DELAUNAY PARTITIONS

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1. INTRODUCTION

Let \( p \geq 0, \, n \geq 0 \) be such that \( 2p - 2 + n > 0 \). The Teichmüller space \( \mathcal{T}(p, n) \) can be identified in various ways as a space of metrics, up to scale factor and isotopy, on a closed oriented surface of genus \( p \) with \( n \) punctures. In this paper the requirements on a metric \( g \) shall be that (a) \( g \) is flat and (b) \( g \) has cone singularities at the punctures with a fixed set of cone angles \( \theta > 0 \) [1]. The cone angles necessarily sum to \((4p - 4 + 2n)\pi\). In all that follows \( M_p \) shall denote a closed oriented surface of genus \( p \). \( S_n, \, n > 0 \), denotes a fixed set of cardinality \( n \) in \( M_p \), and we set \( M_{p, n} = M_p \setminus S_n \).

A cone metric \( g \) determines a unique cell decomposition, called its Delaunay partition, with vertex set the set of punctures. (See [2] and, for a detailed treatment, [3].) Defining properties of the Delaunay partition are (1) every 2-cell \( F \) is isometric to a planar polygon \( P(F) \) which is inscribed in a circle \( C(F) \) and (2) if a 1-cell \( e \) lies on \( \partial F_1 \) and \( \partial F_2 \) (or twice on \( \partial F_1 \)), and if the central angles of the corresponding chords of \( C(F_1) \) and \( C(F_2) \) are denoted \( x_1, \, x_2 \), then \( x_1 + x_2 < 2\pi \). (An edge of an inscribed \( k \)-gon, \( k > 2 \), determines two arcs. "Central angle" is understood in relation to the one of these arcs which is on the opposite side of the edge from the polygon).

Let \( A(p, n) \) be the set of isotopy classes of cell decompositions of a fixed \( M_{p, n} = M_p \setminus S_n, \, |S_n| = n \). If \( \theta \) is a fixed set of cone angles as above, each \( \tau \in \mathcal{T}(p, n) \) determines up to scale an isotopy class of cone metrics \( g \) and therefore an isotopy class of Delaunay partition \( \mathcal{D}(\theta, \tau) \in A(p, n) \). Let \( \mathcal{C}(\mathcal{D}, \theta) \) be the set of \( \tau \) such that \( \mathcal{D}(\theta, \tau) = \mathcal{D} \). Denote by \( \mathcal{C}(\theta) \) the resulting partition of \( \mathcal{T}(p, n) \),

\[
\mathcal{C}(\theta) = \{ \mathcal{C}(\mathcal{D}, \theta) | \mathcal{D} \in A(p, n) \}.
\]

\( \mathcal{C}(\theta) \) is invariant under the pure mapping class group, here denoted \( \text{Map}_0(p, n) \). The pair \( (\mathcal{T}(p, n), \mathcal{C}(\theta)) \) will be described in elementary terms (Theorem 1.1) after a brief digression for purpose of definitions.

Let \( T \) be a nonempty finite set, and let \( A(\cdot, \cdot) \) be a function on \( T \times T \). If \( T_1 \subseteq T \), then we define \( A(T_1) = A|_{T_1 \times T_1} \). Also, define \( |A(T_1)| \) to be sum of values, \(|A(T_1)| = \sum_{s \in T_1} A(s, t) \).

Now define \( \mathcal{X}^+(T) \) to be the set of functions \( A \) such that (a) \( A(\cdot, \cdot) \) is symmetric, (b) \( A(\cdot, \cdot) \geq 0 \), (c) \( |A| = 2|T| \) and (d) if \( T_1 \) is a proper subset of \( T \), then \( |A(T_1)| < 2|T_1| \).

Let \( \mathcal{D} \in A(p, n) \) and let \( E \) and \( T \) denote the set of 1-cells and 2-cells of \( \mathcal{D} \), respectively. If \( \psi \in \mathcal{X}^+(\mathbb{R}^+) \), define a nonnegative symmetric function \( A_\psi \) on \( T \times T \) by

\[
A_\psi(F_1, F_2) = \sum_{\varepsilon \in \partial F_1 \cap \partial F_2} \psi(\varepsilon).
\]
When $F_1 = F_2$, $\psi(e)$ occurs in the sum, and then twice, if, and only if, $e$ occurs twice on $\partial F_1$. Also, define $\theta_\psi \in \mathbb{R}^+$ by

$$\theta_\psi(s) = \pi \sum_{s \in \partial e} (1 - \psi(e)).$$

When $s$ occurs at both ends of $e$, $1 - \psi(e)$ occurs twice in the sum. Euler's formula implies the statements $|\psi_\theta| = 2\pi$ and $|\theta| = (4p - 4 + 2n)\pi$ are equivalent.

With notations as above we define for each $\mathcal{E} \in \Lambda(p, n)$ a relatively compact convex polyhedron $P(\mathcal{E})$ by

$$P(\mathcal{E}) = \{\psi \in \mathbb{R}^{|\mathcal{E}|} | 0 < \psi \leq 1 \text{ and } A_\psi \in \mathcal{N}^+ (T)\}.$$

When $\psi \in P(\mathcal{E})$, then $\theta_\psi > 0$, and $|\theta_\psi| = (4p - 4 + 2n)\pi$. (In fact, it will be true that $\theta_\psi > 0$.) If $\theta \in (\mathbb{R}^+)^\mathcal{E}$ is such that $|\theta| = (4p - 4 + 2n)\pi$, define

$$P(\mathcal{E}, \theta) = \{\psi \in P(\mathcal{E}) | \theta_\psi = \theta\}.$$

Set up the locally compact space

$$\mathcal{H}_0(\theta) = \bigcup_{\mathcal{E} \in \Lambda(p, n)} \{\mathcal{E}\} \times P(\mathcal{E}, \theta).$$

Declare $\mathcal{E}_1, \mathcal{E}_2 \sim (\mathcal{E}_2, \mathcal{E}_3)$ to mean that $E_1 \psi_1^{-1}(1) = E_2 \psi_2^{-1}(1)$ and $\psi_1 |_{E} - \psi_2 |_{E}$, and let $\mathcal{H}(\theta) = \mathcal{H}_0(\theta) \sim$ with the quotient topology. Let $P_0(\mathcal{E}, \theta) \subseteq P(\mathcal{E}, \theta)$ be the set defined by the additional strict inequalities $\psi(\cdot) < 1$ (i.e., instead of $\psi < 1$). The set $\bigcup_{\mathcal{E} \in \Lambda(p, n)} \{\mathcal{E}\} \times P_0(\mathcal{E}, \theta)$ is a complete set of representatives for $\sim$. Therefore if we define $D(\mathcal{E}, \theta) = \pi(\{\mathcal{E}\} \times P_0(\mathcal{E}, \theta))$, the set $\mathcal{H}(\theta) = \{D(\mathcal{E}, \theta) | \mathcal{E} \in \Lambda(p, n)\}$ is a cell decomposition of $\mathcal{H}(\theta)$.

**Theorem 1.1.** Assume $\theta > 0$ satisfies the cone angle condition $\sum \theta(s) = (4p - 4 + 2n)\pi$. There exists a homeomorphism $h: \mathcal{H}(\theta) \to \mathcal{F}(p, n)$ such that $\mathcal{G}(\theta) = h \mathcal{G}(\theta)$. For each $\mathcal{E} \in \Lambda(p, n)$ $\mathcal{G}(\mathcal{E}, \theta)$ is a real analytic submanifold of $\mathcal{F}(p, n)$ and $h^{-1}: P_0(\mathcal{E}, \theta) \to \mathcal{G}(\mathcal{E}, \theta)$ is a real analytic diffeomorphism. With obvious notations

$$\partial \mathcal{G}(\mathcal{E}, \theta) = \bigcup_{\mathcal{F} \sim \mathcal{E}} \mathcal{G}(\mathcal{F}, \theta).$$

It is natural to view Theorem 1.1 in the context of a more general statement. Let $\mathcal{E}'(p, n)$ be the set of all $\theta \in (\mathbb{R}^+)^\mathcal{E}'$ such that $\sum_{s \in \partial e} \theta(s) = (4p - 4 + 2n)\pi$, and let $G^+(p, n)$ be the positive character group of $\pi_1(M_{p,n})$. The product $\mathcal{E}'(p, n) \times \mathcal{F}(p, n) \times G^+(p, n)$ serves as a model for the set $\mathcal{F}(p, n)$ of “admissible” affine complex structures on $M_{p,n}$, up to isotopy. (This model differs insubstantially from the model used in Theorem 1.13 of [40]). Each $w \in \mathcal{E}'(p, n)$ determines $Bw \in \mathcal{F}(p, n)$, a complex holonomy character $\chi(w, \cdot)$, and a complex cone angle $\theta + i\theta' \in \mathcal{E}'(p, n)$, $\sum \theta(s) = 0$. The map from $\mathcal{F}(p, n)$ to the product above assigns $w$ the triple $(\theta, Bw, |\chi(w, \cdot)|)$. It is one-to-one and onto (4 Theorem 1.13). In what follows we identify $w \in \mathcal{E}'(p, n)$ with a triple $(\theta, \tau, v)$, $v(\cdot) = |\chi(w, \cdot)|$.

We shall prove that each $(\theta, \tau, v) \in \mathcal{E}'(p, n)$ admits a unique minimal Delaunay partition. This again is given by $\mathcal{E} \in \Lambda(p, n)$ with each open 2-cell affine complex equivalent to an
inscribed polygon. The condition (2) (on central angles) is the same. If \( \theta \in \ell^+(p,n) \) and \( v \in G^+(p,n) \) are fixed, there arises a partition \( \mathcal{G}(\theta, v) \) of \( \mathcal{F}(p,n) \) just as in the cone metric case. We shall prove

\begin{theorem}
If \( \theta \in \ell^+(p,n), v \in G^+(p,n) \), there exists a homeomorphism \( h_+ : \mathcal{A}(\theta) \to \mathcal{F}(p,n) \) such that \( \mathcal{G}(\theta, v) = h_+ \mathcal{G}(\theta) \). \( \mathcal{G}(\theta, v) \) and \( h_+ \) enjoy the smoothness properties of \( \mathcal{G}(\theta) = \mathcal{G}(\theta, 1) \) and \( h = h_1 \) in Theorem 1.1.
\end{theorem}

We shall give two proofs of the existence of Delaunay partitions for \( w \in \mathcal{F}(p,n) \). One proof is as a corollary of Theorem 1.2. The other proof is direct and is modelled in [3]. A complicating factor in the noncone metric case is the fact an affine geodesic \((\Delta, x(\cdot))\) need not extend continuously to \([0,1]\). However, if \( \Delta = \{|z| < 1\} \), one finds that an affine complex geodesic \((\Delta, x(\cdot))\) does extend continuously to \( \overline{\Delta} \).

If the polyhedra \( \{ \mathbf{e} \} \times P(\mathbf{e}) \) are glued according to the rule of formation for \( \mathcal{A}(\theta) \), the resulting space \( \mathcal{R} = \mathcal{A}(p,n) \) is for each \( v \in G^+(p,n) \) homeomorphic to \( \ell^+(p,n) \times \mathcal{F}(p,n) \times \{v\} \). The homeomorphism defines a partition \( \mathcal{G}(v) \) of \( \ell^+(p,n) \times \mathcal{F}(p,n) \) with properties similar to those in Theorems 1.1 and 1.2. In [5] Penner introduces a "decorated Teichmüller space" which, after factoring out an \( \mathbb{R}^+ \) action is (trivially) topologically the same as \( \ell^+(p,n) \times \mathcal{F}(p,n) \). By an ingenious procedure Penner associates to each point of decorated Teichmüller space a cone metric equipped with a geodesic triaugulation, and Penner's "face conditions" are equivalent to the condition that this triangulation be Delaunay. In principle there exists a homeomorphism between Penner's space and \( \ell^+(p,n) \times \mathcal{F}(p,n) \) which carries Penner's cell decomposition of decorated Teichmüller space to \( \mathcal{G}(1) \). However, since Penner's construction seems not to be conformal, this homeomorphism is somewhat mysterious.

It is a consequence of Theorems 1.1. and 1.2 that the sets \( \mathcal{G}(\theta) \) and \( \mathcal{G}(\theta, v) \) are, in particular, topological convex polyhedra. This latter statement is established by Rivin [6] although the polyhedra are not exhibited in terms of linear equalities and inequalities. It is hoped that the explicit description in the present paper will be of use in settling certain questions of finite volume which are raised in [4].

2. BUILDING BLOCKS

Let \( E \) be a nonempty finite set, and denote the group of permutations of \( E \) by \( \mathfrak{S} \). If \( \xi \in \mathfrak{S} \), then \( \mathcal{C}(\xi) \) is the set of \( \xi \)-orbits and for each \( x \in E \), \( \mathcal{C}(x) \) is the \( \xi \)-orbit of \( x \). Define \( n(\xi) = |\mathcal{C}(\xi)| \) and \( n(\xi, x) = |\mathcal{C}(x)|, x \in E \). The \( P \)-sets below are essentially the same as the fatgraphs of Penner, as formalized in [7]. The latter, in turn, are very similar to the Dessins D'Enfants of Grothendieck [8], as described in [9].

With notations as above a collection \( \mathfrak{P} = (E, \sigma, \tau, \eta) \) is a \( P \)-set if \( \sigma, \tau, \eta \in \mathfrak{S} \) are such that (a) \( n(\sigma, \cdot) \geq 3 \), (b) \( n(\tau, \cdot) \equiv 2 \), (c) \( \eta = \tau \sigma \) and (d) the group (denoted \( \langle \sigma, \tau, \eta \rangle \)) generated by \( \sigma, \tau, \eta \) is transitive on \( E \).

If \( \xi \in \mathfrak{S} \) and \( \varphi \in \mathbb{R}^E \), define \( \varphi_\xi \in \mathbb{R}^E \) by \( \varphi_\xi(x) = \sum_{\varphi \in \mathcal{C}(x)} \varphi(y) \). Associate to the \( P \)-set \( \mathfrak{P} \) polyhedra \( \Pi(\mathfrak{P}) \), defined as

\[ \Pi(\mathfrak{P}) = \{ \varphi \in (0,1)^E \mid \varphi_\sigma \equiv 1, \varphi_\tau \leq 1 \} \]

(2)

\[ \Pi_0(\mathfrak{P}) = \{ \varphi \in \Pi(\mathfrak{P}) \mid \varphi_\xi < 1 \} . \]
It is evident that these sets have the same dimension,

$$\dim \Pi(\mathcal{E}) = 2n(\tau) - n(\sigma).$$

(3)

If \( \phi \in \Pi(\mathcal{E}) \) and \( F \in \mathcal{C}(\sigma) \), \( P(F, \phi) \) will denote a choice from a set of marked polygons. Let \( C \) be any circle, and let \( P \) be an \(|F|\)-gon inscribed in \( C \) with edges labelled, in counterclockwise order, \( x, \sigma x, \ldots \), for some \( x \in F \). If the radius of \( C \) is \( R \), the fact \( \varphi_n(x) = 1 \) makes it possible to choose \( P \) so that the edge with label \( y \) has length \( 2\pi R \varphi(y) \). Now denote \( P \) by \( P(F, \phi) \). If \( (P_1, C_1) \) and \( (P_2, C_2) \) are choices for \( P(F, \phi) \), there is a uniquely determined complex affine map \( T = \alpha z + \beta \) such that \( T C_2 = C_1 \) and \( TP_2 = P_1 \), preserving the labelling. The coefficients \( \alpha \) and \( \beta \) are rational holomorphic functions of the vertices of \( P_1 \) and \( P_2 \).

