A NEW ORBIT CLOSURE IN GENUS 8

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Abstract. We prove that the orbit closure of the unfolding of the $(3, 4, 13)$-triangle is a previously unknown 4-dimensional variety, by using a Hurwitz space of covers with monodromy in a certain solvable group of order 400.

CONTENTS

1. Introduction 1
2. Structure of the proof 4
3. A solvable group of order 400 6
4. Covers of $\mathbb{P}^1$ branched over 7 points 9
5. Differentials on covers branched over 7 points 11
6. A partial compactification 13
7. Covers of $\mathbb{P}^1$ branched over 4 points 17
Appendix A. Explicit surfaces in $\mathcal{N}$ 19
Appendix B. Exploring $\text{GL}(2, \mathbb{R})$-orbit closures with $\text{flatsurf}$ 20
Appendix C. Exceptional billiards 22
References 23

1. Introduction

Given a polygon $P$ in $\mathbb{R}^2$, we consider the associated billiard flow on its unit tangent bundle $T^1P$. When the polygon is rational, that is all its angles are rational multiple of $\pi$, then the 3-dimensional phase space $T^1P$ decomposes into a one-parameter family of invariant surfaces that are the directional flows associated to the vector fields $\omega = e^{i\theta}$ of an Abelian differential $\omega$ on a compact Riemann surface $X$. See the survey [MT02] for this construction, which is called the unfolding of the billiard table $P$.

Let $\mathcal{M}_{g,n}$ denote the moduli space of compact Riemann surface with $n$ distinct marked points. Given a tuple of non-negative numbers $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ which sum to $2g - 2$, we define the stratum $\mathcal{H}_g(\kappa)$ as the moduli space of tuples $(X, p_1, \ldots, p_n, \omega)$ such that $(X, p_1, \ldots, p_n) \in \mathcal{M}_{g,n}$ and $\omega$ is a non-zero Abelian differential on $X$ with zeros of order $\kappa_i$ at $p_i$. The local period coordinates map which, to $(X, p_1, \ldots, p_n, \omega)$ associates

$$[\omega] \in H^1(X, \{p_1, \ldots, p_n\}; \mathbb{C}) \simeq \mathbb{C}^{2g+n-1} \simeq (\mathbb{R}^2)^{2g+n-1},$$

provides each stratum with an atlas of local charts with transition maps in $\text{GL}(2g+n-1, \mathbb{Z})$. 
Each stratum $\mathcal{H}_g(\kappa)$ admits a $\text{GL}^+(2,\mathbb{R})$-action via the action on periods. This $\text{GL}^+(2,\mathbb{R})$-action acts as a renormalization for the directional flows and many results about the combinatorics and the dynamics of billiards in rational polygons are obtained by studying properties of this action. See the surveys [Esk06, For06, HS06, Mas06, Wri15b].

We call a closed, irreducible $\text{GL}^+(2,\mathbb{R})$-invariant subvariety of a stratum of Abelian differentials an invariant subvariety. By [EM18, EMM15, Fil16b], any closed $\text{GL}^+(2,\mathbb{R})$-invariant subset of a stratum is finite union of invariant subvarieties. Moreover, [EM18, EMM15] gives that the image of any invariant $\mathcal{N}$ under the local period coordinates map is locally identified with a finite union of linear subspaces of $H^1(X, Z(\omega); \mathbb{C})$ of the form $V \oplus \sqrt{-1}V$ for some $V \subset H^1(X, Z(\omega); \mathbb{R})$; at typical points of $\mathcal{N}$, only a single subspace is required.

Recall the following definitions: the rank of $\mathcal{N}$ is half the dimension of the image of $V$ in absolute cohomology $H^1(X; \mathbb{R})$; the rel of $\mathcal{N}$ is its (complex) dimension minus two times its rank; and the field of definition $\mathbf{k}(\mathcal{N})$ is the smallest field over which the linear subspace $V$ is defined [Wri14]. Here it is helpful to recall that the image of $V$ in $H^1(X, \mathbb{R})$ is symplectic [AEM17], and that real cohomology has a canonical integral structure.

Similarly to the Abelian case, we define strata of quadratic differentials $Q_g(\mu)$. By a standard construction, a quadratic differential admits a double cover on which the quadratic differential lifts to the square of an Abelian differential. We denote by $\tilde{Q}_g(\mu)$ the locus of all such double covers of elements of $Q_g(\mu)$.

We now state our main result. By default, all dimensions are over $\mathbb{C}$.

**Theorem 1.1.** There exists an invariant subvariety

$$\mathcal{N} \subset \tilde{Q}_4(11, 1) \subset \mathcal{H}_8(12, 2)$$

that is 4-dimensional and has $\mathbf{k}(\mathcal{N}) = \mathbb{Q}(\sqrt{5})$, and which contains all unfoldings of quadrilaterals with angles $\{\frac{2\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10}, \frac{13\pi}{10}\}$.

Rational billiards whose unfoldings have non-dense $\text{GL}^+(2,\mathbb{R})$-orbit closures are rare. We refer the reader to Section C for a survey of the literature and experimental results about triangles and quadrilaterals, emphasizing for now that Theorem 1.1 adds to recent discoveries of the same flavor by McMullen-Mukamel-Wright and Eskin-McMullen-Mukamel-Wright [MMW17, EMMW20].

Before discussing the proof, we point out a few properties of $\mathcal{N}$. Since $\tilde{Q}_4(11, 1)$ has no rel, it follows that $\mathcal{N}$ has no rel. Hence the fact that $\mathcal{N}$ is 4-dimensional implies that it has rank 2. We also note that [AW, Remark 3.4] implies that unfoldings of typical $(\frac{2\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10}, \frac{13\pi}{10})$ quadrilaterals do not cover lower genus translation surfaces, so $\mathcal{N}$ is primitive. Finally, we point out as follows that $\mathcal{N}$ is the orbit closure of many unfoldings.

**Corollary 1.2.** For all but finitely many similarity classes of quadrilaterals with angles $\{\frac{2\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10}, \frac{13\pi}{10}\}$, the orbit closure of the unfolding is equal to $\mathcal{N}$.

The orbit closure of the unfolding of a triangle with angles $\{\frac{3\pi}{20}, \frac{4\pi}{20}, \frac{13\pi}{20}\}$ is equal to $\mathcal{N}$.

**Proof of Corollary.** It follows from [AEM17] that the orbit of any surface in $\mathcal{N}$ is closed or dense. It follows from [EFW18] that $\mathcal{N}$ has only finitely many closed

\[1\]This is not the same as the field of definition of $\mathcal{N}$ in the sense of algebraic geometry.
orbits. This gives the first statement. (The same result with countably rather than finitely many exceptions follows from remarks in \[MW18\].)

The unfolding of the triangle with angles \(\{\frac{3\pi}{20}, \frac{3\pi}{20}, \frac{13\pi}{20}\}\) is also an unfolding of a quadrilateral with angles \(\{\frac{2\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10}, \frac{3\pi}{10}\}\); the quadrilateral can be obtained by doubling the triangle by reflection along the edge joining angles of \(\frac{3\pi}{20}\) and \(\frac{13\pi}{20}\).

Since the triangle does not have closed orbit, this gives the second statement. (The claim that the triangle does not have closed orbit can be rigorously verified for example by the software discussed in the appendices.)

The orbit closures in \[MMW17, EMMW20\] have somewhat miraculous constructions via Hurwitz spaces of maps to \(\mathbb{P}^1\) with dihedral monodromy groups. Similarly, we construct \(\mathcal{N}\) via a Hurwitz space, but now using a certain solvable group of order 400 with structure \(((C_5 \times C_5) \rtimes D_8) \times C_2\). Our construction of \(\mathcal{N}\) exploits many ideas from \[MMW17, EMMW20\], with a key difference, beyond having a more complicated monodromy group, being that we neither establish nor require any global understanding of our subvariety. Instead we rely on a soft codimension argument presented in Section 2. This codimension argument can be used quite broadly, and it would be interesting to investigate which modifications of the group theoretic data in our construction might be associated with orbit closures.

Although surfaces in invariant subvarieties are known to satisfy a host of remarkable properties, there is no general reason that they should admit the sort very special of map to \(\mathbb{P}^1\) that powers our approach. Indeed, the existence or non-existence of such a map is completely invisible from the perspective of flat geometry and dynamics, and although it is related to the algebro-geometric properties of the Jacobian established in special cases by \[M\ddot{o}l06a, M\ddot{o}l06b, McM03, McM06\] and in general by \[Fil16a, Fil16b\], it is not at all guaranteed by them. It is an unexplained mystery why our orbit closure and others, most discovered “in nature” without any indication of having such a nice construction, parametrize surfaces with extremely special maps to \(\mathbb{P}^1\); Why is it that the renormalization for some billiards takes place in a rather special Hurwitz space, especially when that renormalization (namely the \(GL_+(2, \mathbb{R})\) action) is of a highly transcendental nature?

