Interval exchange transformations
Part II: Minimality, Rauzy induction and
Teichmüller flow)

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Rotations

Recall that for rotation we have:

**Theorem**

Let $\alpha$ be irrational, and $X_\alpha$ be the Sturmian shift associated to the rotation $T_\alpha$. Then:

- $p_{X_\alpha}(n) = n + 1$, in particular $X_\alpha$ has 0 entropy;
- the shift $X_\alpha$ is minimal (all orbits are dense);
- (Hecke (1922), Ostrowski (1922)) any clopen $Y \subset X_\alpha$ has bounded remainder: there exists $\mu_Y$ and $C_Y$ so that

\[
\forall x \in X_\alpha, \forall n \geq 0, \quad \left| \sum_{k=0}^{n} (\chi_Y(T_\alpha^k x) - \mu_Y) \right| \leq C_Y.
\]

In particular, the shift $X_\alpha$ is uniquely ergodic.
An interval exchange transformation is a piecewise translation of the interval that is a bijection from $I \{ \alpha_1^{top}, \ldots, \alpha_{d-1}^{top} \}$ to $I \{ \alpha_1^{bot}, \ldots, \alpha_{d-1}^{bot} \}$.

The above interval exchange can be defined from:

- a "permutation" $\pi = (A \ B \ C \ D \ C \ A \ D \ B)$,
- a length vector $\lambda = (\lambda_A, \lambda_B, \lambda_C, \lambda_D)$. 
Interval exchange transformations

An interval exchange transformation is a piecwise translation of the interval that is a bijection from \( I \setminus \{\alpha_{1}^{\text{top}}, \ldots, \alpha_{d-1}^{\text{top}}\} \) to \( I \setminus \{\alpha_{1}^{\text{bot}}, \ldots, \alpha_{d-1}^{\text{bot}}\} \).

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- a "permutation" \( \pi = \begin{pmatrix} A & B & C & D \\ C & A & D & B \end{pmatrix} \),
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Main motivation: rational billiards

(... Sage ...)
As we did for rotations, we could code orbits in \( \{A, B, C, D\}^\mathbb{Z} \) (except the singular ones). We obtain a shift \( X_{\pi, \lambda} \) and a factor map \( p : X_{\pi, \lambda} \to I \).
As we did for rotations, we could code orbits in \{A, B, C, D\}^\mathbb{Z} (except the singular ones). We obtain a shift $X_{\pi, \lambda}$ and a factor map $p : X_{\pi, \lambda} \to I$. All orbits of the iet $T_{\pi, \lambda}$ has one preimage in $X_{\pi, \lambda}$ except the singular ones that have two (i.e. the future orbits of $\alpha_{1}^{bot}, \ldots, \alpha_{2}^{bot}$ and the past orbits of $\alpha_{1}^{top}, \ldots, \alpha_{d-1}^{bot}$).
Connections

\[ \Sigma^{\text{top}} := \{ \alpha_1^{\text{top}}, \ldots, \alpha_{d-1}^{\text{top}} \} \text{ (singularities of } T) \]

\[ \Sigma^{\text{bot}} := \{ \alpha_1^{\text{bot}}, \ldots, \alpha_{d-1}^{\text{bot}} \} \text{ (singularities of } T^{-1}). \]

A connection is a triple \((m, \alpha, \beta)\) where \(m \geq 0, \alpha \in \Sigma^{\text{top}}, \beta \in \Sigma^{\text{bot}}\) and \(T^m \beta = \alpha.\)
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- A rotation has a connection if and only if the angle is rational.
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- A rotation has a connection if and only if the angle is rational.
- If the length data $\lambda$ is rational

$$\dim_{\mathbb{Q}} \left( \mathbb{Q} \frac{\lambda_1}{\lambda_d} + \mathbb{Q} \frac{\lambda_2}{\lambda_d} + \ldots + \mathbb{Q} \frac{\lambda_{d-1}}{\lambda_d} \right) = 1$$

then there are $d - 1$ connections (and all orbits are periodic).
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A **connection** is a triple \((m, \alpha, \beta)\) where \(m \geq 0\), \(\alpha \in \Sigma^{\text{top}}\), \(\beta \in \Sigma^{\text{bot}}\) and \(T^{m} \beta = \alpha\).

\[ \triangleright \text{ A rotation has a connection if and only if the angle is rational.} \]
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then there are \(d - 1\) connections (and all orbits are periodic).

\[ \triangleright \text{ If the length data } \lambda \text{ is maximally irrational} \]
\[ \dim_{\mathbb{Q}} \left( \mathbb{Q} \frac{\lambda_{1}}{\lambda_{d}} + \mathbb{Q} \frac{\lambda_{2}}{\lambda_{d}} + \ldots + \mathbb{Q} \frac{\lambda_{d-1}}{\lambda_{d}} \right) = d - 1 \]

then there are no connection.
Coding

**Theorem**

Let $X_{\pi,\lambda}$ be the shift associated to an interval exchange transformation $T_{\pi,\lambda}$ on $d$ intervals with $\pi$ irreducible. Then the following are equivalent

- $p_{\pi,\lambda}(n) = (d - 1)n + 1$,
- $T_{\pi,\lambda}$ has no connection.
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In general

$$\lim_{n \to \infty} \frac{p_{\pi, \lambda}(n)}{n} = (d - 1) - \#\text{connections}.$$
Minimality: Keane theorem

Theorem

If $T_{\pi,\lambda}$ has no connection then $X_{\pi,\lambda}$ is minimal. In other words, all infinite orbits of $T_{\pi,\lambda}$ are dense.
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Corollary

Let $\pi$ be a primitive substitution. Then for almost every $\lambda$ with respect to the Lebesgue measure, the interval exchange transformation $T_{\pi,\lambda}$ is minimal.
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**Corollary**

In a rational billiard, excepted countably many directions the flow is minimal.
Back and forth between iet and translation surfaces

Poincaré maps

suspensions

I. E. T.

Translation Surfaces
Back and forth between $\text{i.e.t.}$ and translation surfaces

- Rauzy induction
- Poincaré maps
- I. E. T.
- Translation Surfaces
- Teich. flow
- suspensions