We state an elementary fact for later reference:

**Lemma 2.1.** Let \( \phi \in \Pi(\mathcal{E}) \) and \( F \in \mathcal{C}(\sigma) \). If \( x \in F \), the internal angle of \( P(F, \phi) \) at the vertex determined by \( x \) and \( \sigma x \) is \( \rho(x) \), where

$$\rho(x) = \pi(1 - \varphi(x) - \varphi(\sigma x)).$$

(4)

**Proof.** Clear. \( \square \)

Given \( \phi \in \Pi(\mathcal{E}) \) and a choice of \( P(F, \phi) \) form each \( F \in \mathcal{C}(\sigma) \), form the disjoint union \( \bigcup F P(F, \phi) = X' \). If \( x \in E \) and \( \sigma_x(x) = F_1 \), \( \sigma_x(\tau x) = F_2 \), glue \( P(F_1, \phi) \) to the image of \( P(F_2, \phi) \) under \( Tz = \alpha z + \beta \), chosen to send the edge \( \tau x \) to edge \( x \), reversing sense. Extend this gluing relation to an equivalence relation \( \sim \) on \( X' \). If \( x \in E \), the vertex \( v(x) \) determined by edges \( x \) and \( \sigma(x) \) is in an equivalence class with \( n(\eta, x) \) elements. The total angle associated to \( [v(x)] \) is \( \rho_\eta(x) \), where \( \rho(\cdot) \) is defined in (4). For reference in a later section we record

**Lemma 2.2.** Let \( \phi \in \Pi(\mathcal{E}) \), and let \( \rho(x) \) be defined by (4). The total angle, \( \rho_\eta(x) \), for the vertex class \( [v(x)] \) determined by \( x, \sigma x \) is given by

$$\rho_\eta(x) = \pi \left( n(\eta, x) - \sum_{y \in \sigma_x(x)} (\varphi(y) + \varphi(\tau y)) \right).$$

(5)

**Proof.** Because \( x = \tau \eta \), we have \( \sigma \sigma_x(x) = \tau \sigma_x(x) \). Therefore, the sum on the right in (5) also represents \( \rho_\eta(x) \). The lemma is proved. \( \square \)

Given \( \phi \in \Pi(\mathcal{E}) \) and \( X' \) as above, define \( X(\mathcal{E}, \phi) = X'/\sim \). \( X(\mathcal{E}, \phi) \) is a surface with a distinguished finite set \( S \) having \( n(\eta) \) elements. The definition of \( X(\mathcal{E}, \phi) \) equips \( X(\mathcal{E}, \phi) \) with a complex structure and \( X(\mathcal{E}, \phi)/S \) with an affine complex structure. \( X(\mathcal{E}, \phi) \) is connected because \( \langle \{\sigma, \tau, \eta\} \rangle \) is transitive on \( E \). The polygons \( P(F, \phi), F \in \mathcal{C}(\sigma) \), determine a CW-complex on \( X(\mathcal{E}, \phi) \). If \( x \in F \), \( P(F, \phi) \) induces an orientation on the edge labelled \( x \) and \( \sigma x \) is interpreted as assigning to \( x \) its successor edge on \( \partial P(F, \phi) \). Similarly, \( \tau x \) is interpreted as reversing orientation on \( x \). Finally, if \( x \) is now viewed as incoming at \( [v(x)] \in S \), \( \eta x \) represents the succeeding incoming edge in the clockwise sense about \( [v(x)] \).

If \( X_1(\mathcal{E}, \phi) \) and \( X_2(\mathcal{E}, \phi) \) are versions of \( X(\mathcal{E}, \phi) \) resulting from different sets of choices \( P(F, \phi), F \in \mathcal{C}(\sigma) \), the identification with \( \mathcal{E} \) of the associated one skeletons determines a canonical homeomorphism \( f: X_1 \to X_2 \). This map is biholomorphic and affine complex on the complement of the vertex classes.
The construction of $X(\varepsilon, \varphi)$ and Euler's formula imply the Euler characteristic of $X(\varepsilon, \varphi)$ is $n(\sigma) - n(\tau) + n(\eta)$. Accordingly, we define

$$\chi(\varepsilon) = n(\sigma) - n(\tau) + n(\eta).$$

(6)

In most contexts $\chi(\varepsilon)$ will be a fixed value, $\chi(\varepsilon) = 2 - 2p$, and $n(\eta) = n > 0$ will also be fixed.

Since the axioms for $\varepsilon$ imply $3n(\sigma) \leq 2n(\tau)$, we have

**Lemma 2.3.** If $\chi(\varepsilon) = 2 - 2p$ and $n(\eta) = n$ then

$$n(\sigma) \leq 4p - 4 + 2n(\tau)$$

$$n(\tau) \leq 6p - 6 + 3n.$$

(7)

Equality holds if and only if $n(\sigma, \cdot) \equiv 3$.

**Proof.** Clear.


Let $P$ be a simple $v$-gon, $v \geq 3$, which is inscribed in a circle $C$ of radius $R$. If $x$ is an edge of $P$, let $J(x)$ be the arc of $C$ which has $x$ for a chord and which lies on the opposite side of $x$ as $P$. Define $\psi(x) = |J(x)|/2\pi R$. Then $\psi > 0$ and $\Sigma \psi(x) = 1$.

Now let $(P_j, C_j, \psi_j), j = 1, 2$, be as in the preceding paragraph. Let $x_j$ be an edge of $P_j$, $j = 1, 2$, and let $T_j = az + \beta$ be uniquely determined by the requirement that $T_jx_2 = x_1$, reversing the sense induced by $P_2, P_1$. From elementary geometry we have

**Lemma 2.4.** With notations as above $TC_2 = C_1$ if, and only if, $\psi_1(x_1) + \psi_2(x_2) = 1$ and $P_1 \cup TP_2$ is inscribed in $C_1$. No vertex of $P_1$ is interior to $TC_2$ and no vertex of $TP_2$ is interior to $C_1$ if, and only if, $\psi_1(x_1) + \psi_2(x_2) \leq 1$.

**Proof.** Clear.

**Lemma 2.5.** Let $\varphi \in \Pi(\varepsilon)$, and let $X(\varepsilon, \varphi)$ be constructed from choices $P(F, \varphi), F \in \eta(\sigma)$. If $F \in \eta(\sigma)$, and if $D = D(F)$ is the interior of the circle which is circumscribed about $P(F, \varphi)$, there exists a holomorphic map $f: D \rightarrow X(\varepsilon, \varphi)|_S$ such $f|_{P(F, \varphi)}$ is the canonical projection.

**Proof.** This is an application of Lemma 2.4 and the assumption $\varphi_1 \leq 1, \varphi \in \Pi(\varepsilon)$. Let $f_0: P(F, \varphi) \rightarrow X(\varepsilon, \varphi)$ be the canonical projection. Given $x \in F$ it is sufficient to extend $f_0$ to the region bounded by $x$ and the arc $J(x)$ which it determines. Denote the endpoints of this arc by $v_1, v_2$. Choose $Tz = az + \beta$ to send edge $\varepsilon z$ of $P(\varepsilon_1(\tau z), \varphi)$ to $x$, reversing sense. Because $\varphi_1(x) \leq 1$, $T$ sends the edges $\varepsilon^{-1} \tau x$ and $\sigma \tau x$ to line segments, the first emanating from $v_1$ and the second from $v_2$ and neither entirely contained in $D$. Let these line segments exit $\partial D$ at points $w_1, w_2$. (Of course, $D \cap TP(\varepsilon_1(\tau x), \varphi)$ is convex.) Now $f_0$ analytically continues to the convex hull of $P(F, \varphi)$ and the arc of $\partial D(F)$ from $w_1$ to $w_2$. If $w_1 \neq v_1$, for example, the internal angle of $P(F, \varphi) \cap TP(\varepsilon_1(\tau x), \varphi)$ at $v_1$ is greater than the internal angle of $P(F, \varphi)$ at $v_1$ by an amount bounded below by $\min_{\gamma \in E} \rho(y) > 0$. Now repeat this construction for each of the edges $T\sigma^{-1} \tau x$ and $T\tau x$ (requiring use of at most $\tau \sigma^{-1} \tau x$ and $\tau \tau x$). After a finite number of iterations the region between $x$ and $J(x)$ is covered and
the canonical projection extends. Repeat this for every edge of \( P(F, \varphi) \) and the lemma obtains.

\[ \square \]

### 3. Complexes Associated to P-sets

If \( \mathcal{S} \) is a P-set we shall define elementary operations of reduction and augmentation. In terms of \( X(\mathcal{S}, \varphi) \), \( \varphi \in \Pi(\mathcal{S}) \), these operations correspond to deleting an edge common to distinct 2-cells or to adjoining a diagonal to a 2-cell which has more than three edges.

If \( x \in E \), and if \( E_0(x) \neq E_0(rx) \), define \( E_0 - E_0 \{ x, rx \} \), \( \tau_0 = \tau|_{E_0} \), \( \eta_0 \) - first return map of \( \eta \) on \( E_0 \) and \( \sigma_0 = \tau_0 \eta_0 \). \( \sigma_0 \) is defined directly by

\[
\sigma_0(y) = \begin{cases} 
\sigma x & u = \sigma^{-1} x \\
\sigma x & y = \sigma^{-1} rx \\
\sigma y & \text{else.}
\end{cases}
\]

The values of \( n(\sigma_0, y) \) are either values of \( n(\sigma, y) \) or the new value \( n(\sigma, x) + n(\sigma, rx) - 2 \geq 4 \). \( \mathcal{S}_0 = (E_0, \sigma_0, \tau_0, \eta_0) \) is again a P-set. Since \( n(\sigma_0) = n(\sigma) - 1 \), \( n(\tau_0) = n(\tau) - 1 \) and \( n(\eta_0) = n(\eta) \) we have \( \chi(\mathcal{S}_0) = \chi(\mathcal{S}) \).

In the reverse direction if \( F \in \mathcal{C}(\sigma) \) has at least four elements, decompose \( F \) into disjoint \( \sigma \) segments, \( F_1 \) and \( F_2 \), each of length at least two. Adjoin ideal points \( w_1, w_2 \) to \( E \) and extend \( \tau \) to \( \tau_1 \) on \( E_1 = F \cup \{ w_1, w_2 \} \) by \( \tau_1 w_1 = w_2 = \tau_1^{-1} w_1 \). Set \( F_j = F_j \cup \{ w_j \}, j = 1, 2 \), and extend \( \sigma \) to \( \sigma_1 \) so that \( F_j \in \mathcal{C}(\sigma_1) \). Set \( \eta_1 = \tau_1 \tau_1 \) and \( \mathcal{S}_1 = (F_1, \sigma_1, \tau_1, \eta_1) \). Again we find \( n(\eta_1) = n(\eta) \) and \( \chi(\mathcal{S}_1) = \chi(\mathcal{S}) \). Moreover, if the reduction procedure is applied to \( \mathcal{S}_1 \), with \( x = w_1 \), the result is \( \mathcal{S} \).

If \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) are P-sets, we shall write \( \mathcal{S}_1 < \mathcal{S}_2 \) if \( \mathcal{S}_1 \) is obtained from \( \mathcal{S}_2 \) by a sequence of reductions or, equivalently, if \( \mathcal{S}_2 \) is obtained from \( \mathcal{S}_1 \) by a sequence of augmentations.

If \( \mathcal{S}, \mathcal{S}_1, \mathcal{S}_2 \) are P-sets, we shall write \( \mathcal{S}_1 \sim \mathcal{S}_2 \) if \( \mathcal{S} \ll \mathcal{S}_j \), \( j = 1, 2 \), and if the identity map on \( E \) extends to an isomorphism \( \lambda_0: \mathcal{S}_1 \to \mathcal{S}_2 \) (\( \lambda_0 \sigma_1 = \sigma_2 \circ \lambda \), etc.). Let \( \mathbb{F}(\mathcal{S}) \) be the set of equivalence classes for \( \sim_0 \). As \( n(\eta_1) = n(\eta) \) and \( \chi(\mathcal{S}_1) = \chi(\mathcal{S}) \) for \( \mathcal{S} \ll \mathcal{S}_1 \), it is a consequence of (7) that \( \mathbb{F}(\mathcal{S}) \) is finite.

**Lemma 3.1.** Let \( \varphi \in \Pi(\mathcal{S}) \), and define \( E_0 = E \setminus \varphi^{-1} 1 \). If \( \tau_0 = \tau|_{E_0} \), \( \eta_0 \) is the \( \eta \)-first return map on \( F_0 \), and if \( \sigma_0 = \tau_0 \eta_0 \), then \( \mathcal{S}_0 = (F_0, \sigma_0, \tau_0, \eta_0) \) is a P-set. Moreover, \( \mathcal{S} \ll \mathcal{S}_0 \) and \( \varphi_0 = \varphi|_{E_0} \in \Pi(\mathcal{S}_0) \).

**Proof.** Suppose \( \varphi(x) + \varphi(rx) = 1 \). The assumption \( \varphi_x = 1 \), \( \varphi > 0 \) and \( n(\sigma, \cdot) > 2 \) imply \( E_0(x) \neq E_0(rx) \). Let \( \mathcal{S}' = (E', \sigma', \tau', \eta') \) be the reduction determined by \( x, \tau x \), as above. If \( \varphi' = \varphi|_{E'} \), then the sum of \( \varphi' \) along the one new \( \sigma' \)-orbit is \( (1 + 1) - 1 = 1 \), meaning \( \varphi' \in \Pi(\mathcal{S}') \). Now argue by descent. \[ \square \]

**Definition 3.2.** If \( \varphi \in \Pi(\mathcal{S}) \) and if \( (\mathcal{S}_0, \varphi_0) \) is as in Lemma 3.1, then \( (\mathcal{S}_0, \varphi_0) \) will be referred to as the reduced form of \( (\mathcal{S}, \varphi) \).

**Lemma 3.3.** If \( \mathcal{S} \ll \mathcal{S}' \), there is a canonical injection \( J: \Pi(\mathcal{S}) \to \Pi(\mathcal{S}') \) such that for each \( \varphi \in \Pi(\mathcal{S}) \) we have \( E \setminus (J\varphi)^{-1} 1 \subseteq E \) and \( J\varphi|_E = \varphi \).

**Proof.** Reverse the procedure in Lemma 2.1. \[ \square \]
Let \( S = \langle E, \sigma, e, \eta \rangle \) be a P-set. If \( \varrho, \varrho_1 \in \mathcal{F}(S) \), i.e., if \( \varrho \) represents a class in \( \mathcal{F}(S) \), define

\[ \Pi(\varrho, \varrho_1) \]

to be the set of \( \varphi_1 \in \Pi(\varrho) \) such that the reduced form of \( \langle \varrho, \varphi_1 \rangle \), denoted here \( \langle \varrho, \varrho_2 \rangle \) has \( \varrho_2 \in \mathcal{F}(S) \). That is, \( (\varphi_1, e) < 1 \). Choose a complete set of representatives for \( \mathcal{F}(S) \) and define

\[ \Omega_0(\varrho) = \bigcup_{\varrho, \varrho_1 \in \mathcal{F}(S)} \{ \varrho_1 \} \times \Pi(\varrho, \varrho_1). \]  

Of course, \( \Pi(\varrho, \varrho_2) = \Pi_0(\varrho) \) (2).

Now declare that \( \langle \varrho_1, \varrho_2 \rangle \cong \langle \varrho_3, \varrho_2 \rangle \) when the reduced forms \( \langle \varrho_j, \varphi_j \rangle \), \( j = 1, 2 \), satisfy \( \varrho_1 \sim \varrho_3 \) with the canonical isomorphism carrying \( \varphi_1 \) to \( \varphi_3 \). This relation is closed, and each equivalence class is finite, of cardinality at most \( |\mathcal{F}(S)| \). If follows that the space \( \Omega(\varrho) = \Omega_0(\varrho)/\sim \), equipped with the quotient topology, is locally compact and metrizable.

Since \( \Pi_0(\varrho_1) \subseteq \Pi_0(\varrho, \varrho_1) \), \( \varrho_1 \in \mathcal{F}(S) \), and since each \( \langle \varrho_1, \varrho_1 \rangle, \varrho_1 \in \Pi_0(\varrho_1) \) is already a reduced form, the set

\[ \Omega_1(\varrho) = \bigcup_{\varrho, \varrho_1 \in \mathcal{F}(S)} \{ \varrho_1 \} \times \Pi_0(\varrho_1) \]  

is a complete set of representatives for \( \Omega(\varrho) \).

If \( \lambda: \varrho \rightarrow \varrho' \) is an isomorphism of P-sets, \( \lambda \) determines an isomorphism \( \tilde{\lambda}: \mathcal{F}(\varrho) \rightarrow \mathcal{F}(\varrho') \) in the strong sense that if \( \varrho_1 \) is a representative of \( \{ \varrho_1 \} \in \mathcal{F}(\varrho) \), and if \( \varrho_2 \) is a representative of \( \varrho \), then \( \lambda(\varrho_1) = \tilde{\lambda}(\varrho_2) \) meaning \( \varrho_1 \) is included in \( \langle \varrho_2 \rangle \), and each equivalence class is finite, of cardinality at most \( |\mathcal{F}(\varrho)| \). If follows that the space \( \Omega(\varrho) = \Omega_0(\varrho)/\sim \), equipped with the quotient topology, is locally compact and metrizable.

For purposes of later sections we shall now associate to \( \Omega_0(\varrho) \) an open subset \( \Omega^*(\varrho) \) which contains \( \{ \varrho \} \times \Pi_0(\varrho) \), is a union of equivalence classes and which has an additional property, described below.

Let \( \varrho, \varrho_1 \in \mathcal{F}(S) \), \( F \in \mathcal{C}(\varrho) \), \( \varrho_1 \in \Pi_0(\varrho) \). There is a set \( H(F) \subseteq \mathcal{C}(\varrho_1) \) such that \( P(F, \varrho_1) \) is obtained from \( \{ P(H, \varrho_1) \} \) by an appropriate set of gluing maps \( \tilde{T} = ax + \beta \) determined by their relationship between \( \varrho \) and \( \varrho_1 \). Of course, \( P(F, \varrho_1) \) is strictly convex. If \( \psi \in \Pi(\varrho, \varrho_1) \), one can attempt to construct a polygon from \( \{ P(H, \psi) \} \) using the pattern determined by \( \varrho_1 \) and \( \varrho \), but the result may not be a planar polygon. Define \( \Pi^*(\varrho, \varrho_1) \subseteq \Pi(\varrho, \varrho_1) \) to be the open set of \( \psi \) for which the result is a simple strictly convex polygon for each \( F \in \mathcal{C}(\varrho) \). \( \Pi^*(\varrho, \varrho_1) \) contains \( \Pi_0(\varrho) \).

