Windtree for Veech parameters

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These exercises were designed to be part of a mini-course given in Morelia (Mexico) but it should be possible to do them with some knowledge on translation surfaces and the windtree model. To a quick introduction to the windtree model, see the first section of the article [DHL]. The authors will be happy to receive any question, comment or interesting solutions to their exercises.

We study the windtree models for Veech parameters, ie parameters \((a, b)\) such that \(M_{a, b}\) is a Veech surface (recall that \(M_{a, b}\) is a translation surface in the stratum \(\mathcal{H}(2, 2, 2, 2)\) that is an unramified normal cover of \(L_{a, b}\) in \(\mathcal{H}(2)\)).

Section 1 is about the topology of \(M_{a, b}\), Section 2 concerns the special case \(a = b = 1/2\) where all computations can be done explicitly. Section 3 is on general Veech parameters. You can consult the article [DHL] that contains the topological description of \(M_{a, b}\).

1 Topology of \(M_{a, b}\)

Consider the set \(C\) of the 12 curves that are obtained by lifting the circumferences of horizontal and vertical cylinders in \(L_{a, b}\) (implicitly oriented from left to right and from bottom to top):

- \(\alpha_{00},\alpha_{01},\alpha_{10},\alpha_{11}\) from the circumference of the long horizontal cylinder,
- \(\beta_{00},\beta_{01},\beta_{10},\beta_{11}\) from the circumference of the long vertical cylinder,
- \(\gamma_0,\gamma_1\) from the circumference of the short horizontal cylinder,
- \(\delta_0,\delta_1\) from the circumference of the short vertical cylinder.

We note \(\text{Aut}(M_{a, b}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) the translation automorphisms of the surface \(M_{a, b}\). We note \(\tau_h\) and \(\tau_v\) the natural basis of \(\text{Aut}(M_{a, b})\) (that exchanges respectively each copy \(x_0\) with the copy \(x_1\) and each copy \(0x\) with the copy \(1x\)). Let \(E = H_1(S;\mathbb{Z})\) and \(E^{++}\) as the set of \(v \in E\) for which \(\tau_h(v) = +v\) and \(\tau_v(v) = -v\). Similarly we define \(E^{+-}, E^{-+}\) and \(E^{--}\).

**Question 1.**
1. Explain why \(E\) is equal to the direct sum \(E^{++} \oplus E^{+-} \oplus E^{-+} \oplus E^{--}\).
2. Show that the set \(C\) is preserved by \(\text{Aut}(M_{a, b})\), the group of translation automorphisms of \(M_{a, b}\).
3. Write the action of \(\tau_h\) and \(\tau_v\) on \(E\) and find a basis for each eigenspace \(E^{++}, E^{+-}, E^{-+}, E^{--}\).
4. What are the relations between these curves in \(E\)? Note: \(C\) contains 12 curves and \(E\) has dimension 10.
5. What is the dimension of each of the spaces \(E^{++}, E^{+-}, E^{-+}\) and \(E^{--}\)?

2 The case \(a = b = 1/2\)

Recall that \(L_{1/2,1/2}\) is the square-tiled surface made of three squares in \(\mathcal{H}(2)\). In terms of permutations it is represented by \(r = (1, 2)(3)\) and \(u = (1, 3)(2)\). The surface \(M_{1/2,1/2}\) is the unramified normal 4-fold cover of \(L_{1/2,1/2}\) with group \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\). Recall that the windtree surface is a \(\mathbb{Z}^2\) cover of \(M_{1/2,1/2}\).

We denote

\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Let \(G_{i,j} = \langle T^i, U^j \rangle\) and \(PG_{i,j}\) be its image in \(\text{PSL}(2, \mathbb{Z})\). Recall the definition of the principal congruence subgroups:

\[
\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}); a \equiv d \equiv 1 \mod n \text{ and } b \equiv c \equiv 0 \mod n \right\}
\]
and the following congruence subgroups

\[ \Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2,\mathbb{Z}) ; c \equiv 0 \mod n \right\}. \]

**Question 2** (\(G_{i,j}\) and congruence groups).  
1. Prove that \(G_{1,1} = \text{SL}(2,\mathbb{Z})\).
2. Prove that \(G_{1,2} = \Gamma_0(2)\).
3. Prove that \(\langle G_{1,3}, -I \rangle = \Gamma_0(3)\) and \(\langle G_{1,4}, -I \rangle = \Gamma_0(4)\).
4. Prove that \(\langle G_{2,2}, -I \rangle = \Gamma(2)\).