The primary interest in Theorem 1.1 comes from dynamics, but its proof does not involve any dynamics and may also be of interest from the point of view of classical algebraic geometry. The unfoldings under consideration are cyclic covers of \(\mathbb{P}^1\) of the form

\[y^{10} = (z - z_1)^2(z - z_2)^2(z - z_3)^3(z - z_4)^3 \frac{y}{\prod(z - z_i)}\]

endowed with the Abelian differential

As will be apparent from Theorem 2.1, the proof of Theorem 1.1 consists entirely of building an appropriate family of algebraic curves (endowed with Abelian differentials) with special properties; for example their Jacobians have a copy of \(\mathbb{Q}[\sqrt{5}]\) in their endomorphism algebra. This might be compared to the construction of families of algebraic curves with large endomorphism algebra in \[EHL01\], with the main extra difficulty and novelty being that a delicate analysis is required to establish strong restrictions on the divisors of certain Abelian differentials, and that a careful partial compactification must be performed.
It would be valuable to have a more explicit understanding of $\mathcal{N}$. Given $(X, \omega) \in \tilde{Q}_4(11, 1)$, Lemma 5.6 gives a condition which is sufficient to guarantee $(X, \omega) \in \mathcal{N}$. This condition has a similar flavor to the condition used in [EMMW20] and so may serve as a starting point for further progress in this direction. It would be particularly interesting to compute the boundary of $\mathcal{N}$ in the What You See Is What You Get partial compactification used in [MW17, CW19].

Appendix [A] gives examples of surfaces in $\mathcal{N}$, presented via polygons in a particularly nice way, together with the linear equations locally defining $\mathcal{N}$ at these surfaces. Appendix [B] briefly discusses the software that discovered that quadrilaterals with angles $\{\frac{2\pi}{10}, \frac{3\pi}{10}, \frac{13\pi}{10}\}$ are very special and that gave strong evidence for Theorem 1.1.

The discovery that the group $((C_5 \times C_5) \rtimes D_8) \times C_2$ was related to quadrilaterals with angles $\{\frac{2\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10}, \frac{13\pi}{10}\}$ started from the conjecture that Theorem 1.1 is true. Given that the examples in [EMMW20] admit a construction via dihedral maps, it was natural to try to find an analogous construction for any new orbit closure, even though such a construction is not guaranteed to exist. Comparison to [EMMW20], together with (conjectural) knowledge about $\mathcal{N}$, gave a number of likely restrictions on the group. We hope to clarify in the future what these restrictions are and how they were derived, but in the meantime the proof of Theorem 1.1 should provide many hints. A computer search, together with analysis of various candidate groups, eventually led us to $((C_5 \times C_5) \rtimes D_8) \times C_2$. A minor challenge in this process was that, unlike the situation in [EMMW20], the differentials in our $\mathcal{N}$ have zeros of different order. A larger challenge was that, unlike the situation in [EMMW20], the degree of the maps used in our Hurwitz space (20) does not agree with the degree of our cyclic covers (10); this added significantly to the difficulty in discovering the correct group.

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Most of the group theoretic computations in this paper were performed using the GAP [GAP21] software.

2. Structure of the proof

The bulk of this paper is occupied with the proof of the following.

Theorem 2.1. There is a six dimensional irreducible quasi-projective variety $\mathcal{N}_0$ with a morphism $\pi$ to the bundle of non-zero Abelian differentials over $\mathcal{M}_{8,2}$, such that for all $(X, \omega) \in \pi(\mathcal{N}_0)$,

1. the vanishing order of $\omega$ at each of the two marked points is 2 or 6 modulo 10,

2. $X$ has an involution whose fixed point set is the two marked points and which negates $\omega$, and
(3) there is a 4-dimensional subspace in the anti-invariant part of $H^1(X)$ that contains the cohomology class of $\omega$ and is defined over $\mathbb{Q}[\sqrt{5}]$, and such that

(4) $\pi$ has finite fibers, and

(5) $\pi(N_0)$ contains all unfoldings of quadrilaterals with angles $\{\frac{2\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10}, \frac{13\pi}{10}\}$, where the marked points are the two zeros.

Here our marked points are ordered, and for the unfoldings we pick the zero of order 12 as the first marked point. One could equally well phrase condition (5) in terms of cyclic covers.

Proof of Theorem 1.1 assuming Theorem 2.1. Let $\Omega_0$ denote the complement of the zero section in the Hodge bundle over $M_{8,2}$, and let $\Omega_k$ denote the subset of $\Omega_0$ where the differential vanishes to order at least $k$ at the first of the two marked points.

Let $N_k = \pi^{-1}(\Omega_k)$. The subvariety $\Omega_{k+1}$ is locally defined in $\Omega_k$ by a single equation, so it follows that $N_{k+1}$ is locally defined in $N_k$ by a single equation. Hence $N_{k+1}$ is codimension at most 1 in $N_k$ when it is non-empty.

The assumption that $\pi(N_0)$ contains the unfoldings shows in particular that $N_{12}$ is non-empty. We claim that every component of $N_{12}$ has dimension at least 4. Indeed, the condition that $\omega$ can only vanish to order 2 or 6 modulo 10 gives that $N_0 = N_2$, $N_3 = N_6$, and $N_7 = N_{12}$. Since $N_3 \subset N_2$ is codimension at most 1, and $N_7 \subset N_6$ is codimension at most 1, we get the claim.

Since every form in $N_{12}$ has a zero of order 12 and a zero of order 2, and since these two points are the fixed point for an involution negating the Abelian differential, we see that $\pi(N_{12}) \subset \mathbb{Q}_4(11,1) \subset H_8(12,2)$.

For $(X,\omega) \in \pi(N_{12})$, let $\Sigma$ denote the set of the two marked points. Since the zeros are fixed by the involution, the natural map $H^1(X,\Sigma) \to H^1(X)$ is an isomorphism when restricted to the $-1$ eigenspace of the involution. Hence we get that the relative cohomology class of $\omega$ is contained in a 4-dimensional subspace $S$ of $H^1(X,\Sigma)$ defined over $\mathbb{Q}[\sqrt{5}]$.

Let $N$ be the closure of $\pi(N_{12})$ in $H_8(12,2)$. Since $\pi(N_{12})$ is a constructible set, for every $(X,\omega) \in N$ there is a complex disc which maps holomorphically to $N$ in such a way that the origin maps to $(X,\omega)$ and the complement of the origin maps to $\pi(N_{12})$. Since there are only countably many subspaces defined over $\mathbb{Q}[\sqrt{5}]$, one can take the 4-dimensional subspace $S$ to be locally constant along this punctured disc, and hence obtain the existence of such a 4-dimensional subspace at $(X,\omega)$. In conclusion, at every point $(X,\omega) \in \pi(N_{12})$ the relative cohomology class of $\omega$ is contained in a 4-dimensional subspace $S$ of $H^1(X,\Sigma)$ defined over $\mathbb{Q}[\sqrt{5}]$.

Since $\pi$ has finite fibers and $N_{12}$ is at least 4-dimensional, $N$ is at least 4-dimensional. It follows from [MMW17, Theorem 5.1] that $N$ is exactly 4-dimensional and is locally linear in local period coordinates; this is roughly because the four-dimensional subspace must be locally constant since it is defined over a number field, and since period coordinates are locally injective the image of $N$ must locally fill out this subspace. Since the subspaces by construction are defined over $\mathbb{Q}[\sqrt{5}] \subset \mathbb{R}$, it follows that $N$ is an invariant subvariety. □
Remark 2.2. In the argument above, since \( \pi(N_0) \) is only known to be a constructible set, one cannot simply replace \( N_0 \) with its image and work entirely in \( \Omega_0 \). We also note that there are projective varieties with codimension 1 subvarieties not locally defined by one equation, and the preimage of such a codimension 1 subvariety under a morphism need not in general be empty or codimension at most 1.

We will prove Theorem 2.1 using spaces of covers of \( \mathbb{P}^1 \). To start, we will build a very particular space of genus 8 degree 20 covers of \( \mathbb{P}^1 \) branched over 7 points with monodromy group a specific solvable group \( G \) of order 400. These covers will have a fiber which consists of two points; we will call this the special fiber, and mark these two points. Each cover will come with a 2-dimensional space of differentials \( \omega \) as in Theorem 2.1, and since a space of covers of \( \mathbb{P}^1 \) branched over 7 points has dimension 4, this gives rise to a \( 2 + 4 = 6 \) dimensional quasi-projective variety as in Theorem 2.1.

This variety does not however contain the desired unfoldings of quadrilaterals, and it is not even a priori clear it contains any Abelian differentials that vanish to order 12 at the first marked point. To obtain the unfoldings, we must partially compactify our space of covers by allowing limited collisions of branch points.

3. A solvable group of order 400

We consider the group \( G \) with generators \( f_1, f_2, t, d, c \) and relations

\[
\begin{align*}
&f_1^5, f_2^5, [f_1, f_2], \\
t^4, d^2, dt d^{-1} t, \\
t f_1 t^{-1} f_1^{-2}, t f_2 t^{-1} f_2, d f_1 d^{-1} f_2^{-1}, \\
c^2, [c, f_1], [c, f_2], [c, t], [c, d].
\end{align*}
\]

Let \( r := (f_1 f_2)^{-1} c \) and define the following subgroups.

\[
\begin{align*}
G_0 &:= \langle f_1, f_2, t, d \rangle \\
C &:= \langle c \rangle \\
D &:= \langle t, d \rangle \\
F &:= \langle f_1, f_2 \rangle \\
R &:= \langle r \rangle \\
A &:= \langle r, td \rangle \\
H &:= \langle f_1 f_2, t^2 d \rangle \\
H^+ &:= \langle f_1 f_2, t^2 d, r \rangle
\end{align*}
\]

Lemma 3.1. We have the following structure:

- \( C \) is Abelian isomorphic to \( C_2 \) (order 2),
- \( D \) is a dihedral group of order 8,
- \( F \) is Abelian isomorphic to \( C_5 \times C_5 \) (order 25),
- \( R \) is Abelian isomorphic to \( C_{10} \),
- \( A \) is Abelian isomorphic to the direct product \( R \times C \),
- \( H \) is a dihedral group of order 20,
- \( H^+ \) is the direct product \( H \times R \) (order 200),
- \( G_0 \) is the semi-direct product \( F \rtimes D \) (order 200),
- \( G \) is the direct product \( G_0 \times C \) (order 400).
3.1. The group $G_0$ and the action $\varphi_0:\ G_0 \to \text{Sym}_{10}$. Following the standard convention in algebraic topology (explained for example in [Hat02, page 69]), we use the right action of the symmetric group Sym$_n$ on $\{1, 2, \ldots, n\}$. In other words we compose permutations from left to right. In our context, this action is a monodromy action. Later on we will use left actions for deck groups. In the situation of a normal cover with group $G$, these two actions correspond respectively to the right and left action of $G$ on itself by multiplication.