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Now define

\[ \Omega^*(\varrho) = \bigcup_{\varrho, \varrho_1 \in \mathcal{F}(S)} \{ \varrho_1 \} \times \Pi^*(\varrho, \varrho_1). \]  

\( \Omega^*(\varrho) \) is open and a union of equivalence classes. It will be of no importance that \( \Omega^*(\varrho) \) does not, in general, contain \( \Omega_1(\varrho) \), only that it does contain \( \{ \varrho \} \times \Pi_0(\varrho) \). It is true of course that \( \Omega^*(\varrho) \cap \Omega_0(\varrho_1) \subseteq \Omega^*(\varrho_1) \) when \( \Omega_0(\varrho_1) \) is identified as an open subset of \( \Omega_0(\varrho) \).

4. THE DELAUNAY PARTITION OF A CONVEX POLYGON

In this section the letters \( D \) and \( C \) stand for an open disc and the circle which bounds it, respectively.
Let \( Q \) be a convex polygon with vertex set \( V \). We assume \( Q \) is two dimensional. Call a circle \( C \) good for \( Q \) if \( V \cap C = \emptyset \) and \( |V \cap C| \geq 3 \). Let \( \mathcal{C}(Q) = \mathcal{C} \) be the set of all good circles. It is clear that \( \mathcal{C} \) is finite, and we define \( K(C) \) to be the convex hull of \( V \cap C \) for each \( C \in \mathcal{C} \). By definition \( K(C) \) is convex and two dimensional. The set \( \mathcal{H}(Q) = \{ K(C) \mid C \in \mathcal{C}(Q) \} \) is a similarity invariant.

The definition of \( \mathcal{C} = \mathcal{C}(Q) \) implies that for distinct elements \( C_1, C_2 \in \mathcal{C} \) no point of \( V \cap C_1 \) (resp. \( V \cap C_2 \)) is contained in \( C_2 \) (resp. \( C_1 \)). It follows that when \( D_1 \) and \( D_2 \) share a chord, \( K(C_1) \) and \( K(C_2) \) lie on opposite sides of this chord. Therefore, \( \mathcal{H}(Q) \) is a partition of the union of its elements.

Let \( x \) be an edge of \( Q \), and let \( \ell \) be the perpendicular bisector of \( x \). For each \( \zeta \in \ell \) let \( C(\zeta, x) \) be the circle with center \( \zeta \) and chord \( x \). We have \( V \cap D(\zeta, x) = \emptyset \) for \( \zeta \) near \( \infty \) on one end of \( \ell \) and \( V \subseteq D(\zeta, x) \) for \( \zeta \) near \( \infty \) at the other end. There exists a unique value of \( \zeta \in \ell \) such that \( D(\zeta, x) \cap V = \emptyset \) and \( |C(\zeta, x) \cap V| \geq 3 \). Therefore, \( x \) is an edge of \( K(C) \) for some \( C \in \mathcal{C} \). Let \( C \in \mathcal{C} \), and let \( y \) be an edge of \( K(C) \) which is a diagonal of \( Q \), i.e., not an edge of \( Q \). If \( \ell \) is the perpendicular bisector of \( y \), then \( C(\zeta, y) = C \) for some \( \zeta \in \ell \). \( \zeta \) divides \( \ell \) into two open rays \( \ell \pm \zeta \) such that if \( \zeta \in \ell \), then \( V \cap D(\zeta, y) \neq \emptyset \). As \( \zeta \) varies on \( \ell \), \( V \cap D(\zeta, y) \neq \emptyset \) for \( \zeta \) near \( \infty \) while, since \( y \) is a diagonal, \( V \cap D(\zeta, y) \neq \emptyset \) for \( \zeta \) near \( \infty \). It follows there exists a good circle \( C(\zeta, y) \neq C \) such that \( y \) is an edge of \( K(C(\zeta, y)) \). The union \( K(C) \cup K(C(\zeta, y)) \) has \( y \) in its interior.

The discussion just completed implies \( \bigcup_{C \in \mathcal{C}} K(C) \) is an open and closed, nonempty subset of \( Q \). We conclude that \( \mathcal{H}(Q) \) is a decomposition of \( Q \). We shall call it the Delaunay partition.

5. \( \Omega(\mathcal{C}) \) IS A MANIFOLD

Let \( \mathcal{E} \) be a \( P \)-set, and let \( \mathcal{M}(\mathcal{E}) \) be the set of functions \( \ell \) on \( \mathcal{O}(\sigma) \) such that \( \ell(F), F \in \mathcal{O}(\sigma) \), is a complex affine equivalence class of strictly convex \( |F| \)-gons with edges labelled counterclockwise in cyclic correspondence with \( F \). \( \mathcal{M}(\mathcal{E}) \) is a complex manifold of dimension

\[
\dim_{\mathbb{C}} \mathcal{M}(\mathcal{E}) = \sum_{\ell(\sigma)} (|F| - 2)
= 2(n(\tau) - n(\sigma))
= 2(2p - 2 + n).
\]  

(As before we take \( n(\eta) = n \) and \( \chi(\mathcal{E}) = 2 - 2p \).) Discussions of \( \ell \in \mathcal{M}(\mathcal{E}) \) will generally include, without mention, choices of representative polygons \( Q(F, \ell) \in \mathcal{O}(F), \ F \in \mathcal{O}(\sigma) \). For example, one may introduce coordinates by choosing \( x(F) \in F \) for each \( F \) and requiring that \( Q(F, \ell) \) lie in the upper halfplane except for the one edge \( x(F) = [0, 1] \). The transitions associated to a different choice \( x'(F), \ F \in \mathcal{O}(\sigma), \) are rational holomorphic in the vertices of \( Q(F, \ell) \).

We remark that \( \mathcal{E} \) and \( \ell \in \mathcal{M}(\mathcal{E}) \) determine a surface \( X(\mathcal{E}, \ell) \) obtained by gluing representatives \( Q(F, \ell) \) according to \( \tau \).

Associate to \( \ell \in \mathcal{M}(\mathcal{E}) \) and each \( Q(F, \ell) \) the Delaunay partition \( \mathcal{H}(Q(F, \ell)) \). This partition determines (perhaps) certain diagonals \( \delta \) of \( Q(F, \ell) \) such that any distinct pair intersect on \( \partial Q \). Define \( E_{\ell} \supseteq E \) by adjoining to \( E \) two elements for each new diagonal, one for each direction, and extend \( \tau \) to the oriented diagonals by reversing orientation. Extend \( \sigma \) to the edges of each polygon \( K(C), \ C \in \mathcal{C}(Q(F, \ell)) \), assigning each edge its counterclockwise
successor on $\partial K(C)$. If the extensions are denoted $\tau_n, \sigma_n$, define $\eta_n = \tau_n \sigma_n$ and $\mathcal{E}(\ell) = (E(\ell), \tau_n, \sigma_n, \eta_n)$. $\mathcal{E}(\ell)$ is a $\mathcal{P}$-set and clearly $\mathcal{E} \subseteq \mathcal{E}(\ell)$.

The discussion preceding Lemma 2.4 is used to associate to $\ell$ and $\mathcal{E}(\ell)$ a function $\overline{\psi}(\ell) = \psi(\ell, \cdot)$ on $E$, such that $\overline{\psi}_{v}(\ell) \equiv 1$, $\overline{\psi}(\ell) > 0$. Lemma 2.4 implies $\psi(\ell, y) + \psi(\ell, \tau y) < 1$ for each $y \in E \setminus E$. However, if $x \in E$, it is not guaranteed that $\psi(\ell, x) + \psi(\ell, \tau x) \leq 1$. We shall define $\mathcal{L}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{E})$ by

$$\mathcal{L}(\mathcal{E}) = \{ \ell \in \mathcal{M}(\mathcal{E}) | \overline{\psi}_{v}(\ell) < 1 \}.$$  

(12)

By the discussion thus far we have that $\ell \in \mathcal{L}(\mathcal{E})$ if and only if it is true that

$$\psi(\ell, x) + \psi(\ell, \tau x) < 1 \quad (x \in E).$$

(13)

**Lemma 5.1.** Let $\mathcal{E}$ be a $\mathcal{P}$-set, and let $x \in E$. The function $\psi(\cdot, x)$ is continuous on $\mathcal{L}(\mathcal{E})$.

**Proof.** If $k > 2$, let $U_k$ denote set of $v = (v_1, \ldots, v_k) \in C^k$ which are the vertices of a strictly convex polygon, in counterclockwise order. Let $Y$ be the set of unordered triples $v = \{a, b, c\}, 1 \leq a \neq b \neq c \leq k$ and define $C_v(u)$ be the circle determined by $v_a, v_b, v_c$. $C_v(\cdot)$ is continuous for each $v \in Y$. Let $1 \leq j \leq k$ and for each $c \neq j$ or $j + 1$ (mod $k$) let $v_{cj} = \{j, j + 1 \text{ (mod $k$)}\}$. In the set $\{C_v(u) | c \neq j, j + 1\}$ there is one good circle (which may appear more than once). Call this circle $C_j(v)$. If $C_j(v_0) = C_{v_j}(u_0)$ for all $j$ in a set $W = \{1, 2, \ldots, k\}$ then for $v$ close to $v_0$, the circles $C_u(u) | j \in W$, are close to one another and to $C_j(v)$. By the same token if $v$ is close to $v_0$, then $C_j(v) \in \{C_{v_j}(u) | j \in W\}$. We conclude that $C_j(\cdot)$ is continuous on $U_k$. As $C_j(\cdot)$ is equivariant relative to similarities, $C_j(\cdot)$ projects to a continuous function on the set of affine complex equivalence classes of convex $k$-gons. Let $J_j(v)$ be the arc of $C_j(v)$ which has $[v_j, v_{j+1}]$ for a chord and which otherwise contains no vertices in its interior. Let $C_j(v)$ have radius $R_j(v)$, and define $\psi_j(v) = |J_j(v)|/2\pi R_j(v)$. $\psi_j(\cdot)$ is also continuous. In the context of $\mathcal{M}(\mathcal{E})$ this is tantamount to the statement that $\psi(\cdot, x)$ is continuous for each $x \in E$. \hfill \Box

The lemma implies the set where $\psi(\cdot, x) < 1$ is open for each $x \in E$. Therefore,

**Proposition 5.2.** $\mathcal{L}(\mathcal{E})$ is an open set in $\mathcal{M}(\mathcal{E})$.

We return to $\Omega_0^\mathcal{E}(\mathcal{E}) = \bigcup_{\mathcal{E}(\ell)} \{ \mathcal{E}(\ell) \times \mathcal{E}(\mathcal{E}), \mathcal{E}(\ell) \}$ which was defined in (10). To define a natural map $\ell : \Omega_0^\mathcal{E}(\mathcal{E}) \to \mathcal{L}(\mathcal{E})$ we fix $\mathcal{E}(\ell) \in \mathcal{E}(\mathcal{E})$. If $F \in \mathcal{E}(\sigma)$, let $H(F) \equiv \mathcal{E}(\sigma)$ be as in the paragraph which contains (10). Let $x(F) \in E$ be chosen for each $F \in \mathcal{E}(\sigma)$. If $\phi_1 \in \mathcal{E}(\mathcal{E}, \mathcal{E}(\ell))$, the polygons $P(H, \phi_1)$, $H \in H(F)$, are combined to determine a polygon $\tilde{P}(F, \phi_1)$. If we require that $x(F)$, which lies in exactly one $H \in H(F)$, be sent to $[0, 1]$ and that $\tilde{P}(F, \phi_1)$ otherwise lie in the upper halfplane, then $\tilde{P}(F, \phi_1)$ is uniquely determined and varies continuously with $\phi_1 \in \mathcal{E}(\mathcal{E}, \mathcal{E}(\ell))$. In terms of $z_1 = (\mathcal{E}(\ell), \phi_1) \in \Omega_0^\mathcal{E}(\mathcal{E})$ we define $\ell(z_1) \in \mathcal{M}(\mathcal{E})$ to assign to $F \in \mathcal{E}(\sigma)$ the affine complex class of $\tilde{P}(F, \phi_1)$.

The discussion just completed and the definitions imply

**Lemma 5.3.** Let $\ell(\cdot)$ be as above, and set $z(\ell) = (\mathcal{E}(\ell), \overline{\psi}(\ell)), \ell \in \mathcal{L}(\mathcal{E})$. $\ell(\cdot)$ is continuous and $\ell \circ z = 1d$ on $\mathcal{L}(\mathcal{E})$. The map $z \circ \ell$ assigns to each $z_1 \in \Omega_0^\mathcal{E}(\mathcal{E})$ its reduced form.

**Proof.** By construction of $\overline{\psi}$ we have $\overline{\psi}_{v}(\ell) < 1$ and therefore $(\mathcal{E}(\ell), \overline{\psi}(\ell))$ is a reduced form for all $\ell \in \mathcal{L}(\mathcal{E})$. The remaining parts of the statement have been addressed above or are obvious from the definitions. \hfill \Box
Let $\pi: \Omega_0(\mathcal{E}) \to \Omega(\mathcal{E})$ be the canonical projection. $\pi$ is continuous by definition, and we set $\Omega^*\mathcal{E} = \pi\Omega_0(\mathcal{E})$. Since $z$ is injective and $z \circ \ell'$ is the "reduced form" map, $\ell'(z)$ is constant on $\pi$ fibers meaning there is a commutative triangle

\[ \begin{array}{ccc}
\Omega_0^*(\mathcal{E}) & \xrightarrow{\ell} & L(\mathcal{E}) \\
\downarrow & & \\
\Omega^*(\mathcal{E}) & \xrightarrow{m} & L(\mathcal{E})
\end{array} \tag{14} \]

Since $\pi \circ z = m = \text{Id}$, $m(\cdot)$ is one-to-one onto. If $U \subseteq L(\mathcal{E})$ is an open set, then $\ell'^{-1}U$ is an open set which is a union of equivalence classes, i.e., a union of $\pi$ fibers. Therefore, $\pi(\ell'^{-1}U) = m^{-1}U$ is open and $m(\cdot)$ is continuous. If $K \subseteq L(\mathcal{E})$ is a compact set, it is easy to see that for each $\mathcal{E} \in \mathcal{F}(\mathcal{E})$ the set $\ell'^{-1}(K) \cap \{(\mathcal{E}) \times \Omega^*(\mathcal{E},\mathcal{E})\}$ is compact. (If $\ell' \to \ell$, and if for each $n$ and $F \in \mathcal{O}(\mathcal{E})$ $H(F) \subseteq \mathcal{O}(\mathcal{E})$ determines a refinement of the Delaunay partition for $Q(F,\mathcal{E})$, then $H(F)$ also determines a refinement of the Delaunay partition for $Q(F,\mathcal{E})$.) It follows that $m^{-1}(K) = \pi(\ell'^{-1}(K))$ is compact. Therefore, $m(\cdot)$ is continuous, one-to-one onto and proper. We have

**Proposition 5.4.** With notations as above the map $m: \Omega^*(\mathcal{E}) \to L(\mathcal{E})$ is a homeomorphism onto. Moreover, $m^{-1} \circ \ell|_{\mathcal{E}} \times \Pi(\mathcal{E})$ embeds $\Pi(\mathcal{E})$ as a closed submanifold of $\Omega^*(\mathcal{E}) \subseteq L(\mathcal{E})$.

**Proof.** The condition that $\ell \in L(\mathcal{E})$ have $z(\ell) \in \{\mathcal{E}\} \times \Pi(\mathcal{E})$ is that the vertices of $Q(F,\mathcal{E})$ lie on a circle for each $F \in \mathcal{O}(\mathcal{E})$. This condition defines a real analytic submanifold of $L(\mathcal{E})$. The proposition follows. \(\square\)

**Remark 5.5.** If $\mathcal{E} \in \mathcal{F}(\mathcal{E})$, it has been noted that $\Omega^*_0(\mathcal{E}) \cap \Omega_0(\mathcal{E}) \subseteq \Omega^*_0(\mathcal{E})$ holds in a natural sense. On this intersection there are two maps to consider, one to $L(\mathcal{E})$ and one to $L(\mathcal{E})$. In terms of coordinates introduced in the first paragraph of this section the transition from an open set in $L(\mathcal{E})$ to the corresponding open set in $L(\mathcal{E})$ is rational biholomorphic.