**Question 3** (Geometry of \(\text{SO}(2)\)/\(\text{SL}(2,\mathbb{R})/G_{i,j}\)).  
1. * Show that \(G_{i,j}\) has finite index in \(\text{SL}(2,\mathbb{Z})\) if and only if \(ij \leq 4\).
2. For each \(G_{i,j}\) of finite index in \(\text{PSL}(2,\mathbb{Z})\), find representatives of cosets modulo \(G_{i,j}\) and compute the action of the generators \(T\) and \(U\) on these cosets. (Hint: use the projection \(\text{PSL}(2,\mathbb{Z}) \to \text{PSL}(2,\mathbb{Z}/n\mathbb{Z})\) where \(n\) is the level of congruence of \(G_{i,j}\), ie 2,3 or 4).
3. Deduce that

\[
[\text{PSL}(2,\mathbb{Z}) : PG_{i,j}] = \begin{cases} 
1 & \text{if } ij = 1, \\
3 & \text{if } ij = 2, \\
4 & \text{if } ij = 3, \\
6 & \text{if } ij = 4, \\
\infty & \text{otherwise}.
\end{cases}
\]

4. For each finite index \(G_{i,j} < \text{PSL}(2,\mathbb{Z})\) consider the Riemann surface \(X_{i,j} = \text{SO}(2)\)/\(\text{SL}(2,\mathbb{R})/G_{i,j}\). Find the genus, the number of cusps and the number and nature of elliptic points of \(X_{i,j}\). (Hint: for \(G_{1,1} = \text{SL}(2,\mathbb{Z})\) it is well known that there is one cusp (the conjugacy class of \(T\)), one elliptic-point of order two (the conjugacy class of \(T^2U^{-1}\)) and one elliptic-point of order three (the conjugacy class of \(TU^{-1}\)). Then for each group \(G_{i,j}\), considering the ramified covering \(X_{i,j} \to X_{1,1}\)

Define \(\rho^{++} = \rho^{-} : G_{22} \to G_{11}, \rho^{+-} : G_{22} \to G_{13}\) and \(\rho^{-+} : G_{22} \to G_{13}\) by

\[ \rho^{++}(T^2) = T, \quad \rho^{+-}(U^2) = U, \quad \rho^{-+}(U^2) = U^3, \quad \rho^{-+}(T^2) = U^3, \quad \text{and } \rho^{++}(U^2) = T. \]

**Question 4** (Kontsevich-Zorich cocycle for \(M_{1/2,1/2}\)).  
1. Prove that \(\rho^{++}, \rho^{+-}, \rho^{-+}\) and \(\rho^{-}\) are well defined.
2. Prove that the Veech group of \(M_{1/2,1/2}\) contains \(G_{2,2}\) as a subgroup of finite index (more precisely, \(G_{2,2}\) corresponds to a ramified cover of the Teichmüller curve of \(M_{1/2,1/2}\) which is torsion free).
3. Prove that the Kontsevich-Zorich cocycle of \(M_{1/2,1/2}\) seen over \(\text{SL}(2,\mathbb{R})/G_{2,2}\) may be written as the group morphism \(\rho : G_{2,2} \to \text{Sp}(10)\) given by \(\rho : \text{id} \oplus \rho^{++} \oplus \rho^{+-} \oplus \rho^{-+}\).
4. Compute \(\text{covol}(G_{1,1}) / \text{covol}(G_{2,2})\) and \(\text{covol}(G_{1,3}) / \text{covol}(G_{2,2})\).
5. * As shown in [Kap], for each \(\rho \in \{\rho^{++}, \rho^{+-}, \rho^{-+}\}\) there exists an equivariant holomorphic map \(f : \mathbb{H} \to \mathbb{H}\), ie such that for all \(f(\gamma \cdot z) = \rho(\gamma) \cdot f(z)\). By passing to the quotient, we obtain holomorphic maps between Riemann surfaces: \(f^{++} : X_{2,2} \to X_{1,3}, f^{+-} : X_{2,2} \to X_{1,1}\) and \(f^{-+} : X_{2,2} \to X_{1,1}\). Compute