**Lemma 3.2.** There exists a unique morphism $\varphi_0:\ G_0 \to \text{Sym}_{10}$ defined on the generators as
\[
\varphi_0(f_1) = (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)
\]
\[
\varphi_0(f_2) = (5, 4, 3, 2, 1)(6, 7, 8, 9, 10)
\]
\[
\varphi_0(t) = (1, 6)(2, 9, 5, 8)(3, 7, 4, 10)
\]
\[
\varphi_0(d) = (2, 5)(3, 4)
\]
Moreover the stabilizer of 1 under this action is $H$.

**Proof.** It is straightforward to check the relations (3.1) defining the group $G_0$. Similarly, one checks that $H$ stabilizes 1. Since the action of $\varphi_0(G_0)$ is transitive one concludes that the stabilizer is exactly $H$ by cardinality (the index of $H$ in $G_0$ is 10 by Lemma 3.1).\hfill \Box

We did not find an elegant way to present the list of conjugacy classes and characters of the group $G_0$. Instead we rely on the results provided by the GAP software.

**Lemma 3.3.** The group $G_0$ has 14 conjugacy classes with the following list of representatives.

<table>
<thead>
<tr>
<th>representative</th>
<th>order</th>
<th>image under $\varphi_0$</th>
<th>centralizer size</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>()</td>
<td>200</td>
</tr>
<tr>
<td>$d$</td>
<td>2</td>
<td>(2, 5)(3, 4)</td>
<td>20</td>
</tr>
<tr>
<td>$td$</td>
<td>4</td>
<td>(1, 6)(2, 9, 5, 8)(3, 7, 4, 10)</td>
<td>4</td>
</tr>
<tr>
<td>$t^2$</td>
<td>4</td>
<td>(2, 5)(3, 4)(7, 10)(8, 9)</td>
<td>8</td>
</tr>
<tr>
<td>$f_1$</td>
<td>5</td>
<td>(1, 2, 3, 4, 5)(6, 7, 8, 9, 10)</td>
<td>25</td>
</tr>
<tr>
<td>$f_1 f_2$</td>
<td>8</td>
<td>(6, 8, 10, 7, 9)</td>
<td>50</td>
</tr>
<tr>
<td>$f_1 f_2^2$</td>
<td>8</td>
<td>(1, 5, 4, 3, 2)(6, 9, 7, 10, 8)</td>
<td>50</td>
</tr>
<tr>
<td>$f_1 f_2^3$</td>
<td>8</td>
<td>(1, 4, 2, 5, 3)(6, 10, 9, 8, 7)</td>
<td>50</td>
</tr>
<tr>
<td>$f_1 f_2^4$</td>
<td>8</td>
<td>(1, 3, 5, 2, 4)</td>
<td>50</td>
</tr>
<tr>
<td>$f_1 d$</td>
<td>10</td>
<td>(1, 5)(2, 4)(6, 7, 8, 9, 10)</td>
<td>10</td>
</tr>
<tr>
<td>$f_1^2 d$</td>
<td>11</td>
<td>(1, 3)(4, 5)(6, 9, 7, 10, 8)</td>
<td>10</td>
</tr>
<tr>
<td>$f_1 t d$</td>
<td>11</td>
<td>(1, 9, 4, 8, 2, 7, 5, 6, 3, 10)</td>
<td>10</td>
</tr>
<tr>
<td>$f_1^2 t d$</td>
<td>11</td>
<td>(1, 10, 5, 7, 4, 9, 3, 6, 2, 8)</td>
<td>10</td>
</tr>
</tbody>
</table>

We now study the characters of $G_0$. We will always represent them as vectors
\[
(\chi(1), \chi(d), \chi(td), \ldots, \chi(f_1^2 t d)) \in \mathbb{C}^{14}
\]
corresponding to the evaluation on conjugacy classes in the order provided in Lemma 3.3.
Lemma 3.4. The fourteen irreducible characters of $G_0$ are the six characters defined over $\mathbb{Q}$ given by

<table>
<thead>
<tr>
<th>name</th>
<th>character</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{1a}$</td>
<td>1 1 1 1 1 1</td>
</tr>
<tr>
<td>$\chi_{1b}$</td>
<td>1 -1 -1 1 1 1</td>
</tr>
<tr>
<td>$\chi_{1c}$</td>
<td>1 -1 1 -1 1 1</td>
</tr>
<tr>
<td>$\chi_{1d}$</td>
<td>1 1 -1 1 -1 1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>2 0 0 -2 0 0</td>
</tr>
<tr>
<td>$\chi_8$</td>
<td>8 0 0 0 0 0</td>
</tr>
</tbody>
</table>

and the 8 characters defined over $\mathbb{Q}[^{\sqrt{5}}]$ which, up to Galois conjugation, are given by

<table>
<thead>
<tr>
<th>name</th>
<th>character</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{4a}$</td>
<td>4 -2 0 0 0 0</td>
</tr>
<tr>
<td>$\chi_{4b}$</td>
<td>4 2 0 0 0 0</td>
</tr>
<tr>
<td>$\chi_{4c}$</td>
<td>4 0 -2 0 0 0</td>
</tr>
<tr>
<td>$\chi_{4d}$</td>
<td>4 0 2 0 0 0</td>
</tr>
</tbody>
</table>

where $\alpha = \frac{1+i\sqrt{5}}{2}$.

We let $\chi'$ denote the Galois conjugate of a character $\chi$ defined over $\mathbb{Q}[\alpha]$. We set $\xi = \exp(2\pi i/10)$, so $\alpha = \xi + \xi^{-1}$.

Remark 3.5. A representation with character $\chi_{4b}$, can be described concretely as a vector space with basis $\{e_{2,2}, e_{4,1}, e_{3,3}, e_{1,4}\}$ and action

$$f_1 f_2 \cdot e_{a,b} = \xi^{2(i_a + j_b)} e_{a,b}, \quad t \cdot e_{a,b} = e_{a,-3b}, \quad d \cdot e_{a,b} = e_{b,a},$$

where the indices are considered mod 5. The Galois conjugate representation can be described using the same formulas but with basis $\{e_{1,1}, e_{2,3}, e_{4,4}, e_{3,2}\}$. This representation will play a distinguished role in our analysis.

3.2. The group $G$ and the action $\varphi: G \to \text{Sym}_{\pm,10}$. Recall that $G = G_0 \times C$ is a direct product by a cyclic group of order 2. It will be convenient for us to work with signed permutations $\text{Sym}_{\pm,10}$ which are permutations of $\{\pm 1\} \times \{1, 2, \ldots, 10\}$ such that $(i) \cdot \sigma = -(i) \cdot \sigma$. The group $\text{Sym}_{10}$ identifies as the normal subgroup of index 2 of $\text{Sym}_{\pm,10}$ which preserve signs.

Lemma 3.6. There exists a unique morphism $\varphi: G \to \text{Sym}_{\pm,10}$ from $G$ to signed permutations such that

- the restriction to $G_0$ preserves signs and coincide with $\varphi_0$, and
- we have $\varphi(c) = -1 \ (i.e \ \varphi(c)(i) = -i)$.

The stabilizer of 1 under the action $\varphi$ is $H$ and identifies the action with the right action of $G$ on the right cosets of $H$.

Lemma 3.7. Consider $g_0 \in G_0$ and let $\mu = (1^{\mu_1}, 2^{\mu_2}, \ldots, 10^{\mu_{10}})$ be the cycle type of $\varphi_0(g_0)$. Then

- the cycle type of $\varphi(g_0)$ is $(1^{2\mu_1}, 2^{2\mu_2}, \ldots, 10^{2\mu_{10}})$, and
- the cycle type of $\varphi(g_0c) = \varphi(g_0)$ is $(2^{\mu_1+2\mu_2}, 4^{2\mu_4}, 6^{\mu_6}, 8^{\mu_8}, \ldots, 20^{2\mu_{10}})$.

Proof. The action of $c$ simply switches signs. Let $(i_1, i_2, \ldots, i_{m-1}, i_m)$ be a cycle in the cycle decomposition of $\varphi_0(g_0)$. In $\varphi(g_0)$ this cycle “becomes”

$$(i_1, i_2, \ldots, i_{m-1}, i_m)(-i_1, -i_2, \ldots, -i_{m-1}, -i_m).$$
This shows the first item.