### 6. Definition of $\Omega(p,n)$

Let $p \geq 0$ and $n > 0$ be such that $2p - 2 + n > 0$. $M_p$ denotes a fixed closed oriented connected surface of genus $p$. Also fix a puncture set $S \subseteq M_p$, $|S| = n$, and define $M_{p,n} = M_p \setminus S$. Let $\Sigma$ be the set of Jordan paths (more precisely isotopy classes rel $S$) in $M_{p,n}$, which join points of $S$. Define $\gamma = \gamma^{-1}, \gamma \in \Sigma$.

We call a set $E \subseteq \Sigma$ **admissible** if it is $\tau$-invariant and if it determines an ideal cell decomposition of $M_p$ (5). This means (representatives) $\gamma \in E$ have the property that $M_p \setminus \bigcup_{\gamma \in E} |\gamma|$, where $|\gamma|$ is the set of points comprising $\gamma$, is a union of cells $D$ such that $\partial D$, with the induced orientation is a concatenation of elements of $\gamma(|\gamma| = |\gamma^{-1}|)$.

If $E$ is admissible, every cell $D$ above has at least three paths on its boundary because no pair are isotopic rel $S$. The successor map $\sigma$, which assigns each $\gamma \in \partial D$ its counterclockwise successor, and $\eta = \tau\sigma$, determine a $P$-set $\mathcal{E} = (E, \sigma, \tau, \eta)$. The fact that $M_p$ is connected implies $\langle (\sigma, \tau, \eta) \rangle$ is transitive on $E$. 
Definition 6.1. With notations as above define \( \Lambda(p, n) \) to be the set of \( P \)-sets \( \mathcal{E} \) which are determined by admissible sets \( E \subseteq \Sigma \).

Euler characteristic considerations imply \( |E| \leq 12p - 12 + 6n \) (i.e., \( n(\tau) \leq 6p - 6 + 3n \)) for admissible sets \( E \). Therefore, if \( E \) is admissible, the set \( \mathcal{G}(E) = \{ E_1 \subseteq E \mid E \subseteq E_1 \text{ and } E_1 \text{ is admissible} \} \) is finite. If \( E_1, E_2 \in \mathcal{G}(E) \), and if \( \mathcal{E}_1, \mathcal{E}_2 \) are the corresponding \( P \)-sets, the identity map of \( E \) extends to an isomorphism \( \lambda: \mathcal{E}_1 \to \mathcal{E}_2 \) if, and only if, \( E_1 = E_2 \) and \( \lambda = \text{Id} \). It is clear that \( \mathcal{E} \prec \mathcal{E}_1, E_1 \in \mathcal{G}(E) \), and therefore \( \mathcal{G}(E) \) injects as a subset of \( \mathcal{F}(\mathcal{E}) \). Consideration of standard realizations \( X(\mathcal{E}, \varphi), \varphi \in \Pi(\mathcal{E}) \), shows that each element of \( \mathcal{F}(\mathcal{E}) \) is represented in \( \mathcal{G}(E) \). Accordingly, we take

\[ \mathcal{F}(\mathcal{E}) = \{ \mathcal{E}_1 | E_1 \in \mathcal{G}(E) \} \]

as a model for \( \mathcal{F}(\mathcal{E}) \).

Define a locally compact metric space \( \Omega_0(p, n) \) as

\[ \Omega_0(p, n) = \bigcup_{\mathcal{E} \in \Lambda(p, n)} \{ \mathcal{E} \} \times \Pi(\mathcal{E}). \]  

That this notation is slightly at variance with the definition (8) of \( \Omega_0(\mathcal{E}) \) should cause no confusion.

As in Section 3 we define \( z_1 \approx z_2 \in \Omega_0(p, n) \) when the reduced forms are the same. This is a closed relation with a uniform upper bound, depending upon \( (p, n) \), for the cardinality of its equivalence classes. Therefore, \( \Omega(p, n) = \Omega_0(p, n)/\approx \), equipped with the quotient topology, is a locally compact metric space.

If \( \mathcal{E} \in \Lambda(p, n) \), then \( \Omega_0(\mathcal{E}) \subseteq \Omega_0(p, n) \), and \( \Omega_0(\mathcal{E}) \) is both open and a union of equivalence classes. Therefore, the image in \( \Omega(p, n) \) is homeomorphic to \( \Omega(\mathcal{E}) \). Let \( \Omega^*(\mathcal{E}) = \pi_0\Omega^*(\mathcal{E}) \) be as in Section 5, and define a chart \( (\Omega^*(\mathcal{E}), m) \), with values in \( \mathcal{L}(\mathcal{E}) \), as in (14). If \( \mathcal{E}_1, \mathcal{E}_2 \in \Lambda(p, n) \), and if \( \Omega^*(\mathcal{E}_1) \cap \Omega^*(\mathcal{E}_2) \neq \emptyset \), then for each \( w \in \Omega^*(\mathcal{E}_1) \cap \Omega^*(\mathcal{E}_2) \) there exists \( \mathcal{E} \in \Lambda(p, n) \) such that \( \mathcal{E}_1, \mathcal{E}_2 \prec \mathcal{E} \) and \( w \in \Omega^*(\mathcal{E}) \). It follows from Remark 5.5 that \( m_{\mathcal{E}_1} \circ m_{\mathcal{E}_2}^{-1} \) is holomorphic at \( m_{\mathcal{E}_1}(w) \).

**Theorem 6.2.** Let \( \Omega(p, n) \) and \( \mathcal{N} = \{ (\Omega^*(\mathcal{E}), m) | \mathcal{E} \in \Lambda(p, n) \} \) be as above. Then \( (\Omega(p, n), \mathcal{N}) \) is a complex analytic manifold.

The definition of \( \approx \) implies that set \( \Omega_{00}(p, n) \), defined by

\[ \Omega_{00}(p, n) = \bigcup_{\mathcal{E} \in \Lambda(p, n)} \{ \mathcal{E} \} \times \Pi(\mathcal{E}) \]  

is a complete set of representative for \( \approx \). If \( \pi \) is the canonical projection, Proposition 5.4 implies \( \Delta(\mathcal{E}) = \pi((\mathcal{E}) \times \Pi(\mathcal{E})) \) is a real analytic submanifold of \( \Omega(p, n) \) and a closed submanifold of \( \Omega^*(\mathcal{E}) \). We have

**Theorem 6.3.** The set \( \Delta(p, n) = \{ \Delta(\mathcal{E}) | \mathcal{E} \in \Lambda(p, n) \} \) is a real analytic cell decomposition of \( \Omega(p, n) \). For each \( \mathcal{E} \in \Lambda(p, n) \) we have

\[ \partial \Delta(\mathcal{E}) = \bigcup_{\mathcal{E} \prec \mathcal{E}'} \Delta(\mathcal{E}'). \]
We remark that each cell $\Delta(\mathcal{E})$ is real analytically a polyhedron $\Pi_0(\mathcal{E})$. If $\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{F}(\mathcal{E})$, then $\Delta(\mathcal{E}_1)$ and $\Delta(\mathcal{E}_2)$ are real analytically glued along $\Delta(\mathcal{E})$. In terms of $\Pi(\mathcal{E}_1), \Pi(\mathcal{E}_2)$ and $\Pi(\mathcal{E})$, the gluing is linear. Of course, $\partial\Delta(\mathcal{E})$ is the image under $\pi$ of $\{e \times \Pi(\mathcal{E})\} \setminus \Pi(\mathcal{E})$, and as the axioms for $\Pi(\mathcal{E})$ prevent certain faces of $\Pi_0(\mathcal{E})$ from appearing, $\partial\Delta(\mathcal{E})$ is "missing" certain faces.

7. A MAP OF $\Omega(p,n)$ IN $\mathcal{S}(p,n)$

Let $\alpha \in \mathbb{C}$ be such that $\Re \alpha > -1$, and let $\zeta_0 : \mathbb{C} \to \mathbb{C}^*$ be the covering map $\zeta_0(z) = e^{2\pi i z/(\alpha + 1)}$. A branch of $\zeta_0 \circ \zeta_{-1}$ is a branch of $\zeta \to \zeta^{\alpha+1}$. These branches determine an atlas $\mathcal{H}_{\alpha}$ on the punctured $\zeta$-plane. If $\Delta^\alpha = \{\zeta | 0 < |\zeta| < 1\}$, then $(\Delta^\alpha, \mathcal{W}_{\alpha}^{\Delta^\alpha})$ is a model for an affine complex structure with cone singularity and complex cone angle $2\pi(\alpha + 1)$. An affine complex structure $\mathcal{W}$ on $M_{p,n} = M_p \setminus S_a$ is admissible if each $s \in S_a$ admits a deleted neighborhood $U_s$ and $\alpha = \alpha(s), \Re \alpha > -1$, such that $(U_s, \mathcal{W}_{U_s})$ and $(\Delta^\alpha, \mathcal{W}_{\Delta}^{\Delta^\alpha})$ are affine complex equivalent, the equivalence extending continuously to a continuous map between $\Delta$ and $U_s / \{s\}$.

$\mathcal{S}(p,n)$ denotes the set of isotopy classes of admissible $\mathcal{W}$. Each $w \in \mathcal{S}(p,n)$ determines a function $A_w \in C^\infty, \sum_{s \in S_a} A_w(s) = 2p - 2$, where $A_w(s) = \alpha(s)$ above, a point $B_w \in \mathcal{S}(p,n)$ (Teichmüller space) and a monodromy character $\chi \in \text{Hom}(\pi_1(M_{p,n}, \mathbb{C}^*)$. It is proved in [4, Theorem 1.16] that the image $A \times B \times \mathcal{S}(p,n)$ is a closed complex submanifold. $\mathcal{S}(p,n)$ is endowed with the topology and complex structure of this image.

If $\xi \in \Omega(p,n)$, $\xi$ is an equivalence class $[(\mathcal{E}, \phi)], \mathcal{E} \in \Lambda(p,n), \phi \in \Pi(\mathcal{E})$. We recall that because $E \subseteq \Lambda(p,n), X(\mathcal{E}, \phi)$ carries an affine complex structure which is defined uniquely up to isotopy on $M_{p,n}$. Because $X(\mathcal{E}, \phi)$ is defined in terms of simple polygons $(P(F, \psi), F \in \mathcal{O}(\phi))$ and affine gluing maps, it is admissible [4]. Denote its class by $W(\xi) \in \mathcal{S}(p,n)$.

We remark that if $\mathcal{E}_1 \prec \mathcal{E}_2$, then $\mathcal{M}(\mathcal{E}_1)$ is naturally viewed as an open set in $\mathcal{M}(\mathcal{E}_2)$, the inclusion being holomorphic. Moreover, there is a canonical map $Z : \mathcal{M}(\mathcal{E}_1) \to \mathcal{S}(p,n)$ defined in the same way as $W$ and natural relative to inclusions $\mathcal{E}_1 \prec \mathcal{E}_2$ and $\mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_2)$. (See the second paragraph of Section 5.)

Let $\xi_0 = [(\mathcal{E}_0, \phi_0)] \in \Omega(p,n)$ and let $(\mathcal{E}_0, \phi_0)$ be the minimum representative for $\xi_0$. Let $\Omega^\ast(\mathcal{E}_0) \subseteq \Omega(p,n)$ as in Section 6, and let $m_{\mathcal{E}_0} : \Omega^\ast(\mathcal{E}_0) \to \mathcal{S}(\mathcal{E}_0)$ be the associated chart. Of course, $\xi_0 \in \Omega^\ast(\mathcal{E}_0)$. By construction we have $W = Z \circ m_{\mathcal{E}_0}$. Moreover, if $\mathcal{E}_0 \prec \mathcal{E}_1$ is such that $E_1$ is an ideal triangulation, the map $Z : \mathcal{M}(\mathcal{E}_1) \to \mathcal{S}(p,n)$ is a holomorphic injection [4]; take $t = \mathcal{E}_1, \Omega(t) = \mathcal{M}(\mathcal{E}_1)$ and $Z = \sigma_t$ in the notation of that paper. We conclude that $W|_{\mathcal{M}(\mathcal{E}_1)}$ is a holomorphic injection. As $\Omega(p,n)$ and $\mathcal{S}(p,n)$ have equal dimensions, we have

THEOREM 7.1. With notations as above the map $W : \Omega(p,n) \to \mathcal{S}(p,n)$ is holomorphic and locally biholomorphic.

It will develop that $W$ is a biholomorphism of $\Omega(p,n)$ onto $\mathcal{S}(p,n)$.

8. NO HALFPLANES

In this section $\mathcal{W}$ is a fixed admissible structure on $M_{p,n} = M_p \setminus S_a$. If $s \in S_a$, we assume fixed a deleted neighborhood $U_s \subseteq M_{p,n}$ and $\alpha = \alpha(s), \Re \alpha > -1$, such that $(U_s, \mathcal{W}_{U_s})$ and $(\Delta^\alpha, \mathcal{W}_{\Delta}^{\Delta^\alpha})$ are isomorphic (see the first paragraph of Section 7).

A $\mathcal{W}$-line is a path in $M_{p,n}$ whose parameter is affine in $\mathcal{W}$-coordinates. A $\mathcal{W}$-line $\gamma = [(a, b)], x(\cdot))$ is finite if $x(b^\ast)$ exists in $M_p(x(b^\ast) \in S_a$ is allowed).
Lemma 8.1. Assume $\mathcal{U}$ is admissible and $\gamma = ([a, b), x(\cdot))$ is a finite $\mathcal{U}$-line. Then $b < \infty$.

Proof. Extend the puncture set if necessary (and the definition of $\alpha$ by 0) so that $x(b^-) \in S_n$. It is no loss to suppose $\gamma \subseteq U_\delta$ with $x(b^-) = \delta$, and therefore we are reduced to proving that a $\mathcal{W}_\alpha$-line $\gamma = ([a, b), x(\cdot))$ in $\Delta^*$ such that $x(b^-) = 0$ has $b < \infty$. In the notation of the first paragraph of Section 7 the fact $\gamma$ is a $\mathcal{W}_\alpha$-line implies there exists $c, d \in \mathbb{C}$ and a branch of $x(t)^{s+1}$ such that $x(t)^{s+1} = ct + d$. Since $\text{Re} \alpha > -1$, the left side has $x(b^-)^{s+1} = 0$. Therefore, $([a, b), x(\cdot))$ is a finite interval and $b < \infty$.

When $\mathcal{U}$ has transitions in the group of Euclidean motions, corresponding to the case of a Riemannian flat metric with cone singularities, every $\mathcal{U}$-line $\gamma = ([a, b), x(\cdot))$ such that $b < a_\infty$ is finite. This is false in the general (admissible) case. For an elementary example let $P$ be the trapezoid formed by intersecting a triangle $T$ with a line parallel to an edge $e$ and passing through the interior of $T$. Define $\xi \in \mathcal{S}(1, 1)$ by gluing opposite sides of $P$, reversing sense, and letting the vertex class represent the puncture (in this example the affine structure extends through the puncture). For concreteness assume $T$ is equilateral, and let $L$ be the altitude of $T$ which bisects $e$. $L$ passes through infinitely many (similar) copies of $P$ and represents a nonfinite $\mathcal{U}$ line on a finite interval.

Remark 8.2. Let $\gamma \subseteq \Delta^*$ be a finite $\mathcal{W}_\alpha$-line such that $x(b^-) = 0$ ($\text{Re} \alpha > -1$). We have seen that $b < \infty$. It is also true that there exists a neighborhood $U_b$ of $b$ in $\mathcal{C}\setminus(b, \infty)$ such that $x(\cdot)$ extends to be continuous on $U_b$ and $\mathcal{W}_\alpha$-affine complex on $t\in\mathcal{C}\setminus(b)$. 

Lemma 8.3. Let $\mathcal{U}$ be admissible, let $\mathcal{H}$ be a halfplane and let $\varphi: \mathcal{H} \rightarrow M_{p,n}$ be schlicht relative to the complex structure subordinate to $\mathcal{U}$. Then $\varphi$ is not affine in $\mathcal{U}$-coordinates.

Proof. If $\varphi$ is affine, and if $L$ is a line in $\mathcal{H}$ (i.e., parallel to $\partial \mathcal{H}$), Lemma 8.1. implies $\varphi|_L$ does not extend continuously to the point at either $\infty$ of $L$. It follows there is a $\mathcal{U}$-affine disc $(U, f)$, $U \subseteq M_{p,n}$, such that $\varphi(L)$ passes through $U$ infinitely often. If $\mathcal{H}(L) \subseteq \mathcal{H}$ is the halfplane in $\mathcal{H}$ determined by $L$, then for each chord $H$ of $U$ determined by $\varphi(L)$, $\varphi(\mathcal{H}(L))$ contains one component $E$ of $U \setminus H$. Moreover, $\varphi^{-1} E$ is a semidisc based on an interval of $L$. It follows that if $\delta$ is any $\mathcal{U}$-line segment in $U$, then $\varphi(L)$ intersects $\delta$ at most twice. This is incompatible with $\varphi(L)$ passing through $U$ infinitely often, and we conclude $\varphi$ is not affine. The lemma is proved.

