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Diffusion for the periodic wind-tree model

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DIFFUSION FOR THE PERIODIC WIND-TREE MODEL

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ABSTRACT. – The periodic wind-tree model is an infinite billiard in the plane with identical rectangular scatterers placed at each integer point. We prove that independently of the size of scatterers and generically with respect to the angle, the polynomial diffusion rate in this billiard is $2/3$.

RÉSUMÉ. – Le vent dans les arbres périodique est un billard infini construit de la manière suivante. On considère le plan dans lequel sont placés des obstacles rectangulaires identiques à chaque point entier. Une particule (identifiée à un point) se déplace en ligne droite (le vent) et rebondit de manière élastique sur les obstacles (les arbres). Nous prouvons qu'indépendamment de la taille des obstacles et génériquement par rapport à l'angle initial de la particule le coefficient de diffusion polynomial des orbites de ce billard est $2/3$.

1. Introduction

The wind-tree model is a billiard in the plane introduced by P. Ehrenfest and T. Ehrenfest in 1912 ([7]). We study the periodic version studied by J. Hardy and J. Weber [14]. A point moves in the plane $\mathbb{R}^2$ and bounces elastically off rectangular scatterers following the usual law of reflection. The scatterers are translates of the rectangle $[0, a] \times [0, b]$ where $0 < a < 1$ and $0 < b < 1$, one centered at each point of $\mathbb{Z}^2$. We denote the complement of obstacles in the plane by $T(a, b)$ and refer to it as the *wind-tree model* or the *infinite billiard table*. Our aim is to understand dynamical properties of the wind-tree model. We denote by $\phi_t^\theta : T(a, b) \rightarrow T(a, b)$ the billiard flow: for a point $p \in T(a, b)$, the point $\phi_t^\theta(p)$ is the position of a particle after time t starting from position p in direction θ .

It is proved in [14] that the rate of diffusion in the periodic wind-tree model is $\log t \log \log t$ for very specific directions (generalized diagonals which correspond to angles of the form $\arctan(p/q)$ with $p/q \in \mathbb{Q}$). Their result was recently completed by J.-P. Conze and E. Gutkin [6] who explicit the ergodic decomposition of the billiard flow for those directions. K. Frączek and C. Ulcigrai recently proved that generically the billiard flow is non-ergodic. P. Hubert, S. Lelièvre and S. Troubetzkoy [17] proved that for a residual set of parameters a

and b , for almost every direction θ , the flow in direction θ is recurrent. In this paper, we compute the polynomial rate of diffusion of the orbits which is valid for almost every direction θ . We get the following.

THEOREM 1. – *Let $d(.,.)$ be the Euclidean distance on $\mathbb{R}^2$. Then for all parameters $(a, b) \in (0, 1)^2$, Lebesgue-almost all θ and every point p in $T(a, b)$ (with an infinite forward orbit)*

$$\limsup_{T \rightarrow +\infty} \frac{\log d(p, \phi_T^\theta(p))}{\log T} = \frac{2}{3}.$$

By the $\mathbb{Z}^2$ -periodicity of the billiard table $T(a, b)$, our problem reduces to understand deviations of a $\mathbb{Z}^2$ cocycle over the billiard in a fundamental domain. On the other hand, as the barriers are horizontals and verticals, an orbit in $T(a, b)$ with initial angle θ from the horizontal takes at most four different directions $\{\theta, \pi - \theta, -\theta, \pi + \theta\}$ (the billiard is rational). By a standard construction consisting of unfolding the trajectories [29], called the Katok-Zemliakov construction, the billiard flow can be replaced by a linear flow on a (non compact) translation surface which is made of four copies of $T(a, b)$ that we denote $X_\infty(a, b)$ (see Section 3.2 for the construction). The surface $X_\infty(a, b)$ is $\mathbb{Z}^2$ -periodic and we denote $X(a, b)$ the quotient of $X_\infty(a, b)$ under the $\mathbb{Z}^2$ action. As the unfolding procedure of the billiard flow is equivariant with respect to the $\mathbb{Z}^2$ action, $X(a, b)$ can also be seen as the unfolding of the billiard in a fundamental domain of the action of $\mathbb{Z}^2$ on the billiard table $T(a, b)$.

The position of the particle in $X_\infty(a, b)$ can be tracked from $X(a, b)$. More precisely, the position of the particle starting from $p \in X_\infty(a, b)$ in direction θ can be approximated by the pairing of a geodesic $\gamma_t(p)$ of $X(a, b)$ seen as an element of the homology with a cocycle $f \in H^1(X(a, b); \mathbb{Z}^2)$ describing the infinite cover $X_\infty(a, b) \rightarrow X(a, b)$. The growth of pairing of a fixed cocycle with geodesics in a translation surface is equivalent to the growth of certain Birkhoff sums over an interval exchange transformation. The estimation can be obtained from the action of $SL(2, \mathbb{R})$ on strata of translation surfaces $\mathcal{H}_g(\alpha)$ and more precisely of the Teichmüller flow which corresponds to the action of diagonal matrices $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ (see Section 2 for precise definitions). As proved by A. Zorich [35, 36] the Kontsevich-Zorich cocycle over the Teichmüller flow can be used to estimate the deviations of Birkhoff sums for generic interval exchange transformations with respect to the Lebesgue measure. More precisely, he proved that the Lyapunov exponents of the Kontsevich-Zorich cocycle is the polynomial rate of deviations. G. Forni [12] relates this phenomenon to obstructions to solve cohomological equations and extends Zorich's proof to a more general context (see Section 9 of [12]).

The surface $X(a, b)$ is a covering of the genus 2 surface $L(a, b)$ which is a so called L-shaped surface that belongs to the stratum $\mathcal{H}(2)$. The orbit of $X(a, b)$ for the Teichmüller flow belongs to a sub-locus of the moduli space $\mathcal{H}(2^4)$ that we call $\mathcal{G}$.

We now formulate a generalization of A. Zorich's and G. Forni's theorems about deviations of ergodic averages that is a central step in the proof of Theorem 1. Let $\mathcal{H}(\alpha)$ be a stratum of Abelian differentials and $Y \in \mathcal{H}(\alpha)$ a translation surface. The Teichmüller flow (g_t) can be used to renormalize the trajectories of the linear flow on Y . The Kontsevich-Zorich cocycle $B^{(t)}(Y) : H^1(Y; \mathbb{R}) \rightarrow H^1(g_t \cdot Y; \mathbb{R})$ (or KZ cocycle) measures the growth of cohomology vectors along the Teichmüller geodesic $(g_t \cdot Y)_t$. Let μ be a g_t -invariant

ergodic probability measure on $\mathcal{H}(\alpha)$. It follows from [12], that the KZ cocycle is integrable for the measure μ . From Oseledets multiplicative ergodic theorem, there exist real numbers $\nu_1(\mu) > \nu_2(\mu) > \dots > \nu_k(\mu) > 0$, such that for μ -almost every non zero Abelian differential $Y \in \mathcal{H}(\alpha)$ there exists a unique flag

$$H^1(Y; \mathbb{R}) = F_1^u \supset F_2^u \supset \dots \supset F_k^u \supset F_{k+1}^u = F^c \supset F_k^s \supset \dots \supset F_1^s \supset F_0^s = \{0\}$$

such that for any norm $\|\cdot\|$ on $H^1(Y; \mathbb{R})$, for all $1 \leq i \leq k$

1. if $f \in F_i^u \setminus F_{i+1}^u$, then

$$\lim_{t \rightarrow \infty} \frac{\log \|B^{(t)}(Y) \cdot f\|}{\log t} = \nu_i(\mu),$$

2. if $f \in F_i^s \setminus F_{i-1}^s$, then

$$\lim_{t \rightarrow \infty} \frac{\log \|B^{(t)}(Y) \cdot f\|}{\log t} = -\nu_i(\mu),$$

3. if $f \in F^c \setminus F_k^s$, then

$$\lim_{t \rightarrow \infty} \frac{\log \|B^{(t)}(Y) \cdot f\|}{\log t} = 0.$$

There exist also positive integers m_i for $i = 1, \dots, k$ and an integer m such that for μ almost all translation surface Y the filtration satisfies

- the dimension of F_i^s is $m_1 + \dots + m_i$,
- the dimension of F^c is $m_1 + \dots + m_k + 2m$,
- the dimension of F_i^u is $m_1 + \dots + m_{i-1} + 2m_i + \dots + 2m_k + 2m$.