Now, if \( m \) is even the cycle in the product \( \varphi(g_0) \varphi(c) \) “becomes”

\[(i_1, -i_2, \ldots, i_{m-1}, -i_m)(-i_1, i_2, \ldots, -i_{m-1}, i_m)\]

If \( m \) is odd it “becomes” instead

\[(i_1, -i_2, i_3, \ldots, -i_{m-1}, i_m, -i_1, i_2, -i_3, \ldots, i_{m-1}, -i_m)\]

This concludes the proof of the second item. \( \square \)

Given a direct product \( U = U_1 \times U_2 \) of two groups, there is a simple way to define representations of \( U \) from representations of its factors. Namely, given \( \rho_1 : U_1 \rightarrow \text{GL}(V_1) \) and \( \rho_2 : U_2 \rightarrow \text{GL}(V_2) \) we define \( \rho := \rho_1 \otimes \rho_2 : U \rightarrow \text{GL}(V_1 \otimes V_2) \) as \( \rho((u_1, u_2)) = \rho_1(u_1) \otimes \rho_2(u_2) \).

**Lemma 3.8.** There are 28 characters of \( G \). They are the tensor products of the 14 characters of \( G_0 \) and the 2 characters of \( C \).

**Proof.** This is a general feature of direct products: tensor products remain irreducible and give all representations. See [Ser77, Theorem 10]. \( \square \)

4. Covers of \( \mathbb{P}^1 \) branched over 7 points

Here we start the geometric constructions associated to the group \( G \) defined in Section 3. Let \( \{p_i\}_{i=1,\ldots,7} \) be distinct points on \( \mathbb{P}^1 \). Define \( Y \) to be the \( G \)-covering of \( \mathbb{P}^1 \) ramified over the \( p_i \) with the following monodromy

\[(f_1^2 f_2^2 t^2 d, t^2 d, f_1^{-1} f_2 d, d, td, ctd, r).\]

More precisely, each fiber is identified to \( G \) and the action of the \( i \)-th generator of \( \pi_1(\mathbb{P}^1) \) is given by the right multiplication of the \( i \)-th term in the above tuple.

We use the following standard lemma to compute the character of the representation of a group on the cohomology of a cover of \( \mathbb{P}^1 \). See for example [Ell01, Proposition 1.1].

**Lemma 4.1.** Let \( G \) be a finite group and \( X \) be a \( G \)-regular cover of \( \mathbb{P}^1 \) ramified over \( n \) points \( \{p_i\}_{i=1,\ldots,n} \) with monodromy \( \{g_i\}_{i=1,\ldots,n} \). Then the character \( \chi_Y \) of the representation of the \( G \) action on \( H^1(Y; \mathbb{C}) \) is

\[\chi_Y = (n - 2) \chi_{reg} - \sum_{i=1}^{n} \chi_{g_i} + 2 \chi_{triv} .\]

where \( \chi_{reg} \) and \( \chi_{triv} \) are respectively the characters of the regular and trivial representations and \( \chi_{g_i} \) is the character of the left action of \( G \) on the left cosets \( G/\langle g \rangle \) (so \( \chi_1 = \chi_{reg} \)).

**Lemma 4.2.** Let \( Y \) be the \( G \)-cover of \( \mathbb{P}^1 \) as defined above and let \( \chi_Y \) be the character of the \( G \)-action on \( H^1(Y; \mathbb{C}) \). Then, the multiplicities of each irreducible
representation of $G$ in $\chi_Y$ are as in the following array.

<table>
<thead>
<tr>
<th>character $\chi$ of $G_0$</th>
<th>mult. of $1 \otimes \chi$ in $\chi_Y$</th>
<th>mult. of $\epsilon \otimes \chi$ in $\chi_Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{1a}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_{1b}$</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$\chi_{1c}$</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$\chi_{1d}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>$\chi_8$</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>$\chi_{4a}$</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>$\chi_{4b}$</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$\chi_{4c}$</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>$\chi_{4d}$</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

where we used the notations from Lemma 3.3 for the characters of $G_0$ and identified characters of $G$ via Lemma 3.8.

Proof. We start from the general formula from Lemma 4.1

\[(4.2) \quad \chi_Y = 5\chi_{\text{reg}} - \sum_{i=1}^{7} \chi_{g_i} + 2\chi_{\text{triv}}.\]

It remains to compute the multiplicities of each irreducible representations in each of the $\chi_{g_i}$.

Let us first remark that this multiplicity only depends on the conjugacy class of $g_i$. In $G_0$ the elements $f_1^2 f_2 t^2 d, t^2 d, f_1^{-1} f_2 d$ are conjugated to $d$ and $(f_1 f_2^3)^{-1}$ is conjugated to $f_1 f_2^3$ (in $G_0$). Hence the conjugacy classes in $G$ of our 7 elements are respectively $(d, d, d, d, td, ctd, c f_1 f_2^3)$.

Now $\chi_g$ is nothing else but the induced representation $\text{Ind}_{(g)}^G \chi_{\text{triv}}$. Hence, by Frobenius reciprocity, the multiplicity of an irreducible representation $V$ in $\chi_g$ is the dimension of the space of $(g)$-invariants in $\chi_V$.

For an irreducible character $\chi$ of $G_0$, the multiplicities of $s \otimes \chi$ in $\chi_g$ are respectively

- $\frac{1}{2}(\chi(1) + \chi(d))$ for $g = d$,
- $\frac{1}{2}(\chi(1) + \chi(td))$ for $g = td$,
- $\frac{1}{2}(\chi(1) + s(c)\chi(td))$ for $g = ctd$,
- $\frac{2(1 + s(c))}{10}(\chi(1) + 2\chi(f_1 f_2^3) + 2\chi(f_1 f_2^3))$ for $g = f_1 f_2^2$.

Gathering all these contributions, we obtain the multiplicities of each irreducible characters in $\sum_{i=1}^{7} \chi_{g_i}$ as follows.

<table>
<thead>
<tr>
<th>character of $G_0$</th>
<th>mult. of $1 \otimes \chi$</th>
<th>mult. of $\epsilon \otimes \chi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{1a}$</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>$\chi_{1b}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_{1c}$</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_{1d}$</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>$\chi_8$</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>$\chi_{4a}$</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>$\chi_{4b}$</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>$\chi_{4c}$</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>$\chi_{4d}$</td>
<td>16</td>
<td>12</td>
</tr>
</tbody>
</table>
Now plugging these numbers in (4.2) we obtain the statement.

Now we consider the intermediate cover \( Y \to X \to \mathbb{P}^1 \) obtained by quotienting by \( H \).

**Lemma 4.3.** The ramification profile of \( X \to \mathbb{P}^1 \) is
- \([1^{12} 2^4]\) over \( p_1, p_2, p_3, p_4 \),
- \([2^{10}]\) over \( p_5 \) and \( p_6 \), and
- \([10^2]\) over \( p_7 \).

In particular, \( X \) has genus 8.

**Proof.** The ramification profile is given by the cycle types of \( \varphi(g_i) \). Recall that the conjugacy classes of the \( g_i \) are respectively \((d, d, d, d, td, ctd, cf_1 f_2^2)\). From Lemma 3.3 and Lemma 3.7 we obtain the result. \( \square \)

The pullback allows to identify \( H^1(X; \mathbb{C}) \) as the subspace of \( H \)-invariant vectors in \( H^1(Y; \mathbb{C}) \) (endowed with its \( G \)-action). We now compute the dimension of each \( H \)-invariant subspace in each representation in order to complete the description of \( H^1(X; \mathbb{C}) \).

**Lemma 4.4.** For each irreducible character of \( G_0 \) the dimension of the subspace of \( H \)-invariants is given in the following table.

<table>
<thead>
<tr>
<th>character ( \chi ) of ( G_0 )</th>
<th>dimension of ( H )-invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_{1a} )</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_{1b} )</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_{1c} )</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_{1d} )</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_8 )</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_{4a} )</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_{4b} )</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_{4c} )</td>
<td>0</td>
</tr>
<tr>
<td>( \chi_{4d} )</td>
<td>0</td>
</tr>
</tbody>
</table>

**Proof.** For each irreducible representation \( \chi \) of \( G \), the dimension of the space of invariant is simply \( \frac{1}{|H|} \sum_{h \in H} \chi(h) \) which can be re-written as

\[
\frac{1}{|H|} \sum_{h \in H} \chi(h) = \\
\frac{1}{20} \left( \chi(1) + 6\chi(d) + 5\chi(t^2) + 2\chi(f_1 f_2) + 2\chi(f_1 f_3^2) + 2\chi(f_1 d) + 2\chi(f_3^2 d) \right).
\]

Putting together Lemma 4.2 and Lemma 4.3 we obtain the complete description of the 16-dimensional space \( H^1(X; \mathbb{C}) \).

5. **Differentials on covers branched over 7 points**

Let \( \mathcal{E}_0 \) denote the locus of pairs \((X \to \mathbb{P}^1, \omega)\), where:

- \( X \) is a genus 8 Riemann surface.
• $X \to \mathbb{P}^1$ is a branched cover with monodromy given by the 7-tuple (4.1), with normalization $p_5 = 0, p_6 = 1, p_7 = \infty$, and with both preimages of $p_7$ marked.

• $\omega$ is an Abelian differential on $X$ such that, via the identification of $H^1(X)$ with the $H$-invariants of $H^1(Y)$, the cohomology class of $\omega$ is contained in the $H$-invariants of the $\epsilon \otimes \chi^{4b}$ isotypic component of the $G$ action on the homology $H^1(Y, \mathbb{C})$ of the corresponding Galois cover $Y \to \mathbb{P}^1$.

