \text{faces } i \text{ and } j}} \frac{1}{z_i + z_j}$$

## Proof of Laplace transform lemma

$(\ell_e)_{e \in E(G)}$ : edge lengths

$(L_i)_{i=1,\dots,n}$ : boundary length

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$$A \cdot \begin{pmatrix} \ell_1 \\ \ell_2 \\ \dots \\ \ell_{6g-6+3n} \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \\ \dots \\ L_n \end{pmatrix}$$

where  $A$  is the edge-face adjacency matrix.



## Example

## Proof of Laplace transform lemma

Let  $V_A(L_1, \dots, L_n) = \mu_L(V_A(L))$  where  $\mu_L$  is the measure on  $E_L = \{\ell : A\ell = L\}$  such that

$$\mu_L \otimes \prod_{i=1}^n dL_i = \prod_{j=1}^m d\ell_j$$

We have

$$\begin{aligned} \int_{L_i \geq 0} \exp(-z \cdot L) V_A(L) dL &= \int_{L_i \geq 0} \exp(-z \cdot L) \left( \int_{\ell_i \geq 0, A\ell=L} d\mu_L(\ell) \right) dL \\ &= \int_{\ell_i \geq 0} \exp(-z \cdot (A\ell)) d\ell \\ &= \prod_{j=1}^m \frac{1}{A_{*,j} \cdot z} \end{aligned}$$

## Examples

$$\Omega_{0,3}(z_1, z_2, z_3) = \frac{1}{2z_1 z_2 z_3}$$

$$\Omega_{1,1}(z_1) = \frac{1}{48z_1^3}$$

$$\Omega_{0,4}(z_1, z_2, z_3, z_4) = \frac{1}{4z_1 z_2 z_3 z_4} \left( \frac{1}{z_1^2} + \frac{1}{z_2^2} + \frac{1}{z_3^2} + \frac{1}{z_4^2} \right)$$

## Topological recursion for Kontsevich volumes

### Theorem (Kontsevich)

*The  $\Omega_k^{(g)}(z_1, \dots, z_k) = \mathcal{L}(\text{Vol}(\mathcal{M}_k^{(g)}(L_1, \dots, L_k)))$  satisfy a topological recursion.*

## Double scaling limits

## Main result

Recall that our functions  $W_k^{(g)}(z_1, \dots, z_k)$  had poles at  $z_i = \pm 1$  only.

### Theorem

There exists  $\widetilde{W}_n^{(g)}(t; \zeta_1, \dots, \zeta_n)$  such that for any  $\zeta_1, \dots, \zeta_n$ , as  $\delta \rightarrow 0$  we have

$$W_n^{(g)}(t; 1+\delta\zeta_1, \dots, 1+\delta\zeta_n) \sim t^{2-2g-n} \delta^{(2-2g-n)(p+q)-n} \widetilde{W}_n^{(g)}(t; \zeta_1, \dots, \zeta_n)$$

where

- ▶  $(p, q)$  depends on the chosen weights  $(t_3, t_4, \dots)$  and comes from the "nature of the singularity" at  $+1$ ,
- ▶  $\widetilde{W}_k^{(g)}(t; z_1, \dots, z_k)$  satisfies a topological recursion

From  $\widetilde{W}_k^{(g)}(t; z_1, \dots, z_k)$  to asymptotics

Choosing the appropriate scaling  $\delta = (1 - t/t_c)^\nu$  (where  $\nu$  depends on  $(p, q)$ ) B. Eynard deduces precise asymptotics for the number of combinatorial maps.

## Combinatorial maps and Kontsevich volumes

Kontsevich volume admits a natural generalization with weights

### Theorem (Eynard-Orantin)

$\widetilde{W}_n^{(g)}(t; z_1, \dots, z_k)$  is a (generalization of) Kontsevich polynomial  $\omega_n^{(g)}(z_1, \dots, z_k)$ .

(Note: here  $\omega$  is an appropriate rescaling of  $\Omega$ )



Going further

## Link with other known results

- ▶ How to recover E. A. Bender, E. R. Canfield, L. B. Richmond (1993)?
- ▶ How to recover S. R. Carrell, G. Chapuy (2015)?
- ▶ Is there a link with M. Albenque, T. Lepoutre bijective approach?

## Wish list

- ▶ Complete all computations from the course (e.g. the  $u_k$  appearing in the  $z$ -expansion of  $W_0^{(1)}$ )
- ▶ Make computations with other weights  $t_i$  (e.g. triangulations or quadrangulations)
- ▶ In each of the above, analyze the corresponding double scaling limit.
- ▶ Find weights with "exotic" scaling  $\nu$  (or  $(p, q)$ ).
- ▶ Make the double scaling procedure geometric.