On the functional-differential equation of advanced type
\[ f'(x) = af(2x) \] with \( f(0) = 0 \)

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Abstract

In this paper we construct solutions for the equation
\[
\begin{cases}
  f'(x) = af(2x), & -\infty < x < +\infty, \\
  f(0) = 0,
\end{cases}
\]
where \( a \) is a constant with \( a \neq 0 \). The solutions are infinitely differentiable and bounded on \( \mathbb{R} \). Using our method, we can get numerical data easily with a computer. Applying one of the solutions we show that the derivative of order \( k \geq 0 \) of a function \( v \in C^k(\mathbb{R}) \) or \( v \in L_p^k(\mathbb{R}) \) coincides \( \lim_{\epsilon \to 0} v * G_{k,\epsilon} \), where \( L_p^k(\mathbb{R}) \) is the Sobolev space and \( \{G_{k,\epsilon}\}_{\epsilon>0} \) is a family of \( C^\infty \)-functions.

Keywords: Functional-differential equation; Advanced type

1. Introduction

Frederickson [1,2] (1971) investigated functional-differential equations of advanced type
\[ f'(x) = af(\lambda x) + bf(x), \] (1)

Frederickson [1] provided a global existence theorem for equations
\[ f'(x) = F(f(2x)), \quad x \in \mathbb{R} = (-\infty, +\infty), \]
where $F$ is an odd, continuous function with $F(s) > 0$ for $s > 0$. In [1] he applied the Schauder fixed point theorem to the proof. He showed that the absolute value of the solution $|f(x)|$ is periodic for $x \geq 0$. Frederickson [2] also provided a constructive method of solutions for equations of advanced type

$$f'(z) = af(\lambda z) + bf(z),$$

where $a, b \in \mathbb{C}$ and $\lambda > 1$. He gave solutions in the form of a Dirichlet series

$$\varphi(z, \beta) = \sum_{n \in \mathbb{Z}} c_n e^{\beta \lambda z}, \quad \Re(\beta z) \leq 0,$$

where $\beta$ is allowed to vary as a parameter. In the case of $b = 0$ and $\beta = i$, the solution is analytic in the upper half plane $\Im z > 0$, continuous on $\Im z \geq 0$, and the line $\Im z = 0$ is a natural boundary.

In this paper, using another method, we construct solutions for the equation

$$\begin{cases} f'(x) = af(2x), & x \in \mathbb{R} = (-\infty, +\infty), \\ f(0) = 0, \end{cases}$$

(2)

where $a$ is a constant with $a \neq 0$. The solution is not unique. If $f$ is a solution, then a constant times $f$ is also a solution. Our solutions are infinitely differentiable and bounded on $\mathbb{R}$. Using our method we can get numerical data easily with a computer. We also give the Fourier transform of one of our solutions and show the uniqueness of the solution of (2) with a certain condition.

Applying one of the solutions we show that the derivative of order $k \geq 0$ of a function $v \in C^k(\mathbb{R})$ or $v \in L^p_k(\mathbb{R})$ coincides $\lim_{\epsilon \to 0} v \ast G_{k, \epsilon}$, where $L^p_k(\mathbb{R})$ is the Sobolev space and $\{G_{k, \epsilon}\}_{\epsilon > 0}$ is a family of $C^\infty$-functions.

2. Main results

Our main results are the following:

**Theorem 1.** If $a > 0$, then there exists a solution $f \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ of (2) with $f \neq 0$ such that $f$ vanishes on $(-\infty, 0]$ and $|f|$ has period $4/a$ on $[0, +\infty)$. If $a < 0$, then there exists a solution $f \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ of (2) with $f \neq 0$ such that $f$ vanishes on $[0, +\infty)$ and $|f|$ has period $4/|a|$ on $(-\infty, 0]$.

If there exists a solution $f$ of

$$\begin{cases} f'(x) = 4f(2x), & x \in \mathbb{R}, \\ f(0) = 0, \end{cases}$$

(3)

such that $f$ vanishes on $(-\infty, 0]$ and $|f|$ has period 1 on $[0, +\infty)$, then $f(ax/4)$ is a solution of (2). If $a > 0$, then $f(ax/4)$ vanishes on $(-\infty, 0]$ and $|f(ax/4)|$ has period $4/a$ on $[0, +\infty)$. If $a < 0$, then $f(ax/4)$ vanishes on $[0, +\infty)$ and $|f(ax/4)|$ has period $4/|a|$ on $(-\infty, 0]$.