In the second point, when we say “monodromy given by the 7-tuple (4.1)”, we mean that the cover $X \to \mathbb{P}^1$ is isomorphic to one of the covers studied in the previous section. The previous section used implicitly a choice of base point in $\mathbb{P}^1 \setminus \{p_1, \ldots, p_7\}$ and a choice of “standard generators” for $\pi_1(\mathbb{P}^1 \setminus \{p_1, \ldots, p_7\})$ consisting of loops around the punctures, and we do not require these to be fixed.

The purpose of this section is to prove the following.

**Proposition 5.1.** For all $(X \to \mathbb{P}^1, \omega) \in \mathcal{E}_0$,

1. the vanishing order of $\omega$ at each of the two marked points is 2 or 6 modulo 10,
2. $X$ has an involution whose fixed point set is the two marked points and which negates $\omega$, and
3. there is a 4-dimensional subspace in the anti-invariant part of $H^1(X)$ that contains the cohomology class of $\omega$ and is defined over $\mathbb{Q}[\sqrt{5}]$.

Additionally we have

4. $N_0$ is 6-dimensional, and
5. the forgetful map from $\mathcal{E}_0$ to the Hodge bundle over $\mathcal{M}_{8,2}$, defined by

$$(X \to \mathbb{P}^1, \omega) \mapsto (X, \omega),$$

has finite fibers.

We start with the statements that follow immediately from the construction.

**Proof of Proposition 5.1 statements 2, 3, and 4.** Let $Y \to \mathbb{P}^1$ be the Galois closure of $X \to \mathbb{P}^1$. Lemma 4.2 and Lemma 4.4 give that $H$-invariants in the $\epsilon \otimes \chi^{4b}$ isotypic component of $H^1(Y)$ is 2-dimensional. It follows that $\mathcal{E}_0$ is 6-dimensional.

The central element $c \in G$ induces an involution of $X$, whose fixed point set is exactly the special fiber.

The $H$-invariant subspace of the $\epsilon \otimes \chi^{4b}$ isotypic component of $H^1(Y)$ is 4-dimensional and defined over $\mathbb{Q}[\sqrt{5}]$, and by definition it is in the $-1$ eigenspace for $c$. \qed

To address the vanishing order of $\omega$, we will use two straightforward observations. We will make frequent use of the order 10 element $r = (f_1 f_3)^{-1} c$ defined in Lemma 3.1 which appears as the final element in the 7-tuple (4.1). Recall our notation $\xi = \exp(2\pi i/10)$.

**Lemma 5.2.** The $r$ action on $V_{\epsilon \otimes \chi^{4b}}$ has eigenvalues $\xi$ and $\xi^{-1}$, each with multiplicity two. The $r$ action on the Galois conjugate representation $V_{\epsilon \otimes \chi'^{4b}}$ has eigenvalues $\xi^3$ and $\xi^7$, each with multiplicity two.

**Proof.** This follows from Remark 3.3. \qed
In general, we say that a generator $r$ of the stabilizer of a point $p$ is standard when its derivative at $p$ is of the form $\exp(2\pi i/n)$, and we say that $z$ is a standard local coordinate if $r$ acts by multiplication with $\exp(2\pi i/n)$ in this coordinate.

**Lemma 5.3.** Suppose that a Riemann surface $Y$ has an order $n$ automorphism $r$, and moreover that $r$ fixes $p \in Y$ and is a standard generator of the stabilizer at $p$. Suppose $\omega_0$ is contained in an $r$-invariant subspace $V \subset H^1(Y)$. Then, in the Taylor expansion of $\omega_0$ in a standard local coordinate at $p$, all the non-zero terms must have degree equal to $\ell - 1$ modulo $n$ for some $\ell$ such that $\exp(2\pi i/n)^\ell$ is an eigenvalue for the $r$ action on $V$.

In particular, the order of vanishing of $\omega_0$ at $p$ is equal to one of these values $\ell - 1$ modulo $n$.

**Proof.** This follows by expressing $\omega_0$ as a sum of eigenvectors for the action of $r$ on $V$, and using that, in the standard local coordinate, $r^*(z^i dz) = \xi^i + 1 z^i dz$. □

Lemmas 5.2 and 5.3 combine to give the following.

**Proof of Proposition 5.1 statement (1).** The standard generator for any point in the special fiber of $Y \to \mathbb{P}^1$ is a conjugate of $r$. Thus Lemmas 5.2 and 5.3 give that any differential in the $\epsilon \otimes \chi'$ isotopic component of $H^1(Y)$ has vanishing order equal to 2 or 6 modulo 10 at each point of the special fiber. Since $Y \to X$ is unramified over the special fiber, the same holds for the pushforward to $X$ of any $H$-invariant form. □

It remains only to consider the forgetful map. To this end, recall that we defined $H^+$ to be the subgroup generated by $H$ and $r$, and that $H^+$ has index two in $G$.

**Lemma 5.4.** Consider a regular cover $Y \to \mathbb{P}^1$ with monodromy given by the 7-tuple (4.1). Then

1. the map $\zeta : H^+ \setminus Y \to G \setminus Y = \mathbb{P}^1$ has degree two and its branch points are $p_5 = 0$ and $p_6 = 1$,
2. $H^+ \setminus Y$ has genus 0,
3. the map $\rho : X = H \setminus Y \to H^+ \setminus Y$ is degree 10 and is totally ramified at both points of the special fiber, and
4. for $i \in \{1, 2, 3, 4\}$, the map $\rho$ is branched over both points of $\zeta^{-1}(p_i)$.

**Proof.** The first claim holds because $H^+$ has index 2 in $G$, and in the 7-tuple (4.1) it is exactly the 5-th and 6-th elements that are not contained in $H^+$. The second claim follows from the first, and the third claim is left to the reader.

For the fourth claim, we start by noting the following general result.

**Claim 5.5.** Let $Y \to \mathbb{P}^1$ be any Galois cover with Deck group $G$. Let $H < H^+$ be subgroups of $G$, and let $\zeta$ denote the map $H^+ \setminus Y \to \mathbb{P}^1$. Let $p \in \mathbb{P}^1$ be a branch point with local monodromy conjugate to an involution $d \in G$. Then the map $H - d^{-1} \zeta^{-1}(p)$ is branched over more than one point of $\zeta^{-1}(p)$ if and only if there exist $g_1, g_2 \in G$ such that $g_1^{-1} d g_1^{-1}, g_2^{-1} d g_2^{-1} \in H^+ \setminus H$ and $g_2 g_1^{-1} \notin H^+$. 
Proof of Claim. Fix a preimage $\hat{p}$ of $p$ in $Y$ with stabilizer $\langle d \rangle$. Two ramification points $g_1\hat{p}$ and $g_2\hat{p}$ have distinct images in $H^+ \setminus Y$ if $g_2g_1^{-1} \notin H^+$.

The stabilizers of these points in $Y$ are generated by the involutions $g_1dg_1^{-1}$ and $g_2dg_2^{-1}$ respectively. For the corresponding points in $H \setminus Y$ to be branch points for the map to $H^+ \setminus Y$, we need these involutions to lie in $H^+ \setminus H$. □

To conclude the proof of the fourth claim, recall that the first four elements in the 7-tuple are all conjugate to $d$, which is conjugate to $f_1f_2^{-1}t^2d \in H^+ \setminus H$. Conjugating that element by $f_1t$ gives another element in $H^+ \setminus H$. Since $f_1t \notin H^+$, this gives the result. □

Proof of Proposition 5.1 statement (5). Given $(X, \omega)$, we wish to show there are only finitely many $(X \to \mathbb{P}^1, \omega) \in \mathcal{E}_0$. Note that because of the two marked points, we know what the special fiber of $X \to \mathbb{P}^1$ must be.

Let $Z = Y/H^+$. Up to Mobius transformations of $Z$ that fix the two points of $\zeta^{-1}(\infty)$, there is at most one map $\rho : X \to Z$ that is totally ramified at the pair of marked points, and maps these two points bijectively to $\zeta^{-1}(\infty)$ in a prescribed way. (Otherwise, one could construct a non-constant holomorphic function on $X$, using an isomorphism $Z \to \mathbb{P}^1$ that sends the two points of $\zeta^{-1}(\infty)$ to 0 and $\infty$.)

The map $X \to \mathbb{P}^1$ is the composition of the map $X \to Z$ with the map $\zeta$. To get a valid choice of $X \to \mathbb{P}^1$, the Mobius transformation must be chosen to map a pair of branch points to a fiber of $\zeta$, and one can see there are only finitely many such choices. (After using the same isomorphism $Z \to \mathbb{P}^1$, the relevant Mobius transformations of $\mathbb{P}^1$ are $z \mapsto az$, and the fibers of $\zeta$ can be arranged to be the pairs $\{z, 1/z\}$. For any $z_1, z_2$, there are only two solutions $a$ to $az_1 = 1/(az_2)$.) □

We conclude with one observation which, although not used in this paper, may facilitate further investigation.

Lemma 5.6. Let $X \to \mathbb{P}^1$ be a cover of the type considered in this section, and let $\omega$ be an anti-invariant Abelian differential that vanishes to order 12 at one of the marked points. Then $(X, \omega) \in \mathcal{E}_0$.

