Contribution of maximal-cylinder surfaces to the Masur-Veech volumes of the minimal strata of Abelian differentials

Ivan Yakovlev

Contents

1 Translation surfaces 1
  1.1 Motivation: billiards in polygons ................................. 1
  1.2 Three definitions of a translation surface ....................... 2
  1.3 Moduli spaces of Abelian differentials ........................... 4
  1.4 Period coordinates ............................................ 4

2 Masur-Veech volumes and square-tiled surfaces 5
  2.1 Masur-Veech volumes ........................................... 5
  2.2 Computation of volumes via square-tiled surfaces .............. 5
  2.3 Geometry of square-tiled surfaces ............................. 7
  2.4 Examples of computations ....................................... 8

3 Contribution of maximal-cylinder surfaces to the volumes of
  minimal strata 9
  3.1 Motivation .................................................... 9
  3.2 Generic values exist ........................................... 12
  3.3 Recurrence for the generic values of $Q_k$ ....................... 16
  3.4 Analysis for each connected component of the minimal strata 18
    3.4.1 Preliminaries ............................................. 19
    3.4.2 Existence of generic values ............................... 22
    3.4.3 Recurrences for generic values ............................ 23

References 27

1 Translation surfaces

In this section we define translation surfaces, their moduli spaces and a special set of coordinates on the latter. The material we present in this section is classic, and has been discussed in many wonderful surveys, see, for example [Wri16a], [Wri16b], [Zor06], [MT02].

1.1 Motivation: billiards in polygons

A simple (and visual) motivation for the study of translation surfaces comes from billiards in rational polygons.

A billiard consists of a (not necessarily convex) polygon in the plane and a point in its interior which moves in a straight line, with constant (unit) speed, and reflects ideally from the sides of the polygon (the angle of incidence is equal
to the angle of reflection). If the point gets to the vertex of the polygon, its
trajectory ends there — “the billiard ball falls into the pocket”.

Consider now the (simplest) billiard in the form of a rectangle and a traject-
ory inside it. Follow this trajectory until it reaches the side. Then, instead of
reflecting the trajectory, reflect the billiard table. Repeat this procedure each
time the trajectory reaches the side. In this way we “straighten” the trajectory,
while tiling the plane with the copies of our rectangle. Note that there are 4
different types of these copies, corresponding to 4 possible directions in which
this trajectory can travel. Since the corresponding points in rectangles of
the same type represent the same points and directions in the initial billiard, we
can identify these rectangles by translation, and by doing so we obtain a flat torus
(glued from 4 copies of the initial rectangle). The trajectories of the billiard
now correspond to the geodesics of the flat metric on this torus.

It turns out that this construction works for any initial polygon whose angles
are all rational multiples of $\pi$. Indeed, this condition ensures that a single
trajectory can travel in only a finite number of directions. Then, taking a finite
number of copies of our polygon (each corresponding to a possible direction
of travel), we identify their sides in pairs by translation, according to how the
direction changes when we reach the corresponding side. In this way we obtain
a surface with flat metric, whose geodesics correspond to the trajectories of the
initial billiard.

Note however, that in general this flat metric will have a finite number
of conical singularities, which come from the vertices of the polygons. For
example, a billiard in the triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$ gives rise to a regular
octagon, whose opposite sides should be identified by translation. Identification
produces a surface of genus 2 with flat metric and one conical singularity of total
angle $6\pi$, which comes from the vertices of the octagon (they are all identified
between themselves). An $\varepsilon$-neighborhood of this singularity is isometric to 6
half-discs of radius $\varepsilon$ which are glued along their radii in cyclic order.

The surfaces we get by this procedure are special cases of translation surfaces.

1.2 Three definitions of a translation surface

Definition 1.1. A translation surface is (an equivalence class of) a collection
of polygons in the plane $\mathbb{C}$ whose sides are divided into pairs of equal and parallel
sides which are identified by translation (the corresponding polygons should be
“on different sides” of these parallel sides). Two collections of polygons represent
the same translation surface (are equivalent) if one can be obtained from the
other by the following operations:

- translate a polygon in the plane;

- cut a polygon along a straight line into two parts and identify the two new
equal and parallel sides;

- glue two polygons along a pair of identified sides.
Obviously, these “cut and glue” operations do not change the underlying flat surface. What changes is the way we “cut” and “unfold” the surface on the plane.

Note that such polygon representation gives an additional structure on the underlying flat surface. Namely, since we cannot turn the polygons, and since all the side identifications are made by translation, the canonical vertical direction \( i \) at each point of \( \mathbb{C} \) descends to a consistent choice of “vertical” direction at each (non-singular) point of the surface. This gives rise to the following intrinsic geometric definition of a translation surface (for the proof of equivalence to Definition 1.1, see [Wri16b], Section 1).

**Definition 1.2.** A translation surface is a compact oriented flat surface \( X \) with a finite set of conical singularities \( \Sigma \), trivial holonomy and a consistent choice of “vertical” direction at each non-singular point.

Since the holonomy is trivial, all the conical singularities should have angles which are integer multiples of \( 2\pi \) (this can also be easily seen from a polygon representation). However, the converse is not true: even if all the conical singularities of a flat surface have angles which are integer multiples of \( 2\pi \), it can have a non-trivial holonomy, see Figure ??? for an example. This shows that translation surfaces are rather special instances of general flat surfaces.

Observe that \( X - \Sigma \) possesses an atlas of charts to \( \mathbb{C} \) such that the transition functions are all translations \( z \mapsto z + c \). Indeed, around any non-singular point \( X \) is isometric to \( \mathbb{C} \) and the isometry can be chosen to preserve the vertical direction. The transitions between such charts are translations, because they are the only isometries of \( \mathbb{C} \) that preserve the vertical direction.

Obviously, such an atlas equips \( X - \Sigma \) with a complex structure. Moreover, since the 1-form \( dz \) on \( \mathbb{C} \) is invariant under translations, the 1-forms \( dz \) in each chart glue together to give a holomorphic 1-form \( \omega \) on \( X - \Sigma \).

Moreover, both the complex structure and the 1-form \( \omega \) can be extended to all of \( X \): around any singularity with angle \( 2\pi(k+1) \) there is a coordinate \( z \) such that the map \( z \mapsto z^{k+1} \) is a local isometry onto a punctured neighborhood of 0 in \( \mathbb{C} \), and preserves the vertical direction at each non-singular point; it is easy to show that this chart is compatible with the previous charts; the expression for \( \omega \) in this chart is \( (k+1)z^k dz \) (it is the pullback of \( dz \) by \( z \mapsto z^{k+1} \)), so \( \omega \) extends to \( X \), the singularities \( \Sigma \) corresponding to its zeros.

**Definition 1.3.** A translation surface is a pair \( (X, \omega) \), where \( X \) is a Riemann surface and \( \omega \) is a non-zero holomorphic 1-form on \( X \) (also called an Abelian differential).

We have seen above that the structure of Definition 1.2 gives the structure of Definition 1.3. Conversely, given \( \omega \), at each point that is not a zero of \( \omega \) there is a unique coordinate \( z \) such that \( \omega = dz \), and at each zero of multiplicity \( k \) there is a coordinate \( z \) such that \( \omega = z^k dz \). Transition functions between the charts of the first group are necessarily translations, and so the flat metrics in each chart glue together to form a flat metric on \( X - \text{Zeroes}(\omega) \), with a consistent
choice of vertical direction. The zeroes of \( \omega \) of multiplicity \( k \) then correspond to conical singularities with angle \( 2\pi(k+1) \).

Beware of the following correspondence between the flat-geometric and complex-analytic terms: “translation surface” \( \leftrightarrow \) “Abelian differential”, “conical singularity of angle \( 2\pi(k+1) \)” \( \leftrightarrow \) “zero of \( \omega \) of multiplicity \( k \)”, “(complex) length of a geodesic \( \gamma \)” \( \leftrightarrow \) “\( \int_{\gamma} \omega \)”. In what follows we will use these terms interchangeably.

1.3 Moduli spaces of Abelian differentials

Consider now the problem of parametrizing all Abelian differentials (i.e. translation surfaces). The space of all Abelian differentials on surfaces of fixed genus is called a moduli space of Abelian differentials.

First of all, due to the Gauss-Bonnet type restriction on curvature, for a translation surface of genus \( g \geq 1 \) with conical singularities of angles \( 2\pi(k_1+1), \ldots, 2\pi(k_s+1) \) we must have \( k_1 + \ldots + k_s = 2g - 2 \). It means that the moduli spaces of Abelian differentials are naturally stratified according to the number and the angles of conical singularities. We denote each such stratum by \( \mathcal{H}(k_1, \ldots, k_s) \).

Recall a regular octagon whose opposite sides are identified (Figure ?????). It represents a surface from \( \mathcal{H}(2) \). It is easy to see that its small deformations are parametrized by the vectors (or elements of \( \mathbb{C} \)) representing pairs of identified sides. This suggests that \( \mathcal{H}(2) \) is locally \( \mathbb{C}^4 \).