We shall next remove the hypothesis that $\varphi$ be one-to-one.

Lemma 8.4. Let $\mathcal{U}$ be admissible, let $\mathcal{H}$ be a halfplane and let $\varphi: \mathcal{H} \rightarrow M_{p,n}$ be affine holomorphic in $\mathcal{U}$-coordinates. Then $\varphi$ is constant.

Proof. Assume $\varphi$ is affine but not constant. By Lemma 8.3 there exist $z_1 \neq z_2$ such that $\varphi(z_1) = \varphi(z_2)$. Since $\varphi$ is nonconstant affine, $\varphi$ is locally one-to-one. There exist open sets $V_j \subseteq \mathcal{H}$, $z_j \in V_j$, $j = 1, 2$, and $Tz = az + b$, $a \neq 0$, such that $TV_1 = V_2$ and $\varphi \circ T = \varphi$ on $\mathcal{H} \setminus T^{-1} \mathcal{H} \equiv V_1$. We separate the cases $a \neq 1$ and $a = 1$.

Case 1 ($a \neq 1$). If $z_0 = b/(1 - a)$, then $Tz_0 = z_0$. We may vary $z_1, z_2$, if necessary, and suppose the triangle $\Delta(z_0, z_1, z_2)$ is not degenerate. The facts that $\varphi$ is locally one-to-one and $\varphi \circ T = \varphi$ imply $z_0 \notin \mathcal{H}$. Let $L_j \subseteq \mathcal{H}$ be the ray $L_j = \{(1 - t)z_0 + tz_j | t \geq 1\}$. With $T$ as above the facts $Tz_0 = z_0$ and $Tz_1 = z_2$ imply $TL_1 = L_2$. Let $K$ be the convex hull of $L_1$ and $L_2$ and let $V = K/T$ be the Riemann surface got by using $T$ to glue $L_1$ to $L_2$. $V$ is
biholomorphic to a punctured disc: if \( e' = a \), then \( \psi(z) = e^{2\pi i \log(z - z_0)} \), \( z \in \mathcal{H} \), maps \( V \) biholomorphically to a neighborhood of \( \infty \) with

\[
\lim_{z \to \infty} \psi(z) = \infty.
\]

Now \( V \) and \((M_{p,n}, \mathcal{U})\) are hyperbolic Riemann surfaces (in this regard we recall that the complex structure subordinate to \( \mathcal{U} \) extends through \( S_\kappa \) and \( 2p - 2 + n > 0 \)). It follows that \( \phi: V \to M_{p,n} \) extends to the point at \( \infty \), taking on a value at \( \infty \) which may be in \( S_\kappa \). In particular, \((L_\kappa, \phi)\) is a finite \( \mathcal{U} \)-line. We have reached a contradiction.

Case 2 (\( a = 1 \)). In this case let \( L_j \) be rays in \( \mathcal{H} \) such that \( L_1 \cap L_2 = \emptyset \) and \( TL_1 = L_2 \). Form \( K \) and \( V \) and argue as in the case above.

We have reached a contradiction to the assumption that \( \phi \) is nonconstant affine and the lemma obtains.

9. \( \mathcal{U} \)-DISCS ARE FINITE

By a \( \mathcal{U} \)-disc we shall understand an open disc \( D \) of finite radius and an affine complex map \( f: D \to M_{p,n} \) relative to a given admissible structure \( \mathcal{U} \). We shall see that, in contrast to the situation for \( \mathcal{U} \)-lines \( \gamma = ([a, b), x(\cdot)) \), \( b < \infty \) it is the case that every \( \mathcal{U} \)-disc is finite, in the sense that \( f \) extends continuously to \( \overline{D} \). Moreover if \( f \) is the extended function, \( f^{-1} S_\kappa \) is a finite set in \( \partial D \).

In preparation of Proposition 8.1. we make some assumptions and notations. \( \mathcal{U} \) is admissible, and \((D, f)\) is a \( \mathcal{U} \)-disc with \( D \) centered at 0. If \( w \in \partial D \), \( \gamma_w \) is the \( \mathcal{U} \)-line \( \gamma_w = ([0, 1), x(\cdot) = f^w(\cdot)) \). When \( \gamma_w \) is finite \( x(\cdot) \) extends to a neighborhood of \( t = \text{im} \ C \setminus (1, \infty) \). In particular, \( \gamma_w \) is also finite for all \( w' \in \partial D \) in a neighborhood of \( w \). We conclude that if \( \gamma_w \) is finite, then \( f \) extends to be continuous on a neighborhood of \( w \) in \( D \).

**Proposition 9.1.** With notations as above every \( \mathcal{U} \)-disc \((D, f)\) is finite.

**Proof.** If the statement is false, the discussion proceeding the proposition implies there exists \( w \in \partial D \) such that \( \gamma_w \) is not finite. We shall prove this leads to a contradiction.

First, when \( \gamma_w \) is not finite, the cluster set of \([x, (t)] \mid t \to 1\) is connected and contains at least two points. We may therefore select an element \( x \) in this set such that \( x \notin S_\kappa \). Fix \((U, h) \in \mathcal{U} \) such that \( x \in U \) and \( h(U) \) is a disc centered at \( h(x) \). Let \( \frac{1}{2} U \) denote the pullback by \( h \) of the concentric disc with one-half the radius of \( h(U) \). If \( y \in \frac{1}{2} U \), there exists an open set \( V_y \subseteq U \) such that \( x, y \in V_y \) and \( h(V_y) \) is a disc with center \( h(y) \).

Let a sequence \( 0 < t_k < t_{k+1} < 1, \ t_k \to 1 \), be chosen so that \( \gamma_k = x_w(t_k) \in \frac{1}{2} U \), \( k \geq 1 \). Denote by \( D_k \) the disc centered at \( t_kw \) which is internally tangent to \( D \) (at \( w \)). We claim that by definition of \( V_y \) we have \( V_y \subseteq f(D_k) \). To see this we first observe that on the component \( W_k \) of \( D \cap f^{-1} U \) which contains \( t_kw \) the "transition" \( h = f \) has the form \( T_k z = a_k z + b_k, a_k \neq 0 \). Now \( h^{-1} |_{T_k \gamma_k} = f^{-1} |_{T_k \gamma_k} \) cannot be continued to any disc choose center is \( h(y_k) \) and which contains \( T_k w \) in its interior. For otherwise \( (h^{-1} |_{T_k \gamma_k}) = T_k = f^{-1} |_{T_k \gamma_k} \) would continue to a disc centered at \( t_kw \) which contains \( w \) in its interior. This is impossible, and \( V_y \subseteq f(D_k) \), as claimed.

It follows from the last paragraph that the component \( W_k \) of \( D \cap f^{-1} U \) which contains \( t_kw \) also contains \( T_k^{-1} h(V_k) \). Recalling that \( x \in V_y \), define \( \zeta_k = T_k^{-1} h(x) \), so that \( f(\zeta_k) = x \), and let \( E_k \) be the largest disc with center \( \zeta_k \) to which \( f \) admits analytic continuation. As \( \zeta_k \in D_k \) and \( w \) cannot be interior to \( E_k \), we have \( \lim_{k \to \infty} \text{radius } (E_k) = 0 \).
By construction $T_k E_k$ is a disc centered at $T_k \zeta_k = h(x)$, and the relations between $T_k$, $h$ and $f$ imply $T_k E_k$ is independent of $k$. In terms of $T_k z = a_k z + b_k$, the fact the radius of $E_k$ tends to zero implies $a_k \to \infty$. As $h^{-1} \circ T_k = f$ on $E_k$, $h^{-1}$ admits analytic continuation to $h(U) \cup T_k D$ for all $k$. As $\bigcap_{k \geq 1} T_k D$ contains $h(x)$, $h^{-1}$ admits analytic continuation to $\bigcup_{k \geq 1} T_k D = \Omega$. The fact $a_k \to \infty$ implies $\Omega$ contains a halfplane. As $h^{-1} : \Omega \to M_{p,n}$ is affine holomorphic, Lemma 8.4. implies $h^{-1}$ is constant. This contradicts the fact $(U, h)$ is a chart, and the proposition follows.

10. DELAUNAY PARTITION ASSOCIATED TO $\mathcal{U}$

$\mathcal{U}$ denotes a fixed admissible affine complex structure. If $x \in M_{p,n}$, we can speak of a maximal $\mathcal{U}$-disc $(D_x, f)$ at $x$: Begin with any $\mathcal{U}$-disc $(D, f)$ such that $f^{-1} x$ is the center of $D$, and let $D_x$ be the union of all concentric discs to which $f$ admits analytic continuation. Since $f$ is not constant, $D_x$ is a finite disc. This defines a $\mathcal{U}$-disc $(D_x, f_x)$.

Proposition 9.1. implies $f_x$ admits a continuous extension of $f$. Remark 8.2 implies that if $z \in \partial D_x$, $z$ has a deleted neighborhood in $\partial D_x$ which is disjoint from $f^{-1} S$. Therefore, if $v(x) = \operatorname{Card}(f_x^{-1} S)$, $v(x) < \infty$. The discussion in the preceding paragraph implies $v(x)$ depends only upon $\mathcal{U}$ and $x$.

With notations as above let $\zeta \in f_x^{-1}(S)$, and let $\rho(\zeta)$ be the open ray which connects $\zeta$ to $\infty$ in $\mathbb{C} \setminus D_x$ and is collinear with the radius of $D_x$ at $\zeta$. Define $U = \mathbb{C} \setminus \bigcup_{\zeta \in f_x^{-1} S} \rho(\zeta)$. If $\varepsilon > 0$, and if $r$ is the radius of $D_x$, $(1 + \varepsilon) D_x$ will denote the concentric disc of radius $(1 + \varepsilon) r$.

Remark 8.2. implies there exists $\varepsilon > 0$ such that if $U_\varepsilon = (1 + \varepsilon) D_x \cap U$, then $f_x$ admits a continuous extension to $U_\varepsilon$ which is analytic on $U_\varepsilon \setminus f_x^{-1} S$. (By definition $\zeta \notin \rho(\zeta)$ so that $f_x^{-1} S \subset U_\varepsilon$.)

**Lemma 10.1.** If $x \in M_{p,n}$, then $v(x) > 0$.

**Proof.** If $v(x) = 0$, then $U_\varepsilon = (1 + \varepsilon) D_x$, and $D_x$ is not the largest disc upon which $f_x$ exists. Therefore, $v(x) > 0$.

**Lemma 10.2.** The set $v^{-1}(1)$ is an open set which contains a deleted neighborhood of $s$ for each $s \in S$.

**Proof.** For convenience assume $D_x$ is centered at 0. Let $U_\varepsilon$ be defined above, and let $(\zeta_0) = f_x^{-1} S$. If $|z| < \varepsilon/2$, the disc $D(z)$ with center $z$ and radius $|z - \zeta_0|$ satisfies $D(z) \subset U_\varepsilon$. Then $f(z) = f|_{D(z)} D(z) = D(z)$ give a version of $(D(f(z)), f(z))$. Therefore, $v(f(z)) = 1$, $|z| < \varepsilon/2$. As $f$ is an open map, $x$ is interior to $v^{-1}(1)$. Consideration of the models $(A^*, \mathcal{U}_{A^*}), \Re \varepsilon > -1$, defined in Section 7 establishes the fact $v^{-1}(1)$ contains a deleted neighborhood of each $s \in S$. The lemma is proved.

**Lemma 10.3.** If $v(x) = 2$, there exists a $\mathcal{U}$-disc $(D_x, f)$ centered at $x$ (i.e., $f^{-1}(x)$) such that $f^{-1} \circ v^{-1}(2)$ is a diameter of $D$.

**Proof.** Let $f_x^{-1} S = \{\zeta_0, \zeta_1\}$, $\zeta_0 \neq \zeta_1$, and let $L$ be the diameter of $D_x$ which bisects the chord $[\zeta_0, \zeta_1]$. If $U_\varepsilon$ is as above, and if $|z| < \varepsilon/2$, define $D(z)$ to be the disc centered at $z$ of
radius \min(|x - \zeta_0|,|x - \zeta_1|). Then \((D(x), f_\lambda)|_{\partial D(x)}\) is a maximal \(\mathcal{W}\)-disc at \(f(x)\) and \(v(f(x)) = 2\) if, and only if, \(z \in L\).

**Lemma 10.4.** If \(x \in M_p,\) there exists a \(\mathcal{W}\)-disc \((D, f)\) centered at \(f^{-1}x = 0\) such that \(f^{-1} \circ v^{-1}\{3, 4, \ldots\} \subseteq \{0\}\). If \(v(x) > 2\), then \(f^{-1} \circ v^{-1}(2)\) is a union of \(v(x)\) radii of \(D\).

**Proof.** Let \(D_x\) be centered at 0, and let \(U_x\) be as before. We may suppose \(v(x) > 2\). If \(|z| < \varepsilon/2\), define \(D(z)\) to be the disc with radius \(\text{dist}(z, f^{-1}S_x)\). Then \(D(z) \subseteq U_x\) and \(\partial D(z)\) contains more than two points of \(f^{-1}S_x \subseteq \partial D_x\) if and only if \(D(z) = D_x\) and \(z = 0\). It is clear that \(f_{x}^{-1}v^{-1}(2) \cap \{|z| < \varepsilon/2\}\) is a union of open radii, one for each edge of \(w(f^{-1}S_x)\), i.e., one for each adjacent pair in \(f_x^{-1}S_x\). The lemma is proved.

Let \(G = G(\mathcal{W}) = \{x | v(x) > 2\}\). Thus far we known that \(G\) is a finite set. Indeed, Lemma 10.3 implies the set \(\{v > 1\}\) is compact in \(M_p,\), and Lemma 10.4 implies it has an open cover each element of which contains at most one point at which \(v > 2\).

**Lemma 10.5.** Each component of \(v^{-1}(2)\) in \(v^{-1}\{(2, 3, \ldots)\}\) is a finite \(\mathcal{W}\)-line.

**Proof.** Lemma 10.3 implies each component of \(v^{-1}(2)\) is a \(\mathcal{W}\)-line. Fix a component \(1\) and use analytic continuation of a \(\mathcal{W}\)-chart \(\phi\) to realize \(\lambda\) as a line \(L \subseteq C\). Each \(z \in L\) is centered at a maximal disc \(D(z)\) to which \(\phi^{-1}\) admits analytic continuation. The proof of Lemma 10.3 shows that there is a fixed line segment \(\delta\) such that for each \(z \in L\) \(D(z)\) contains \(\delta\) as a chord. If \(\Omega = \bigcup_{z \in L} D(z)\), then \(\phi^{-1}\) admits analytic (affine) continuation to a map \(f: \Omega \to M_p,\). If \(L\) is infinite at either end, then \(\Omega\) contains a halfplane. Since \(f\) is not constant, Lemma 8.4 implies \(L\) is finite. The lemma is proved.

The lemma implies that if \(\lambda \neq \emptyset\) is a component of \(v^{-1}(2)\), then \(\lambda\) is a \(\mathcal{W}\) line joining points \(x_1, x_2 \in G(\mathcal{W})\), possibly with \(x_1 = x_2\). We shall look at this more closely below.

**Lemma 10.6.** \(v^{-1}(2) \neq \emptyset\), and, in particular \(G(\mathcal{W}) \neq \emptyset\).

**Proof.** Let \(x \in M_p,\). If \(v(x) > 1\), there is nothing to prove. If \(v(x) = 1\), let \(\zeta_0 \in \partial D_x\) be such that \(f_\lambda(\zeta_0) \in S_x\) and let \(\rho(\zeta_0)\) be the complement of the ray (defined in paragraph three) \(\rho(\zeta_0)\) in the line which it determines. If \(z \in \rho(\zeta_0)\), let \(D(z)\) be the disc with radius \(|z - \zeta_0|\). If \(z\) lies on the radius determined by \(\zeta_0\), then \(D(z) \subseteq D_x\) and \((D(z), f_\lambda)|_{\partial D(z)}\) is a maximal \(\mathcal{W}\)-disc. Let \(\ell \subseteq \rho(\zeta_0)\) be the maximal segment such that \(f_\lambda\) admits continuation to \(\bigcup_{z \in \ell} D(z)\). If \(\ell\) is finite, and if \(\zeta_1 \neq \zeta_0\) is an endpoint of \(\ell\), \(f_\lambda(\zeta_1)\) must have \(v(f_\lambda(\zeta_1)) > 1\) if \(\ell\) is infinite, there exists a nontrivial \(\mathcal{W}\)-halfplane, contradicting Lemma 8.4. The lemma follows.

We shall now extend the Masur–Smillie construction [3] to \(\mathcal{W}\). For each \(x \in G(\mathcal{W})\) select a maximal \(\mathcal{W}\)-disc \((D_x, f_x)\) with center \(f_x^{-1}(x)\), and define a nondegenerate convex \(v(x)\)-gon \(P(x) = P(x, f_x)\) by \(P(x) = \text{conv}(f_x^{-1}S_x)\). Let \(E\) be the set consisting of the oriented (counterclockwise) edges of \(P(x), x \in G,\) and define \(\sigma\) on \(E\) by counterclockwise succession. We shall define \(\tau\) and \(\eta = \tau\sigma\) below.