To construct a solution of (3), we use the function $u$ in the next lemma.

**Lemma 2.** There exist a nonnegative function $u$ with supp $u \subset [0, 1]$ such that

(i) $u \in C^\infty(\mathbb{R}),$

(ii) $u(x)$ satisfies (3) for $0 \leq x \leq 1/2,$

(iii) $u(x) = u(1 - x)$ for $0 \leq x \leq 1.$
(iv) \( u(x) + u(1/2 - x) = 2 \) for \( 0 \leq x \leq 1/2 \),
(v) \( u(x) \) is increase for \( 0 \leq x \leq 1/2 \),
(vi) \( u^{(k)}(0) = u^{(k)}(1) = 0 \) for \( k = 0, 1, 2, \ldots \), and
(vii) \( \int_0^1 u(x) \, dx = 1 \).

Remark 3. The function \( u \) in Lemma 2 is unique. If \( v \) satisfies (ii), (iii) and \( v(1/4) = b \) \((b \in \mathbb{R})\), then \( v(x) = bu(x) \) for \( 0 \leq x \leq 1 \) (see Theorem 11 and Remark 12).

The graph of the function \( u(x) \) is in Fig. 1.

Theorem 4. A solution \( f(x) \) of (3) is expressed by

\[
f(x) = \sum_{k=1}^{\infty} (-1)^n u(x - k + 1),
\]

where \( u \) is in Lemma 2 and

\[
\begin{aligned}
n_1 &= 0, \\
n_2 &= 1, \\
n_{2k-1} &= 1, \\n_{2k} &= 0, & \text{if } n_k = 1 \,(k \geq 2), \\
n_{2k-1} &= 0, \\n_{2k} &= 1, & \text{if } n_k = 0 \,(k \geq 2).
\end{aligned}
\]

Remark 5. Remark 3 shows that, if \( g(x) \) is a solution of (3) with \( g(x) = g(1 - x) \) for \( 0 \leq x \leq 1 \) and \( g(1/4) = b \), then \( g(x) = bf(x) \) for \( 0 \leq x \leq 1 \), where \( f \) is in (4). If \( g \) is a solution of (3), then the value of \( g(x) \) on \([2^k, 2^{k+1}]\) is determined by the value of \( g'(x) \) on \([2^{k-1}, 2^k]\), \( k = 0, 1, 2, \ldots \). Therefore, \( g(x) = bf(x) \) for \( x \geq 0 \).

The solution \( f(x) \) satisfies

\[
f(2^n + t) = (-1)^n f(2^n - t), \quad t \in [0, 2^{n+1}], \quad n = 1, 2, \ldots,
\]

and \( |f(x)| \) has period 1 on \([0, +\infty)\) as Frederickson mentioned in [1]. Actually, \( f(x) = f(1 - x) \) \((x \in [0, 1/2])\) implies \( f(1/2 + t) = f(1/2 - t) \) \((t \in [0, 1/2])\). Then \( 4f(1 + 2t) = f'(1/2 + t) = -f'(1/2 - t) = -4f(1 - 2t) \) \((t \in [0, 1/2])\), i.e. \( f(1 + t) = -f(1 - t) \) \((t \in [0, 1])\). This is the equality (5) for \( n = 1 \). In the same way we have (5) for \( n = 2, 3, \ldots \). Moreover, \( f \in C^\infty(\mathbb{R}) \).

The graph of the solution \( f(x) \) of (3) is in Fig. 2.

The next theorem is another expression of the solution \( f \). Let

\[
\hat{f}(\xi) = \mathcal{F} f(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx,
\]
and \( \text{sinc} \xi = \sin(\pi \xi)/(\pi \xi) \). Let \( S \) be the space of all rapidly decreasing functions and \( S' \) be the space of all tempered distributions.