Likewise, the (complex) lengths of identified sides in a polygon representation of a general \((X, \omega)\) determine it uniquely. However, these lengths are not all independent in general, so they are not proper coordinates on \( \mathcal{H}(k_1, \ldots, k_s) \). To fix this problem, take a set of curves \( \gamma_i \) representing a basis of the relative homology group \( H_1(X, \Sigma, \mathbb{Z}) \) (this basis can be naturally transferred to the nearby surfaces \( X \)). Then the integrals \( \int_{\gamma_i} \omega \) give the local coordinates on \( \mathcal{H}(k_1, \ldots, k_s) \) around \((X, \omega)\). Indeed, these integrals determine the integral of \( \omega \) over any element of the relative homology, in particular, over any curve representing a side in any polygon decomposition. Such integrals are exactly the complex lengths of corresponding sides.

To sum up, we see that each stratum \( \mathcal{H}(k_1, \ldots, k_s) \) is locally \( \mathbb{C}^{2g+s-1} \). In fact, \( \mathcal{H}(k_1, \ldots, k_s) \) is a complex orbifold of dimension \( 2g+s-1 \), the orbifold points corresponding to the surfaces with non-trivial symmetry groups (for example, the regular octagon above represents an orbifold point).

1.4 Period coordinates

The coordinates on \( \mathcal{H}(k_1, \ldots, k_s) \) we have constructed are called period coordinates. Let us take a closer look at them. There was a degree of freedom in their construction, namely, the choice of the basis of \( H_1(X, \Sigma, \mathbb{Z}) \). Hence, each choice of the basis gives another chart around the same point. Since the change of basis in \( H_1(X, \Sigma, \mathbb{Z}) \) is given by a matrix in \( GL(2g+s-1, \mathbb{Z}) \), the transition function between the corresponding charts on \( \mathcal{H}(k_1, \ldots, k_s) \) will also be given by a matrix from \( GL(2g+s-1, \mathbb{Z}) \).
Note that when the (complex) sides of a polygon representation do give coordinates on the stratum, the “cut and glue” operations of Definition 1.1 correspond exactly to the changes of basis in $H_1(X, \Sigma, \mathbb{Z})$.

Thus, we have

**Proposition 1.4.** Each stratum $\mathcal{H}(k_1, \ldots, k_s)$ has an atlas of charts to $\mathbb{C}^{2g+s-1}$ (away from the orbifold points) with transition functions in $GL(2g+s-1, \mathbb{Z})$.

This fact has several important implications, some of which we will encounter in the next section.

## 2 Masur-Veech volumes and square-tiled surfaces

In this section we define the Masur-Veech volumes of strata of Abelian differentials, present an approach to the computation of these volumes based on counting of square-tiled surfaces, and motivate the work carried out in Section 3.

### 2.1 Masur-Veech volumes

Proposition 1.4 says that the transition functions between the charts are linear maps with determinant 1. It means that the standard volume forms in different charts are compatible and so they glue together to give a volume form on $\mathcal{H}(k_1, \ldots, k_s)$, which is called the *Masur-Veech volume form*. We denote it by $d\nu$.

Obviously, the total volume of any stratum is infinite. However, consider the hypersurface $\mathcal{H}_1(k_1, \ldots, k_s) \subset \mathcal{H}(k_1, \ldots, k_s)$ of translation surfaces of area 1. The volume form $d\nu$ induces a volume form $d\nu_1$ on $\mathcal{H}_1(k_1, \ldots, k_s)$ (since this hypersurface is a level set of a real function on $\mathcal{H}(k_1, \ldots, k_s)$ which computes the area of a translation surface). Even though $\mathcal{H}_1(k_1, \ldots, k_s)$ is never compact, it turns out that its volume is always finite.

**Theorem 2.1** (H. Masur [Mas82], W. Veech [Vee82]). For every stratum $\mathcal{H}(k_1, \ldots, k_s)$, the total volume
\[
\int_{\mathcal{H}_1(k_1, \ldots, k_s)} d\nu_1
\]is finite.

By convention, it is this number that we call the *Masur-Veech volume* of the corresponding stratum.

### 2.2 Computation of volumes via square-tiled surfaces

We now explain the basic idea of computation of the Masur-Veech volumes. We start with a motivating example.

Let $B_d(R)$ be the ball if radius $R$ in $\mathbb{R}^d$. Let $N_d(R)$ be the number of integer points inside it. Obviously, $N_d(R)$ is asymptotic (as $R \to \infty$) to the volume
Vol(B_d(R)). Note that Vol(B_d(R)) is a homogeneous function of R of degree d, and so its asymptotics coincides with the formula for Vol(B_d(R)). Hence, if we know the asymptotics of N_d(R), we know the formula for Vol(B_d(R)), and we can compute the volume (“hyperarea”) of the unit sphere S_d(1) as:

$$\text{Vol}(S_d(1)) = \frac{d}{dR} \left( \text{Vol}(B_d(R)) \right)_{R=1}.$$ 

Similarly to \( \mathbb{R}^d \), each stratum \( \mathcal{H}(k_1, \ldots, k_s) \) has a natural notion of an integer point: a point is an integer point of a stratum, if it is represented in (some) period coordinates by a point from \( (\mathbb{Z} \oplus \mathbb{Z}i)^{2g+s-1} \subset \mathbb{C}^{2g+s-1} \). Since the transition functions between different period coordinates are integer matrices, this notion is independent of the choice of the chart containing the point in question.

Hence, we can apply the same strategy to the computation of the Masur-Veech volumes. Let \( \mathcal{H}_{\leq R}(k_1, \ldots, k_s) \) be the set of translation surfaces of area at most \( R \). The volume \( \nu(\mathcal{H}_{\leq R}(k_1, \ldots, k_s)) \) is a homogeneous function of \( R \) of degree \( \dim \mathcal{H}(k_1, \ldots, k_s) / 2 \) (because for the area of the translation surface to be \( R \) times larger, the linear dimensions should be \( R^{1/2} \) times larger), and is asymptotic to the number of integer points inside \( \mathcal{H}_{\leq R}(k_1, \ldots, k_s) \). Thus, knowing the asymptotics of this number of integer points, we know the formula for \( \nu(\mathcal{H}_{\leq R}(k_1, \ldots, k_s)) \), and we can compute the Masur-Veech volume as:

$$\nu_1(\mathcal{H}_1(k_1, \ldots, k_s)) = 2 \frac{d}{dR} \nu(\mathcal{H}_{\leq R}(k_1, \ldots, k_s))_{R=1}.$$ 

Let us now study the surfaces \((X, \omega)\) represented by the integer points of the strata. Fix a conical singularity \( P_0 \in X \) and consider a map from \( X \) to the standard torus \( \mathbb{C} / (\mathbb{Z} \oplus \mathbb{Z}i) \) given by

$$P \mapsto \int_{P_0}^{P} \omega \pmod{\mathbb{Z} \oplus \mathbb{Z}i}.$$ 

This map is well-defined because any two paths from \( P_0 \) to \( P \) differ by an element of \( H_1(X, \Sigma, \mathbb{Z}) \), and the integral of \( \omega \) over any such element is in \( \mathbb{Z} \oplus \mathbb{Z}i \) (by the definition of an integer point of the stratum).

It is easy to see that this map is a covering away from the zeroes of \( \omega \), so it is a ramified covering with ramification points being the zeroes of \( \omega \). Moreover, the pushforward of \( \omega \) by this map is the standard 1-form \( dz \). It means that this map is also a local isometry of the corresponding translation surfaces, preserving the vertical direction.

If we think about \((\mathbb{C} / (\mathbb{Z} \oplus \mathbb{Z}i), dz)\) as a unit square with opposite sides identified, the ramified covering constructed above induces a tiling of \((X, \omega)\) by unit squares. Conversely, if we take a finite number of unit squares with sides labelled as top, bottom, left and right, and we glue them, respecting this labels (top to bottom, left to right), to form a closed oriented surface, this surface will represent an integer point of some stratum.

We call such surfaces the square-tiled surfaces.
Proposition 2.2. The problem of computing the Masur-Veech volume of a stratum is equivalent to the problem of computing the asymptotics (as $N \to \infty$) of the number of square-tiled surfaces with at most $N$ squares belonging to this stratum.

As noted in [Zor06], this approach to the computation of volumes was suggested by A. Eskin and A. Okounkov, and independently by M. Kontsevich and A. Zorich. The problem was solved by A. Eskin and A. Okounkov by methods of representation theory ([EO01], [EOP08]). However, we will describe another (more geometric) approach, based on the geometry of square-tiled surfaces.

2.3 Geometry of square-tiled surfaces

Consider a square-tiled surface $(X, \omega)$ in $\mathcal{H}(k_1, \ldots, k_s)$. The vertical direction of the surface allows us to distinguish the sides of each square as top, bottom, left and right. Orient the top and bottom sides of each square in the positive horizontal direction ($\omega > 0$).

The conical singularity of angle $2\pi(k_i + 1)$ has $4(k_i + 1)$ squares around it. Since the squares are glued consistently (top to bottom, left to right), there are $2(k_i + 1)$ horizontal sides of squares incident to this singularity. The incoming and the outgoing sides alternate, and the angle between consecutive sides is $\pi$.

