

Windtree for Veech parameters

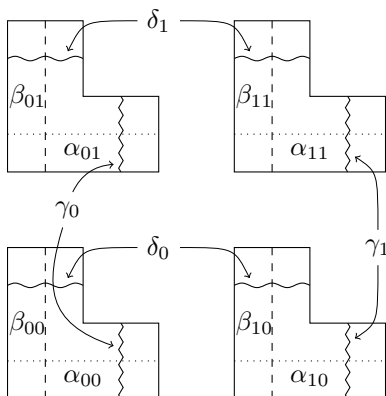
V. Delecroix (Université Paris Diderot, delecroix@mail.math.jussieu.fr)
 S. Lelièvre (Université Paris-Sud, Orsay, samuel.lelievre@gmail.com)

29 July - 2 August 2013, Morelia (Mexico)
 International conference and workshop on
 surfaces of infinite type

These exercises were designed to be part of a mini-course given in Morelia (Mexico) but it should be possible to do them with some knowledge on translation surfaces and the windtree model. To a quick introduction to the windtree model, see the first section of the article [DHL]. The authors will be happy to receive any question, comment or interesting solutions to their exercises.

We study the windtree models for Veech parameters, ie parameters (a, b) such that $M_{a,b}$ is a Veech surface (recall that $M_{a,b}$ is a translation surface in the stratum $\mathcal{H}(2, 2, 2, 2)$ that is an unramified normal cover of $L_{a,b}$ in $\mathcal{H}(2)$). Section 1 is about the topology of $M_{a,b}$, Section 2 concerns the special case $a = b = 1/2$ where all computations can be done explicitly. Section 3 is on general Veech parameters. You can consult the article [DHL] that contains the topological description of $M_{a,b}$.

1 Topology of $M_{a,b}$



Consider the set C of the 12 curves that are obtained by lifting the circumferences of horizontal and vertical cylinders in $L_{a,b}$ (implicitly oriented from left to right and from bottom to top):

- $\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}$ from the circumference of the long horizontal cylinder,
- $\beta_{00}, \beta_{01}, \beta_{10}, \beta_{11}$ from the circumference of the long vertical cylinder,
- γ_0, γ_1 from the circumference of the short horizontal cylinder,
- δ_0, δ_1 from the circumference of the short vertical cylinder.

We note $\text{Aut}(M_{a,b}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ the translation automorphisms of the surface $M_{a,b}$. We note τ_h and τ_v the natural basis of $\text{Aut}(M_{a,b})$ (that exchanges respectively each copy x_0 with the copy x_1 and each copy $0x$ with the copy $1x$). Let $E = H_1(S; \mathbb{Z})$ and E^{+-} as the set of $v \in E$ for which $\tau_h(v) = +v$ and $\tau_v(v) = -v$. Similarly we define E^{+-} , E^{-+} and E^{--} .

- Question 1.**
1. Explain why E is equal to the direct sum $E^{++} \oplus E^{+-} \oplus E^{-+} \oplus E^{--}$.
 2. Show that the set C is preserved by $\text{Aut}(M_{a,b})$, the group of translation automorphisms of $M_{a,b}$.
 3. Write the action of τ_h and τ_v on E and find a basis for each eigenspace $E^{++}, E^{+-}, E^{-+}, E^{--}$.
 4. What are the relations between these curves in E ? Note: C contains 12 curves and E has dimension 10.
 5. What is the dimension of each of the spaces E^{++}, E^{-+}, E^{+-} and E^{--} ?

2 The case $a = b = 1/2$

Recall that $L_{1/2,1/2}$ is the square-tiled surface made of three squares in $\mathcal{H}(2)$. In terms of permutations it is represented by $r = (1, 2)(3)$ and $u = (1, 3)(2)$. The surface $M_{1/2,1/2}$ is the unramified normal 4-fold cover of $L_{1/2,1/2}$ with group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Recall that the windtree surface is a \mathbb{Z}^2 cover of $M_{1/2,1/2}$.

We denote

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $G_{i,j} = \langle T^i, U^j \rangle$ and $PG_{i,j}$ be its image in $\text{PSL}(2, \mathbb{Z})$. Recall the definition of the principal congruence subgroups:

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}); a \equiv d \equiv 1 \pmod{n} \text{ and } b \equiv c \equiv 0 \pmod{n} \right\}$$

and the following congruence subgroups

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}); c \equiv 0 \pmod{n} \right\}.$$

Question 2 ($G_{i,j}$ and congruence groups). 1. Prove that $G_{1,1} = \mathrm{SL}(2, \mathbb{Z})$.