**Theorem 6.** The Fourier transform of the solution \( f \) in (4) is expressed by

\[
\hat{f}(\xi) = \sum_{k=1}^{\infty} (-1)^n k e^{-i(k-1)\xi} \hat{u}(\xi), \quad \text{in } S',
\]

where

\[
\hat{u}(\xi) = e^{-i\xi/2} \lim_{n \to \infty} \prod_{k=1}^{n} \text{sinc}(\xi/(2^{k+1}\pi)) \cdot \text{sinc}(\xi/(2^{n+1}\pi)), \quad \text{uniformly on } \mathbb{R}.
\]

For a nonnegative integer \( k \) and \( 1 \leq p \leq \infty \), let \( L^p_k(\mathbb{R}) \) be the Sobolev space. Applying \( f \) in (4), we have the following.

**Theorem 7.** Let \( f \) be the solution in (4) and

\[
G_{k,\epsilon}(x) = \left(2^{k(k-1)/2}e^{k+1}\right)^{-1} \left(f \chi_{[0,2^k]}\right)(x/\epsilon).
\]

If \( v \in C^k(\mathbb{R}) \) or \( v \in L^p_k(\mathbb{R}) \) (\( k \geq 0, \ 1 \leq p < \infty \)), then

\[
\frac{d^k v}{dx^k} = \lim_{\epsilon \to 0} v \ast G_{k,\epsilon},
\]

uniformly on each compact subset in \( \mathbb{R} \) or in \( L^p(\mathbb{R}) \), respectively.

**Remark 8.** \( G_{k,\epsilon} \) is in \( C^\infty(\mathbb{R}) \) with compact support. To prove the theorem, we use Friedrichs’ mollifier \( \frac{d^k v}{dx^k} \ast u_\delta = v \ast \frac{d^k u_\delta}{dx^k} \), where \( u_\delta = u(x/\delta)/\delta, \ \delta > 0, \) and \( u \) is the function in Lemma 2.

We construct a solution of (3) and prove Lemma 2 and Theorem 4 in the next section. We prove Theorems 6 and 7 in Sections 4 and 5, respectively.
3. Construction of a solution of (3)

In this section we prove Lemma 2 and Theorem 4 to construct a solution of (3). The initial value problem (3) is equivalent to the integral equation

\[ f(x) = 2 \int_0^{2x} f(t) \, dt, \quad x \in \mathbb{R}. \]  

(6)

If \( f \) is a solution of (3) and satisfies \( f(x) = f(1-x) \) (\( 0 \leq x \leq 1 \)), then we have

\[ 2 \int_0^{2x} f(t) \, dt = f(x) = f(1-x) = 2 \int_0^{2(1-x)} f(t) \, dt = 2 \int_0^{2(1-x)} f(1-t) \, dt = 2 \int_{2x-1}^1 f(t) \, dt, \quad \frac{1}{2} \leq x \leq 1. \]

Then we define a function space \( X \) and an operator \( T : X \to X \) as follows:

\[ X = \{ u \in L^1(\mathbb{R}) : \text{supp} u \subset [0, 1], \ u(x) = u(1-x) \}, \]

\[ Tu(x) = \begin{cases} 2 \int_0^{2x} u(t) \, dt, & x \in [0, 1/2], \\ 2 \int_{2x-1}^1 u(t) \, dt, & x \in (1/2, 1], \\ 0, & x \notin [0, 1]. \end{cases} \]

We construct a function \( u \in X \) such that \( u = Tu \) in the proof of Lemma 2. Then \( u(x) \) satisfies (6) for \( x \in [0, 1/2] \). The function \( u(x) \) satisfies (6) for \( x \in (-\infty, 0] \) clearly. In the proof of Theorem 4 the function \( u(x) \) is extended uniquely to the right (increasing value of \( x \)) by using the equality \( f'(x) = 4f(2x) \).

3.1. Proof of Lemma 2 (Step 1)

We state two lemmas. Let \( \chi_I \) be the characteristic function of the interval \( I \subset \mathbb{R} \).

**Lemma 9.** The operator \( T \) is expressed by

\[ Tu(x) = 2(\chi_{[0,1]} * u)(2x), \quad u \in X, \]  

(7)

and satisfies

\[ \int_0^1 u(x) \, dx = \int_0^1 Tu(x) \, dx, \]  

(8)

\[ Tu(x) \geq 0 \quad \text{if} \ u(x) \geq 0, \]  

(9)

\[ |Tu(x)| \leq 2 \int_0^1 |u(x)| \, dx. \]  

(10)
Proof. From \( \text{supp} \ u \subseteq [0, 1] \) and \( \chi_{[0,1]}(2x - t) = \chi_{[2x-1,2x]}(t) \) it follows that

\[
2(\chi_{[0,1]} * u)(2x) = 2 \int_{0}^{1} \chi_{[0,1]}(2x - t)u(t) \, dt = 2 \int_{0}^{1} \chi_{[2x-1,2x]}(t)u(t) \, dt.
\]

Then we have (7). The properties (8)–(10) follows from (7) and the definition of \( T \).