Leave one of the singularities along an outgoing horizontal side and follow the sides of the squares. Since the number of squares is finite, we will eventually reach a singularity (possibly the same one). We obtain an oriented geodesic starting and ending at a conical singularity, which we call a (horizontal) separatrix.

Thus, all the incoming and outgoing sides at all singularities split into pairs, with sides in each pair connected by a separatrix. It means that singularities together with all horizontal separatrices form an oriented multigraph $G$ (possibly disconnected, loops and multiple edges are allowed). However, since $G$ is embedded into $(X, \omega)$, it inherits additional structure: the edges at each vertex are circularly ordered. A graph with such additional data is often called a ribbon graph. The terminology comes from the fact that the circular ordering of edges at each vertex allows us to recover the topology of a neighborhood $O$ of $G$ in $(X, \omega)$. Indeed, a small enough neighborhood $O$ can be glued from several rectangles (corresponding to the edges) and several disks (corresponding to the vertices); the circular orders tell us exactly in which order to glue the rectangles to the boundary of a particular disk (the gluing must also respect the orientations – “the ribbons do not twist”). Thus, a ribbon graph can also be viewed as the resulting topological space.

Note also that in $(X, \omega)$ each edge of $G$ has a “top” side (to the left of the edge) and a “bottom” side (to the right of the edge). Since the incoming and outgoing edges at each vertex alternate, “top” sides are glued to “top” sides and the same for “bottom” sides. This implies that these sides together form several “top” and “bottom” circular boundary components of the ribbon graph $G$. 
Since $X - G$ does not contain any singularities, it admits a (non-singular) flat metric. Gauss-Bonnet theorem implies that $X - G$ is a disjoint union of several flat (square-tiled) cylinders. Each cylinder has a “top” and a “bottom” boundary component which should be glued to one of the “bottom” and one of the “top” boundary components of $G$ respectively. Consequently, the “top” and “bottom” boundary components of $G$ are divided into pairs.

The resulting ribbon graph $G$ with the pairing of its boundary components is called the separatrix diagram of the corresponding square-tiled surface $(X, \omega)$. Abstractly, we have

**Definition 2.3.** A separatrix diagram is a (possibly disconnected) oriented ribbon graph $G$ such that:

- at each vertex the incoming and the outgoing edges alternate (with respect to the circular ordering at this vertex);
- the boundary components of $G$ to the left (to the right respectively) of its edges are marked as “top” (“bottom” respectively);
- the boundary components of $G$ are paired in such a way that each pair has one “top” and one “bottom” boundary component.

The discussion above sums up to the following

**Proposition 2.4.** Topologically, a square-tiled surface can be obtained from its separatrix diagram $G$ by gluing a cylinder to every two paired boundary components. Geometrically, to specify a square-tiled surface we also need to:

- assign integer lengths to the edges of $G$ in such a way that the boundary components of $G$ in each pair have equal lengths (this condition is equivalent to a system of linear equations);
- assign an integer “height” to each cylinder;
- assign an integer “twist” to each cylinder (if the circumference of a cylinder is $w$, we can glue it to $G$ in $w$ different ways).

### 2.4 Examples of computations

Armed with the above description of square-tiled surfaces, we can compute the asymptotics of the number of square-tiled surfaces with at most $N$ squares in a particular stratum by computing separately the contributions of surfaces with common separatrix diagrams (the number of diagrams corresponding to a particular stratum is finite).

For example, square-tiled surfaces in $H(0)$ are simply square-tiled tori, and there is only one separatrix diagram corresponding to this stratum. If $h$ is the height of the unique cylinder and $w$ is its width (circumference), then the
number of tori with at most $N$ squares is

$$\sum_{w,h \in \mathbb{N} \atop wh \leq N} w \approx \sum_{h \in \mathbb{N}} \frac{1}{2} \cdot \left( \frac{N}{h} \right)^2 = \frac{N^2}{2} \cdot \zeta(2) = \frac{N^2 \pi^2}{6}. \quad (1)$$

This leads to the volume $\nu_1(\mathcal{H}_1(0)) = \frac{\pi^2}{3}$. In the stratum $\mathcal{H}(2)$, we have several possible separatrix diagrams. Take, for instance, the one from Figure ????. If $l_1$ is the length of the upper and the lower loops, and $l_2$ is the length of the middle loop, then the circumferences of the cylinders we have to glue are $l_1$ and $l_1 + l_2$. If $h_1, h_2$ are the heights of these cylinders, then the number of surfaces with at most $N$ squares corresponding to this separatrix diagram is

$$\sum_{l_1, l_2, h_1, h_2 \in \mathbb{N} \atop l_1 h_1 + (l_1 + l_2) h_2 \leq N} l_1 (l_1 + l_2) = \sum_{l_1, l_2, h_1, h_2 \in \mathbb{N} \atop l_1 (h_1 + h_2) + l_2 h_2 \leq N} l_1^2 + l_1 l_2. \quad (2)$$

One can compute the asymptotics of this sum in terms of multiple zeta values $\zeta(1, 3)$ and $\zeta(2, 2)$ (see the paper [Zor02] for a more detailed computation). We get the asymptotics

$$\sum_{l_1, l_2, h_1, h_2 \in \mathbb{N} \atop l_1 (h_1 + h_2) + l_2 h_2 \leq N} l_1^2 + l_1 l_2 \sim \frac{N^4}{24} (2 \cdot \zeta(1, 3) + \zeta(2, 2)) = \frac{N^4}{24} \cdot \frac{5}{4} \cdot \frac{\pi^4}{90}. \quad (3)$$

Doing a similar computation for the other diagrams and summing up the contributions, one can obtain the volume $\nu_1(\mathcal{H}_1(2)) = \frac{\pi^4}{150}$.

The disadvantage of this approach is that, when genus grows, the number of diagrams grows rapidly, and it becomes more and more difficult to express the asymptotics of the corresponding sums in reasonable terms (multiple zeta values).

3 Contribution of maximal-cylinder surfaces to the volumes of minimal strata

3.1 Motivation

The following is known from the works of Eskin and Okounkov ([EO01], [EOP08]):

**Theorem 3.1.** The volume of every connected component of every stratum $\mathcal{H}(k_1, \ldots, k_s)$ is a rational multiple of $\pi^{2g}$, where $g$ is the corresponding genus, given by $k_1 + \ldots + k_s = 2g - 2$.

However, the following stronger phenomenon is observed experimentally for the so-called minimal strata of Abelian differentials:
Conjecture 3.2. Let $C$ be a connected component of a minimal stratum $\mathcal{H}(2g-2)$ of Abelian differentials and $k \geq 0$. Then the contribution to the Masur-Veech volume of the square-tiled surfaces in $C$ made of $k$ cylinders is a rational multiple of $\pi^2$.

Note however, that this phenomenon is not present for general strata: for example, it is known ([DGZ+20]) that the contribution of 1-cylinder surfaces to the volume of $\mathcal{H}(1,1)$ is a rational multiple of $\zeta(5)$, and it is believed that $\zeta(5)$ and $\pi^2$ are algebraically independent over $\mathbb{Q}$ (it is one of a series of important conjectures that $\{\zeta(2)\} \cup \{\zeta(2k+1)\}_{k \geq 1}$ are algebraically independent over $\mathbb{Q}$).

A statement similar to Conjecture 3.2 was proven in the paper [DGZZ] for the principal strata of quadratic differentials (these objects are similar to Abelian differentials, only their holonomy is allowed to be quadratic):

Theorem 3.3. Let $Q(1^{4g-4+n},-1^n)$ be the principal stratum of quadratic differentials in genus $g$ with $n$ poles. Then for each stable graph $\Gamma$ of type $(g,n)$ the contribution of square-tiled surfaces of type $\Gamma$ to the volume of $Q(1^{4g-4+n},-1^n)$ is a rational multiple of $\pi^{6g-6+2n}$.

In this theorem the authors group square-tiled surfaces according to their stable graphs. We will not give a formal definition of a stable graph, but we just mention that for the square-tiled surfaces from the minimal stratum $\mathcal{H}(2g-2)$ the grouping by stable graphs is equivalent to the grouping by the number of cylinders.

Consider a square-tiled surface in $\mathcal{H}(2g-2)$ with $k$ cylinders. Clearly, its separatrix diagram has a single vertex, $k$ “top” and $k$ “bottom” boundary components. From now on, we label the cylinders, as well as the corresponding “top” and “bottom” boundary components, by numbers from 1 to $k$.

Similarly to what has been explained in Subsection 2.3, any such square-tiled surface can then be specified by a choice of positive integer “heights” $(h_i)_{i=1,\ldots,k}$ and “circumferences” $(L_i)_{i=1,\ldots,k}$ of the $k$ cylinders, the twists $t_i \in \{1,\ldots,L_i-1\}$ of the cylinders, and a choice of a separatrix diagram $G$ with one vertex, $2g-1$ edges with positive integer lengths, and $k$ labelled “top” and $k$ labelled “bottom” boundary components of lengths $L_1,\ldots,L_k$.