Let \(x \in G,\) and let \(\gamma\) be an edge of \(P(x),\) \(\gamma\) determines two arcs of \(\partial D_x,\) and we let \(J(\gamma)\) be the arc which is free of vertices of \(P(x)\). The ray from the center of \(D_x\) which bisects \(J(\gamma)\) contains a maximal open initial segment \(L\) to which \(f_x\) extends with \(v(f_x(z)) = 2\). (See the
proof of Lemma 10.4.) For each \( z \in L \) if \( D(z) \) is the open disc with center \( z \) having \( y \) for a chord, then \( f_z \) admits continuation to the union \( \bigcup_{z \in L} D(z) = \Omega \). Denote the continuation by \( f \). If \( z_1 \) is the endpoint of \( L \) which is not the center of \( D_z \), then \( \partial D(z_1) \) contains at least three preimages of points of \( S_n \). Then \( (D(z_1), f|_{D(z_1)}) \) is a maximal \( \mathcal{W} \) disc centered at \( z_1 \) with \( f(z_1) \in G(\mathcal{W}) \). Let \( Q(z_1) = \text{co}(f^{-1} S_n \cap \partial D(z_1)) \). If \( x_1 = f(z_1) \), then \( Q(z_1) \) is canonically complex affine identified with \( P(x_1) \). As \( f^{-1} S_n \cap \Omega = \emptyset \), \( P(x) \) and \( Q(z_1) \) lie on opposite sides of \( y \). Let \( \tau \gamma \) be the edge of \( P(x_1) \) which is determined by \( y \) and \( Q(z_1) \).

**Lemma 10.7.** \( \tau \) is fixed point free involution of \( E \).

**Proof.** It is clear that \( \tau^2 = \gamma \). We must prove that if \( x_1 = x \), then \( \tau \gamma \neq y \). In all cases there exists \( Tz = az + b \) such that \( TD_x = D(z_1) \) and \( f \circ T = f_x \) on \( D_x \cap T^{-1} D(z_1) \). Since \( f \) is defined on \( D_x \cap D(z_1) \), and \( f|_{D(z_1)} = f_{x_1} \), we can express this relation as \( f \circ T = f \) on \( D_x \cap T^{-1} D(z_1) \). The fact \( D(z_1) \neq D_x \) implies \( T \neq \text{Id} \). Now to say \( \tau \gamma = \gamma \) is to say \( \tau \gamma = \gamma \), and therefore \( T \) fixes the midpoint of \( y \). Since \( f \circ T = f \) and \( f \) is locally injective, \( T = \text{Id} \). This is a contradiction, and we have \( \tau \gamma \neq y \).

Define \( \delta(\mathcal{W}) = \mathcal{E} = (E, \sigma, \tau, \eta) \), \( \eta = \tau \sigma \). We do not assume \( \langle \sigma, \tau, \eta \rangle \) is transitive on \( E \). Since each polygon \( P(x), x \in G \), is inscribed in a circle, there is defined on \( E \) a function \( \psi(\cdot, \cdot) = \psi(\mathcal{E}, \cdot) \) such that \( \psi_x \equiv 1 \). The definition of \( \tau \) and Lemma 2.4 imply \( \psi_x < 1 \). Therefore, \( \psi \in \Pi_0(\delta) \).

Let \( X(\mathcal{E}, \psi) \) be the standard realization of \( (\mathcal{E}, \psi) \) (Section 2). The maps \( f_x : P(x) \to M_{p,n} \) and the relations \( f_x \circ T = f_x \), associated to edges \( \gamma, \tau \gamma \) of \( P(x_1) \), \( P(x_2) \) imply there is a canonical affine holomorphic map

\[
f : X(\mathcal{E}, \psi) \to (M_{p,n}, \mathcal{W}).
\]

(17)

We remark that since \( \langle \sigma, \tau, \eta \rangle \) has not yet been proved transitive, \( X(\mathcal{E}, \psi) \) is not yet known to be connected. This will be established once \( f \) is proved to be a biholomorphism.

**Theorem 10.8.** With notations as above \( f \) is a biholomorphism.

**Proof.** \( f \) is locally nonconstant and holomorphic, and therefore \( f(X) \) is open and closed in \( M_p \). It follows \( f \) is onto. Let \( y_1, y_2 \in X(\mathcal{E}, \psi) \) be such that \( x = f(y_1) = f(y_2) \). Assume \( x \notin S_n \). Choose \( x_1, x_2 \in G \) such that \( P(x_i) \) contains \( z_j \) with \( f_{x_i}(z_j) = x_i \), \( i = 1, 2 \). There exists \( Tz = az + b \) such that \( Tz = z_2 \) and \( f_{x_1} \circ T = f_{x_2} \) on \( D_{x_1} \cap T^{-1} D_{x_2} \). Let \( \Omega = D_{x_2} \cup T D_{x_1} \), and define \( h \) on \( \Omega \) by

\[
h(z) = \begin{cases} f_{x_2}(z) & z \in D_{x_2} \\ f_{x_1} \circ T^{-1}(z) & z \in TD_{x_1}. \end{cases}
\]

If \( x_1 = x_2 \), and if \( TD_{x_1} = D_{x_1} \), then \( f_{x_1} \circ T = f_{x_2} \) on \( D_{x_1} \). Since \( T \) fixes the center of \( D_{x_1} \), and \( f \) is locally one-to-one, \( T = \text{Id} \) and \( z_1 = z_2 \), \( y_1 = y_2 \). Now suppose \( x_1 \neq x_2 \), or if \( x_1 = x_2 \), \( T \neq \text{Id} \). The fact \( h^{-1} S_n \cap \Omega = \emptyset \) implies \( P(x_0) \) and \( TP(x_0) \) lie on opposite sides of a chord common to \( D_{x_2} \) and \( TD_{x_1} \). The line \( L \) of centers between \( D_{x_1} \) and \( TD_{x_1} \) determines a component of \( v^{-1}(2) (z = h(L)) \), and therefore \( y_1, y_2 \) lie on the image of an edge of \( P(x_1) \) under the canonical projection. The proof that \( \tau \) is fixed point free implies that \( f_{x_1} \) is one-to-one on each edge of \( P(x_1) \), and the lemma follows. \( \square \)
Remark 10.9. Define $f_a$, $a \in C$, by

$$f_a(z) = \begin{cases} \frac{e^{a-1}}{z} & a \neq 0 \\ z & a = 0. \end{cases}$$

If $\Lambda \subseteq C$ is a lattice, $f_a$ determines an affine complex structure on the torus $C/\Lambda$, and we declare $[0]$ to be a cone point, albeit with trivial cone angle $2\pi$. A Delaunay decomposition of $C/\Lambda$ with vertex set $[0]$ determines a fundamental domain $U$ for $\Lambda$ and a simple quadrilateral $Q$ such that $f: U \to Q$ is schlicht. A diagonal of $Q$ determines a pair of triangles, $T_1$ and $T_2$, such that any pair of opposed angles (in $C/\Lambda$) have sum at most $\pi$. Let $T_1$ be marked by a counterclockwise order of edges $x_1, x_2, x_3$. Let $\Delta(0, 1, \zeta)$, $\Im \zeta > 0$, be the similar triangle with $x_1$ corresponding to $[0, 1]$, and let $\theta_0, \theta_1$ denote the angles of $\Delta$ at $0$ and $1$. Let $J$ be the arc of the circumscribed circle which lies in the lower halfplane, and let $\mathcal{O}(\zeta)$ be the region in the lower halfplane which is bounded by $J$ and by the rays $R^{-e^{i\theta_1}} = R^{-e^{i\theta_0}}$ and $1 + R^{-e^{-i\theta_1}}$. The map $S(c) = \zeta(1 - z)/(\zeta - z)$ sends $\mathcal{O}(\zeta)$ to $T_1$, preserving angles. Moreover, if the chart $T_1 \to \Delta(0, 1, \zeta)$ is continued across $x_1$, $T_2$ is mapped to $\Delta(0, 1, \zeta)$, and we can encode $(T_1, T_2)$ as a pair $(\zeta, S(c)) \in \{\zeta \times \Delta(0, 1, \zeta)\}$.

A finite branched cover of $\Delta(1, 1)$ therefore occurs as the "universal triangle" over the space $\mathcal{X}^+$ of similarity classes of triangles $\Delta(0, 1, \zeta)$.

11. $W$ is a biholomorphism

In the last section we associated to an admissible affine complex structure $\mathcal{U}$ on $M_{p,n}$ a $P$-set $\mathcal{F}(\mathcal{U})$ and $\psi(\mathcal{U}, \cdot) : P_0(\mathcal{U}) \to \Omega(p,n)$. The set $E = E(\mathcal{U})$ is a set of Jordan paths joining punctures, and (the projection of) $E$ determines a set $E \subseteq \Lambda(p,n)$. The fact $f:X(\mathcal{F}, \psi) \to M_p$ is a homeomorphism implies $E$ is admissible. The pair $(\mathcal{F}(\mathcal{U}), \psi(\mathcal{U}, \cdot))$ belongs to $\Omega_0(p,n)$, in fact to $\{\mathcal{F}\} \times \Pi_0(\mathcal{F})$.

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are admissible and isotopic, then as elements of $\Lambda(p,n)$ $E(\mathcal{U}_1) = E(\mathcal{U}_2)$. Moreover, $\psi(\mathcal{U}_1, \cdot) = \psi(\mathcal{U}_2, \cdot)$. In this way we have constructed a map $Z$,

$$Z: \mathcal{F}(p,n) \to \Omega(p,n)$$

(i.e., project $(\mathcal{F}(\mathcal{U}), \psi(\mathcal{U}, \cdot))$ to $\Omega(p,n)$). Also, since $f:X(\mathcal{F}, \psi) \to (M_{p,n}, \mathcal{U})$ is an affine biholomorphism (remove the vertex classes from $X(\mathcal{F}, \psi)$), the definition of $W$ in Section 7 implies

$$W \circ Z = \text{Id.} \quad (18)$$

Since $W$ is locally a holomorphic injection, (18) implies $Z(\cdot)$ is a global holomorphic injection.

Lemma 11.1 With notations as above

$$Z \circ W = \text{Id.} \quad (19)$$

Proof. Let $[\mathcal{F}, \psi] \in \Omega_{p,n}$, and let $(\mathcal{F}, \psi)$ be a minimal representative. The polygons $P(F, \phi), F \in \mathcal{O}(\sigma)$, used to define $X(\mathcal{F}, \psi)$ are inscribed in circles $C_F$. Lemma 2.5 implies that for each such circle the canonical projection of $P(F, \phi)$ in $X(\mathcal{F}, \psi)$ extends to a holomorphic map $f_F : D_F \to X(\mathcal{F}, \psi)$, where $C_F = D_D$. Let $\mathcal{V}$ be the subordinate admissible affine complex structure on $X(\mathcal{F}, \psi) \setminus \{\text{vertex classes}\}$. If $y_F = f_F(z_F), z_F$ the center of $D_F$, then $y_F \in \mathcal{G}(\mathcal{V})$ and
(\(D_F, f_F\)) = (D_g, f_g) \) (Section 10). As the images of \(f_F(p(F, \phi))\) exhaust \(X(\delta, \phi)\), \(G(\gamma') = \{y_F | F \in \mathfrak{O}(\sigma)\}\). As noted in Section 7, \(\gamma'\) determines a class in \(\mathfrak{P}(p, n)\). The identification of \(G(\gamma')\) implies the Delaunay partition associated to \([\gamma']\) is \(E\). Therefore, \(Z([\gamma]) = [(\delta, \phi)]\). As \(W([((\delta, \phi)]) = [\gamma']\) we have \(Z \circ W = \text{Id}\), as claimed. 

Collecting results, we have

**Theorem 11.2.** The maps \(W\) and \(Z = W^{-1}\) determine a biholomorphism between \(\Omega(p, n)\) and \(\mathfrak{P}(p, n)\).

**12. A Chart on \(\Pi(\phi)\)**

Fix a \(\phi\)-set \(\phi\). If \(E \in G(\delta)\), we use \(W_E\) to denote the space of real \(\xi\)-invariant elements of \(E\). \(W_\xi\) is spanned by the characteristic functions \(\chi_u, u \in \mathfrak{O}(\xi)\), and \(\dim W_\xi = n(\xi)\). As usual \(\delta = (E, \sigma, \tau, \eta)\).

**Lemma 12.1.** Define \(W = W_\sigma + W_\tau = W(\delta)\). Then \(\dim W = n(\sigma) + n(\sigma) - 1\).

**Proof.** The linear relation

\[
\sum_{u \in \mathfrak{O}(\xi)} Y_u = \sum_{F \in \mathfrak{O}(\sigma)} Y_F
\]

implies the dimension is at most \(n(\sigma) + n(\tau) - 1\). Now suppose \(\sum_{u \in \mathfrak{O}(\xi)} a_u \chi_u + \sum_{F \in \mathfrak{O}(\sigma)} b_F \chi_F = 0\). We shall prove \(a_u = - b_F\) for all \(u \in \mathfrak{O}(\tau)\) and \(F \in \mathfrak{O}(\sigma)\). To this end fix \(c \in \mathbb{R}\) and let \(Z_c = \{F \in \mathfrak{O}(\sigma) | b_F = c\}\) be nonempty. If \(F \in Z_c\) and \(x \in F\), and if \(u = \{x, \tau x\}\), then necessarily \(a_u = - b_F\). If \(\tau x \in F\), then \(b_F = - a_u = b_F\). It follows that \(E_\xi = \bigcup_{F \in Z_c} F\) is invariant under \(\sigma\) and \(\tau\). Therefore, \(E_\xi = E\) when \(Z_c \neq \emptyset\). It follows \(a_u = - c\) for all \(u \in \mathfrak{O}(\tau)\), and the lemma obtains. \(\square\)

Let \(\psi = \sum_{x \in E} \psi(x)\psi(x)\) be the standard inner product. We define \(V = V(\phi) \subseteq \mathbb{R}^E\) by

\[
V(\phi) = W(\phi)\perp.
\]

Lemma 12.1 implies \(\dim V = 2n(\tau) - (n(\sigma) + n(\tau) - 1) = n(\tau) - n(\sigma) + 1\). We assume \(n(\eta) = n\) and \(\chi(\phi) = 2 - 2p, 2p - 2 + n > 0\). In terms of \(p, n\) we have

\[
\dim V(\phi) = 2p - 1 + n.
\]

Associate to \(\phi \in \Pi(\delta)\) a positive character \(m(\phi, \cdot)\) on \(V\), defined for \(\mu \in V\) by

\[
m(\phi, \mu) = \prod_{x \in E} (\sin \pi \phi(x))^{m(x)}.
\]

Since \(0 < \phi(x) < 1, x \in E\), we can define (22) using principal branches of the exponentials. Define \(P(\phi) \subseteq W_\tau\) to be the image of \(\Pi(\phi)\) under \(\varphi \rightarrow \varphi_v\). \(P_0(\phi)\) denotes the corresponding image of \(\Pi_0(\delta)\). In Section 16 we will give a concrete description of \(P(\phi)\) and \(P_0(\phi)\). Let \(\mathbb{R}^+ = (0, \infty)\).

**Theorem 12.2.** Let \(H : \Pi(\phi) \rightarrow P(\phi) \times \text{Hom}(V, \mathbb{R}^+)\) be defined by \(H(\phi) = (\phi_v, m(\phi, \cdot))\). Then \(H(\cdot)\) is a homeomorphism (onto). \(H|_{\Pi_0(\phi)}\) is a real analytic diffeomorphism onto \(P_0(\phi) \times \text{Hom}(V, \mathbb{R}^+)\).
The proof of the theorem will occupy the rest of this section. If $\varphi \in \Pi(\mathcal{E})$, define $U(\varphi)$ to be the polyhedron

$$U(\varphi) = (\varphi + V) \cap \Pi(\mathcal{E})$$

(23)

where $V = V(\mathcal{E})$. If $\psi \in \varphi + V$, then $\psi_x \equiv \varphi_x$ and $\psi_t \equiv \varphi_t$. Therefore, $U(\varphi)$ can be characterized by $U(\varphi) = (\varphi + V) \cap \mathcal{Q}$, $\mathcal{Q} = (0,1)^{\mathcal{F}}$, and by $U(\varphi) = \{ \psi \in \Pi(\mathcal{E}) | \psi_t = \varphi_t \}$.

**Lemma 12.3.** If $\varphi \in \Pi_0(\mathcal{E})$, then $m|_{U(\varphi)}$ is a real analytic diffeomorphism of $U(\varphi)$ onto $\text{Hom}(V, \mathbb{R}^+)$. 