Let \( u_0 = \chi_{[0,1]} \in X \) and \( u_{n+1} = Tu_n, \ n = 0, 1, 2 \ldots \). Then \( \{u_n\}_{n=0}^{+\infty} \) is a sequence of functions in \( X \). It follows from Lemma 9 that

\[
\int_{0}^{1} u_n(x) \, dx = 1, \quad 0 \leq u_n(x) \leq 2, \quad x \in \mathbb{R}.
\] (11)

We note that

\[
Tu_n(x) + Tu_n(1/2 - x) = 2 \quad x \in [0, 1/2], \ n = 0, 1, 2, \ldots
\] (12)

Actually,

\[
2 \int_{0}^{2x} u_n(t) \, dt + 2 \int_{0}^{1-2x} u_n(t) \, dt = 2 \int_{0}^{2x} u_n(t) \, dt + 2 \int_{2x}^{1} u_n(t) \, dt = 2.
\]

Lemma 10. There exists a function \( u \in X \) such that \( u_n(x) \to u(x) \) as \( n \to +\infty \) uniformly on \( \mathbb{R} \).

Proof. Let \( g_n = u_n - u_{n-1}, \ n = 1, 2, \ldots \). Then \( g_{n+1} = Tg_n \) and

\[
\begin{aligned}
g_n(x) + g_n(1/2 - x) &= 0, \quad x \in [0, 1/2], \\
g_n(x) &= g_n(1 - x), \quad x \in [0, 1], \quad n = 1, 2, \ldots \\
g_n(x) + g_n(3/2 - x) &= 0, \quad x \in [1/2, 1].
\end{aligned}
\] (13)

For example,

\[
g_1 = u_1 - u_0 = (4x \chi_{[0,1/2]} + (-4x + 4) \chi_{(1/2,1)}) - \chi_{[0,1]}
\]

\[
= (4x - 1) \chi_{[0,1/2]} + (-4x + 3) \chi_{(1/2,1)},
\]

\[
g_2 = Tg_1 = (16(x - 1/8)^2 - 1/4) \chi_{[0,1/4]} + (16(x - 3/8)^2 + 1/4) \chi_{(1/4,1/2]}
\]

\[
+ (-16(x - 5/8)^2 + 1/4) \chi_{(1/2,3/4]} + (16(x - 7/8)^2 - 1/4) \chi_{[3/4,1]}.
\]

We show

\[
\begin{aligned}
g_n(x) &\leq 0, \quad x \in [0, 1/4] \cup [3/4, 1], \quad n = 1, 2, \ldots, \\
g_n(x) &\geq 0, \quad x \in [1/4, 3/4],
\end{aligned}
\] (14)

by induction. At first, \( g_1 \) satisfies (14) clearly. Assume that \( g_n \) satisfies (14). From \( g_n(x) \leq 0, \ x \in [0, 1/4], \) and \( g_n(x) + g_n(1/2 - x) = 0, \ x \in [0, 1/2], \) it follows that

\[
g_{n+1}(x) = Tg_n(x) = 2 \int_{0}^{2x} g_n(t) \, dt \leq 0, \quad x \in [0, 1/8],
\]

and
\[ g_{n+1}(x) = T g_n(x) = 2 \int_0^{2x} g_n(t) \, dt = 2 \int_0^{1/4} g_n(t) \, dt - 2 \int_{1/4}^{2x} g_n(1/2 - t) \, dt = 2 \int_0^{1/4} g_n(t) \, dt - 2 \int_{1/2 - 2x}^{1/4} g_n(t) \, dt = 2 \int_0^{1/4} g_n(t) \, dt - 2 \int_{1/4}^{1/2 - 2x} g_n(t) \, dt \leq 0, \quad x \in [1/8, 1/4]. \]

Using (13) for \( g_{n+1} \), we have (14) for \( n + 1 \) instead of \( n \).