2. Prove that $G_{1,2} = \Gamma_0(2)$.
3. Prove that $\langle G_{1,3}, -I \rangle = \Gamma_0(3)$ and $\langle G_{1,4}, -I \rangle = \Gamma_0(4)$.
4. Prove that $\langle G_{2,2}, -I \rangle = \Gamma(2)$.

Question 3 (Geometry of $\mathrm{SO}(2) \backslash \mathrm{SL}(2, \mathbb{R}) / G_{i,j}$). 1. * Show that $G_{i,j}$ has finite index in $\mathrm{SL}(2, \mathbb{Z})$ if and only if $ij \leq 4$.

2. For each $G_{i,j}$ of finite index in $\mathrm{PSL}(2, \mathbb{Z})$, find representatives of cosets modulo $G_{i,j}$ and compute the action of the generators T and U on these cosets. (Hint: use the projection $\mathrm{PSL}(2, \mathbb{Z}) \rightarrow \mathrm{PSL}(2, \mathbb{Z}/n\mathbb{Z})$ where n is the level of congruence of $G_{i,j}$, ie 2,3 or 4).
3. Deduce that

$$[\mathrm{PSL}(2, \mathbb{Z}) : PG_{i,j}] = \begin{cases} 1 & \text{if } ij = 1, \\ 3 & \text{if } ij = 2, \\ 4 & \text{if } ij = 3, \\ 6 & \text{if } ij = 4, \\ \infty & \text{otherwise.} \end{cases}$$

4. For each finite index $G_{i,j} < \mathrm{PSL}(2, \mathbb{Z})$ consider the Riemann surface $X_{i,j} = \mathrm{SO}(2) \backslash \mathrm{SL}(2, \mathbb{R}) / G_{i,j}$. Find the genus, the number of cusps and the number and nature of elliptic points of $X_{i,j}$. (Hint: for $G_{1,1} = \mathrm{SL}(2, \mathbb{Z})$ it is well known that there is one cusp (the conjugacy class of T), one elliptic-point of order two (the conjugacy class of T^2U^{-1}) and one elliptic-point of order three (the conjugacy class of TU^{-1}). Then for each group $G_{i,j}$, considering the ramified covering $X_{i,j} \rightarrow X_{1,1}$.)

Define $\rho^{++} = \rho^{--} : G_{2,2} \rightarrow G_{1,1}$, $\rho^{+-} : G_{2,2} \rightarrow G_{1,3}$ and $\rho^{-+} : G_{2,2} \rightarrow G_{1,3}$ by

$$\rho^{++}(T^2) = T, \quad \rho^{++}(U^2) = U, \quad \rho^{+-}(T^2) = T, \quad \rho^{+-}(U^2) = U^3, \quad \rho^{-+}(T^2) = U^3, \quad \text{and} \quad \rho^{-+}(U^2) = T.$$

Question 4 (Kontsevich-Zorich cocycle for $M_{1/2,1/2}$). 1. Prove that ρ^{++} , ρ^{+-} , ρ^{-+} and ρ^{--} are well defined.

2. Prove that the Veech group of $M_{1/2,1/2}$ contains $G_{2,2}$ as a subgroup of finite index (more precisely, $G_{2,2}$ corresponds to a ramified cover of the Teichmüller curve of $M_{1/2,1/2}$ which is torsion free).
3. Prove that the Kontsevich-Zorich cocycle of $M_{1/2,1/2}$ seen over $\mathrm{SL}(2, \mathbb{R}) / G_{2,2}$ may be written as the group morphism $\rho : G_{2,2} \rightarrow \mathrm{Sp}(10)$ given by $\rho : \mathrm{id} \oplus \rho^{++} \oplus \rho^{+-} \oplus \rho^{-+} \oplus \rho^{--}$.
4. Compute $\mathrm{covol}(G_{1,1}) / \mathrm{covol}(G_{2,2})$ and $\mathrm{covol}(G_{1,3}) / \mathrm{covol}(G_{2,2})$.
5. * As shown in [Kap], for each ρ in $\{\rho^{++}, \rho^{+-}, \rho^{-+}\}$ there exists an equivariant holomorphic map $f : \mathbb{H} \rightarrow \mathbb{H}$, ie such that for all $f(\gamma \cdot z) = \rho(\gamma) \cdot f(z)$. By passing to the quotient, we obtain holomorphic maps between Riemann surfaces: $f^{+-} : X_{2,2} \rightarrow X_{1,3}$, $f^{++} : X_{2,2} \rightarrow X_{1,1}$ and $f^{--} : X_{2,2} \rightarrow X_{1,1}$. Compute