Consider a separatrix diagram $G$ as above. Glue a topological disk to each boundary component of $G$. What we get is a closed surface of genus $g(G) := g-k$ (because when we glue $k$ cylinders instead of the disks, the genus increases by $k$, and we get a surface of genus $g$). The graph $G$ is embedded into this surface in such a way that the faces are topological disks. This is an instance of a combinatorial map. Moreover, the faces inherit the labels and the types (“top” or “bottom”) from the boundary components of $G$. The faces of the same type are not adjacent. The edges also inherit their positive integer lengths.

Definition 3.4. Let $g \geq 0$, $k,l \geq 1$ be positive integers. For $L_1,\ldots,L_k,L'_1,\ldots,L'_l \in \mathbb{N}$ we define $P^g_{k,l}(L_1,\ldots,L_k;L'_1,\ldots,L'_l)$ as the number of metric combinatorial maps of genus $g$ such that: there is a single vertex, the edges have positive integer lengths, there are $k$ labeled “black” faces of perimeters $L_1,\ldots,L_k$ and $l$
labeled “white” faces of perimeters $L'_1, \ldots, L'_l$, with the map being face-bipartite (faces of the same color are not adjacent).

The above considerations imply that the number of square-tiled surfaces in $H(2g - 2)$ with $k$ cylinders and at most $N$ squares is equal to:

$$\frac{1}{k!} \cdot \sum_{h, L_i \leq N} L_1 \cdots L_k \cdot P^g_{k,k}(L_1, \ldots, L_k; L_1, \ldots, L_k), \quad (1)$$

where the factor $\frac{1}{k!}$ accounts for the arbitrary numbering of the cylinders, and the factor $L_1 \cdots L_k$ accounts for the twists of the cylinders.

In the context of Theorem 3.3, the paper [DGZZ] gives a formula similar to (1), but there instead of $P^g_{k,k}$ one has to use the function $N^g_{g,n}$ which counts trivalent metric ribbon graphs of genus $g$ with edges of positive integer lengths, and with $n$ labelled boundary components of specified perimeters. The following is a famous result of M. Kontsevich appearing in his proof [Kon92] of Witten’s conjecture [Wit91].

**Theorem 3.5.** Let $L_1, \ldots, L_n$ be positive integers such that $\sum_i L_i$ is even. Then

$$N^g_{g,n}(L_1, \ldots, L_n) = N^g_{g,n}(L_1, \ldots, L_n) + \text{lower order terms},$$

where $N^g_{g,n}$ is a homogeneous polynomial of degree $3g - 3 + n$, whose coefficients are certain intersection numbers of psi-classes on the moduli spaces of Riemann surfaces.

Moreover, the “lower order terms” has been given a formal sense in the works [Nor10], [CMS11] by showing that the $N^g_{g,n}$ are actually quasi-polynomials, often called *local quasi-polynomials*. One can then compute the asymptotics of the sum similar to (1) in an elementary manner, because the highest order term $N^g_{g,n}$ is a polynomial and because of the following standard fact:

**Lemma 3.6.** As $N \to \infty$,

$$\sum_{\sum_{i=1}^k a_i L_i \leq N} L_1^{a_1} \cdots L_k^{a_k} \sim \frac{N^{a+k}}{(a+k)!} \cdot \prod_{i=1}^k a_i! \cdot \zeta(a_i + 1),$$

where $a_i \in \mathbb{N}$ and $a = a_1 + \ldots + a_k$.

If we want now to compute the asymptotics of the sum (1), we have to understand the functions $P^g_{k,k}$. We present the following conjecture:

**Conjecture 3.7.** The functions $P^g_{k,k}$ are quasi-polynomials of degree $2g$.

The aim of the rest of this section is to study the functions $P^g_{k,k}$, which correspond to the case $k = g$ in Conjecture 3.2, i.e. to the square-tiled surfaces with the maximal possible number of cylinders for the fixed genus $g$, as well as their specifications for each connected component of the minimal stratum.
3.2 Generic values exist

Consider the functions $P_{0,k,l}^0$. According to Conjecture 3.7, they must be quasi-polynomials of degree 0, i.e. (quasi-)constants. Let us define

$$Q_k(L_1, \ldots, L_k) := P_{0,k}^0(L_1, \ldots, L_k; L_1, \ldots, L_k).$$

It is these functions that we are interested in, in view of the formula (1). However, we start by analyzing the more general functions

$$P_{0,k,l}^0(L_1, \ldots, L_k; L'_1, \ldots, L'_l).$$

**Remark.** In the Definition 3.4 we allowed the arguments $L_i, L'_j$ to only be positive integers, and the function $P_{0,k,l}^0$ to only count maps with edges of positive integer lengths. However, for convenience, we now allow the arguments $L_i, L'_j$ and the edge lengths to be arbitrary real numbers, and the function $P_{0,k,l}^0$ to count maps with edges of arbitrary positive length. Lemma 3.9 below ensures that we will not get any infinite values of $P_{0,k,l}^0$ in this way.

**Definition 3.8.** We call a metric combinatorial map (a metric plane tree, for example) positive (non-negative) if all of its edges have positive (non-negative) lengths.

By Definition 3.4, $P_{0,k,l}^0$ counts the number of face-bipartite positive metric planar maps with one vertex, and with $k$ marked black faces of perimeters $L_1, \ldots, L_k$ and $l$ marked white faces of perimeters $L'_1, \ldots, L'_l$.

Passing to the dual map, we see that, alternatively, $P_{0,k,l}^0$ counts the number of bipartite positive metric plane trees, with $k$ marked black vertices and $l$ marked white vertices with the sum of lengths of incident edges equal to $L_1, \ldots, L_k$ and $L'_1, \ldots, L'_l$, respectively. In what follows, we call the numbers $L_i$ and $L'_i$ the vertex lengths.

A standard double-counting argument implies the equality

$$L_1 + \ldots + L_k = L'_1 + \ldots + L'_l.$$

Hence, we will always work inside the linear subspace $S$ of $\mathbb{R}^{k+l}$ defined by (2). Moreover, $P_{0,k,l}^0$ is zero outside $(\mathbb{R}^+)^{k+l}$, so we will restrict our attention to the values of $P_{0,k,l}^0$ on the (convex) subset $S^+ = S \cap (\mathbb{R}^+)^{k+l}$.

**Lemma 3.9.** Suppose an edge in a bipartite tree divides the set of vertices into two groups with lengths

$$L_1, \ldots, L_m; L'_1, \ldots, L'_n \text{ and } L_{m+1}, \ldots, L_k; L'_{n+1}, \ldots, L'_l.$$

If the black vertex of this edge is in the first group, then the length of this edge is equal to

$$(L_1 + \ldots + L_m) - (L'_1 + \ldots + L'_n) = (L'_{n+1} + \ldots + L'_l) - (L_{m+1} + \ldots + L_k).$$

In particular, the vertex lengths of a tree uniquely determine its edge lengths.
Proof. Hang the first group by the black vertex incident to the edge. Then, starting from the leaves, progressively compute the lengths of all edges. The equality of two differences follows from equation (2).

Let $\mathcal{L}$ be the set of linear forms on $S$ of the form $\sum \varepsilon_i L_i - \sum \varepsilon'_j L'_j$, where $\varepsilon_i, \varepsilon'_j \in \{0, 1\}$, $\varepsilon_i$ are not all equal, and the same for $\varepsilon'_j$. In view of Lemma 3.9, linear forms from $\mathcal{L}$ correspond to a priori possible lengths of edges not incident to the leaves in a bipartite tree with vertex lengths $L_1, \ldots, L_k; L'_1, \ldots, L'_l$ (edges which are incident to the leaves have lengths of the form $L_i$ or $L'_j$).

Note that $f \in \mathcal{L}$ implies $-f \notin \mathcal{L}$. Indeed, if, without loss of generality, $f = (L_1 + \ldots + L_m) - (L'_1 + \ldots + L'_n)$, then on $S$ we have:

$$-f = -f + (L_1 + \ldots + L_k) - (L'_1 + \ldots + L'_l)$$
$$= (L'_1 + \ldots + L'_n) - (L_1 + \ldots + L_m) + (L_1 + \ldots + L_k) - (L'_1 + \ldots + L'_l)$$
$$= (L_{m+1} + \ldots + L_k) - (L'_{n+1} + \ldots + L'_l) \in \mathcal{L}.$$

The linear form $-f$ corresponds to the edge that produces the same partition of the vertices into two groups as $f$ does, except the black vertex of the edge is now in the other group.

Let also $\mathcal{V}$ be the set of linear subspaces of $S$ which are the intersections of kernels of functions from $\mathcal{L}$.

The edge length formula of Lemma 3.9 suggests that the value of $P^0_{k,l}$ depends on the signs (+, − or 0) of linear forms from $\mathcal{L}$, that is, on the position of the point $(L_1, \ldots, L_k; L'_1, \ldots, L'_l)$ with respect to the subspaces from $\mathcal{V}$.

Lemma 3.10. For every subspace $V \in \mathcal{V}$, $P^0_{k,l}$ is constant on $V' \cap S^+$, where $V' := V - \bigcup_{W \in V'} W$.