**Proof.** We shall prove that $m|_{U(\varphi)}$ is (i) an immersion and (ii) proper. Together, (i) and (ii) establish the lemma.

Proof of (i). Let $\psi \in U(\varphi)$. If $\lambda \in V$, the directional derivative of $m|_{U(\varphi)}$ at $\psi$ in direction $\lambda$ is denoted $m_{\lambda}(\psi, \lambda)$ and is computed to be

$$m_{\lambda}(\psi, \mu) = \pi m(\psi, \mu) \sum_{x \in \mathcal{E}} \lambda(x) \mu(x) \cot \pi \psi(x).$$

(24)

If $m_{\lambda}(\psi, \lambda) \equiv 0$, then $m_{\lambda}(\psi, \lambda) = 0$. The terms $x$ and $\tau x$ in (24) contribute the value

$$\pi m(\psi, \lambda) \lambda^2(x) (\cot \pi \psi(x) + \cot \pi \psi(\tau x)) = \pi m(\psi, \lambda) \lambda^2(x) \frac{\sin \pi \psi(x)}{\sin \pi \psi(\tau x)}. $$

In order that this be zero for all $x$ it is necessary, because $0 < \psi_t(x) < 1$, that $\lambda(x) \equiv 0$. We have proved that $Dm|_{U(\varphi)}$ is injective for all $\psi$, and as $\dim U(\varphi) = \dim V = \dim \text{Hom}(V, \mathbb{R}^+)$, $m|_{U(\varphi)}$ is open and a local diffeomorphism.

Proof of (ii). If $\psi = \varphi + \lambda \in U(\varphi)$, then $-1 < \lambda < 1$. In particular, $U(\mathcal{E})$ is a compact polyhedron in $\varphi + V$. Let $\psi \in \partial U(\varphi) = \overline{U(\varphi)} \setminus U(\varphi)$. As $\psi_x \equiv 1$ and $\psi_t \equiv \varphi_t < 1$, the fact $\psi \not\in \Pi(\mathcal{E})$ implies $\psi^{-1}(0) \not\equiv 0$. Let $\psi = \varphi + \lambda \varphi$, and observe that $\lambda \varphi < 0$ on $\psi^{-1}(0)$. Choose a compact neighborhood $D(\psi)$ of $\psi$ in $\overline{U(\varphi)}$ such that if $\psi_1 \in D(\psi)$, then $\psi_1^{-1}(0) \subseteq \psi^{-1}(0)$. This is possible because $\psi$ has a positive lower bound on $E \setminus \psi^{-1}(0)$. We observe that on $D(\psi)$ the function

$$m(\psi_1, - \lambda \varphi) = \prod_{x \in \mathcal{E}} (\sin \pi \psi_1(x))^{-\lambda \varphi(x)}$$

is continuous and vanishes on $D(\psi) \cap \partial U(\varphi)$. Choose a finite set $\{ \psi_j : 1 \leq j \leq N \} \subseteq \partial U(\varphi)$ such that $\partial U(\varphi) \subseteq \bigcup_{j=1}^{N} D(\psi_j)$. Now map $U(\varphi)$ to $(\mathbb{R}^+)^N$ by $\psi \to \{ m(\psi, - \lambda \varphi) \}_{1 \leq j \leq N}$. This map is clearly proper, and therefore $m|_{U(\varphi)}$ is proper. The lemma is proved.

**Remark 12.4.** Let $\varphi \in \Pi(\mathcal{E})$ have reduced form $(\mathcal{E}_0, \varphi_0)$. It is easy to see that $V(\mathcal{E})|_{\mathcal{E}_0} = V(\mathcal{E}_0)$, and the map $\varphi + \lambda \to \varphi|_{\mathcal{E}_0} + \lambda|_{\mathcal{E}_0}$ is an affine isomorphism of $U(\varphi)$ onto $U(\varphi_0)$. The fact $\sin \pi \psi(x)/\sin \pi \psi(\tau x) - 1$ when $\psi_t(x) - 1$ implies $m(\psi, \lambda) = m(\psi, - \lambda)$. Therefore, $m|_{U(\varphi)}$ is a diffeomorphism onto $\text{Hom}(\mathbb{R}^+)$ even for $\varphi \in \Pi(\mathcal{E}) \setminus \Pi_0(\mathcal{E})$.

**Proof of Theorem 11.5.** If $H(\varphi') = H(\varphi'')$, then $\varphi'_t = \varphi''_t$ and $\varphi'' \in U(\varphi')$. The fact $m(\varphi', \cdot) = m(\varphi'', \cdot)$ implies $\varphi' = \varphi''$. Therefore, $H(\cdot)$ is one-to-one. Any compact set $K \subseteq \mathcal{P}(\mathcal{E}) \times \text{Hom}(V, \mathbb{R}^+)$ is contained in a compact rectangle $K_1 \times K_2$, and one sees easily.
that \( H^{-1}(K_1 \times K_2) \) is compact in \( \Pi(\mathcal{E}) \). Therefore, \( H \) is proper and \( H \) is a homeomorphism. If \( \varphi \in \Pi_0(\mathcal{E}) \), then \( \ker DH_0 \subseteq V \) because the projection of \( P_0(\mathcal{E}) \) has this property. However, \( D_{\varphi}\mid_{\mathcal{E}} \) is injective on \( V \), and therefore \( H \) is an immersion. We conclude that \( H_{|_{\mathcal{E}}(\Omega)} \) is a real analytic diffeomorphism onto \( P_0(\mathcal{E}) \times \text{Hom}(V, \mathbb{R}^+) \).

\[ \square \]

13. THE HOLONOMY CHARACTER

For convenience of notation we suppose \( \mathcal{E} \in \Lambda(p, n) \). \( E \) is an isotopy class of cell decompositions of \( M_p \) with vertex set \( S_p \). \( \Gamma = \Gamma(E) \) denotes the graph which is dual to \( E \). \( \Gamma \) has one vertex for each face, i.e., each \( F \in \mathcal{C}(\sigma) \), and for each \( x \in F \) there is an edge of \( \Gamma \) which crosses \( \{x, tx\} \) joining the nodes \( \partial_x(x) \) and \( \partial_x(tx) \).

Let each edge \( \gamma \) of \( \Gamma \) be given an orientation, and define \( \lambda_\gamma \in W_+^1 \) by \( \lambda_\gamma(y) = 1 = -\lambda_\gamma(tx) \) if \( \gamma \) crosses \( x, tx \) from \( \partial_x(x) \) to \( \partial_x(tx) \). Otherwise, set \( \lambda_\gamma(y) = 0 \). The map \( T(\Sigma, \lambda_\gamma) = \Sigma \lambda_\gamma \) is a linear isomorphism of \( C_1(E) \) onto \( W_+^1 \). One has \( T(\Sigma, \lambda_\gamma) \in V(\mathcal{E}) \) if, and only if, \( c \in Z_1(E) \). Therefore, \( T: Z_1(E) \to V(\mathcal{E}) \) is also an isomorphism. Setting \( G^+(p, n) = \text{Hom}(\pi_1(M_p, s), \mathbb{R}^+) \), we have

\[ T^* \text{Hom}(V, \mathbb{R}^+) = G^+(p, n) \tag{25} \]

since \( \Gamma \) is homotopy equivalent to \( M_p, s \).

If \( z' = (\mathcal{E}, \varphi) \in \Omega_0(p, n) \), define \( z = \pi(z') \in \Omega(p, n) \) and \( w = W(z) \in \mathcal{E}(p, n) \). Let \( \chi(w, \cdot) \in \text{Hom}(\pi_1(M_p, s), \mathbb{C}^*) \) be the holonomy character (Section 7 or [4]).

**Lemma 13.1.** With notations as above we have

\[ |\chi(w, c)| = m(\varphi, Tc). \tag{26} \]

That is, \( |\chi(w, \cdot)| = T^*m(\varphi, \cdot) \).

**Proof.** \( m(\varphi, \cdot) \) is defined in (22) Let \( c \in Z_1(E) \) have the special form \( c = \sum_{j=1}^n \gamma_j \), where for each \( j \), \( \gamma_j \) crosses \( x_j, tx_j \) from \( \partial_x(x_j) \) to \( \partial_x(tx_j) \) and \( \partial_x(tx_j) = \partial_x(x_{j+1}) \). Let \( P(\partial_x(x), \varphi) \) be polygons associated to a standard realization and with circumradius \( R(x) \). If \( a \in E \), \( \chi(w, c) \) is the coefficient \( a \) in \( t(z) = az + b \), where \( t(z) \) is the affine complex transformation of \( P(\partial_x(x_j), \varphi) \) obtained from the developing map along \( c \). One sees easily that \( |a| \) has the form

\[ |a| = \prod_{j=1}^n \frac{R(x_j) \sin \pi \varphi(x_j)}{R(tx_j) \sin \pi \varphi(tx_j)} = \frac{R(x_j)}{R(tx_j)} \cdot m(\varphi, Tc). \]

Since \( \partial_x(x_j) = \partial_x(tx_j) \) by assumption, we have \( R(x_j) = R(tx_j) \) and \( |\chi(w, c)| = m(\varphi, Tc) \) for \( c \in Z_1(E) \), of a special form. As these special cycles span \( Z_1(E) \) as a vector space, the lemma follows. \( \square \)

14. REALIZATION OF \( \Omega(p, n) \) AS A PRODUCT

We have defined \( P(\mathcal{E}) \) (resp. \( P_0(\mathcal{E}) \)) to be image of \( \Pi(\mathcal{E}) \) (resp. \( \Pi_0(\mathcal{E}) \)) under the projection \( \varphi \to \varphi_0 \). Set up a space \( \mathcal{R}_0(p, n) \) as \( \mathcal{R}_0(p, n) = \bigcup_{\mathcal{E} \in \Lambda(p, n)} (\mathcal{E}) \times P(\mathcal{E}) \), and use \( \rho_0 \) to denote the
Define \( \mathcal{R}(p,n) = \Omega_0(p,n)/\sim \), and let \( \pi \) denote the canonical projection. In view of the lemma the rectangle

\[
\begin{array}{ccc}
\Omega_0(p,n) & \stackrel{\rho_0}{\rightarrow} & \mathcal{R}_0(p,n) \\
\downarrow{x} & & \downarrow{x} \\
\Omega(p,n) & \stackrel{\rho}{\rightarrow} & \mathcal{R}(p,n)
\end{array}
\]

is defined naturally to be commutative with \( \rho \) continuous.

We next construct a map

\[
\Omega(p,n) \rightarrow \mathcal{R}(p,n) \times G^+(p,n)
\]

by setting \( r(z) = (\rho(z), |\chi(W(z), \cdot)|) \). Theorem 12.2, Remark 12.4, Lemma 13.1 and the continuity of \( \rho \) imply

**Theorem 14.2.** The map \( r \) in (13.4) is a homeomorphism onto.

**Proof.** At the level

\[
\Omega_0(p,n) \rightarrow \mathcal{R}_0(p,n) \times G^+(p,n)
\]

Theorem 12.2 implies \( r_0 \) is a homeomorphism onto. Lemma 14.1 implies \( r_0 \) descends to a homeomorphism \( r \). \( \square \)
15. A LEMMA ON STOCHASTIC MATRICES

Let $S$ be a nonempty finite set. We adopt some notational conventions for real matrices $A$ indexed by $S \times S$, i.e., real functions on $S \times S$: If $S_1, S_2 \subseteq S$, then $A(S_1 \times S_2)$ denotes the rectangular matrix $A_{S_1 \times S_2}$. When $S_1 = S_2$, we write $A(S_1)$ instead of $A(S_1 \times S_1)$. Finally, we set $|A(S_1 \times S_2)| = \sum_{s \in S_1} A(s, s_2)$.

**Definition 15.1.** Let $S$ be a finite nonempty set. $\mathcal{X}(S)$ shall be the set of matrices $A$ on $S \times S$ such that

(i) $A( \cdot, \cdot ) \geq 0$,

(ii) $A$ is symmetric,

(iii) $|A| = |S|$, 

(iv) $|A(T)| \leq 2|T|$, $T \subseteq S$.

The set $\mathcal{X}(S)$ will be characterized in terms of stochastic matrices (Lemma 15.5). $B(\cdot, \cdot ) \geq 0$ is **stochastic** if $\sum_{s \in S} B(s', s) = 1$, $s' \in S$. $B'$ denotes transpose.

**Lemma 15.2.** If $B$ is stochastic matrix, then $A = B + B'$ belongs to $\mathcal{X}(S)$. Moreover, if $A$ also satisfies $|A(T)| < 2|T|$ for every proper subset $T \subset S$, there is a (possibly different) stochastic matrix $B_1$ such that (a) $B_1 + B'_1 = A$ and (b) $B_1(s, t) > 0$ whenever $A(s, t) > 0$.

**Proof.** The first statement is obvious. As for the second the condition $|A(T)| < 2|T|$ for every proper subset implies $B(T)$ is not a stochastic matrix whenever $T$ is proper. Let $s, t$ be such that $A(s, t) = A(t, s) > 0$ but $B(s, t) = 0$. We shall produce a stochastic matrix $B_1$ such that (a) is true and, moreover, $B_1(s, t) > 0$ and $B_1(s', s'') > 0$ whenever $B(s', s'') > 0$. If we then replace $B$ by $B_1$, reletter and repeat this procedure we eventually obtain a stochastic matrix $B_1$ which satisfies (a) and (b).

To produce $B_1$ for $s, t$ given as above let $S_1$ be the set of $t' \in S$ such that there exist $a > 0$ and $t_1, \ldots, t_a \in S$ with

$$B(s, t_1) B(t_1, t_2) \cdots B(t_{a-1}, t_a) B(t_a, t') > 0.$$  

It is clear $S_1 \neq \emptyset$ and $B(S_1)$ is a stochastic matrix. As observed in the preceding paragraph, $S_1 = S$. Let $t' = t$ above, and choose $a, t_1, \ldots, t_a$. Let $2\varepsilon = \min\{B(s, t_1), B(t_m, t), B(t_p, t_{p+1})\}$. Define $B_1$ by **subtracting** $\varepsilon$ from each of the entries $B(s, t_1), B(t_j, t_{j+1}), 1 \leq j < a$, and $B(t_m, t)$. Next, **add** $\varepsilon$ to each of the entries $B(s, t) (= 0$ by assumption), $B(t, t_a), B(t_j, t_{j-1}), 2 \leq j \leq a$, and $B(t_1, s)$. By definition of $\varepsilon$ $B_1(\cdot, \cdot ) \geq 0$. Also, $B_1(s, t) > 0$ and $B_1(s', s'') > 0$ whenever $B(s', s'') > 0$. Finally, each row sum of $B_1$ is one. The lemma is proved. 

**Lemma 15.3.** Let $A \in \mathcal{X}(S)$. Let $S_1 \subseteq S$ be a set such that (a) $|A(S_1)| = 2|S_1|$ and (b) if $S_1 \not\subseteq S_1 \not\subseteq S$, then $|A(S_1)| < 2|S_1|$. If $R \subseteq S$ is such that $R \cap (S \setminus S_1)$ is a proper subset of $S \setminus S_1$, then $|A(R)| < 2|R|$.

**Proof.** If $S_1 = \emptyset$, there is nothing to prove. Therefore, we may assume $S_1 \neq \emptyset$. Define a sequence $S_2, S_3, \ldots$, inductively, as $S_{k+1} = \{s \in S \setminus \bigcup_{j=1}^{k} S_j \setminus \sum_{s \in S_j} A(s, t) > 0\}$. This sequence has the property that $A(S_i \times S_j)$ is a zero matrix for $|i - j| > 1$. Let $N \geq 1$ be the least integer such that $S_{N+1} = \emptyset$. 


Let $R$ be as in the statement of the lemma and define $R_j = R \cap S_j$. Also, set $R_{N+1} = R \cap (\bigcup_{j=1}^N S_j)$. The properties of $S_j$ imply

$$|A(R)| = |A(R_1)| + \sum_{j=2}^N (|A(R_j)| + 2|A(R_j \times R_{j-1})|) + |A(R_{N+1})|.$$  

In order that this be $2|R|$ it is necessary that

$$\sum_{j=2}^N (|A(R_j)| + 2|A(R_j \times R_{j-1})|) + |A(R_{N+1})| = 2|R|.$$  

(30)

Define $S'_1 = S_1 \cup \bigcup_{j=2}^N R_j$. By assumption $S'_1$ is a proper subset of $S$. The expression for $|A(S'_1)|$ differs from that for $|A(R)|$ only in that $|A(R_j)|$ is replaced by $|A(S'_1)| = 2|S_1|$ and $|A(R_2 \times R_1)|$ is replaced by the possibly larger quantity $|A(S_2 \times S_1)|$. Therefore, (30) implies $|A(S'_1)| = 2|S_1|$. As $S_1$ is assumed to be maximal, we have reached a contradiction and the lemma obtains.