\[
\frac{\deg(f^{++}) \text{vol}(X_{1,3})}{\text{vol}(X_{2,2})}, \quad \frac{\deg(f^{+-}) \text{vol}(X_{1,1})}{\text{vol}(X_{2,2})} \quad \text{and} \quad \frac{\deg(f^{-+}) \text{vol}(X_{1,1})}{\text{vol}(X_{2,2})}.
\]

Show that these numbers are the Lyapunov exponents of the Kontsevich-Zorich cocycle \(\Gamma\), namely 1/3 and 2/3.

Recall that in \(\text{PSL}(2,\mathbb{Z})\) the relations between \(T\) and \(U\) are generated by

\[
(T^{-1}UT^{-1})^2 \quad \text{and} \quad (T^{-1}U)^3.
\]

**Question 5** (Kernel of \(\rho^{++}\) and \(\rho^{-+}\)).  
1. Check that the two products in \([\Gamma]\) are relations in \(\text{PSL}(2,\mathbb{Z})\).
2. * Find the kernels of \(P\rho^{++}\) and \(P\rho^{-+}\) \(^2\) What is their intersection? (Hint: use question 3).

**Question 6** (Possible diffusion rates for \(T_{1/2,1/2}\)). Diffusion rate in the windtree corresponds to Lyapunov exponents in \(E^{++}\) (for the horizontal diffusion) or \(E^{-+}\) (for the vertical diffusion). For a hyperbolic \(g \in G_{2,2}\), the corresponding positive Lyapunov exponent in \(E^{++}\) is the ratio of the logarithms of the dominant eigenvalues of respectively \(\rho^{++}(g)\) and \(g\).

\(^1\)This is actually a way to compute Lyapunov exponents in \(M_{1/2,1/2}\) that can be extended to other similar situations see [Kap].

\(^2\)By a result of B. Weiss and P. Hooper [HW], these kernels are not trivial.
1. Find a direction $\theta$ associated to a pseudo-Anosov diffeomorphism such that the horizontal and vertical diffusion rates differ. In other words, find a hyperbolic $g \in G_{2,2}$ such that $\rho^+(g)$ and $\rho^-(g)$ have different eigenvalues.
2. Show that the diffusion rates coming from pseudo-Anosovs are dense among all diffusion rates.
3. Deduce that the possible diffusion rates form an interval.
4. What is the minimum diffusion rate?
5. Show that for a large set of directions, we have diffusion rate 0 in the windtree model.
6. * Show that the maximum diffusion rate is less than 1 (Hint: use Forni’s theorem [For02] which states that the expansion in the orthogonal of the tautological subspace is uniformly bounded on compact sets, together with the fact that any geodesic in a hyperbolic surface spends most of its time in a compact part).
7. * What is the maximal diffusion rate?

3 General Veech parameters

Recall (Calta [Cal04], McMullen [McM03]) that $L_{a,b}$ is Veech if and only if it admits both a horizontal and a vertical Dehn-multitwist.

**Question 7.**
1. For which $(a, b)$ does the surface $L_{a,b}$ admit a horizontal multitwist? a vertical one? both?
2. Show that $L_{a,b}$ is a Veech surface if and only if $M_{a,b}$ is a Veech surface.

**Question 8.**
1. Assume that $M_{a,b}$ admits a multitwist in the horizontal direction. Write down the action of the multitwists on each component $E^{++}, E^{+-}, E^{--}$.
2. * For which Veech parameters $(a, b)$ is the group generated by the horizontal and vertical multitwists of finite index in the Veech group?
3. For these values, compute the morphisms $\Gamma(M_{a,b}) \rightarrow \text{Sp}(E^{+-})$ and $\Gamma(M_{a,b}) \rightarrow \text{Sp}(E^{-+})$ associated to the Kontsevich-Zorich cocycle.
4. Compute the two kernels and their intersection.

References