Proof. Obviously, $P^0_{k,l}(L_1, \ldots, L_k; L'_1, \ldots, L'_l)$ is equal to the number of positive trees among trees with vertex lengths $L_1, \ldots, L_k; L'_1, \ldots, L'_l$.

First we show that $P^0_{k,l}$ is locally constant on $V' \cap S^+$. By the edge length formula of Lemma 3.9, positivity of any particular marked bipartite plane tree is equivalent to a system of linear inequalities of type $f > 0$, where $f \in \mathcal{L}$ (these are the conditions on the edges which are not incident to the leaves; the latter always have positive lengths, since we work in $S^+$). Thus $P^0_{k,l}$ remains constant when each such linear form preserves its sign (positive, negative or zero). This condition is satisfied locally on $V'$, hence on $V' \cap S^+$.

Next we show that the value of $P^0_{k,l}$ does not change when we traverse (inside $S^+$) a codimension 1 subspace $W \in \mathcal{V}$ in $V$ (this is enough to prove the Lemma, since $V \cap S^+$ is connected (as a convex set), and subspaces of greater codimension do not divide $V$). For convenience, we suppose that we traverse $W$ along a linear path.

Suppose $V$ is defined by independent linear equations $f_1 = 0, \ldots, f_d = 0$, and $W$ — additionally by an equation $f_{d+1} = 0$, where $f_i \in \mathcal{L}$. When we reach $W = V \cap \{f_{d+1} = 0\}$ from the half-space $V \cap \{f_{d+1} > 0\}$, certain positive trees degenerate — some of their edges become zero-length. Specifically, these are the edges whose lengths are represented by linear forms $f \in \mathcal{L}$ such that the
value of $f$ goes from positive to zero when we reach $W$ from $V \cap \{f_{d+1} > 0\}$. Let $F$ be the set of such forms.

Likewise, reaching $W$ from the half-space $V \cap \{f_{d+1} < 0\}$, the lengths of the edges that become zero-length are represented by linear forms $f$ such that the value of $f$ goes from positive to zero when we reach $W$ from $V \cap \{f_{d+1} < 0\}$. It is easy to see that these are exactly the forms from the set $-F := \{-f : f \in F\}$. Indeed, since the path is linear, the restrictions of all the forms to the path are also linear, so, if the form $f$ was positive in $V \cap \{f_{d+1} > 0\}$ and zero at the point of traversal, it will be negative in $V \cap \{f_{d+1} < 0\}$, so $-f$ will be positive in $V \cap \{f_{d+1} < 0\}$.

Now it remains to prove that the numbers of degenerate trees on both sides are equal, since, when we reach $W$, the value of $P_{k,l}^0$ decreases by exactly the number of degenerate trees.

Each degenerate tree consists of several smaller positive trees connected by zero-length edges, whose corresponding linear forms are in $F$ (on the side $f_{d+1} > 0$), or in $-F$ (on the side $f_{d+1} < 0$). Recall that the edges corresponding to linear forms $f \in F$ and $-f \in -F$ induce the same partition of the vertices into two groups, except the black vertex of the edge must be in different groups for $f$ and $-f$. This allows us to establish a bijection between degenerate trees on different sides by the following procedure. Given a degenerate tree, replace each zero-length edge (joining vertices $v$ and $w$, say) by a zero-length edge joining vertices $v'$ and $w'$, where $v'$ is a vertex following $v$ when we go counter-clockwise along the boundary of the corresponding positive tree, and $w'$ is defined analogously (see Figure 1). The inverse procedure is the same, except “counter-clockwise” should be replaced by “clockwise”.

![Figure 1: Modification of the zero-length edge.](image)

Corollary 3.11. For every $k \in \mathbb{N}$, $Q_k$ is constant on $(\mathbb{R}^+)^k$ minus the union of hyperplanes of the form $\sum \varepsilon_i L_i = 0$, where $\varepsilon_i \in \{-1, 0, +1\}$.

Proof. Apply Lemma 3.10 to the subspace $V$ defined by the equations

$$L_1 - L'_1 = 0, \ldots, L_k - L'_k = 0.$$
Figure 2: At the bottom of the figure we see the projectivization of the three-dimensional cone \( \{L_1 = L'_1, L_2 = L'_2, L_3 = L'_3, L_1 > 0, L_2 > 0, L_3 > 0 \} \). It is divided by 6 codimension 1 subspaces (“walls”) into 12 “cells”. The value of \( P_{3,3}^0 \) in each cell is the same (11 in this case). At the top of the figure are given the sets of realizable marked plane trees corresponding to two points of the cone. Going from one of the points to the other, we must cross the “wall” \( \{L_1 = L_2\} \). When we reach the “wall” from each side, certain trees degenerate — some of their edges become zero-length (these edges are marked by dotted lines). Degenerate trees on both sides split into several groups (according to the trees left after removing the zero-length edges, and the way these trees are connected). The bijection of Lemma 3.10 maps each such group on one side into the corresponding group on the other side.
3.3 Recurrence for the generic values of $Q_k$

To establish the recurrence relation for the generic values of $Q_k$, we will study positive trees with vertex lengths that are particularly easy to handle.

**Theorem 3.12.** Let $q_k$ be the generic value of $Q_k$, $k \geq 1$. Then the sequence $(q_k)_{k \geq 1}$ is given by the following recurrence relation:

$$q_1 = 1,$$

$$q_k = \sum_{\substack{a+b=k \cr a,b \geq 1}} \binom{k-1}{a} (2a - 1)^2 q_a q_b, k \geq 2. \quad (3)$$

**Proof.** First note that, for every $k \geq 1$ and for every $0 \leq i_1 < i_2 < \ldots < i_k$, the point

$$(2^{i_1}, 2^{i_2}, \ldots, 2^{i_k}; 2^{i_1}, 2^{i_2}, \ldots, 2^{i_k}) \quad (4)$$

lies in the subspace defined by equations $L_i = L'_i, 1 \leq i \leq k$, and does not lie in any subspace from $\mathcal{V}$ of smaller dimension, because the equality $\sum_{i \in I} L_i = \sum_{j \in J} L'_j$ with $I, J \subset \{1, \ldots, k\}$ implies $I = J$ (this is the uniqueness of representation in base 2). Hence, by Lemma 3.10, for any such set of numbers $i_1, \ldots, i_k$ we have $q_k = Q_k(2^{i_1}, 2^{i_2}, \ldots, 2^{i_k}) = P_{k,k}^0(2^{i_1}, 2^{i_2}, \ldots, 2^{i_k}; 2^{i_1}, 2^{i_2}, \ldots, 2^{i_k}).$

Now we will establish a recurrence relation for the numbers $q_k$ by studying positive trees with vertex lengths as in (4). The equality $q_1 = 1$ is obvious. Let now $k \geq 2$.

Consider a positive tree $T$ with vertex lengths $1, 2, \ldots, 2^{k-1}; 1, 2, \ldots, 2^{k-1}$. Since the vertex lengths are integers, the lengths of the edges must also be (positive) integers. This fact implies that the black and the white vertices with labels 1 must always be leaves (we call them the 1-leaves). Consider the (unique) oriented path between the black and the white 1-leaves. Let $e$ be the first edge of length 1 in this path, whose first vertex is black, and which is different from the first edge of the path. There is such an edge because, for example, the last edge of the path satisfies all these conditions. Let also $v_b$ and $v_w$ be respectively the black and the white vertices adjacent to $e$.

The edge $e$ induces a partition of the vertices into two groups $V_1, V_2$ (the first contains the black 1-leaf, and the second contains the white 1-leaf). We claim that the set of vertex labels (not equal to 1) in each group is symmetric, in the sense that a black vertex with label $i$ is in the group if and only if the white vertex with label $i$ is in the group. It is enough to prove the claim for $V_1$. Let $V_1$ contain the black vertices with labels $\{1\} \cup I$ and white vertices with labels $J$, where $I, J \subset \{2, \ldots, k\}$. The edge-length formula of Lemma 3.9 applied to the edge $e$ gives:

$$1 = 1 + \sum_{i \in I} 2^i - \sum_{j \in J} 2^j,$$

which implies $I = J$ by the uniqueness of base 2 representation.
Figure 3: Example of a decomposition of a tree \((k = 7)\). Vertices are labeled with their lengths. The paths between 1-leaves are highlighted in grey. The edges of length 1 whose first vertex is black are marked by dashed lines.

Let now \(T_1\) be the subtree of \(T\) induced by the vertices in \(V_1 \cup \{v_w\}\), and let \(T_2\) be the subtree of \(T\) induced by the vertices in \(V_2 \cup \{v_b\}\). \(T_1\) and \(T_2\) inherit edge lengths from \(T\). The vertex lengths remain the same, except the vertex lengths of \(v_w\) in \(T_1\) and of \(v_b\) in \(T_2\) both become equal to 1 (see Figure 3 for an example).

By the above arguments, \(T_1\) and \(T_2\) are positive trees with vertex lengths of the form (4), which both have 1-leaves, and whose vertex lengths distinct from 1 together form a partition of \(\{2, \ldots, 2^{k-1}\}\). In addition, \(T_1\) has the property that the oriented path between its black and its white 1-leaves has no edge of length 1 that starts with a black vertex, except the first and the last edge.