Next we show \( \| g_{n+1} \|_{L^\infty} \leq 1/2 \| g_n \|_{L^\infty}, \quad n = 1, 2, \ldots. \) (15)

We note that \( \text{supp } g \subset [0, 1] \). From (13) for \( g_{n+1} \) it follows that

\[
\begin{align*}
\sup_{x \in [0, 1/2]} |g_{n+1}(x)| &= \sup_{x \in [1/2, 1]} |g_n(x)|, \\
\sup_{x \in [0, 1/4]} |g_{n+1}(x)| &= \sup_{x \in [1/4, 1/2]} |g_{n+1}(x)|.
\end{align*}
\]

Then we have \( \| g_{n+1} \|_{L^\infty} = \sup_{x \in [0, 1/4]} |g_{n+1}(x)| \). For \( x \in [0, 1/4] \) we have

\[
0 \geq g_{n+1}(x) = T g_n(x) = 2 \int_0^{2x} g_n(t) \, dt \geq 2 \int_0^{1/4} g_n(t) \, dt = g_{n+1}(1/8),
\]

since \( g_n(t) \leq 0 \) (\( t \in [0, 1/4] \)) and \( g_n(t) \geq 0 \) (\( t \in [1/4, 1/2] \)). Hence

\[
\| g_{n+1} \|_{L^\infty} = |g_{n+1}(1/8)| = 2 \int_0^{1/4} g_n(t) \, dt \leq \frac{1}{2} \| g_n \|_{L^\infty}.
\]

Therefore, we obtain (15). Thus \( \{u_n\} \) converge uniformly on \( \mathbb{R} \). \( \square \)

By Lemma 10, we have \( u = T u \). Moreover, from Lemma 9, (11) and (12) it follows that \( u \) satisfies (ii)–(v) and (vii) in Lemma 2.

3.2. Proof of Lemma 2 (Step 2)

We show that \( u \) satisfies (i) and (vi) in Lemma 2, i.e., \( u \in C^\infty(\mathbb{R}) \) and \( u^{(k)}(0) = u^{(k)}(1) = 0 \) for \( k = 0, 1, 2, \ldots \) by induction.

Fig. 3. \( g_1(x) \) and \( g_2(x) \).
The equality $u = Tu$, i.e.,
$$u(x) = \begin{cases} 2\int_0^{2x} u(t) \, dt, & x \in [0, 1/2], \\ 2\int_{2x-1}^{1} u(t) \, dt, & x \in (1/2, 1], \\ 0, & x \notin [0, 1] \end{cases}$$
(16)
implies that $u \in C^0(\mathbb{R})$ and $u(0) = u(1) = 0$.
Assume that $u \in C^k(\mathbb{R})$ and $u^{(k)}(0) = u^{(k)}(1) = 0$. Then the equality (16) implies that $u \in C^{k+1}(\mathbb{R} \setminus \{0, 1/2, 1\})$. From $u^{(k)}(0) = 0$ and the continuity of $u^{(k)}$ it follows that
$$\begin{cases} u^{(k+1)}(x) = 0 & (x < 0), \\ u^{(k+1)}(x) = (4u(2x))^{(k)} = 2^{k+2}u^{(k)}(2x) \to 0 & (x \to +0), \end{cases}$$
i.e.,
$$\lim_{x \to \pm 0} u^{(k+1)}(x) = 0.$$ By the mean value theorem, we have that
$$\frac{u^{(k)}(h) - u^{(k)}(0)}{h} = u^{(k+1)}(\theta h) \quad (0 < \theta < 1).$$
As $h \to 0$ we have that $u^{(k+1)}(0) = 0$ and $u^{(k+1)}$ is continuous at 0. By $u(x) = u(1-x)$ we have that $u^{(k+1)}(1) = 0$ and $u^{(k+1)}$ is continuous at 1. By
$$\begin{cases} u^{(k+1)}(x) = (4u(2x))^{(k)} = 2^{k+2}u^{(k)}(2x) \to 0 & (x \to 1/2 - 0), \\ u^{(k+1)}(x) = (-4u(2x-1))^{(k)} = -2^{k+2}u^{(k)}(2x-1) \to 0 & (x \to 1/2 + 0), \end{cases}$$
we have that $u^{(k+1)}(1/2) = 0$ and $u^{(k+1)}$ is continuous at 1/2. Therefore $u \in C^{k+1}(\mathbb{R})$ and $u^{(k+1)}(0) = u^{(k+1)}(1) = 0$.