From the definition of the Teichmüller flow and the KZ cocycle, it follows that $\nu_1 = 1$. Forni proved that $m_1 = 1$ [12]. The *Lyapunov spectrum of the KZ cocycle* is the multiset of numbers

$$\nu_1 = 1 \quad \underbrace{\nu_2 \dots \nu_2}_{m_2 \text{ times}} \dots \underbrace{\nu_k \dots \nu_k}_{m_k \text{ times}} \quad \underbrace{0 \dots 0}_{2m \text{ times}} \quad \underbrace{-\nu_k \dots -\nu_k}_{m_k \text{ times}} \dots \underbrace{-\nu_2 \dots -\nu_2}_{m_2 \text{ times}} - 1 = -\nu_1.$$

The numbers $\nu_i(\mu)$ for $i = 1, \dots, k$ are called the *positive Lyapunov exponents (with respect to μ)*. The subspace $F^s = F_k^s$ is called the *stable space (at Y)* of the KZ cocycle.

In order to state a precise statement for deviations, one needs genericity with respect to Lyapunov exponents but also an extra assumption on recurrence. Let μ be a g_t ergodic measure on some stratum $\mathcal{H}(\alpha)$. We say that a surface $Y \in \mathcal{H}(\alpha)$ is *generic recurrent* for μ if there exist compact neighborhoods $U_i \subset \mathcal{H}(\alpha)$ of Y such that $\bigcap_i U_i = \{Y\}$ and

$$\lim_{t \rightarrow \infty} \frac{\text{Leb}(\{s; s \in [0, t] \text{ and } g_s Y \in U\})}{t} = \mu(U).$$

Birkhoff theorem ensures that this condition is satisfied for almost every surface.

THEOREM 2. – *Let μ be a g_t -ergodic measure on a stratum of Abelian differentials. Let ν_i for $i = 1, \dots, k$ denote the positive Lyapunov exponents of the KZ cocycle for μ and denote, for an Oseledets generic surface Y , $F_i^u(Y)$, $F^c(Y)$ and $F_i^s(Y)$ the components of the flag of the Oseledets decomposition.*

Then, for a surface $Y \in \mathcal{H}(\alpha)$ which is generically recurrent and Oseledets generic for μ , for every point $p \in Y$ with an infinite forward orbit

1. *along the unstable space the growth is polynomial: for all $1 \leq i \leq k$, for all $f \in F_i^u \setminus F_{i+1}^u$*

$$\limsup_{T \rightarrow \infty} \frac{\log |\langle f, \gamma_T(p) \rangle|}{\log T} = \nu_i,$$

2. *along the central space the growth is sub-polynomial: for all $f \in F^c \setminus F_k^s$*

$$\limsup_{T \rightarrow \infty} \frac{\log |\langle f, \gamma_T(p) \rangle|}{\log T} = 0,$$

3. *along the stable space the growth is bounded: there exists a constant C such that for all $f \in F^s$*

$$\forall T \geq 0, |\langle f, \gamma_T(p) \rangle| \leq C \|f\|.$$

Theorem 2 has first been proved by A. Zorich [34, 35, 36] for the Lebesgue measure on a connected component of a stratum or equivalently for a generic interval exchange transformation. G. Forni [12] extended the theorem for a very large class of functions and for certain measures. More precisely, his proof of the lower bound relies on the existence of a particular translation surface in the support of the measure. A. Bufetov [3] gave a proof of Case 1 of Theorem 2 (when the cocycle f is associated with a positive Lyapunov exponent) in the general context of symbolic dynamics which applies in particular to translation flows (Propositions 2 and 5 of [3]). Our approach uses Veech's zippered rectangles [31] and gives a concrete version of the renormalization process by the Teichmüller flow and the Kontsevich-Zorich cocycle in the flavor of [35, 36] and [12].

On the other hand, from results of A. Eskin, M. Kontsevich and A. Zorich [9] about sum of Lyapunov exponents in hyperelliptic loci, we deduce that the Lyapunov exponent for $X(a, b)$ which controls the deviation in the wind-tree model equals $2/3$. The value $2/3$ comes from algebraic geometry. More precisely, it corresponds to the degree of a subbundle of the Hodge bundle over the moduli space of complex curves (or Riemann surfaces) in which the wind-tree cocycle belongs.

Using only Birkhoff and Oseledets theorems, one can prove that the conclusion of Theorem 1 holds for almost every parameters a, b . In order to obtain all parameters we use a recent result of J. Chaika and A. Eskin [5] which asserts that Birkhoff theorem for regular functions and Oseledets theorem for the Kontsevich-Zorich cocycle are more regular for $SL(2, \mathbb{R})$ -invariant measures: they hold for all surfaces in almost every directions. The work of Chaika and Eskin strongly relies on previous work of A. Eskin and M. Mirzakhani [10] and A. Eskin, M. Mirzakhani and M. Mohamadi [11] on $SL(2, \mathbb{R})$ -invariant measures on strata of Abelian differentials.

The paper is organized as follows. In Section 2 we introduce the tools from Teichmüller theory which are involved in our proof of Theorem 1. In Section 3, we detail the unfolding procedure and prove that the distance in Theorem 1 corresponds to a pairing between a geodesic in $X(a, b)$ with an integer cocycle. Then we reformulate Theorem 1 in the language of translation surfaces (see Theorem 6). In Section 4 we compute the Lyapunov exponents relative to every measure on $\mathcal{H}(2^4)$ which is supported on the closure of the $SL(2, \mathbb{R})$ -orbit of a surface $X(a, b)$. Section 5 is devoted to the proof of Theorem 2.

The preprint [5] appeared after preliminary versions of this paper. In earlier versions, the conclusion of Theorem 1 was weaker and we rely heavily on classification of $\mathrm{SL}(2, \mathbb{R})$ -invariant measures in genus 2 by K. Calta [4] and McMullen [24, 25, 26].

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2. Background

The main objects in this paper are:

- closed compact translation surfaces—equivalently, closed compact Riemann surfaces endowed with a holomorphic 1-form;
- infinite-area periodic translation surfaces.

For general references on translation surfaces and interval exchange transformations we refer the reader to the survey of A. Zorich [37], J.-C. Yoccoz [33] or the notes of M. Viana [32].

A *translation surface* is a surface which can be obtained by edge-to-edge gluing of polygons in the plane using translations only. Such a surface is endowed with a flat metric (the one from $\mathbb{R}^2$) and a canonical direction. There is a one to one correspondence between compact translation surfaces and compact Riemann surfaces equipped with a non-zero holomorphic 1-form. If (Y, ω) is a Riemann surface together with a holomorphic one-form, the flat metric corresponds to $|\omega|^2$. In particular, the area of (Y, ω) is $i/2 \int \omega \wedge \bar{\omega}$.

In a translation surface, directions are globally defined. Hence the geodesic flow in a direction can be defined on the surface. There is a canonical vertical direction in each translation surface and we refer to the flow in this direction as the *linear flow*. The flow in the direction $\theta \in [0, 2\pi)$ for the differential ω on Y is the linear flow of $e^{-i\theta}\omega$ on Y . Note that the flow is not defined at the zeros of ω .

Now we define the moduli space of translation surfaces. We use a marking in order to avoid symmetries which create singularities in the moduli space. Let $\alpha = (\alpha_1, \dots, \alpha_s)$ and g be integers such that $\alpha_1 + \dots + \alpha_s = 2g - 2$. Let S be a compact (topological) surface of genus g and let $\Sigma = \{x_1, x_2, \dots, x_s\}$ be a set of s points in S . The stratum $\mathcal{H}_g(\alpha)$ is the set of (equivalence classes) translation structure on S with

1. zeros of degree α_i at x_i , and regular out of Σ ,
2. a horizontal separatrix is fixed at each x_i .

Two translation structures ω and η on S are identified if there exists a diffeomorphism $\phi: S \rightarrow S$ that fixes pointwise the set Σ maps ω to η and maps the marked horizontal separatrix of ω at x_i to the marked horizontal separatrix of η at x_i . We often use exponential notation for α , for example $\mathcal{H}(2^4)$ means $\mathcal{H}(2, 2, 2, 2)$ in our context. These strata can have up to three connected components, which were classified by M. Kontsevich and A. Zorich [20], and distinguished by two invariants: hyperellipticity and parity of spin structure. We denote by $\mathcal{H}^{(1)}(\alpha) \subset \mathcal{H}(\alpha)$ the codimension 1 subspace which consists of area 1 translation surfaces.

Each stratum $\mathcal{H}_g(\alpha)$ carries a natural affine structure which makes it a manifold (if we forget markings, we obtain an orbifold). The affine structure is obtained from the association of $(S, \omega) \in \mathcal{H}_g(\alpha) \mapsto [\omega] \in H^1(S, \Sigma; \mathbb{C})$ (the period map). On translation surfaces obtained

by polygon gluings, this map may be seen as the edges (as element of $\mathbb{C}$). In that model, two surfaces are nearby if they are obtained from the same gluings and the edges are nearby for the natural topology on finite dimensional vector spaces.