Conversely, given any two trees \(T_1\) and \(T_2\) with the properties as in the preceding paragraph, we can glue the edge incident to the white 1-leaf of \(T_1\) with the edge incident to the black 1-leaf of \(T_2\) to get a positive tree with vertex lengths \(1, 2, \ldots, 2^{k-1}\).

This establishes a bijection between the set of positive trees \(T\) with vertex lengths \(1, 2, \ldots, 2^{k-1}, 1, 2, \ldots, 2^{k-1}\) and the set of pairs of such trees \((T_1, T_2)\).

The cardinality of the first set is \(q_k\). The cardinality of the second set is

\[
\sum_{a+b=k+1} \sum_{a \geq 2, b \geq 1} \tilde{q}(I) q_b,
\]

where \(a\) and \(b\) correspond to the number of black vertices of \(T_1\) and \(T_2\) respectively, the inner sum corresponds to the choice of vertex lengths from \(\{2, \ldots, 2^{k-1}\}\) which are present in \(T_1\), \(\tilde{q}(I)\) is the cardinality of the set of possible trees \(T_1\) with vertex lengths \(I\), and \(q_b\) is the cardinality of the set of possible trees \(T_2\).

Lemma 3.13 below implies that if \(|I| = a - 1\), then \(\tilde{q}(I) = (2a - 3)^2 q_{a-1}\). Substituting this into (5) and making a change of variable \(a - 1 =: a\), we get (3).
Lemma 3.13. Let $k \geq 2$ and $0 = i_1 < i_2 < \ldots < i_k$. Then there are exactly $(2k - 3)^2 q_{k-1}$ positive trees with vertex lengths $(2^{i_1}, 2^{i_2}, \ldots, 2^{i_k}; 2^{i_2}, 2^{i_3}, \ldots, 2^{i_k})$ such that the oriented path between their black and white 1-leaves has no edge of length 1 that starts with a black vertex, except the first and the last edge.

**Proof.** Consider a positive tree satisfying the conditions of the Lemma. Alternately add $-1$ or $+1$ (starting with $-1$) to the lengths of the edges along the path joining the two 1-leaves. In this way we get a tree with two zero-length edges incident to the leaves. After removing them, we get a positive tree with vertex lengths $(2^{i_2}, \ldots, 2^{i_k}; 2^{i_2}, \ldots, 2^{i_k})$.

Conversely, given a positive tree with vertex lengths $(2^{i_2}, \ldots, 2^{i_k}; 2^{i_2}, \ldots, 2^{i_k})$, we glue two zero-length edges to one black and one white vertex in one of the $(2k - 3)^2$ ways, then we alternately add $+1$ or $-1$ (starting with $+1$) to the lengths of edges along the path joining two new vertices. We will not get an edge of length zero because all edge lengths are positive and even (all vertex lengths are even), hence at least 2. Obviously, the resulting tree satisfies the conditions of the Lemma.

Thus, the map we have described is $(2k - 3)^2$-to-one, and maps onto the set of positive trees with vertex lengths $(2^{i_2}, \ldots, 2^{i_k}; 2^{i_2}, \ldots, 2^{i_k})$, whose cardinality is $q_{k-1}$. The conclusion of the Lemma follows.

We present in Table 1 the values of $q_k$ for small $k$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_k$</td>
<td>1</td>
<td>1</td>
<td>11</td>
<td>335</td>
<td>19449</td>
<td>1815089</td>
<td>247145107</td>
<td>46121543775</td>
</tr>
</tbody>
</table>

Table 1: The values of $q_k$ for small $k$.

### 3.4 Analysis for each connected component of the minimal strata

Connected components of the strata of Abelian differentials have been classified by Kontsevich and Zorich in [KZ03]. For minimal strata $\mathcal{H}(2g - 2)$ we have the following results:

- for $g \geq 4$, $\mathcal{H}(2g - 2)$ always has 3 connected components: the hyperelliptic one $\mathcal{H}^{hyp}(2g - 2)$, and two other components $\mathcal{H}^{even}(2g - 2)$ and $\mathcal{H}^{odd}(2g - 2)$;
- for $g = 3$, $\mathcal{H}(2g - 2) = \mathcal{H}(4)$ has two connected components $\mathcal{H}^{hyp}(4)$ and $\mathcal{H}^{odd}(4)$;
- for $g \in \{1, 2\}$, $\mathcal{H}(2g - 2)$ has one connected component $\mathcal{H}^{hyp}(2g - 2)$.

In this section we will analyse the contribution of maximal-cylinder surfaces to the volumes of each individual connected component.
3.4.1 Preliminaries

First, we explain how to distinguish different connected components (see [KZ03] for more details) and the trees corresponding to each component.

Recall that a Riemann surface $X$ is hyperelliptic if it admits a ramified double cover $X \to \mathbb{C}P^1$. Such a surface $X$ always admits a hyperelliptic involution $\sigma : X \to X$, $\sigma^2 = \text{Id}$, which interchanges the sheets of the cover.

The hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$ consists of Abelian differentials $\omega$ on hyperelliptic Riemann surfaces $X$ of genus $g$ which have a single zero of multiplicity $2g-2$. Necessarily, $\sigma^*\omega = -\omega$.

Note that $(X, -\omega)$ corresponds to the same translation surface as $(X, \omega)$ except the direction to the north is changed to the opposite one. Hence, $\sigma$ is an isometry of $(X, \omega)$ as a flat surface. Obviously, the zero of $\omega$ is fixed by $\sigma$, and, since $\sigma^2 = \text{Id}$, near this zero $\sigma$ must act as a central symmetry.

Suppose now that $(X, \omega)$ is a square-tiled surface in $\mathcal{H}^{hyp}(2g-2)$. Then $\sigma$ must send each cylinder of $X$ to itself, interchanging the top and the bottom boundaries. Indeed, if $\sigma$ interchanges two distinct cylinders, we can deform one of them by altering its height slightly, and we run into a contradiction, because, on the one hand, the deformed surface is still in $\mathcal{H}^{hyp}(2g-2)$, while on the other hand, the hyperelliptic involution no longer exists (this is the reasoning of Lemma 8 in [KZ03]). The boundaries are interchanged because the direction to the north is changed to the opposite one.

Now we are ready to describe the trees corresponding to the hyperelliptic component.

**Lemma 3.14.** A positive marked plane bipartite tree corresponds to a square-tiled surface from the hyperelliptic component if and only if it is symmetric with respect to one of its edges, in the sense that the tree on one side of this edge is isomorphic (as a plane tree) to the tree on the other side, with the colors of the vertices inverted and the labels preserved (the edge lies in the correspondent corners of the boundaries of these trees).

**Proof.** Consider a separatrix diagram of a square-tiled maximal-cylinder surface from $\mathcal{H}^{hyp}(2g-2)$ (see Figure 4). The hyperelliptic involution $\sigma$ induces an automorphism of this diagram that interchanges opposite (with respect to the unique vertex) edges and faces. Opposite faces have the same labels and are of different colors.

The corresponding automorphism of the dual plane tree interchanges the vertices and the edges that are opposite on the boundary of the unique face. The opposite vertices have the same labels and are of different colors. To conclude the “only if” part, it is enough to note that in a tree with an even number of vertices there is always an edge that has an equal number of other edges on both of its sides. Two sides of this one edge will be opposite on the boundary of the unique face, hence the automorphism will map this edge to itself, interchanging its endpoints, hence also the two trees on each side of it.

For the “if” part, given such a tree, the corresponding square-tiled surface comes equipped with an involution that extends the local central symmetry near
the unique singularity. This involution has $2g + 2$ fixed points (the vertex, two points inside each of the $g$ cylinders, midpoint of the separatrix corresponding to the central edge of the tree). By Riemann-Hurwitz formula, the quotient of the surface by the involution has genus 0, so the canonical projection is a ramified double cover of $\mathbb{C}P^1$. Hence the surface is hyperelliptic (again, this is the reasoning of Lemma 8 in [KZ03]).

To distinguish other connected components, we need an invariant called the parity of the spin structure.

Recall that the canonical class $K_X \in \text{Pic}(X)$ of a compact Riemann surface $X$ is the class of divisors corresponding to the sections of the cotangent bundle $T^*X$. A spin structure on a compact Riemann surface $X$ is a choice of a half of the canonical class, i.e. an element $\alpha \in \text{Pic}(X)$ such that $2\alpha = K_X$. The parity of the spin structure $\alpha$ is the residue modulo 2 of the dimension $\dim \Gamma(X, L)$ of the space of sections of a line bundle $L$ corresponding to $\alpha$.

Suppose $\omega \in \mathcal{H}(2g - 2)$ is an Abelian differential on a surface $X$, with the zero of multiplicity $2g - 2$ at a point $P$. Then the divisor $(2g - 2)P$ represents the canonical class, and we have a canonical spin structure on $X$ defined by the divisor $(g - 1)P$.