3.3. Proof of Theorem 4

If $f$ is a solution of (3), then the value of $f(x)$ on $[2^k, 2^{k+1}]$ is determined by the value of $f'(x)$ on $[2^{k-1}, 2^k]$.
We define a function $f \in L^{\infty}(\mathbb{R})$ as follows:
$$\begin{cases} f(x) = 0, & x \in (-\infty, 0), \\ f(x) = u(x), & x \in [0, 1], \\ f(x) = f'(x/2)/4, & x \in (1, 2], \\ f(x) = f'(x/2)/4, & x \in (2^k, 2^{k+1}], & k = 1, 2, \ldots. \end{cases}$$
By induction we show that $f$ is a solution of (3) and that $f$ is expressed by (4). Then from $u^{(k)}(0) = u^{(k)}(1) = 0$ for $k = 0, 1, 2, \ldots$ it follows that $u \in C^{\infty}(\mathbb{R})$.
First we show that
$$\begin{cases} f(x) = u(x) - u(x - 1), & x \in [0, 2], \\ f'(x) = 4f(2x), & x \in [0, 1]. \end{cases}$$
(17)
From $u(x) = u(1-x)$, $u'(x) = -u'(1-x)$ and $u(x) = u'(x/2)/4$ for $x \in [0, 1]$, it follows that, for $x \in [1, 2]$,
$$f(x) = f'(x/2)/4 = u'(x/2)/4 = -u'(1-x/2)/4 = -u(2-x) = -u(x-1).$$
Then we have (17). By (17) we have
\[ u'(x) = f'(x) = 4f(2x) = 4u(2x) - 4u(2x - 1), \quad x \in [0, 1]. \quad (18) \]
Assume that, for general \( m \geq 2, \)
\[
\begin{align*}
& \left\{ \begin{array}{l}
  f(x) = \sum_{k=1}^{2(m-1)} (-1)^n u(x - k + 1), \quad x \in [0, 2(m-1)], \\
  f'(x) = 4f(2x), \\
  x \in [0, m-1].
\end{array} \right.
\end{align*}
\]
Then
\[
\begin{align*}
& \left\{ \begin{array}{l}
  f(x) = (-1)^{n_m} u(x - m + 1), \\
  f'(x) = (-1)^{n_m} u'(x - m + 1), \\
  x \in [m - 1, m].
\end{array} \right.
\end{align*}
\] (20)
If \( x \in \{2(m-1), 2m\}, \) then \( x/2 \in [m - 1, m] \) and \( x/2 - m + 1 \in [0, 1]. \) By (20) and (18) we have
\[
\begin{align*}
  f(x) &= \frac{1}{4} f'(x/2) = \frac{1}{4} (-1)^{n_m} u'(x/2 - m + 1) \\
  &= (-1)^{n_m} u(2(x/2 - m + 1)) - (-1)^{n_m} u(2(x/2 - m + 1) - 1) \\
  &= (-1)^{n_m} u(x - (2m - 1) + 1) - (-1)^{n_m} u(x - 2m + 1) \\
  &= (-1)^{n_{2m-1}} u(x - (2m - 1) + 1) + (-1)^{n_{2m}} u(x - 2m + 1).
\end{align*}
\]
This shows that (19) holds for \( m \) instead of \( m - 1. \) Therefore we have (4) and
\[ f'(x) = 4f(2x), \quad x \in [0, +\infty). \]
Clearly
\[ f'(x) = 4f(2x), \quad x \in (-\infty, 0). \]
Then \( f \) is a solution of (3).