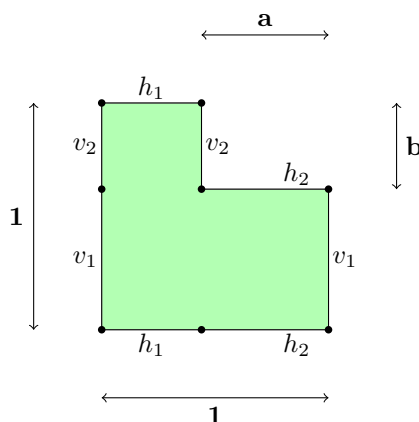
There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on components of strata $\mathcal{H}(\alpha)$ coming from the linear action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$. More precisely, let (Y, ω) be a translation surface obtained by gluing a finite family of polygons (P_i) and $g \in \mathrm{SL}_2(\mathbb{R})$. Then the surface $g \cdot (Y, \omega)$ is the surface obtained by gluing the polygons $(g \cdot P_i)$. The *Teichmüller geodesic flow* on $\mathcal{H}_g$ is the action of the diagonal matrices $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. The image of the orbits $(g_t \cdot (X, \omega))_t$ in $\mathcal{M}_g$ is geodesic with respect to the Teichmüller metric. Each stratum $\mathcal{H}_g(\alpha)$ carries a natural *Lebesgue measure*, invariant under the action of $\mathrm{SL}(2, \mathbb{R})$. Moreover, this action preserves the area and hence $\mathcal{H}^{(1)}(\alpha)$. H. Masur [23] and independently W. Veech [31] proved that on each component of a normalized stratum $\mathcal{H}^{(1)}(\alpha)$ the total mass of the Lebesgue measure is finite and the geodesic flow acts ergodically with respect to this measure. Another important one parameter flow on $\mathcal{H}(\alpha)$ is the *horocycle flow* given by the action of $h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

More generally, one can consider the strata of quadratic differentials with at most simple poles $\mathcal{Q}_g(\alpha)$ where α is an integer partition of $4g - 4$. The degree α_i corresponds to a conic point of angle $(2 + \alpha_i)\pi$. A translation surface associated to a quadratic differential may have non trivial holonomy with value in $\{1, -1\}$. The action of $\mathrm{SL}(2, \mathbb{R})$ on Abelian differentials extends to quadratic differentials.

Stabilizers for the action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{H}_g$ or $\mathcal{Q}_g$, called *Veech groups*, are discrete non-cocompact subgroups of $\mathrm{SL}_2(\mathbb{R})$; they are trivial (i.e., either $\{\mathrm{Id}\}$ or $\{\mathrm{Id}, -\mathrm{Id}\}$) for almost every surface in each stratum component, and in exceptional cases are lattices (i.e., finite-covolume subgroups) in $\mathrm{SL}_2(\mathbb{R})$. In such cases, the surface satisfies the Veech dichotomy: in every direction, the linear flow is either uniquely ergodic, or decomposes the surface into a finite union of cylinders of periodic trajectories (see [31]). Closed compact translation surfaces with a lattice Veech group are exactly those whose $\mathrm{SL}_2(\mathbb{R})$ -orbit is closed in the corresponding stratum component. They are called *Veech surfaces*. Their orbits project to *Teichmüller curves* in the moduli space $\mathcal{M}_g$ of closed compact Riemann surfaces of genus g . A translation surface is a *square-tiled surface* if it is a ramified cover of the torus $\mathbb{R}^2/\mathbb{Z}^2$ with only 0 as ramification point. Square-tiled surfaces are examples of Veech surfaces. Their Veech groups are commensurable to $\mathrm{SL}_2(\mathbb{Z})$.

The simplest stratum besides the one of tori is $\mathcal{H}(2)$ which consists of equivalence classes of 1-forms with a double zero (in flat surfaces terms a cone point of angle 6π) on Riemann surfaces of genus two. Important examples of such surfaces are given by the family of surfaces $L(a, b)$ with $0 < a < 1, 0 < b < 1$ which are built as follows (see also Figure 1). Let $0 < a < 1$ and $0 < b < 1$. Consider the polygon with extremal points $(0, 0), (1 - a, 0), (1, 0), (1, 1 - b), (1 - a, 1 - b), (1 - a, 1), (0, 1), (0, 1 - b)$ and glue the opposite sides together:

1. $[(0, 0), (1 - a, 0)]$ with $[(0, 1), (1 - a, 1)]$ (the side h_1 labeled on Figure 1),
2. $[(1 - a, 0), (1, 0)]$ with $[(1 - a, 1 - b), (1, 1 - b)]$ (the side h_2),
3. $[(0, 0), (0, 1 - b)]$ with $[(1, 0), (1, 1 - b)]$ (the side v_1),
4. $[(0, 1 - b), (0, 1)]$ with $[(1 - a, 1 - b), (1 - a, 1)]$ (the side v_2).


 FIGURE 1. The surface $L(a, b)$ built from a L-shaped polygon.

The stratum $\mathcal{H}(2)$ is connected and is the best understood. It was proven that the Teichmüller curves are generated by surfaces of the form $L(a, b)$.

THEOREM 3 (Caltà [4], McMullen [24, 25]). – *The surface $L(a, b)$ is a Veech surface if and only if*

1. *either $a, b \in \mathbb{Q}$ in which case $L(a, b)$ is square-tiled,*
2. *or there exist $x, y \in \mathbb{Q}$ and $D > 1$ a square-free integer such that $1/(1 - a) = x + y\sqrt{D}$ and $1/(1 - b) = (1 - x) + y\sqrt{D}$.*

Moreover, any Teichmüller curve in $\mathcal{H}(2)$ contains (up to rescaling the area) a surface of the form $L(a, b)$.

In his fundamental work, C. McMullen [26] proved a complete classification theorem for $SL_2(\mathbb{R})$ -invariant measures and closed invariant set.

THEOREM 4 (McMullen, [26] Theorems 10.1 and 10.2 p. 440–441)

The only $SL(2, \mathbb{R})$ -invariant irreducible closed subsets of $\mathcal{H}(2)$ are the Teichmüller curves and the whole stratum. The only $SL(2, \mathbb{R})$ -invariant probability measures are the Haar measure carried on Teichmüller curves and the Lebesgue measure on the stratum.

Let $g \geq 2$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s)$ an integer partition of $2g - 2$. The *Hodge bundle* E_g is the real vector bundle of dimension $2g$ over $\mathcal{H}_g(\alpha)$ where the fiber over $(S, \omega) \in \mathcal{H}_g(\alpha)$ is the real cohomology $H^1(S; \mathbb{R})$. Each fibre $H^1(X; \mathbb{R})$ has a natural lattice $H^1(X; \mathbb{Z})$ which allows identification of nearby fibers and definition of the Gauss-Manin (flat) connection. The holonomy along the Teichmüller geodesic flow provides a symplectic cocycle called the *Kontsevich-Zorich cocycle*. It is formally defined as a map $B^{(t)} : H^1(X; \mathbb{R}) \rightarrow H^1(g_t \cdot X; \mathbb{R})$. As each holonomy element corresponds to the action in homology of an element of the mapping class group of X , the map $B^{(t)}$ is symplectic. For a small transversal U of the Teichmüller flow on $\mathcal{M}_g$ for which there exists a trivialization of the Hodge bundle, we identify all fibers with a fixed $H^1(X; \mathbb{R})$ where $X \in U$. The sequence of first return times

$0 = t_0 < t_1 < \dots$ of $g_t \cdot X$ in U gives a sequence of symplectic matrices in $H^1(X; \mathbb{R})$ that we still denote $B^{(t_n)}(X)$.

For each g_t -invariant ergodic probability measure for the Teichmüller geodesic flow on $\mathcal{H}_g$, this cocycle has associated Lyapunov exponents. Based on computer experimentations, M. Kontsevich [21] conjectured a formula for the sum of positive Lyapunov exponents of the cocycle for Lebesgue measures on strata as well as for Veech surfaces. These formulas are now fully proven [8, 9].