It is known that the parity of the spin structure is invariant under continuous deformations. Hence, on each connected component of $\mathcal{H}(2g - 2)$ the parity of the correspondent canonical spin structure is constant. Finally, connected components $\mathcal{H}^{\text{odd}}(2g - 2)$ and $\mathcal{H}^{\text{even}}(2g - 2)$ consist of differentials on non-hyperelliptic surfaces with odd and even canonical spin structures respectively.\(^1\)

There exists an alternative (topological) definition of the spin structure on a Riemann surface which we omit here. However, in Section 3 of [KZ03] it is used to find the following effective way to compute the parity $\varphi(\omega)$ of the spin structure defined by $\omega$.

Consider a surface $X$ with an Abelian differential $\omega$. Recall that the flat structure defined by $\omega$ has a consistent choice of the direction to the north at each point of $X$ except the zeros of $\omega$. Let now $\alpha$ be a smooth closed oriented path on $X$ that omits the zeros of $\omega$. Then we can define its index $\text{ind}_\alpha \in \mathbb{Z}$\(^1\)

---

\(^1\)The spin parity for the hyperelliptic component depends (simply) on $g$. 

Figure 4: Separatrix diagram of a square-tiled surface from the hyperelliptic component and the corresponding tree.
as the total change of the angle between its tangent vector and the direction to the north divided by $2\pi$. Index is extended to linear combinations of curves by linearity.

Finally, let $(\alpha_i, \beta_i)_{i=1,...,g}$ be a symplectic basis of $H_1(X,\mathbb{Z})$. Seen as elements of $H_1(X,\mathbb{Z}/2\mathbb{Z})$, $\alpha_i$ and $\beta_i$ can be represented as a sum of smooth closed curves:

$$\alpha_i = \sum_{n=1}^{n_i} [\alpha_{i,n}], \beta_i = \sum_{m=1}^{m_i} [\beta_{i,m}], 1 \leq i \leq g.$$  

Then, we have the equality

$$\varphi(\omega) = \sum_{i=1}^{g} (\text{ind}_{\alpha_i} + n_i)(\text{ind}_{\beta_i} + m_i) \pmod{2}. \quad (6)$$

**Lemma 3.15.** Let a positive marked plane bipartite tree with $g$ black and $g$ white vertices be given. For each $i = 1,...,g$ choose an oriented path $\gamma_i$ between the white and the black vertices with labels $i$ along the boundary of the unique face. Then the parity of the spin structure of the surface corresponding to this tree is equal to

$$\sum_{i=1}^{g} \frac{l(\gamma_i) + 1}{2} + |S| \pmod{2}, \quad (7)$$

where $l(\gamma)$ is the (graph-theoretic) length of the path $\gamma$, and $S = \{(i,j) : i < j, \text{ exactly 1 endpoint of } \gamma_i \text{ lies inside } \gamma_j\}$.

**Proof.** First, we choose a (non-symplectic) basis $(\alpha_i, \beta_i)_{i=1,...,g}$ of $H_1(X,\mathbb{Z})$ as follows. $\alpha_i$ is a horizontal waist curve of the $i$-th cylinder. Let $0 < \varepsilon_1 < ... < \varepsilon_g$ be sufficiently small numbers. For each $i$, let $\beta_i$

- go around the singularity at a constant distance $\varepsilon_i$ from it, through the corners corresponding to the inner vertices of the path $\gamma_i$;
- when it reaches the bottom (black) boundary of the $i$-th cylinder, it goes vertically upwards inside the $i$-th cylinder and comes back to the point on the top (white) boundary where it started (we can assume this since twisting the cylinders does not change the parity).

See Figure 5 for an illustration.

Let $\langle \cdot, \cdot \rangle$ be the intersection form on $H_1(X)$. Obviously, for all $i,j$ we have $\langle \alpha_i, \alpha_j \rangle = 0$ and $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$. When $i < j$, $\varepsilon_i < \varepsilon_j$, so the curves $\beta_i$ and $\beta_j$ can intersect (transversally) only at a point where $\beta_i$ goes vertically inside $i$-th cylinder and $\beta_j$ goes circularly around the singularity. Each such point corresponds to the endpoint of $\gamma_i$ lying inside $\gamma_j$.  

21
Figure 5: Construction of the curve $\beta_i$.

Consider the basis $(\alpha_i, \beta'_i)_{i=1, \ldots, g}$ of $H_1(X)$, where

$$\beta'_1 = \beta_1 - \langle \beta_1, \beta_2 \rangle \alpha_2 - \langle \beta_1, \beta_3 \rangle \alpha_3 - \ldots - \langle \beta_1, \beta_g \rangle \alpha_g,$$

$$\beta'_2 = \beta_2 - \langle \beta_2, \beta_3 \rangle \alpha_3 - \ldots - \langle \beta_2, \beta_g \rangle \alpha_g,$$

$$\ldots$$

$$\beta'_{g-1} = \beta_{g-1} - \langle \beta_{g-1}, \beta_g \rangle \alpha_g,$$

$$\beta'_g = \beta_g.$$

Then, for $i < j$ we have $\langle \beta'_i, \beta'_j \rangle = \langle \beta_i, \beta_j \rangle - \langle \beta_i, \beta_j \rangle \langle \alpha_j, \beta_j \rangle = 0$. So this new basis is symplectic. We apply formula (6) to this basis. Note that, since $\alpha_i$ are horizontal curves, $\text{ind}_{\alpha_i} = 0$. The curve $\beta_i$ is vertical inside $i$-th cylinder and turns by $\pi$ inside each corner around the singularity. The number of such corners is $l(\gamma_i) - 1$, so $\text{ind}_{\beta_i} = \frac{l(\gamma_i) - 1}{2}$. In formula (6) we have $n_i = 1$ and $m_i = 1 + |\{ j : j > i, \langle \beta_i, \beta_j \rangle \text{ is odd} \}|$, since we look at $\alpha_i$ and $\beta'_i$ as elements of $H_1(X, \mathbb{Z}/2\mathbb{Z})$. Substituting, we get (7).

Let $Q_k^{hyp}(L_1, \ldots, L_k), Q_k^{even}(L_1, \ldots, L_k), Q_k^{odd}(L_1, \ldots, L_k)$ be the functions counting the number of positive trees with vertex lengths $(L_1, \ldots, L_k)$ which are hyperelliptic, even-spin-parity and odd-spin-parity, respectively. First we will show that each of these functions is constant on the same sets as $Q_k$.

Then, we will derive the recurrence relation for the generic value $q_k^{hyp}$ and the difference of generic values $q_k^{odd} - q_k^{even}$.

3.4.2 Existence of generic values

Lemma 3.16. $Q_k^{hyp}, Q_k^{even}$ and $Q_k^{odd}$ are constant on $(\mathbb{R}^+)^k$ minus the union of hyperplanes of the form $\sum \varepsilon_i L_i = 0$, where $\varepsilon_i \in \{-1, 0, +1\}$.

Proof. Recall the proof of Lemma 3.10. To prove our statement, it is enough to show that the bijection between degenerate trees on different sides of a hyperplane from $\mathcal{V}$ sends hyperelliptic trees to hyperelliptic trees and preserves the spin parity of the tree (i.e. of the corresponding surface).
Consider a degenerate hyperelliptic tree. By Lemma 3.14, the tree must be symmetric with respect to one of its edges. Note that if an edge corresponding to a linear form $f = \sum_{i \in I} L_i - \sum_{j \in J} L'_j$ is zero-length, then its symmetric edge, which corresponds to the linear form $f^{\text{sym}} := \sum_{i \in J} L_i - \sum_{j \in I} L'_j$, is also zero-length, since $f + f^{\text{sym}} = \sum L_i - \sum L'_j = 0$. Hence, the zero-length edges are also symmetric, and when we perform the modifications of the zero-length edges, the tree remains symmetric. So the bijection preserves hyperellipticity.

Let us show now that the bijection preserves spin parity. Actually, we will prove a stronger statement: the modification of a single zero-length edge preserves spin parity.

Refer to Figure 1 for notation. For each $i = 1, \ldots, k$ choose a path $\gamma_i$ along the boundary of the tree between the white and the black vertices with labels $i$. We put two restrictions on the choice of these paths:

- the path that starts at the white vertex $w$ should not start inside the corner $w'vw$;
- the path that ends at the black vertex $v$ should not end inside the corner $v'vw$.

When we perform the modification of the zero-length edge $vw$, the only change to the circular order of labels along the boundary is that the labels of $v$ and $w$ corresponding to the corners $v'vw$ and $w'wv$, respectively, are changed to the labels of $w'$ and $v'$, respectively (see Figure 6). Since none of our paths started or ended inside one of this corners, each $\gamma_i$ will remain a valid path joining vertices with labels $i$. Moreover, their lengths and the position of their endpoints relative to each other will not change. Hence, both summands in formula (7) for the spin parity remain constant, so the spin parity is preserved.

\[ q^{\text{hyp}}_k = (2^k - 3)!!. \]

3.4.3 Recurrences for generic values

**Theorem 3.17.** Let $q^{\text{hyp}}_k$ be the generic value of $Q^{\text{hyp}}_k$, $k \geq 1$. Then

\[ q^{\text{hyp}}_k = (2k - 3)!!. \]
Proof. Recall the proof of Theorem 3.12. We considered there the trees with vertex lengths $1, 2, \ldots, 2k-1, 1, 2, \ldots, 2k-1$, and decomposed them into two smaller trees by looking at a particular edge $e$ of length 1 on the (unique) oriented path between the black and the white 1-leaves. We proved there that this edge divides the set of vertices into two groups which are symmetric, i.e. for each $i \geq 2$ the black and the white vertices with labels $i$ belong to the same group.