$\mathcal{K}(S)$ is a compact convex set, and so to understand $\mathcal{K}(S)$ it is sufficient to understand its set of extreme points:

**Lemma 15.4.** Let $A$ be an extreme point of $\mathcal{K}(S)$, and let $S_1$ be maximal for the properties $S_1 \neq S$ and $|A(S_1)| = 2|S_1|$. If $S_2 = S_1$, and if $U = (S_2 \times S) \cup (S \times S_2)$, then there exist $s, t \in S_2$, possible with $s = t$, such that the nonzero entries of $A(U)$ occur at $(s, t)$ and $(t, s)$.

**Proof:** If $A(U)$ has more than one symmetric set of nonzero entries, there exist distinct symmetric pairs $u = \{(s_1, t_1), (t_1, s_1)\}$ and $v = \{(s_2, t_2), (t_2, s_2)\}$ such that $A(s_1, t_1) A(s_2, t_2) > 0$. It is possible that $s_1 = t_1$, or $s_2 = t_2$. We may view the characteristic functions $\chi_u, \chi_v$ as symmetric matrices on $S \times S$.

Let $\epsilon_1, \epsilon_2 > 0$ be such that $|\epsilon_1| = |\epsilon_2| < 1$, and set up matrices $A^\pm = A \pm \epsilon_1 \chi_u \mp \epsilon_2 \chi_v$. We have $|A^\pm| = |A| \pm \epsilon_1 |\chi_u| \mp \epsilon_2 |\chi_v| = |A|$. Also $A^\pm(s, t) > 0$ for $\epsilon_1, \epsilon_2$ sufficiently small. If $R \subseteq S$ is any set, then $|A^\pm(R)| \leq |A(R)| + 2(\epsilon_1 + \epsilon_2)$. Let $c = \min(2|R| - |A(R)|)$ over sets $R$ such that $R \cap (S \setminus S_1)$ is a proper subset of $S \setminus S_1$. If $2(\epsilon_1 + \epsilon_2) < c$, then $|A^\pm(R)| < 2|R|$ for such sets. If $R \supseteq S \setminus S_1$, then $|A^\pm(R)| = |A(R)| \leq 2|R|$, and, of course, the same is true for $R \subset S_1$. Therefore, $A^\pm \in \mathcal{K}(S)$, $\epsilon_1, \epsilon_2$ sufficiently small. As $A = \frac{1}{2}(A^+ + A^-)$, $A$ cannot be extreme. The lemma is proved.

**Lemma 15.5.** $A \in \mathcal{K}(S)$ if, and only if, $A$ can be expressed as $A = B + B'$ for some stochastic matrix $B$. If $A$ satisfies the additional condition that $|A(T)| < 2|T|$ for all proper subsets $T \subseteq S$, then the stochastic matrix $B$ can be chosen to satisfy $B(s, t) > 0$ whenever $A(s, t) > 0$.

**Proof:** The statement is trivial for $|S| = 1$, and therefore we assume $|S| > 1$ and the statement has been proved for sets with fewer than $|S|$ elements. Assume $A \in \mathcal{K}(S)$ is an extreme point, and let $S_1$ be a maximal subset such that $S_1 \neq S$ and $|A(S_1)| = 2|S_1|$. As $A(S_1) \in \mathcal{K}(S_1)$, the induction assumption implies there exists a stochastic matrix $B_1$ on $S_1$ such that $A(S_1) = B_1 + B'_1$. Let $S_2 = S_1$ and $U = S_2 \times S \cup S \times S_2$, as before. Lemma 15.4 implies there exists $(s, t) \in U$ such that $A(U)$ is zero except possibly at $(s, t)$ and $(t, s)$. As $|A(S_1)| = 2|S_1|$ ($< 2|S|$), we have $|A(U)| = 2(|S| - |S_1|) > 0$. If $S_2$ is a singleton, then
|A(U)| - 2 and either s - t with A(s, s) = 2 or s ≠ t with A(s, t) = 1 - A(t, s). In either case B_1 admits an extension to a stochastic matrix B on S with A = B + B'.

To deal with the case |S_2| ≥ 2, we observe that for any s ∈ S if U_s = \{s\} ≺ S_j, then |A(U)| = 2|S| - |A(S, \{s\})| ≥ 2. Therefore, if |S_2| ≠ 1 it must be that |S_2| = 2. Let U = S_2 × S_j ≺ S and so now |A(U)| = 4. If S_2 = \{s, t\}, the fact neither row A(s, ·), A(t, ·) is zero implies A(s, t) = A(t, s) > 0. Now A(s, t) = 2 = A(t, s). Again we can extend B_1 to a stochastic matrix B on S with A = B + B'.

Since \mathcal{K}(S) is the convex hull of its extreme points and the set of stochastic matrices is a convex set, the existence follows. The remainder of the theorem follows from Lemma 15.2.

16. CHARACTERIZATION OF P(\sigma)

Let \sigma be a P-set, and let W_\sigma ≺ R^E, as before, be the set of \tau-invariant functions. If \psi \in W_\sigma, define a matrix A_\psi on \mathcal{C}(\sigma) × \mathcal{C}(\sigma) by

\[ A_\psi(t_1, F_2) = \sum_{x \in F_1 \cap \tau F_2} \psi(x). \]  

Since \psi ◦ \tau = \psi and \tau(F_1 \cap \tau F_2) = \tau F_1 \cap F_2, A_\psi is symmetric.

In what follows \mathcal{K}^+(\mathcal{C}(\sigma)) denotes the subset of \mathcal{K}(\mathcal{C}(\sigma)) for which the strict inequalities |A(T)| < 2|T| are true for every proper subset T ≺ S.

THEOREM 16.1. Let \sigma be a P-set, and let P(\sigma) be the image of \Pi(\sigma) under the map \varphi → \varphi_\tau. Then P(\sigma) is a polyhedron defined by

\[ P(\sigma) = \{ \psi \in W_\sigma | 0 < \psi \leq 1, A_\psi \in \mathcal{K}^+(\mathcal{C}(\sigma)) \}. \]  

Proof. If \psi \in \Pi(\sigma) and \psi = \varphi_\tau, then 0 < \psi \leq 1 and A_\psi = A_\varphi + A_\varphi', where A_\varphi is also a stochastic matrix such that A_\varphi(T) is stochastic for no proper subset T ≺ \mathcal{C}(\sigma). (This uses the fact \langle \langle \sigma, \tau, \eta \rangle \rangle) is transitive on \mathcal{E}.) Therefore, A_\psi \in \mathcal{K}^+(\mathcal{C}(\sigma)).

Conversely, suppose 0 < \psi \in W_\sigma is such that A_\psi \in \mathcal{K}^+(\mathcal{C}(\sigma)). By Theorem 15.5 there exists a stochastic matrix B such that A_\psi = B + B' and B(F_1, F_2) > 0 whenever A_\varphi(F_1, F_2) > 0. Choose a complete set \mathcal{U} of representatives of symmetric pairs \{(F_1, F_2), (F_2, F_1)\} for which \tau F_1 \cap \tau F_2 ≠ \emptyset (⇔ A_\varphi(F_1, F_2) > 0). For each (F_1, F_2) \in \mathcal{U} define \varphi on \tau F_1 \cap \tau F_2 to satisfy

\[ 0 < \varphi(x) < \psi(x) \]  

\[ \sum_{x \in F_1 \cap \tau F_2} \varphi(x) = B(F_1, F_2) (\varphi_\varphi(F_1, F_2)). \]  

Define \varphi on \tau F_1 \cap \tau F_2 by \varphi(y) = \psi(y) - \varphi(\tau y). We have

\[ \sum_{y \in \tau F_1 \cap \tau F_2} \varphi(y) = \sum_{y \in \tau F_1 \cap \tau F_2} (\psi(y) - \varphi(\tau y)) \]  

\[ = \sum_{x \in F_1 \cap \tau F_2} (\psi(x) - \varphi(x)) \]
\[ w_\mathcal{L}(F_1, F_2) - B(F_1, F_2) \]

\[ B(F_2, F_1). \]

\( \varphi \) is defined on \( E \), and \( \varphi \) satisfies \( 0 < \varphi < \psi \), \( \varphi_a = \psi \) and

\[
\varphi_a(x) = \sum_{y \in \mathcal{C}_a(x)} \varphi(y) = \sum_{F \in \mathcal{C}(x)} B(\mathcal{C}_a(x), F) = 1.
\]

Now when \( \psi \leq 1 \), we also have \( \varphi_a \leq 1 \) and \( \varphi \in \Pi(\mathcal{E}) \). The theorem is proved.

\section{17. PROOFS OF THE MAIN RESULTS}

If \( w \in \mathcal{S}(p, n) \), the \( Aw = a \in \mathcal{C}^\mathbb{R} \) satisfies \( \text{Re} \, a(\cdot) > -1 \) and \( \sum_{\delta(s)} \alpha(s) = 2p - 2 \). The complex cone angles associated to \( w \) are the numbers \( \theta(s) = 2\pi(\alpha(s) + 1) \) which satisfy

\[
\text{Re} \theta(s) > 0
\]

\[
\sum_{\delta(s)} \theta(s) = (4p - 4 + 2n)\pi. \tag{34}
\]

If one is given \( \{\text{Re} \theta(s), s \in S_n\} \) and \( |\chi(w, \cdot)| \) and the image \( Bw \in \mathcal{S}(p, n) \), one can reconstruct \( w \) [4]. However, \( \chi(w, \cdot) \) is determined by a simple formula only on (say) simple loops \( \delta(s) \) of index one about \( s \in S_n \), for which

\[
\chi(w, \delta(s)) = e^{i \text{Re} \theta(s)} |\chi(w, \delta(s))|. \]

In this section we shall let \( \mathcal{L}^+(p, n) \) be the set of \( \text{real} \ \theta(\cdot) \) which satisfy (34) and we shall consider \( \mathcal{S}(p, n) \) as a product

\[
\mathcal{S}(p, n) \equiv \mathcal{L}^+(p, n) \times \mathcal{F}(p, n) \times G^+(p, n) \tag{35}
\]

where the map is \( w \rightarrow (\text{Re} 2\pi(Aw(\cdot) + 1), Bw, |\chi(w, \cdot)|) \). It is a slightly form of (35) which is used to define topology and real analytic structure on \( \mathcal{S}(p, n) \) [4, Theorem 1.13].

Let \( W \) and \( r \) be the maps of \( \Omega(p, n) \) to \( \mathcal{S}(p, n) \) and \( \mathcal{R}(p, n) \times G^+(p, n) \), respectively, and consider then map \( W \circ r^{-1} \). In terms of the isomorphism in (35) \( W \circ r^{-1} \) defines for each \( v \in G^+(p, n) \) a homeomorphism

\[
\mathcal{R}(p, n) \times \{v\} \xrightarrow{W \circ r^{-1}} \mathcal{L}^+(p, n) \times \mathcal{F}(p, n) \times \{v\}. \tag{36}
\]

If we drop the trivial factor \( v \), it is necessary to record dependence upon \( v \) of the map (36). We use \( h \), in place of \( W \circ r^{-1} \):

\[
\mathcal{R}(p, n) \xrightarrow{h} \mathcal{L}^+(p, n) \times \mathcal{F}(p, n). \tag{37}
\]
The sets \( \{ \delta \} \times P_0(\delta) \subseteq \mathcal{R}_0(p, n) \) (Section 12) comprise a complete set of reduced representations for the relation \( \sim \). Let \( D(\delta) = \pi(\{ \delta \} \times P_0(\delta)) \). Then \( \mathcal{D} = \{ D(\delta) | \delta \in \Lambda(p, n) \} \) is a cell decomposition of \( \mathcal{R}(p, n) \) with

\[
\partial D(\delta) = \bigcup_{\varepsilon \subseteq \sigma} D(\varepsilon).
\]

If \( v \in G^+(p, n) \) define

\[
C(\varepsilon, v) = h_c(D(\varepsilon)) \ (\varepsilon \in \Lambda(p, n))
\]

and let \( \mathcal{C}(v) = \{ C(\varepsilon, v) | \varepsilon \in \Lambda(p, n) \} \).

Fix \( v \in G^+(p, n) \), \( \varepsilon \in \Lambda(p, n) \), and define \( \tilde{C}(\varepsilon, v) \subseteq C^+(p, n) \times \mathcal{I}(p, n) \) to be the set of \((\theta, \tau)\) such that when \( w \in \mathcal{I}(p, n) \) and \( \theta = 2\pi(\text{Re} \ A(w) + 1) \), \( \tau = Bw \) and \( v(\cdot) = |\chi(w, c\delta)| \), then \( \delta \) is the class of the Delaunay partition associated to \( w \). Let

\[
\hat{C}(v) = \{ \tilde{C}(\varepsilon, v) | \varepsilon \in \Lambda(p, n) \}.
\]

**Theorem 17.1** If \( v \in G^+(p, n) \) then \( \mathcal{C}(v) = \hat{C}(v) \). For each \( \varepsilon \in \Lambda(p, n) \) the set \( \tilde{C}(\varepsilon, v) \) is a real analytic submanifold of \( C^+(p, n) \times \mathcal{I}(p, n) \). The map \( h_c \circ \pi: \{ \delta \} \times P_0(\delta) \rightarrow \tilde{C}(\varepsilon, v) \) is a real analytic diffeomorphism. Moreover,

\[
\partial \tilde{C}(\varepsilon, v) \bigcup_{\varepsilon \subseteq \sigma} \tilde{C}(\varepsilon, v).
\]

If \( \theta \in C^+(p, n) \) is fixed, we define for \( v \in G^+(p, n) \) and \( \varepsilon \in \Lambda(p, n) \) a set \( \tilde{C}(\varepsilon, v, \theta) \subseteq \mathcal{I}(p, n) \) as

\[
\tilde{C}(\varepsilon, v, \theta) = \{ \tau \in \mathcal{I}(p, n) | (\theta, \tau) \in \tilde{C}(\varepsilon, v) \}.
\]

For certain \( \varepsilon \) and \( \theta \) it is true that \( \tilde{C}(\varepsilon, v, \theta) = \emptyset \). Let \( \hat{C}(v, \theta) \) be the partition of \( \mathcal{I}(p, n) \) defined by \( \tilde{C}(\varepsilon, v, \theta) \), \( \varepsilon \in \Lambda(p, n) \).

To identify the sets \( \tilde{C}(\varepsilon, v, \theta) \), we define polyhedra \( P_0(\varepsilon, \theta) \subseteq P_0(\varepsilon) \) as

\[
P_0(\varepsilon, \theta) = \{ \psi \in \psi_{\varepsilon} \} | \theta(s) = (1 - \psi)_{\varepsilon} x, x \in s \}.
\]

The notation \( x \in s \) is understood to mean that \( s \in S_{\psi}(x) \) is identified with \( \theta_{\psi}(x) \). Let \( D(\varepsilon, \theta) \) be the subset of \( \mathcal{R}(p, n) \) which is the image of \( \{ \delta \} \times P_0(\varepsilon, \theta) \) under the canonical projection. For later reference let \( \mathcal{D}(\theta) = \{ D(\varepsilon, \theta) | \varepsilon \in \Lambda(p, n) \} \). We have

**Theorem 17.2**. With notations as above we have

\[
\tilde{C}(\varepsilon, v, \theta) = h_c(D(\varepsilon, \theta)).
\]

For each pair \( v, \theta \) the partition \( \hat{C}(v, \theta) \) determined by (40) is a partition of \( \mathcal{I}(p, n) \) by real analytic polyhedra.

The case \( v = 1 \) of the theorem is the flat cone metric case.
Theorem 17.3. Let $\theta \in \mathcal{I}^1(p, n)$. If $\mathcal{D} \in \Delta(p, n)$ define $\Delta(\mathcal{D}, \theta)$ to be the set of $\tau \in \mathcal{I}(p, n)$ such that the associated isotopy class of volume one flat cone metrics with cone angles $\theta(s)$, $s \in S_\mathcal{D}$, has $\mathcal{D}$ for its Delaunay partition. The corresponding partition $\Delta(\theta)$ satisfies $\Delta(\theta) = h_! \mathcal{D}(\theta)$, where $\mathcal{D}(\theta)$ is the partition defined in the paragraph preceding Theorem 17.2.

Of course, $\Delta(\theta) = \mathcal{D}(1, \theta)$. Unlike $\mathcal{D}(v, \theta)$, $v \neq 1$, $\Delta(\theta)$ is invariant under the natural action of the pure mapping class group of $M_{p, n}$. Moreover, if $\mathcal{R}(p, n, \theta) = h_!^{-1}(\{\theta\} \times \mathcal{I}(p, n))$, then $\mathcal{R}(p, n, \theta)$ admits a natural action of the pure mapping class group and $\mathcal{R}(p, n, \theta) \xrightarrow{h_!} \{\theta\} \times \mathcal{I}(p, n)$ is equivariant. Observe that for all $v$ we have $\mathcal{R}(p, n, \theta) = h_!^{-1}(\{\theta\} \times \mathcal{I}(p, n))$. However, the maps $h_!$ are all different.

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