An *automorphism* of a translation surface (S, ω) is a diffeomorphism $\phi : S \rightarrow S$ that preserves ω (in other words, it acts by translations in the natural charts of ω). We warn the reader that with our convention of markings, even if a point (S, ω) in $\mathcal{H}_g(\alpha)$ admits some automorphisms, it has trivial stabilizer as element of $\mathcal{H}_g(\alpha)$. In some concrete situations, as the one of the wind-tree model described in that article, the existence of automorphisms provides an $\mathrm{SL}_2(\mathbb{R})$ -equivariant splitting of the Hodge bundle. Under suitable assumptions for the $\mathrm{SL}(2, \mathbb{R})$ -subbundles (relative to variations of Hodge structure), it appears that for each of them there is a formula for the sum of positive Lyapunov exponents of the restricted Kontsevich-Zorich cocycle. Sometimes even individual Lyapunov exponents can be computed (see [2], [13], [8]). We recall a theorem of [9] which is a formula for the sum of Lyapunov exponents for the so called hyperelliptic loci of a stratum.

Quadratic differentials with at most simple poles on a Riemann surface are natural generalizations of translation surfaces. In that case, the holonomy is not necessarily trivial and may have values in $\{+1, -1\}$. On a Riemann surface with a quadratic differential, directions are still globally defined and there is an action of $\mathrm{PSL}(2, \mathbb{R})$. The stratum $\mathcal{Q}(d_1, \dots, d_n)$ denotes the moduli space of area 1 quadratic differentials with singularities of angles $(2 + d_1)\pi, \dots, (2 + d_n)\pi$ which are not squares of an Abelian differential.

Let q be a quadratic differential on some Riemann surface S . The foliation on S in some direction θ is not orientable. There is a canonical way to define a double cover $\pi : \tilde{S} \rightarrow S$ ramified at the singularities for which the degree d_i is odd and for which π^*q is the square of an Abelian differential on $\tilde{S}$. The locus of such double covers when the pair (S, q) moves in its stratum gives an $\mathrm{SL}(2, \mathbb{R})$ invariant locus in an Abelian stratum called *orientation cover locus*. When S is a sphere, the orientation cover locus is called a *hyperelliptic locus*. For Lyapunov exponents of hyperelliptic loci, the following general theorem holds.

THEOREM 5 (Eskin-Kontsevich-Zorich [9], Corollary 1 p. 14). – *Let μ be an $\mathrm{SL}(2, \mathbb{R})$ -invariant ergodic probability measure on a stratum $\mathcal{H}_g(\alpha)$ of Abelian differential. Assume that μ comes from the orientation covering morphism of an $\mathrm{SL}(2, \mathbb{R})$ -invariant (regular) measure $\bar{\mu}$ on a stratum of quadratic differentials on the sphere $\mathcal{Q}(d_1, d_2, \dots, d_n)$. Then, the sum of positive Lyapunov exponents $\nu_1 \geq \dots \geq \nu_g$ for the measure μ is given by*

$$\nu_1 + \dots + \nu_g = \frac{1}{4} \sum_{j \text{ with } d_j \text{ odd}} \frac{1}{d_j + 2}.$$

In particular the value of the sum does not depend on the measure but only on the stratum $\mathcal{Q}(d_1, d_2, \dots, d_n)$. For the condition of *regular measure* which appears in the statement of Theorem 5 we refer to Definition 1 p. 9 of [9]. We emphasize that all known $\mathrm{SL}(2, \mathbb{R})$ -ergodic measures on strata of Abelian differentials are regular.

For infinite-area translation surfaces, it is not clear what the good notions of moduli spaces are. However, the action of $\mathrm{SL}_2(\mathbb{R})$ still makes sense, and Veech groups can be defined [27, 30]. An *infinite periodic translation surface* is an infinite area translation surface which is an infinite normal cover of a finite area translation surface. We say Γ -infinite translation surface to specify the Deck group Γ . Examples of $\mathbb{Z}$ -infinite translation surfaces are studied by P. Hubert and G. Schmithüsen in [18] and a general formalism is introduced by P. Hooper and B. Weiss in [16]. For some particularly symmetric examples, it is possible to get a very complete picture of the dynamics [15]. The family of surfaces $X_\infty(a, b)$ obtained by unfolding the billiard tables $T(a, b)$ are $\mathbb{Z}^2$ -infinite translation surfaces.

3. From infinite billiard table to finite surface

First of all, the flow in the billiard table $T(a, b)$ is invariant under $\mathbb{Z}^2$ translation. Secondly, the angles between the scatterers are multiples of $\pi/2$ and the Katok-Zemliakov construction conjugates the billiard flow on $T(a, b)$ to a linear flow on an infinite translation surface $X_\infty(a, b)$. Using these two ingredients, we reduce the study of the billiard flow into the study of a $\mathbb{Z}^2$ -cocycle over the linear flow of a finite translation surface $X(a, b)$. The surface $X(a, b)$ obtained by unfolding a fundamental domain of the table $T(a, b)$ is an intermediate cover between the finite surface $L(a, b)$ of genus 2 and the infinite surface $X_\infty(a, b)$. The surface $X(a, b)$ is the main actor of this paper.

Notation. – For the whole section, we fix $0 < a < 1$ and $0 < b < 1$.

3.1. Unfolding the fundamental domain

A fundamental domain for the $\mathbb{Z}^2$ action on the infinite billiard $T(a, b)$ can be seen either as a torus with a square obstacle inside (see Figure 2a) or as a surface $L = L(a, b)$ with barriers on its boundary (see Figure 2b). The Katok-Zemliakov construction (or unfolding procedure) of the billiard in the fundamental domain gives a surface $X(a, b)$ made of 4 reflected copies of the fundamental domain (see Figure 3). The surface $X(a, b)$ was studied in the particular case $a = b = 1/2$ by different authors [22], [28], [13], [8] and is called in this particular case the 6-escalator (see Figure 3b for the origin of the name).

LEMMA 1. – *The surface $X(a, b)$ is a genus 5 surface in $\mathcal{H}(2^4)$. It is a normal unramified cover of the surface $L(a, b)$ with a Deck group K isomorphic to the Klein four-group $K = \mathbb{Z}/2 \times \mathbb{Z}/2$.*

Proof. – The billiard table $T(a, b)$ is invariant under horizontal and vertical reflections as well as the billiard in a fundamental domain. It is then straightforward to show that $X(a, b)$ is an unramified normal cover of $L(a, b)$ with group $\mathbb{Z}/2 \times \mathbb{Z}/2$. A direct computation shows that $X(a, b)$ has 4 singularities of angle 6π (see Figure 3). $\square$

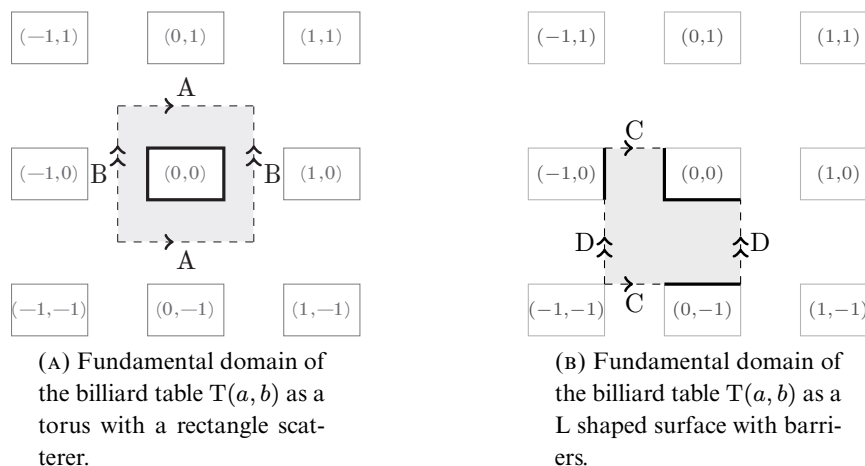


FIGURE 2. Two versions of the fundamental domains for the billiard table $T(a, b)$. The boundaries of the scatterers are thick and the arrows together with letters indicate the gluings.

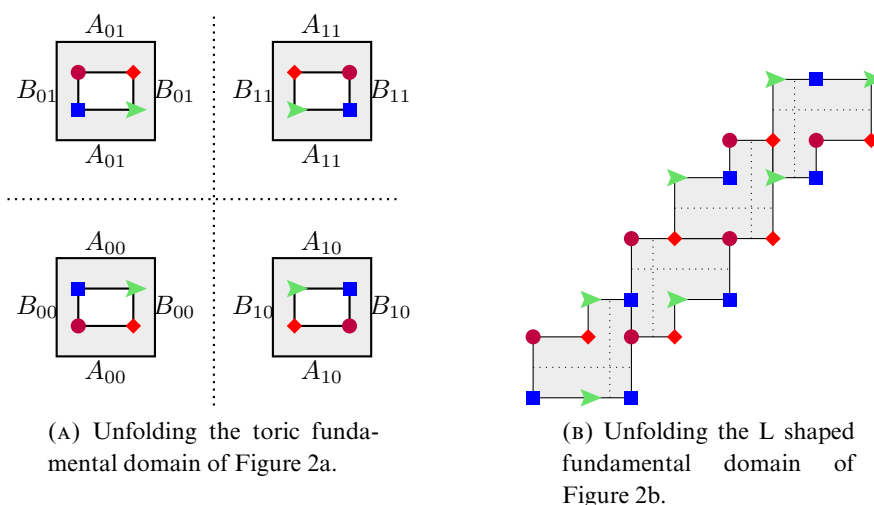


FIGURE 3. Two versions of the surface $X(a, b)$ obtained by unfolding the billiard in a fundamental domain. The gluings of edges are indicated by labels in case of ambiguity.