It is easy to see that, due to its symmetry (Lemma 3.14), a hyperelliptic tree possesses only two such edges $e$: the two edges incident to the 1-leaves. It means that we are directly in the conditions of Lemma 3.13. The procedure of this lemma simply removes these two edges, and we get a hyperelliptic tree again. Viewed as a map between hyperelliptic trees of sizes $2^k$ and $2^k-2$, it is $(2^k-3)$-to-one (and not $(2^k-3)^2$-to-one), because of the symmetry. Hence $q_{k}^{hyp} = (2^k-3)/q_{k-1}^{hyp}$. Since $q_1^{hyp} = 1$, we get the desired formula.

**Theorem 3.18.** For $k \geq 1$, let $q_k^{even}$ and $q_k^{odd}$ be the generic values of $Q_k^{even}$ and $Q_k^{odd}$, respectively. Then the difference $d := q_k^{odd} - q_k^{even}$ satisfies the following recurrence relation:

$$d_1 = 1,$$

$$d_k = \sum_{a+b=k, a,b \geq 1} \binom{k-1}{a}(2a-1)d_ad_b, k \geq 2.$$

**Proof.** $d_1 = 1$ because $q_1^{odd} = 1$ and $q_1^{even} = 0$.

For the recurrence relation, again, we revisit the proof of Theorem 3.12. Refer to Figure 3 for an illustration.

First of all, we want to understand how the spin parity of a tree $T$ depends on the spin parities of its parts $T_1$ and $T_2$. For each $i \geq 2$, choose a path $\gamma_i$ along the boundary of the tree between the white and the black vertices with labels $i$, such that it does not pass through either side of the edge $e$ (this is possible because the endpoints of $\gamma_i$ lie on the same side of $e$). This ensures that the paths $\gamma_i$ can be viewed as paths in $T_1$ or $T_2$.

Choose also an arbitrary path $\gamma_1$ along the boundary between the 1-leaves. Its restriction to $T_i$ gives a path $\gamma_{1,i}$ between the 1-leaves of $T_i$ for $i = 1, 2$.

Obviously, $l(\gamma_{1,1}) + l(\gamma_{1,2}) = l(\gamma_1) + 1$. The paths on different sides of $e$ do not intersect. Hence, using formula (7) for the spin parity of the tree, we get

$$\varphi(T) = \varphi(T_1) + \varphi(T_2) + 1 \pmod{2}.$$

Now, the tree $T_1$ is constructed (Lemma 3.13) by gluing two edges of length 1 to one black and one white vertex of a smaller tree $T'_1$. The formula (7) for the spin parity then gives

$$\varphi(T'_1) = \varphi(T_1) - \frac{l(\gamma_{1,1}) + 1}{2} - \frac{2N}{2} = \varphi(T_1) - \frac{l(\gamma_{1,1}) + 1}{2} \pmod{2},$$

where $N$ is the number of paths $\gamma_i$ passing through the black 1-leaf. The term $\frac{2N}{2}$ corresponds to the total change of lengths of such paths, and the term $N$ corresponds to the loss of their intersections with the path $\gamma_{1,1}$.
To sum up, $\phi(T) = \phi(T_1') + \phi(T_2) + \frac{l(\gamma_{1,1})+1}{2} + 1 \ (\text{mod } 2)$. Let $a$ be the number of black vertices in $T_1'$. There are $(2a-1)(a-1)$ ways to glue the two edges of length 1 so that $l(\gamma_{1,1}) = 1 \ (\text{mod } 4)$, and $(2a-1)a$ ways to do it so that $l(\gamma_{1,1}) = 3 \ (\text{mod } 4)$. This gives the following recurrence relations:

$$q_{k}^{\text{odd}} = \sum_{a+b=k, a, b \geq 1} \binom{k-1}{a} (q_a^{\text{even}} q_b^{\text{odd}} + q_a^{\text{odd}} q_b^{\text{even}})(2a-1)(a-1) + (q_a^{\text{even}} q_b^{\text{even}} + q_a^{\text{odd}} q_b^{\text{odd}})(2a-1)a],$$

$$q_{k}^{\text{even}} = \sum_{a+b=k, a, b \geq 1} \binom{k-1}{a} [(q_a^{\text{even}} q_b^{\text{even}} + q_a^{\text{odd}} q_b^{\text{odd}})(2a-1)(a-1) + (q_a^{\text{even}} q_b^{\text{odd}} + q_a^{\text{odd}} q_b^{\text{even}})(2a-1)a].$$

Taking the difference, we get the desired formula.

Now we will deduce the asymptotics of $d_k$, by using the method of singularity analysis (see, for instance [FS09], Part B).

**Theorem 3.19.** The differences $d_k = q_k^{\text{odd}} - q_k^{\text{even}}$ have the asymptotics

$$d_k = \frac{1}{4\sqrt{2\pi e}} k!(2e)^k (k^{-5/2} + O(k^{-7/2})), \text{ as } k \to \infty.$$

**Proof.** Consider the exponential generating function

$$F(z) = \sum_{k=0}^{\infty} d_{k+1} \frac{z^k}{k!}.$$

The recurrence relation of Theorem 3.18 can be rewritten as

$$F(z) = 1 + 2zF(z)^2 - (fF)(z)F(z),$$

with $F(0) = 1$.

Differentiation gives $F'(z) = F(z)^2 + 4zF(z)F'(z) - (fF)(z)F'(z)$ or

$$F'(z)(1 + (fF)(z)) = F(z)^2 + 4zF(z)F'(z).$$

Note that (8) implies $F(z)(1 + (fF)(z)) = 1 + 2zF(z)^2$. Hence, multiplying both sides of equation (9) by $F(z)$ and rearranging, we finally get:

$$F'(z) = \frac{F(z)^3}{1 - 2zF(z)^2}, F(0) = 1.$$

This first-order differential equation can be integrated to give $F$ as an implicit function:

$$F(z) = e^{F(z)^2}.$$
Recall the Cayley tree function $T(z) = \sum_{k=1}^{\infty} k^{k-1} z^k$, which is the exponential generating function of non-plane rooted labelled trees. It satisfies the equation $T(z) = z e^{T(z)}$, $T(0) = 0$. It is easy to see that we must have $2zF(z)^2 = T(2z)$, and hence

$$F(z) = \sqrt{\frac{T(2z)}{2z}}.$$ 

Figure 7: The curve $C$ (solid line) is the image of the unit circle centered at 0 by the map $y \mapsto z = ye^{-y}$. The circle $|z| = e^{-1}$ (dashed line) is contained in the interior of $C$.

Since the Cayley tree function satisfies $T(z) = ze^{T(z)}$, it is an inverse function of $y \mapsto z = ye^{-y}$. Let $C$ be the image of the radius 1 circle centered at 0 by this map. $C$ is a curve that has a cusp at the point $e^{-1}$ (see Figure 7) and contains a circle of radius $e^{-1}$ centered at 0 in its interior $C^{int}$. It is known (see Example VI.8 in [FS09]), that $T$ is analytic in $C^{int}$ and has a square-root type singularity at the point $e^{-1}$. Note also that, since $T$ maps $C^{int}$ biholomorphically onto $\{ |z| < 1 \}$, its only (simple) zero is at the point 0. It means that $T(2z)/2z$ is analytic in $\frac{1}{2}C^{int}$ and does not have any zeroes there. Moreover, the limit value of $T$ at the point $e^{-1}$ is finite (equal to 1).

The above arguments imply that $F(z) = \sqrt{T(2z)/2z}$ is also analytic in $\frac{1}{2}C^{int}$ and still has a square-root type singularity at the point $(2e)^{-1}$. It means that $F$ satisfies the necessary conditions for the application of singularity analysis (Theorem VI.4 in [FS09]). The singular expansion

$$T(z) = 1 - \sqrt{2}(1 - ez)^{1/2} + O(1 - ez)$$

for $z \to e^{-1}, z \in C^{int}$, gives the singular expansion

$$F(z) = \sqrt{\frac{T(2z)}{2z}} = \sqrt{e} - \sqrt{\frac{e}{2}}(1 - 2ez)^{1/2} + O(1 - 2ez)$$

26
for $z \to (2e)^{-1}, z \in \frac{1}{2}C^{\text{int}}$.

By singularity analysis, this translates to the asymptotics of coefficients:

$$\frac{d_{k+1}}{k!} = [z^k] F(z) = \frac{\sqrt{e}}{2\sqrt{2\pi}} (2e)^k (k^{-3/2} + O(k^{-5/2})).$$

Changing $k + 1$ to $k$, we get the desired asymptotics. □

References