4. Fourier transform of the solution \( f \) in (4)

Let \( w = \chi_{[0,1]} \). Then \( Tv(x) = 2(w * v)(2x) \) for \( v \in L^1(\mathbb{R}) \) with \( \text{supp} \ v \subset [0, 1] \). Hence
\[
\mathcal{F}(Tv)(\xi) = \hat{w}(\xi/2) \hat{v}(\xi/2), \quad \hat{w}(\xi) = e^{-i\xi/2} \text{sinc}(\xi/(2\pi)).
\]
By
\[
\mathcal{F}(T^{k+1}v)(\xi) = \hat{w}(\xi/2) \mathcal{F}(T^k v)(\xi/2),
\]
we have
\[
\mathcal{F}(T^n v)(\xi) = \prod_{k=1}^{n} \hat{w}(\xi/2^k) \cdot \hat{v}(\xi/2^n) = \prod_{k=1}^{n} \left( e^{-i\xi/2^{k+1}} \text{sinc}(\xi/(2^{k+1}\pi)) \right) \cdot \hat{v}(\xi/2^n),
\]
this is
\[
\mathcal{F}(T^n v)(\xi) = e^{-i\xi(1/2^{-n+1})} \prod_{k=1}^{n} \text{sinc}(\xi/(2^{k+1}\pi)) \cdot \hat{v}(\xi/2^n). \quad (21)
\]

**Proof of Theorem 6.** Using (21), we have
\[
\hat{u}_n(\xi) = \mathcal{F}(T^n \chi_{[0,1]})(\xi) = e^{-i\xi/2} \prod_{k=1}^{n} \text{sinc}(\xi/(2^{k+1}\pi)) \cdot \text{sinc}(\xi/(2^{n+1}\pi)).
\]
Lemma 10 shows that \( u_n = T^n \chi_{[0,1]} \) converges to \( u \) in \( L^1(\mathbb{R}) \). Hence \( \hat{u}_n \) converges to \( \hat{u} \) uniformly on \( \mathbb{R} \). Using the fact that the right-hand side of (4) converges to \( f \) in \( S' \), we have the conclusion.

**Theorem 11.** Let \( v \in L^1(\mathbb{R}) \) and \( \text{supp} \, v \subset [0,1] \). Then \( T^n v(x) \) converges to \( \hat{v}(0)u(x) \) uniformly on \( \mathbb{R} \), where \( u \) is in Lemma 10.

**Proof.** By (21) and Theorem 6 we have that \( \| \mathcal{F}(T^n v)\chi_{[-R,R]} \|_{L^1} \) is small uniformly for large \( R > 0 \) and that
\[
\mathcal{F}(T^n v)(\xi) = \hat{u}_n(\xi) \frac{e^{i \xi / 2^n + 1} \hat{v}(\xi / 2^n)}{\text{sinc}(\xi / (2^{n+1} \pi))} \rightarrow \hat{u}(\xi) \hat{v}(0), \quad \text{uniformly on } [-R, R].
\]

Hence \( \mathcal{F}(T^n v) \) converges to \( \hat{v}(0) \hat{u}(\xi) \) in \( L^1(\mathbb{R}) \). Therefore we have the conclusion.

**Remark 12.** If \( v \) satisfies (1), (2), \( \text{supp} \, v \subset [0,1] \) and \( v(1/4) = b \) (\( b \in \mathbb{R} \)), then \( v = T v \) and
\[
b = v(1/4) = 2 \int_0^{1/2} v(t) \, dt = \int_0^1 v(t) \, dt = \hat{v}(0).
\]

Hence \( v = bu \).

5. **Proof of Theorem 7**

Let \( u_\delta(x) = u(x/\delta)/\delta \). Since \( \int_{\mathbb{R}} u(x) \, dx = 1 \),
\[
v * \frac{d^k u_\delta}{dx^k} = \frac{d^k v}{dx^k} * u_\delta \rightarrow \frac{d^k v}{dx^k} \quad (\delta \rightarrow 0),
\]
uniformly on each compact subset in \( \mathbb{R} \) or in \( L^p(\mathbb{R}) \). From \( u(x) = f(x) \chi_{[0,1]}(x) \) and \( f'(x) = 4f(2x) \) it follows that
\[
\frac{d^k u}{dx^k}(x) = 2^{(k^2 + 3k)/2} f(2^k x) \chi_{[0,1]}(x) = 2^{(k^2 + 3k)/2} (f \chi_{[0,2^k]})(2^k x).
\]

Hence
\[
\frac{d^k u_\delta}{dx^k}(x) = \frac{1}{\delta^{k+1}} \frac{d^k u}{dx^k}(x/\delta) = \frac{2^{(k^2 + 3k)/2}}{\delta^{k+1}} (f \chi_{[0,2^k]})(2^k x/\delta).
\]

Let \( \epsilon = \delta / 2^k \). Then \( \frac{d^k u_\delta}{dx^k}(x) \equiv G_{k,\epsilon}(x) \).

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**References**