3.2. The surface $X_\infty(a, b)$ as a $\mathbb{Z}^2$ cover of $X(a, b)$

As we did for unfolding the fundamental domain of the infinite billiard, we consider the unfolding of the whole billiard table $T(a, b)$. The unfolding leads to a non-compact surface that we denote $X_\infty(a, b)$ which is made of four copies of the initial billiard. As the unfolding commutes with the action of $\mathbb{Z}^2$ the surface $X(a, b)$ is also the $\mathbb{Z}^2$ quotient of $X_\infty(a, b)$. We use

this description to rewrite the distance in Theorem 1 as a pairing between a geodesic segment in $X(a, b)$ with a cocycle in $H^1(X; \mathbb{Z}^2)$.

We first build a system of generators for the homology of $X(a, b)$. We label each copy of the torus fundamental domain in $X(a, b)$ by 00, 01, 10 and 11 (see Figures 3a and 4). For $\kappa \in \{00, 10, 01, 11\}$ let h_κ (resp. v_κ) be the horizontal (resp. vertical) simple closed curve that delimit each copy (the exterior boundary). The curves h_κ (resp. v_κ) have holonomy 1 (resp. i). The automorphism group $K \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ of $X(a, b)$ acts on the indices of h_κ and v_κ by addition (where we consider 0 and 1 as elements of $\mathbb{Z}/2$). The intersection form $\langle \cdot, \cdot \rangle$ on $X(a, b)$ is such that $\langle h_\kappa, v_{\kappa'} \rangle = \delta_{\kappa, \kappa'}$ where δ_{ij} is the Kronecker symbol. In other words, the module generated by the elements h_κ and v_κ is a symplectic submodule and $\{(h_\kappa, v_\kappa)\}_\kappa$ is a symplectic basis. Moreover, this $\mathbb{Z}$ -submodule is invariant under the action of K (but not irreducible, see Lemma 4 below).

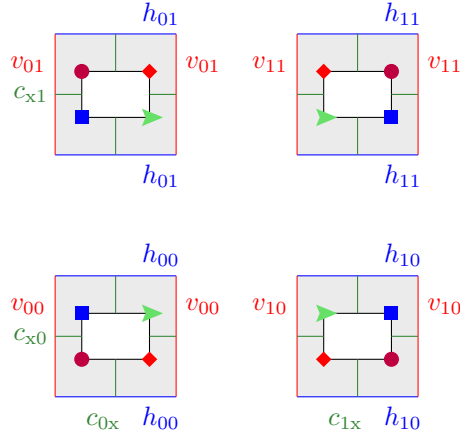


FIGURE 4. Homology generators for $X(a, b)$.

We consider four more elements of $H_1(X; \mathbb{Z})$. Let c_{x0} (resp. c_{x1}) be the circumferences of the horizontal cylinder that intersects the two copies 00 and 10 (resp. 01 and 11) of the torus fundamental domain. The curves c_{x0} and c_{x1} have both holonomy $(2 - 2a, 0)$. We define as well the curves c_{0x} and c_{1x} with respect to the vertical cylinders. The curves c_{0x} and c_{1x} have both holonomy $(0, 2 - 2b)$. As before the action of K as automorphism group of $X(a, b)$ corresponds to an action on indices of c_{ij} if we set $0 \cdot x = 1 \cdot x = x$.

There are two relations in $H_1(X; \mathbb{Z})$ among the curves defined above

$$(1) \quad \begin{aligned} c_{x0} - c_{x1} &= h_{00} - h_{01} + h_{10} - h_{11} \\ c_{0x} - c_{1x} &= v_{00} - v_{10} + v_{01} - v_{11}. \end{aligned}$$

We have the following elementary

LEMMA 2. — *The relations (1) are the only relations in the family $\{h_{ij}, v_{ij}, c_{xj}, c_{ix}\}$. Let E_h (resp. E_v) be the span of $\{h_{ij}, c_{xk}\}_{i,j,k \in \{0,1\}}$ in $H_1(X(a, b); \mathbb{Z})$ (res. of $\{v_{ij}, c_{kx}\}_{i,j,k \in \{0,1\}}$ in $H_1(X(a, b); \mathbb{Z})$); then $H_1(X(a, b); \mathbb{Z}) = E_h \oplus E_v$ and the sum is orthogonal with respect to the intersection form.*

The infinite cover $X_\infty(a, b) \rightarrow X(a, b)$ corresponds to a certain subgroup H of $\pi_1(X(a, b))$ such that $\pi_1(X(a, b))/H \simeq \mathbb{Z}^2$. But as the cover is normal and $\text{Deck}(X_\infty(a, b)/X(a, b)) \simeq \mathbb{Z}^2$ is an Abelian group, there exists a factorisation through the Abelianisation $H_1(X(a, b); \mathbb{Z})$ of $\pi_1(X(a, b))$ (see also [16] for the description of $\mathbb{Z}$ -cover). In other terms the cover is defined by an element of $H^1(X(a, b); \mathbb{Z}^2)$ and more precisely we have the following explicit description.

LEMMA 3. – *The $\mathbb{Z}^2$ covering $T(a, b)/X(a, b)$ is given by the dual $f \in H^1(X; \mathbb{Z}^2)$ with respect to the intersection form of the cycle*

$$\begin{pmatrix} v_{00} - v_{10} + v_{01} - v_{11} \\ h_{00} - h_{01} + h_{10} - h_{11} \end{pmatrix} \in H_1(X; \mathbb{Z}^2).$$

In other words, the subgroup of $\pi_1(X(a, b))$ associated to the covering is the kernel of

$$\pi_1(X(a, b)) \xrightarrow{Ab} H_1(X(a, b); \mathbb{Z}) \xrightarrow{f} \mathbb{Z}^2.$$

Proof. – As before we consider the surface decomposed into four copies of the torus fundamental domain labeled 00, 10, 01 and 11. The labeling fits the action of the Klein 4 group on the surface. Let γ be a smooth curve in $T(a, b)$ which follows the law of reflection when it hits an obstacle. Let $\bar{\gamma}$ be its image in $X(a, b)$. There is an ambiguity for the starting point of $\bar{\gamma}$ and we assume that we start in the copy 00. Each time the curve $\bar{\gamma}$ hits a side associated to a vertical (resp. horizontal) scatter the curve $\bar{\gamma}$ switches from the copy κ to $(1, 0) \cdot \kappa$ (resp. $(0, 1) \cdot \kappa$). At the same time, in the infinite table $T(a, b)$ the curve γ is reflected vertically (resp. horizontally). When the curve crosses a vertical (resp. a horizontal) boundary of the fundamental domain (labelled A (resp. B) in Figure 2a) the curve $\bar{\gamma}$ remains in the same copy. In other words, the endpoint of γ in $T(a, b)$ only depends on the monodromy of $\bar{\gamma}$ with respect to X_∞/X and we need to consider only the case of the curves $\gamma = h_{ij}, v_{ij}$ for $i = 0, 1$ and $j = 0, 1$.

As the copies 00 and 01 in $X(a, b)$ correspond to the absence of vertical reflection, the monodromy of v_{00} and v_{01} is $(1, 0)$. Whereas for the copies 10 and 11, the curve γ has been reflected and the monodromy of h_{10} and h_{11} is $(-1, 0)$. The same analysis can be made for the curves v_{ij} and the lemma follows from duality between $\{h_\kappa\}$ and $\{v_\kappa\}$. $\square$

Now, we use the description of $X_\infty(a, b) \rightarrow X(a, b)$ in terms of homology to approximate the distance $d(p, \phi_t^\theta(p))$ of Theorem 1 in terms of intersection. But first of all, we need to approximate the geodesic segment by elements of $H_1(X, \mathbb{Z})$.

For each triple $(p, \theta, t) \in X \times S^1 \times \mathbb{R}_+$ we define an element $\gamma_t^\theta(p) \in H_1(X; \mathbb{Z})$ as follows. Consider the geodesic segment of length t from p in the direction θ and close it by a small piece of curve that does not intersect any curve h_κ nor v_κ . The curve used to close the geodesic segment can be chosen to be uniformly bounded.