Proof. The anti-invariants in $H^{1,0}(X)$ are the direct sum of the $\epsilon \otimes \chi_{4b}$ and $\epsilon \otimes \chi'_{4b}$ isotypic components of $H^{1,0}(Y)$. Lemmas 5.2 and 5.3 thus give the result. □

6. A PARTIAL COMPACTIFICATION

We say that a cover is evenly ramified over a point if all the pre-images of that point have the same ramification index. Consider the space $\mathcal{P}$ of degree 20 covers $f : X \to \mathbb{P}^1$ from a (smooth) Riemann surface $X$ of genus 8 to $\mathbb{P}^1$, such that

1. $f$ is evenly ramified over 0, 1, $\infty$ with 10 pre-images of 0, 10 pre-images of 1, and 2 pre-images of $\infty$, and

2. there are at most 4 other branch points.

The other branch points are allowed to collide with each other but not with 0, 1 or $\infty$, as long as such a collision does not cause the the cover $X$ to become singular. As we discuss at the end of this section, $\mathcal{P}$ can be endowed with the structure of a quasi-projective variety.

We define $\mathcal{N}_0$ as the closure of $\mathcal{E}_0$ in the bundle of non-zero Abelian differentials over $\mathcal{P}$. Marking the two pre-images of $\infty$, we obtain a map $\pi : \mathcal{N}_0 \to \Omega_0$ defined by $\pi(X \to \mathbb{P}^1, \omega) = (X, \omega)$. 
With $\mathcal{N}_0$ and $\pi$ thus defined, in this section will will prove all but the last statement of Theorem 2.1. We will then conclude the proof of Theorem 2.1 in the next section by proving that the unfoldings are contained in $\pi(\mathcal{N}_0)$.

All the statements we require have already been established on $\mathcal{E}_0 \subset \mathcal{N}_0$ by Proposition 5.1. Our task in this section is to extend these statements to $\mathcal{N}_0$. We start by noting that two of the statements extend automatically.

**Proof of Theorem 2.1 statements 2 and 3.** The locus with an involution negating $\omega$ and fixing the two marked points is closed, and so we obtain such an involution on $\mathcal{N}_0$. One can produce the 4-dimensional subspace for points of $\mathcal{N}_0$ as in the proof of Theorem 1.1 assuming Theorem 2.1. □

The next easiest statement to address is the finiteness of fibers, which we do by pointing out that the proof we used for $\mathcal{E}_0$ also applies for $\mathcal{N}_0$.

**Proof of Theorem 2.1 statement 4.** We will use the details of Lemma 5.4 and the proof of Proposition 5.1 statement (5). Recall that $\zeta : Z \to \mathbb{P}^1$ is the double cover branched over 0 and 1.

Any point $(X \to \mathbb{P}^1, \omega) \in \mathcal{N}_0$ is by definition a limit of points in $\mathcal{E}_0$. By taking limits, we see that the map $X \to \mathbb{P}^1$ factors as a map $X \to Z$ followed by the map $\zeta$. Although the points $p_1, p_2, p_3, p_4$ may not longer be distinct for $X \to \mathbb{P}^1$, they are still branch points, and it is still true that both of their pre-images on $Z$ are branch points for $X \to Z$.

The result now follows as in the proof of Proposition 5.1 statement (5). □

It remains to consider the order of vanishing. For this we give two proofs.

**First proof of Theorem 2.1 statement 1.** Consider a holomorphic map from a disc $D$ to $\mathcal{N}_0$, written as $t \mapsto (X_t \to \mathbb{P}^1, \omega_t) \in \mathcal{N}_0$, such that $(X_t \to \mathbb{P}^1, \omega_t) \in \mathcal{E}_0$ when $t \neq 0$. It suffices to prove that $\omega_0$ vanishes to order 2 or 6 mod 10 at the first marked point; the second marked point can be treated in exactly the same way.

Recall that, for a branched cover $X \to \mathbb{P}^1$, we say that a local coordinate around a ramification point in $X$ is a standard local coordinate if, in that local coordinate, the map is given by quotienting by multiplication by $\exp(2\pi i/k)$ for some $k$. This is equivalent to the existence of a local coordinate for $\mathbb{P}^1$ for the branch point for which the map is locally $w \mapsto w^k$. Here we think of the local coordinates in the domain and co-domain as being maps from small discs $U, U'$ about 0 ∈ $\mathbb{C}$ to the Riemann surface which map 0 to the point of interest.

\[
\begin{array}{ccc}
U' & \longrightarrow & X \\
\downarrow & & \downarrow \\
U & \longrightarrow & \mathbb{P}^1
\end{array}
\]

By shrinking $D$ if necessary, we first assume that our maps $X_t \to \mathbb{P}^1$ do not have any branch points close to $\infty$ except for $\infty$ itself. We can now fix a local coordinate $U$ at $\infty \in \mathbb{P}^1$ that never contains any branch points of $X_t \to \mathbb{P}^1$ except $\infty$ itself. We define $U'$ to be the preimage of $U$ under the map $w \mapsto w^{10}$. For each $t$, the composite map $U' \to \mathbb{P}^1$ can be lifted to $X_t$ to provide a standard local coordinate at the first marked point. In fact there are 10 choices of such a lift, but passing to a 10-fold cover of $D$ allows to make a choice continuously in $t$. 
To conclude the proof we consider the Taylor series expansion of $\omega_t$ in this this local coordinate. Lemma 5.3 and the proof of Proposition 5.1 statement (1) show that only terms of degree 2 or 6 mod 10 may have non-zero coefficients with $t \neq 0$. Continuity gives the same statement at $t = 0$. □

For the second proof we will work with the Deligne-Mumford compactification $\overline{\mathcal{M}}_{381}$ of $\mathcal{M}_{381}$. (Without substantive changes one may replace $\overline{\mathcal{M}}_{381}$ with a moduli space of $G$-covers, which was constructed in [ACV03] and is the normalization of the space of admissible covers. Any reader familiar with this beautiful moduli space will undoubtedly prefer to make this substitution.)

**Second proof of Theorem 2.1 statement (1)**. As in the first proof we begin with a disc $t \mapsto (X_t \to \mathbb{P}^1, \omega_t) \in \mathcal{N}_0$. For each $t \neq 0$, we can consider the Galois closure $Y_t \to \mathbb{P}^1$, so $Y_t \in \mathcal{M}_{381}$. Let $\eta_t$ denote the pull-back of $\omega_t$ to $Y_t$, so $\eta_t$ is in the $H$-invariants of the $\epsilon \otimes \chi_4^b$-isotypic component of $H^1$. Since the projectivization of the Hodge bundle over Deligne-Mumford compactification is a projective variety, after rescaling the differentials if necessary, the map $t \mapsto (Y_t, \eta_t)$, initially only defined when $t \neq 0$, can be extended to $t = 0$, where now $Y_0$ is a nodal Riemann surface and $\eta_0$ a priori may have simple poles at the nodes. At the cost of passing to a finite cover of the disc one can assume that the $Y_t$ form a family over the punctured disc and that the $G$ actions arise from a $G$ action on the total space of this family.

The central surface $X_0$ can be obtained by contracting the unstable components of the nodal surface $Y_0/H$. Additionally $\eta_0$ is the pull back of $\omega_0$.

By continuity we get that $\eta_0$ is in the $\epsilon \otimes \chi_4^b$-isotypic component of the fiber of the Hodge bundle over $Y_0$. With straightforward notation changes, the proof of Proposition 5.1 statement (1) now applies to give the result. □

The final detail to address is why $\mathcal{P}$ is a quasi-projective variety. Two approaches for this arose in conversation with Dawei Chen.

In the first approach, one starts with a Kontsevich moduli space of degree 20 stable maps to $\mathbb{P}^1$. (See the survey [FP97] for an introduction to such moduli spaces.) Our space $\mathcal{P}$ is a subset of this moduli space, and its algebraicity then follows from the version of Chow’s Theorem in [Mum95, Theorem 4.5]. (Chen has also pointed out to us that the algebraicity of the ramification condition over 0, 1 and $\infty$ also follows from [Gat02].)

In the second approach, one uses a slight generalization of [Deo14] to allow more general weights, as in [Has03], to directly obtain a moduli space one of whose irreducible components is $\mathcal{P}$. (Deopurkar has confirmed to us that his projective coarse moduli space exists even with distinct weights. For our purposes one should take four weights of $\epsilon$ and three weights of 1.)

We end this section with two tangential remarks:

1. We do not know of any elementary proof that even the classical Hurwitz space of simply branched covers of $\mathbb{P}^1$ is a quasi-projective variety. Before Fulton’s scheme-theoretic construction [Ful69], the only proof we are aware of requires the Riemann Existence Theorem of [GR58].

2. It is likely possible to rephrase our arguments in this section and the next section to work almost entirely in the space of $G$-covers of $\mathcal{A} \mathcal{V} \mathcal{V} _{03}$, invoking the Stein factorization to rectify the fact that otherwise the natural map to $\Omega_0$ would have some infinite fibers.
7. Covers of \( \mathbb{P}^1 \) branched over 4 points

In this section, we show that the desired unfoldings are contained in the partial compactification constructed in the previous section.

Consider the 4-tuple of elements of \( G \)

\[
(f_1 f_2^3, td, ctd, (f_1 f_2^3)^{-1} c) = (r^{-6}, td, r^5 td, r)
\]

All notations come from Section 3. The first element in the tuple above is obtained by multiplying the first four elements in the 7-tuple (4.1). This 4-tuple generates the Abelian subgroup \( A \subset G \) of order 20, see Lemma 3.1. In this section we consider genus 8 regular covers \( X' \to \mathbb{P}^1 \) with Deck group \( A \) branched over a set \( \{ p_1, p_2, p_3, p_4 \} \) of 4 points with monodromy given by the above 4-tuple.