$$\frac{\deg(f^{+-}) \mathrm{vol}(X_{1,3})}{\mathrm{vol}(X_{2,2})}, \quad \frac{\deg(f^{++}) \mathrm{vol}(X_{1,1})}{\mathrm{vol}(X_{2,2})} \quad \text{and} \quad \frac{\deg(f^{--}) \mathrm{vol}(X_{1,1})}{\mathrm{vol}(X_{2,2})}.$$

Show that these numbers are the Lyapunov exponents of the Kontsevich-Zorich cocycle¹, namely 1/3 and 2/3.

Recall that in $\mathrm{PSL}(2, \mathbb{Z})$ the relations between T and U are generated by

$$(T^{-1}UT^{-1})^2 \quad \text{and} \quad (T^{-1}U)^3. \tag{1}$$

Question 5 (Kernel of ρ^{+-} and ρ^{-+}). 1. Check that the two products in (1) are relations in $\mathrm{PSL}(2, \mathbb{Z})$.

2. * Find the kernels of $P\rho^{+-}$ and $P\rho^{-+}$ ². What is their intersection? (Hint: use question 3).

Question 6 (Possible diffusion rates for $T_{1/2,1/2}$). Diffusion rate in the windtree corresponds to Lyapunov exponents in E^{+-} (for the horizontal diffusion) or E^{-+} (for the vertical diffusion). For a hyperbolic $g \in G_{2,2}$, the corresponding positive Lyapunov exponent in E^{+-} is the ratio of the logarithms of the dominant eigenvalues of respectively $\rho^{+-}(g)$ and g .

¹This is actually a way to compute Lyapunov exponents in $M_{1/2,1/2}$ that can be extended to other similar situations see [Kap].

²By a result of B. Weiss and P. Hooper [HW], these kernels are not trivial.

1. Find a direction θ associated to a pseudo-Anosov diffeomorphism such that the horizontal and vertical diffusion rates differ. In other words, find a hyperbolic $g \in G_{2,2}$ such that $\rho^{+-}(g)$ and $\rho^{-+}(g)$ have different eigenvalues.
2. Show that the diffusion rates coming from pseudo-Anosovs are dense among all diffusion rates.
3. Deduce that the possible diffusion rates form an interval.
4. What is the minimum diffusion rate?
5. Show that for a large set of directions, we have diffusion rate 0 in the windtree model.
6. * Show that the maximum diffusion rate is less than 1 (Hint: use Forni's theorem [For02] which states that the expansion in the orthogonal of the tautological subspace is uniformly bounded on compact sets, together with the fact that *any* geodesic in a hyperbolic surface spends most of its time in a compact part).
7. * What is the maximal diffusion rate?

3 General Veech parameters

Recall (Caltà [Cal04], McMullen [McM03]) that $L_{a,b}$ is Veech if and only if it admits both a horizontal and a vertical Dehn-multitwist.

Question 7. 1. For which (a, b) does the surface $L_{a,b}$ admit a horizontal multitwist? a vertical one? both?
 2. Show that $L_{a,b}$ is a Veech surface if and only if $M_{a,b}$ is a Veech surface.

Question 8. 1. Assume that $M_{a,b}$ admits a multitwist in the horizontal direction. Write down the action of the multitwists on each component E^{-+} , E^{+-} , E^{--} .
 2. * For which Veech parameters (a, b) is the group generated by the horizontal and vertical multitwists of finite index in the Veech group?
 3. For these values, compute the morphisms $\Gamma(M_{a,b}) \rightarrow \mathrm{Sp}(E^{+-})$ and $\Gamma(M_{a,b}) \rightarrow \mathrm{Sp}(E^{-+})$ associated to the Kontsevich-Zorich cocycle.
 4. Compute the two kernels and their intersection.

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