The proposition below shows that the distance of the particle in the billiard $T(a, b)$ can be reduced to the study of the pairing of the approximative geodesic $\gamma_t^\theta(p)$ in $X(a, b)$ and the cocycle $f \in H^1(X; \mathbb{Z}^2)$ defined in Lemma 3.

PROPOSITION 1. – *Let $\|\cdot\|_2$ be the Euclidean norm on $\mathbb{R}^2$. Let $p \in X(a, b)$ and $\tilde{p} \in T(a, b)$ be the lift of p which belongs to the translate of the fundamental domain that contains the point $(0, 0)$. Let $f \in H^1(X; \mathbb{Z}^2)$ be as in the previous lemma. Then*

$$\|\langle f, \gamma_t^\theta(p) \rangle - \phi_t^\theta(\tilde{p})\|_2 \leq \sqrt{2}.$$

In particular

$$|\|\langle f, \gamma_t^\theta(p) \rangle\|_2 - d(\tilde{p}, \phi_t^\theta(\tilde{p}))| \leq \sqrt{2}.$$

Proof. – The distance between the point $\phi_t^\theta(p) \in \mathbb{R}^2$ and the associated level $\langle f, \gamma_t^\theta(p) \rangle \in \mathbb{Z}^2$ is bounded from above by the diameter of the fundamental domain. The latter is uniformly bounded by $\sqrt{2}$ (with respect to the parameters a and b). $\square$

As a consequence of the above proposition we reformulate our main result (Theorem 1).

THEOREM 6. – *Let $0 < a < 1$, $0 < b < 1$ and $\gamma_T^\theta(p)$ be the approximative geodesic starting from p in direction θ in $X(a, b)$.*

1. *If a and b are rational numbers or can be written as $1/(1-a) = x + y\sqrt{D}$ and $1/(1-b) = (1-x) + y\sqrt{D}$ with $x, y \in \mathbb{Q}$ and $D > 1$ a positive square-free integer, then for almost every θ and every point p in $X(a, b)$ (with an infinite forward orbit under the linear flow)*

$$\limsup_{T \rightarrow \infty} \frac{\log |\langle f, \gamma_T^\theta(p) \rangle|}{\log T} = \frac{2}{3}.$$

2. *For almost all $(a, b) \in (0, 1)^2$, for almost all θ and for every point p in $X(a, b)$ (with an infinite forward orbit)*

$$\limsup_{T \rightarrow \infty} \frac{\log |\langle f, \gamma_T^\theta(p) \rangle|}{\log T} = \frac{2}{3}.$$

3.3. Deck group action on $X(a, b)$

We study the covering $X(a, b)/L(a, b)$ which is normal with Deck group the Klein four group $K = \mathbb{Z}/2 \times \mathbb{Z}/2$ by Lemma 1.

Let v_{ij} , h_{ij} , c_{xj} and c_{ix} for $i, j \in \{0, 1\}$ be the generators of $H_1(X; \mathbb{Z})$ defined in Section 3.2. We identify them to vectors in the cohomology $H^1(X; \mathbb{Z})$ by duality. The action of the Klein four group K on $X(a, b)$ splits the homology in four subspaces. For the generators $\tau_v = (1, 0)$ and $\tau_h = (0, 1)$ of K we define the subspace E^{+-} to be the set of vectors $v \in H_1(X; \mathbb{Z})$ such that $\tau_v(v) = +1$ and $\tau_h(v) = -1$. We define similarly E^{++} , E^{-+} and E^{--} .

We denote $h_K = h_{00} + h_{01} + h_{10} + h_{11}$ and $v_K = v_{00} + v_{01} + v_{10} + v_{11}$.

LEMMA 4. – *The action of the Deck group of $X(a, b) \rightarrow L(a, b)$ splits the cohomology into four subspaces*

$$H^1(X(a, b); \mathbb{Q}) = E^{++} \oplus E^{+-} \oplus E^{-+} \oplus E^{--},$$

where each subspace E^κ is defined over $\mathbb{Q}$ as follows

- $E^{++} = \mathbb{Q}[h_K] \oplus \mathbb{Q}[c_{x0} + c_{x1}] \oplus \mathbb{Q}[v_K] \oplus \mathbb{Q}[c_{0x} + c_{1x}] \simeq H_1(L(a, b); \mathbb{Q})$,
- $E^{+-} = \mathbb{Q}[h_{00} - h_{01} + h_{10} - h_{11}] \oplus \mathbb{Q}[v_{00} - v_{01} + v_{10} - v_{11}]$,
- $E^{-+} = \mathbb{Q}[h_{00} + h_{01} - h_{10} - h_{11}] \oplus \mathbb{Q}[v_{00} + v_{01} - v_{10} - v_{11}]$,
- $E^{--} = \mathbb{Q}[h_{00} - h_{01} - h_{10} + h_{11}] \oplus \mathbb{Q}[v_{00} - v_{01} - v_{10} + v_{11}]$.

We emphasise that the invariant part of $H^1(X(a, b); \mathbb{Z})$ under the subgroup $\langle \tau_v \rangle \subset K$ is identified with $H^1(X(a, b)/\langle \tau_v \rangle; \mathbb{Z})$. This is the main reason for which we consider each quotient of $X(a, b)$ by the three subgroups of order two generated by τ_v , τ_h and $\tau_h \tau_v$.

LEMMA 5. – *The surfaces $X(a, b)/\langle \tau_v \rangle$ and $X(a, b)/\langle \tau_h \rangle$ belong to the hyperelliptic component $\mathcal{H}^{\text{hyp}}(2, 2)$ while the surface $X(a, b)/\langle \tau_h \tau_v \rangle$ belongs to the hyperelliptic locus $\mathcal{L} \subset \mathcal{H}^{\text{odd}}(2, 2)$.*

Proof. – We see on the two figures below that the central symmetry in each polygonal representation of the surfaces $X(a, b)/\langle \tau_v \rangle$ and $X(a, b)/\langle \tau_v \tau_h \rangle$ gives rise to an involution that does not preserve the directions: the direction θ is sent to $-\theta$. The quotient by such involution gives rise to quadratic differentials. In both cases the quotient is a sphere endowed

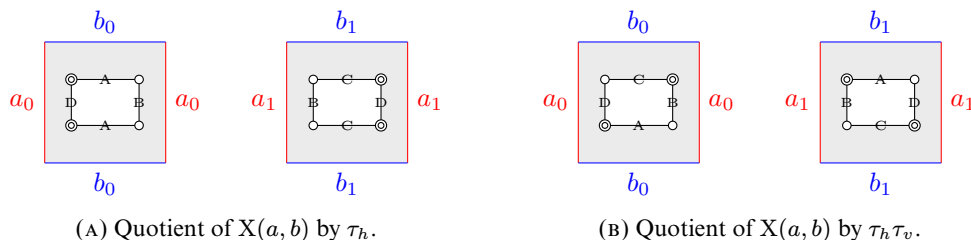


FIGURE 5. The quotients of degree 2 of $X(a, b)$.

with a quadratic differential and hence the surfaces $X(a, b)/\langle \tau_v \rangle$ and $X(a, b)/\langle \tau_v \tau_h \rangle$ are hyperelliptic. In the first quotient, the two singularities are exchanged and hence $X(a, b)/\langle \tau_v \rangle$ belongs to $\mathcal{H}^{\text{hyp}}(2, 2)$ which corresponds to the orientation cover of quadratic differentials in $\mathcal{Q}(4, -1^8)$ (this is the definition of $\mathcal{H}^{\text{hyp}}(2, 2)$, see [20]). While for $X(a, b)/\langle \tau_v \tau_h \rangle$ the zeros are fixed by the involution and the surface belongs to the hyperelliptic locus $\mathcal{L} \subset \mathcal{H}^{\text{odd}}(2, 2)$ which corresponds to the orientation cover of quadratic differentials in $\mathcal{Q}(1^2, -1^6)$. $\square$

4. Moduli space and Lyapunov exponents

In this section, using McMullen classification of $\text{SL}(2, \mathbb{R})$ -invariant closed set and probability measures in $\mathcal{H}(2)$, we classify the possible closures $\overline{\text{SL}(2, \mathbb{R}) \cdot X(a, b)}$ of $\text{SL}(2, \mathbb{R})$ -orbits of the surfaces $X(a, b)$ in $\mathcal{H}(2^4)$. Each closure carries a unique $\text{SL}(2, \mathbb{R})$ -invariant ergodic probability measure and we compute the Lyapunov exponents of the Kontsevich-Zorich cocycle with respect to it.