The main result in this section is the following.

Proposition 7.1. All unfoldings \( (X', \omega') \) of quadrilaterals with angles

\[
\begin{pmatrix}
2\pi & 2\pi & 3\pi & 13\pi \\
10 & 10' & 10' & 10
\end{pmatrix}
\]

admit \( \mathbb{P}^1 \) as regular \( A \)-quotient \( X' \to \mathbb{P}^1 \) with monodromy given by the 4-tuple above in such a way that \( r^* (\omega') = \xi^3 \omega' \), where \( \xi = \exp(2\pi i/10) \).

The above proposition allows us to conclude the proof of Theorem 1.1 as follows. (As for Theorem 2.1 statement (1), different approaches are possible, but this time we omit the more modular approach and just give the more elementary proof.)

Proof of Theorem 2.1 statement (5) assuming Proposition 7.1. Let \( (X', \omega') \) be an unfolding. Proposition 7.1 gives that \( X' \) admits a regular \( A \) cover \( X' \to \mathbb{P}^1 \) with monodromy given by the 4-tuple above.

Because the 4-tuple was obtained by replacing the first 4 elements of the 7-tuple with their product, the map \( X' \to \mathbb{P}^1 \) arises as a limit of maps \( X \to \mathbb{P}^1 \) with monodromy given by the 7-tuple.

By taking limits, we see that there exists a two dimensional subspace \( V \) of the anti-invariant part of \( H^1,0(X') \) that arises as limits of forms in \( E_0 \). We know from the first proof of Theorem 2.1 statement (1) that in a standard local coordinate at the first marked point, only the terms that are 2 or 6 mod 10 in the Taylor series are non-zero.

The only forms whose Taylor series satisfy this condition on \( X' \) are in the \( \xi^3 \) or \( \xi^7 \) eigenspaces for \( r \). The \( \xi^7 \) eigenspace in \( H^{1,0}(X') \) is zero and the \( \xi^3 \) eigenspace is 2-dimensional, so we get that \( V \) is equal to the \( \xi^3 \) eigenspace. (See for example [MW18, Lemma 6.1] for the computation of eigenspaces.) Since the form \( \omega' \) is in this eigenspace, this gives the result.

We will prove Proposition 7.1 by exhibiting additional symmetry in the unfoldings of the billiard.

Lemma 7.2. Suppose \( a, b \in \mathbb{Z} \) satisfy \( 2a + 2b \equiv 0 \mod k \). Let \( f_1 : X' \to \mathbb{P}^1 \) be the regular \( \mathbb{Z}/k\mathbb{Z} \)-cover ramified over four points \( \{ q_1, q'_1, q_4, q'_4 \} \) with local monodromy \((a, a, b, b)\). Let \( f_2 : \mathbb{P}^1 \to \mathbb{P}^1 \) be the quotient by the unique holomorphic involution \( \mathbb{P}^1 \to \mathbb{P}^1 \) that exchanges \( q_1 \) with \( q'_1 \) and \( q_4 \) with \( q'_4 \). Then \( f_2 \circ f_1 : X' \to \mathbb{P}^1 \) is a regular \( \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)-cover with monodromy \((a, 0), (b, 0), (0, 1), (a + b, 1)) \).
Proof. Abelian covers with Deck group \( \mathbb{Z}/k\mathbb{Z} \) of a surface \( S \) ramified over \( \Sigma \) are in bijection with surjective elements in \( H^1(S \setminus \Sigma; \mathbb{Z}/k\mathbb{Z}) \) (or equivalently, elements of \( H_1(S, \Sigma; \mathbb{Z}/k\mathbb{Z}) \)). The following figure illustrates an element of \( H^1(\mathbb{P}^1, \{q_1, q_1', q_2, q_3, q_4, q_4'\}; \mathbb{Z}/k\mathbb{Z}) \); the evaluation of this element of on a loop is calculated by summing the indicated numbers for each black segment that the loop crosses.

The element in the above picture corresponds to a cover with
- monodromy \(+a\) around \( q_1 \) and \( q_1' \),
- monodromy \(+b\) around \( q_4 \) and \( q_4' \),
- unramified over \( q_2 \) and \( q_3 \) (over \( q_3 \) this is because \( 2a + 2b \equiv 0 \mod k \)).

In other words, the above picture describes the cohomology element corresponding to \( f_1 : X' \to \mathbb{P}^1 \). This picture has an order two symmetry exchanging \( q_1 \) with \( q_1' \), \( q_4 \) with \( q_4' \) and fixing \( q_2 \) and \( q_3 \). This allows to describe \( X' \) as a \( \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) regular covering as the following cohomology class

\[
(+a, 0) \quad (+b, 0)
\]

where \( p_1, p_2, p_3, p_4 \) denote respectively the images of \( \{q_1, q_1'\}, q_2, q_3 \) and \( \{q_4, q_4'\} \) under the quotient by the involution. (This analysis may be somewhat easier in reverse; one can start with \( \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) cover illustrated, and check that it admits the original cover as a factor.)

Proof of Proposition 7.1. Let us consider a billiard unfolding \( X' \to \mathbb{Z} \). Let \( \omega' \) be the corresponding Abelian differential on \( X \) and let \( q_1, q_1', q_2, q_3 \) and \( q_4, q_4' \) be the points in \( Z \) corresponding to respectively the \( \frac{2\pi}{10}, \frac{13\pi}{10}, \frac{4\pi}{10} \) and \( \frac{7\pi}{10} \). The unfolding \( (X', \omega') \) comes equipped with automorphism \( \rho \) which rotates by \( 2\pi/10 \) in the counter-clockwise direction, by the definition of unfolding, so \( \rho^*(\omega) = \xi \omega \).
Over the points arising from the $\frac{2\pi}{10}$ angles, the local standard generator is $\rho^2$. Over the points arising from the $\frac{4\pi}{10}$ or $\frac{13\pi}{10}$ angles, the standard generator is $\rho^3$. So the monodromy of the map to $\mathbb{P}^1$ is $(\rho^2, \rho^2, \rho^3, \rho^3)$.

If we use $r = \rho^3$, so $\rho = r^{-3}$, this becomes $(r^{-6}, r^{-6}, r, r)$.

Now, applying Lemma 7.2 we obtain a $A$-cover whose monodromy corresponds to the 4-tuple presented at the beginning of the section. □

Appendix A. Explicit surfaces in $\mathcal{N}$

By construction, there is an invariant subvariety $\overline{\mathcal{N}} \subset \mathcal{Q}_4(11,1)$ such that $\mathcal{N}$ is the locus of double covers of quadratic differentials in $\overline{\mathcal{N}}$. In this section, which give especially nice polygonal presentations for some surfaces in $\overline{\mathcal{N}}$, and we give the explicit linear equations that locally define $\mathcal{N}$ near these surfaces.

In order to describe linear equations in relative cohomology one needs to pick a specific surface in $\overline{\mathcal{N}}$ and a specific homology basis. Our choice is shown in Figure 1.

![Figure 1](image-url)

**Figure 1.** A quadratic differential in $\overline{\mathcal{N}} \subset \mathcal{Q}_4(11,1)$ obtained from the gluing of 6 polygons. The sides $B$, $C$, $D$, $S$, $T$, $U$ and $V$ are identified by translations $z \mapsto z + c$ while the sides $A$ and $E$ are identified by "half translations" $z \mapsto -z + c$, respectively.

The complex dimension of $\mathcal{Q}_4(11,1)$ is 8, so our four-dimensional subvariety $\overline{\mathcal{N}}$ is locally described by 4 equations. On the surface illustrated, these equations become particularly elegant. In the theorem below, we write the equations directly in the homology of the surface that supports the quadratic differential even though these equations live in the double cover. The integration of the square root of a quadratic differential is only well defined up to sign, but Figure 1 makes it clear which sign to choose.
Theorem A.1. Given the relative homology basis \{A, B, C, D, E, S, T, U, V\} for the surface depicted in Figure 1, a set of defining equations for the tangent space to \(\mathcal{N}\) is given by

\[
\int_B \sqrt{q} = \pm \varphi \cdot \int_A \sqrt{q} \quad \int_C \sqrt{q} = \pm \varphi \cdot \int_D \sqrt{q} \\
\int_S \sqrt{q} = \pm \varphi \cdot \int_T \sqrt{q} \quad \int_U \sqrt{q} = \pm \varphi \cdot \int_V \sqrt{q}
\]

where \(\varphi = \frac{1 + \sqrt{5}}{2}\).

Proof. The proof is a rather straightforward computation with the flatsurf software suite that is described in Appendix B. All the steps of the computation are explained in a Jupyter notebook available at https://git.io/Ju2lU. Here we only provide a sketch of the steps and leave all computational details to the software suite (or the determined reader).