4.1. Moduli space and $X(a, b)$

We recall that $X(a, b) \in \mathcal{H}(2^4)$ is a cover of $L(a, b) \in \mathcal{H}(2)$ (Lemma 1). This property is preserved by the action of $\text{SL}(2, \mathbb{R})$ and more precisely the action of $\text{SL}(2, \mathbb{R})$ is equivariant: for any $g \in \text{SL}(2, \mathbb{R})$ the surface $g \cdot X(a, b)$ is a cover of $g \cdot L(a, b)$. Hence, all $\text{SL}(2, \mathbb{R})$ -orbits of $X(a, b)$ belong to the sublocus of $\mathcal{H}(2^4)$ which corresponds to the particular covering of surfaces in $\mathcal{H}(2)$. This locus, which we denote by $\mathcal{G}$, is a closed $\text{SL}(2, \mathbb{R})$ -invariant subvariety of $\mathcal{H}(2^4)$ which is a finite cover of $\mathcal{H}(2)$. In particular, McMullen's classification Theorem for $\text{SL}(2, \mathbb{R})$ -invariant closed subset and probability measures (Theorem 4) holds for closures of $\text{SL}(2, \mathbb{R})$ -orbits of $X(a, b)$.

The action of the Klein four-group K on surfaces $X(a, b)$ and the splitting of Lemma 4 hold for any surface Y in $\mathcal{G}$. For any $Y \in \mathcal{G}$ we denote by $\bar{Y} = Y/K$ its quotient in $\mathcal{H}(2)$. We have maps $H_1(Y; \mathbb{R}) \rightarrow H_1(\bar{Y}; \mathbb{R})$ (resp. $H^1(\bar{Y}; \mathbb{R}) \rightarrow H^1(Y; \mathbb{R})$) which are equivariant with respect to the Kontsevich-Zorich cocycle. In particular the explicit decomposition in the first part of Lemma 4 remains valid for any surface Y in $\mathcal{G}$ as it depends only of the

action of K . In particular, we get an $\mathrm{SL}_2(\mathbb{R})$ -equivariant splitting of the Hodge bundle. But, as $H_1(X(a, b); \mathbb{Z})$ and $H_1(Y; \mathbb{Z})$ can only be identified locally, the explicit basis of homology we have exhibited for $X(a, b)$ has no meaning for Y .

4.2. Computation of Lyapunov exponents

In this section we compute the individual Lyapunov exponents of the KZ cocycle for all $\mathrm{SL}(2, \mathbb{R})$ -invariant ergodic measures on $\mathcal{G}$. We denote by $E \rightarrow \mathcal{G}$ the Hodge bundle over $\mathcal{H}(2^4)$ restricted to $\mathcal{G}$.

Recall, that the KZ cocycle is symplectic. Hence, the Lyapunov exponents come by pair of opposites $(\nu, -\nu)$. In the following we call *non negative spectrum* of the KZ cocycle the non-negative numbers $1 = \nu_1 > \nu_2 \geq \dots \geq \nu_g$ such that the multiset $(\nu_1, \nu_2, \dots, \nu_g, -\nu_g, \dots, -\nu_1)$ is the Lyapunov exponents of the KZ cocycle. In our case, for any surface Y in $\mathcal{G}$ the Oseledets decomposition of $H_1(Y; \mathbb{R})$ respects the splitting $E = E^{++} \oplus E^{+-} \oplus E^{-+} \oplus E^{--}$. Moreover, the maximal Lyapunov exponent of the KZ cocycle, which equals 1, belongs to E^{++} . Hence the non-negative spectrum can be written $\{1, \nu^{++}, \nu^{+-}, \nu^{-+}, \nu^{--}\}$ where $\{1, \nu^{++}\}$ (resp. $\{\nu^{+-}\}$, $\{\nu^{-+}\}$ and $\{\nu^{--}\}$) is the non negative Lyapunov spectrum of the KZ cocycle restricted to E^{++} (resp. E^{+-} , E^{-+} and E^{--}).

THEOREM 7. – *For any $\mathrm{SL}(2, \mathbb{R})$ -invariant ergodic probability measure on $\mathcal{G}$:*

$$\nu^{++} = \nu^{--} = 1/3 \quad \text{and} \quad \nu^{+-} = \nu^{-+} = 2/3.$$

Proof. – We first consider the case of the rank 4 subbundle E^{++} which corresponds to invariant vectors under the action of the Klein four group K . E^{++} identifies with the pullback of the Hodge bundle over $\mathcal{H}(2)$ and in particular, we deduce from results of M. Bainbridge [1] (see also Theorem 5) that $\nu^{++} = 1/3$.

We now consider the case of the rank 2 subbundles E^κ for $\kappa \in \{--, +-, -+\}$. Lemma 5 implies that the subbundle $E^{++} \oplus E^{--}$ (resp. $E^{++} \oplus E^{+-}$ and $E^{++} \oplus E^{-+}$) can be identified to a pullback of bundles over different covering loci in $\mathcal{H}^{\mathrm{hyp}}(2, 2)$ and $\mathcal{L} \subset H^{\mathrm{odd}}(2, 2)$. More precisely, the quotient map $Y \mapsto Y/\langle \tau_h \tau_v \rangle$ (resp. $Y \mapsto Y/\langle \tau_v \rangle$ and $Y \mapsto Y/\langle \tau_h \rangle$) induces an isomorphism between $E^{++} \oplus E^{--}$ (resp. $E^{++} \oplus E^{+-}$ and $E^{++} \oplus E^{-+}$) and respectively $H_1(Y/\langle \tau_h \tau_v \rangle; \mathbb{Z})$ (resp. $H_1(Y/\langle \tau_h \rangle; \mathbb{Z})$ and $H_1(Y/\langle \tau_v \rangle; \mathbb{Z})$).

To compute the remaining Lyapunov exponents, we use Theorem 5 twice. The loci $\mathcal{H}^{\mathrm{hyp}}(2, 2)$ and $\mathcal{L}$ comes from orientation coverings of surfaces in the quadratic strata respectively $\mathcal{Q}(4, -1^8)$ and $\mathcal{Q}(1^2, -1^6)$. For those two components, using Theorem 5, we get that the sum of positive Lyapunov exponents are respectively

$$\begin{aligned} \nu_1 + \nu_2 + \nu_3 &= \frac{2/3 + 6}{4} = 5/3 & \text{for } \mathcal{L} \subset \mathcal{H}^{\mathrm{odd}}(2, 2) \\ \nu_1 + \nu_2 + \nu_3 &= \frac{8}{4} = 2 & \text{for } \mathcal{H}^{\mathrm{hyp}}(2, 2). \end{aligned}$$

By subtracting $4/3 = 1 + 1/3$ to each of them that corresponds to the contribution of E^{++} we get that $\nu^{--} = 1/3$ and $\nu^{+-} = \nu^{-+} = 2/3$.

Notice that the assumption of *regularity* for the measures in Theorem 5 is automatically satisfied. The regularity property is stable under taking covering locus and is satisfied for Haar measures on Veech surfaces and Lebesgue measure on a stratum (see [9]). $\square$

Applying the main result of Chaika-Eskin [5] one obtains directly from Theorem 7 the following.

COROLLARY 1 ([5]). – *For all surfaces Y in $\mathcal{G}$, for almost every $\theta \in [0, 2\pi)$, the surface $e^{i\theta}Y$ is Oseledets generic: the Lyapunov exponents exist and their values are described by Theorem 7.*

4.3. A remark on the equality $\nu^{+-} = \nu^{-+}$

In Theorem 7, we have equality of Lyapunov spectrum in E^{+-} and E^{-+} . In that section we deduce the equality from the symmetries of the wind-tree model.

An *automorphism* (respectively *anti-automorphism*) of a translation surface (S, ω) is an affine diffeomorphism $\phi : S \rightarrow S$ such that $\phi^*\omega = \omega$ (resp. $\phi^*\omega = -\omega$). Quotient by automorphisms or anti-automorphisms corresponds to cover of translation surfaces (with either an Abelian or a quadratic differential). Automorphisms and anti-automorphisms commute with the $\mathrm{SL}(2, \mathbb{R})$ actions (they are precisely the affine diffeomorphisms for which the derivative belongs to the center of $\mathrm{SL}(2, \mathbb{R})$ which is $\pm id$) and this was used in the preceding sections to build the decomposition $E = E^{++} \oplus E^{+-} \oplus E^{-+} \oplus E^{--}$ (see Lemma 4).

An *elliptic symmetry* of a translation surface (S, ω) is a diffeomorphism $\phi : S \rightarrow S$ such that $\phi^*\omega = e^{i\theta}\omega$ for some $\theta \in [0, 2\pi) \setminus \{0, \pi/2, \pi, 3\pi/2\}$. Such map ϕ necessarily preserves the underlying complex structure of ω and implies that ϕ is of finite order. Identically, a translation surface has an elliptic symmetry if it comes from an orientation cover of higher order differentials. Elliptic symmetries *do not* commute with the $\mathrm{SL}(2, \mathbb{R})$ -action but only with $\mathrm{SO}(2)$.