First, starting from the unfolding of the triangle with angles \((\frac{3\pi}{20}, \frac{4\pi}{20}, \frac{13\pi}{20})\), we compute tangent vectors to the orbit closure, see Appendix B. In this case, we find four tangent vectors. By Theorem 1.1 and Corollary 1.2 we know that the generated subspace coincides with the full tangent space. Next, we change our base point by following the available four-dimensional space and arrive at the surface presented in Figure 1. The tangent space is then expressed in the new homology basis shown in Figure 1. \(\square\)

Appendix B. Exploring \(\text{GL}(2, \mathbb{R})\)-orbit closures with flatsurf

The discovery that the orbit closure of the unfolding of the triangle with angles \((\frac{3\pi}{20}, \frac{4\pi}{20}, \frac{13\pi}{20})\) was likely of rank 2 was made using by the flatsurf suite. Largely inspired by Alex Eskin’s polygon program, this free software suite is maintained by Vincent Delecroix and Julian Rüth. It consists of the libraries

- e-antic [DKR21] and exact-real [Rü21a] for computations with exact real numbers,
- intervalxt [DR21b] to work with interval exchange transformations,
- flatsurf [DR21a], surface-dynamics [DFJ+21] and sage-flatsurf [DHR21] for translation surfaces, and
- vue-flatsurf [Rü21c] and ipyvue-flatsurf [Rü21b] for visualization.

All libraries have Python interfaces, mostly through cppy [LD16] which provides automatic Python bindings for C++ libraries.

We now outline how this software suite rigorously gives “lower bounds” for the size of orbit closures; a more in-depth discussion will be in [DER, DR]. Concrete examples of such computations in Python are available in the sage-flatsurf documentation at https://flatsurf.github.io/sage-flatsurf/.

An invariant subvariety \(\mathcal{M}\) in a stratum of Abelian differentials is completely determined by a translation surface \((X, \omega)\) and a vector subspace \(V\) of \(H^1(X, \mathbb{Z}(\omega); \mathbb{R})\). This vector subspace is such that \(V \oplus \sqrt{-1}V\) can be identified with the tangent space of \(\mathcal{M}\) at \((X, \omega)\) via the period map \(\omega \mapsto [\omega] \in H^1(X, \mathbb{Z}(\omega); \mathbb{C})\). Our approach to compute the orbit closure consists of building a sequence of subspaces \(V_0 \subset V_1 \subset \ldots \) of \(V\). We start with the tautological space \(V_0 = \mathbb{R} \text{Re}(\omega) \oplus \mathbb{R} \text{Im}(\omega)\). By the \(\text{GL}(2, \mathbb{R})\)-invariance the two vectors \(\text{Re}(\omega)\) and \(\text{Im}(\omega)\) belong to \(V\) and so \(V_0 \subset V\). We then apply the following procedure to obtain \(V_{i+1}\) from \(V_i\).
(1) Find a direction in \((X, \omega)\) that contains a cylinder and apply the cylinder deformation theorem [Wri15a] to get tangent vectors in the orbit closure.

(2) Set \(V_{i+1}\) to the sum of \(V_i\) and the newly discovered tangent vectors.

Figure 2. This unfolding of the triangle with angles \((\frac{3\pi}{20}, \frac{4\pi}{20}, \frac{13\pi}{20})\) decomposes into 8 cylinders in horizontal direction. Application of the cylinder deformation theorem to this decomposition does, however, not produce any new tangent vectors that were not already in \(V_0\).

Of course, there is an issue with this approach: the algorithm has no stopping time, unless \(V_n = H^1(X, Z(\omega); \mathbb{R})\) for some \(n\), in which case the orbit closure is dense. At each step, it is guaranteed that \(V_n \oplus \sqrt{-1} V_n\) is a subspace of the tangent space but we have no way yet to automatically certify that a candidate subspace is indeed the full tangent space. An even more serious theoretical issue is the following question.

**Question B.1.** If we consider all directions that contain cylinders, do we ever get \(V_n = V\)? That is, does the cylinder deformation theorem eventually discover the entire tangent space?

P. Apisa communicated to us some evidence that Question B.1 may have a negative answer. More specifically, during a summer school at ICERM, his students B. Harper, H. Wan, and H. Yang discovered a specific quadrilateral whose unfolding seems to have \(\dim V_n \leq 3\) for all \(n\) even though the \(\text{GL}(2, \mathbb{R})\)-orbit is known to be dense in its 6-dimensional stratum (so \(\dim V = 6\)).

The following modification of the algorithm turns out to be useful in practice to make the algorithm more efficient, and it may additionally circumvent a negative answer to Question B.1 in some cases. At step \(i\) of the algorithm, instead at looking for cylinders in \((X, \omega)\) one can look for cylinders in a deformation of \((X, \omega)\) along \(V_i + \sqrt{-1} V_i\).

**Question B.2.** When deformations are performed in each step of the algorithm, do we ever get \(V_n = V\)?

Compare Questions B.1 and B.2 to [SW07, Question 5], [MW17, Remark 6.1], and [LNW17, Theorem 1.1], and note in particular that if there exists a completely parabolic surface that is not Veech, both questions will have a negative answer.
Appendix C. Exceptional billiards

We now survey the various rational billiards whose unfoldings have non-dense orbit closures. Most of them were already known in the literature. In addition to the triangle with angles \((\frac{3\pi}{20}, \frac{4\pi}{20}, \frac{13\pi}{20})\), the flatsurf suite allowed us to find two other exceptional orbit closures. We conjecture that the finite list of exceptions in Theorem C.1 and Theorem C.2 is complete. Partial results in this direction are obtained in \cite{MWT18, LNZ}.

We say that an \(n\)-tuple of positive integers \((a_1, \ldots, a_n)\) is admissible if

- \(a_1 \leq a_2 \leq \ldots \leq a_n\),
- \(\gcd(a_1, \ldots, a_n) = 1\), and
- \((n-2)a_i \neq a_1 + \ldots + a_n\) for all \(i \in \{1, \ldots, n\}\).

To the tuple \((a_1, \ldots, a_n)\) we associate the moduli space of flat metrics on the sphere with conical singularities of angles \((a_1\alpha, a_2\alpha, \ldots, a_n\alpha)\) where \(\alpha = \frac{(n-2)\pi}{a_1+\ldots+a_n}\). Such locus of flat metrics admits a cover of degree \(a_1 + \ldots + a_n\) ramified over the conical points which belongs to a stratum of Abelian or quadratic differentials if \(a_1 + \ldots + a_n\) is even or odd, respectively.

In this notation, this article considers the billiard in the \((3,4,13)\) triangle and the \((2,2,3,13)\) quadrilateral. Namely, the \(\GL(2,\RR)\)-orbit closures of the unfoldings of these two polygons is the invariant subvariety \(\mathcal{N}\) appearing in Theorem 1.1 and Corollary 1.2.

We say that an admissible tuple \((a_1, \ldots, a_n)\) is reducible if it satisfies one of the following conditions

- \(n = 3\) and \(a_1 = a_2 < a_3\) or \(a_1 < a_2 = a_3\), or
- \(n = 4\) and \(a_1 = a_2 < a_3 = a_4\).

The reducible tuples correspond to triangles and quadrilaterals that admit a reflection symmetry (independently of the side lengths). In particular their unfoldings are double covers of unfoldings of their “halves”. For example

- the isosceles \((2,2,3)\)-triangle unfolds to \(H_3(2,1^2)\) while its half, the right \((3,4,7)\)-triangle unfolds to \(Q_0(2,1,-1^7)\),
- the isosceles \((3,3,5)\)-triangle unfolds to \(H_5(4,2^2)\) while its half, the right \((5,6,11)\)-triangle unfolds to \(Q_0(4,3,-1^11)\),
- the isosceles \((3,3,4)\)-triangle unfolds to \(Q_2(2,1^2)\) while its half, the right \((2,3,5)\)-triangle unfolds to \(Q_0(1,-1^5)\).

An admissible tuple which is not reducible is called reduced. An analysis using the the flatsurf suite has rigorously proven the following.

**Theorem C.1.** All admissible and reduced triples \((a,b,c)\) with \(a + b + c \leq 58\) with the exception of the list below correspond to triangular billiards whose unfoldings have dense \(\GL(2,\RR)\)-orbit closures in their ambient strata.

- \{1, 2, 2k+1\}_{k \geq 2}, \{1, k, k+1\}_{k \geq 2}, \{2, 2k+1, 2k+3\}_{k \geq 1}: Teichmüller curves \cite{Vee89, War98},
- (1, 4, 7), (2, 3, 4), (3, 4, 5), (3, 5, 7): Teichmüller curves \cite{Hoo13, KS00, Vee90, Vor06},
- (1, 3, 6), (1, 3, 8): rank-one but not Teichmüller curves (eigenform loci inside a Prym subvariety) \cite{AAD},
- (1, 4, 11), (1, 4, 15), (3, 4, 13): rank 2 orbit closures \cite{EMMW20} and this article.
The (1, 3, 6) and (1, 3, 8) triangles were discovered as part of the exhaustive search of the flatsurf suite on small triangles.

In [EMMW20] the authors exhibited 6 quadrilaterals with rank 2 orbit closures and this article adds one to them. This gives the following list of exceptional quadrilaterals

\[(C.1) \quad (1, 1, 1, 7), (1, 1, 1, 9), (1, 1, 2, 8), (1, 1, 2, 12), (1, 2, 2, 11), (1, 2, 2, 15), (2, 2, 3, 13).\]

**Theorem C.2.** All admissible and reduced quadruples \((a, b, c, d)\) with \(a + b + c + d \leq 32\) with the exception of the list \((C.1)\) correspond to billiards in quadrilaterals whose unfoldings have dense \(\text{GL}(2, \mathbb{R})\)-orbit closures in their ambient strata.

**References**