An *elliptic locus* is an orientation cover of a connected component of differential of order larger than > 2 . Equivalently, it is a subset of a connected component of stratum with an elliptic symmetry which is locally constant.

In $\mathcal{H}(2^4)$, we consider the locus of surfaces which is cover of L-shaped surfaces with a rotation symmetry of order 4. Examples of such surfaces are the surfaces $X(a, a)$. The quotient of an element of that locus is a quartic differential on the sphere with angles $\pi, \pi, -3/4\pi$ and $-1/4\pi$. The real dimension of that locus is 4 (or 3 in $\mathcal{H}^{(1)}(2^4)$). The following lemma may be used to conclude to equality of Lyapunov exponents in E^{+-} and E^{-+} .

LEMMA 6. – *Let $X \subset \mathcal{H}_g(\alpha)$ be a closed $\mathrm{SL}(2, \mathbb{R})$ -invariant affine locus in $\mathcal{H}_g(\alpha)$ of real dimension $2d$ that supports an ergodic measure μ . Let $Y \subset X$ be an elliptic locus of real dimension d and denote by G the associated action on $H^1(S, \mathbb{R})$ over an element (S, ω) in Y . Let E_i and E_j be two subbundles of the Hodge bundle that are $\mathrm{SL}(2, \mathbb{R})$ invariant and such that for some $g \in G$ we have $gE_i = E_j$ in the intersection of a neighborhood of (S, ω) and Y . Then the spectrum of the Kontsevich-Zorich cocycle restricted to E_i and E_j coincides.*

Proof. – We prove that in the intersection of the neighborhood of (S, ω) with Y there is a generic surface for μ .

Let ϕ be an elliptic symmetry of (S, ω) . In other words

$$\phi^*(\omega) = e^{i\theta}\omega = \cos(\theta) \operatorname{Re}(\omega) - \sin(\theta) \operatorname{Im}(\omega) + i(\sin(\theta) \operatorname{Re}(\omega) + \cos(\theta) \operatorname{Im}(\omega)).$$

It is clear that if $\sin(\theta) \neq 0$, then this equation is not preserved along the stable variety of ω (i.e., along a deformation of $\operatorname{Im}(\omega)$ only) and along the geodesic flow. Hence, at each point (S, ω) of Y the subspace Y , the stable variety of (S, ω) and its geodesic are transverse. From the dimension condition, almost all geodesics encounter the stable variety of some element in Y . In particular, almost all points in Y (for its $d - 1$ -dimensional Lebesgue measure) have generic Teichmüller geodesics.

To conclude to equality, let us consider a generic surface in Y . There exists a generic direction θ' in G such that $\theta + \theta'$ is also generic (θ is such that $\phi^*\omega = e^{i\theta}\omega$). Because $\phi^*E_i = \phi^*E_j$ at (S, ω) we do have equality of exponents in E_i and E_j . $\square$

5. Deviations for translation surfaces

In this section, we prove Theorem 2 which concerns growth of geodesics.

We recall the notation from the introduction. Let $\mathcal{H}(\alpha)$ be a stratum of Abelian differential and μ a g_t -invariant ergodic measure on $\mathcal{H}(\alpha)$. We denote by $1 = \nu_1 > \nu_2 > \dots > \nu_k$ the positive Lyapunov exponents of the KZ cocycle and for $X \in \mathcal{H}(\alpha)$ which is Oseledets generic

$$(2) \quad H^1(X; \mathbb{R}) = F_1^u \supset F_2^u \supset \dots \supset F_k^u \supset F_{k+1}^u = F^c \supset F_k^s \supset \dots \supset F_1^s \supset F_0^s = \{0\}$$

the associated Oseledets flag. By Oseledets theorem, the decomposition (2) is measurable and is invariant under the Teichmüller flow.

We want to prove the following statement: for μ -almost all X which are Oseledets generic, for all $p \in X$ with infinite forward orbit and any norm on $H^1(X; \mathbb{R})$

1. for all $f \in F_i^u \setminus F_{i+1}^u$

$$\limsup_{T \rightarrow \infty} \frac{\log |\langle f, \gamma_T(p) \rangle|}{\log T} = \nu_i,$$

2. for $f \in F^c$

$$\limsup_{T \rightarrow \infty} \frac{\log |\langle f, \gamma_T(p) \rangle|}{\log T} = 0,$$

3. there exists a constant C , such that for $f \in F_i^s \setminus F_{i-1}^s$

$$\forall T \geq 0, |\langle f, \gamma_T(p) \rangle| \leq C.$$

We first notice that to prove Theorem 2, by ergodicity of the Teichmüller flow, it is enough to prove it for surfaces X belonging in a small open set of $\mathcal{H}(\alpha)$ of positive measure. The strategy is as follows. We build a small open set in which we have uniform estimates for the linear flows. Next, for a surface in this small open set we consider long pieces of trajectory under the linear flow that we decompose using the KZ cocycle. Then, using the uniform estimates, we get the lower and upper bounds.

5.1. Transversals for the Teichmüller flow

In order to code geodesics in an individual surface we use Veech's construction of zippered rectangles [31]. We recall our convention of markings (see Section 2) that each translation structure in $\mathcal{H}(\alpha)$ carries a choice of a horizontal separatrix at each singularity.

A surface in $\mathcal{H}(\alpha)$ is called *regular* if there is no saddle connection in both horizontal and vertical directions. In a regular surface the linear flow in vertical direction is minimal (Keane's Theorem [19]). If there is a connection in vertical (resp. horizontal) direction then the forward (resp. backward) orbit for the Teichmüller flow goes to infinity. In particular, using Poincaré recurrence theorem, we get that the set of regular surfaces is a set of full measure for μ .

Let X be a regular surface in the support and $\Sigma \subset X$ the finite set of singularities of X . Following [31], we decompose the surface into zippered rectangles. Recall that there is a marked outgoing separatrix in X . We consider the initial segment of length 1 on this separatrix that we identify with $[0, 1]$. The Poincaré map of the linear flow in this segment is an interval exchange transformation. There exists a canonical segment $I \subset X$ built from Rauzy induction ([31, Proposition 9.1], see also [32]). The rectangles above each domain of continuity of the interval exchange transformation on I give a decomposition $X = \bigcup R_j$ where R_j are geodesic rectangles with horizontal sides inside I and vertical sides which contain singularities or hit a singularity in the future. The number of rectangles is $d = 2g - 2 + s - 1$ where g is the genus of X and s the number of singularities.

Let X be a regular surface in the support of μ and $X = \bigcup R_j$ its decomposition into zippered rectangles. The parameters of the zippered rectangles (lengths and heights of the rectangles) give local coordinates for $\mathcal{H}(\alpha)$ in a neighborhood of X . In particular, we get zippered rectangles construction for surfaces near X that are *not* regular. Let $U \subset \mathcal{H}(\kappa)$ be an open set which contains X and for which the zippered rectangles obtained from X give a chart of $\mathcal{H}(\alpha)$. In U , we have a trivialization of the Hodge bundle and we identify all fibers with $H^1(X; \mathbb{R})$.

To each rectangle R_j of a zippered rectangle decomposition of a surface Y in U is associated a curve $\zeta_j \subset Y \setminus \Sigma$ (up to homotopy in $Y \setminus \Sigma$) which corresponds to the Poincaré map on the canonic interval of Y . The vertical holonomy of ζ_j is the height of R_j . The following is a classical fact.

LEMMA 7. — *The set $\{\zeta_j\}_{j=1,\dots,d}$ forms a basis of $H_1(Y \setminus \Sigma; \mathbb{Z})$.*

Let $Y \in U$ and $I \subset Y$ be the canonical transversals for the linear flow of Y . To a point p in I , we associate the sequence of return times $T_n = T_n(p)$ of the linear flow into I . Each curve $\gamma_{T_n}(p)$ has both ends in I and we close it using a small piece of the horizontal segment contained in I . For any $p \in I$ with infinite orbit and any n we have a unique decomposition as concatenation of curves

$$\gamma_{T_n}(p) = \zeta_{j_1}(p) \zeta_{j_2}(p) \zeta_{j_3}(p) \cdots \zeta_{j_n}(p)$$

and hence, there exist some non negative numbers $m_{T_n,j}(p)$ for $j = 1, \dots, d$ such that, in homology

$$\gamma_{T_n}(p) = \sum_{j=1}^d m_{T_n,j}(p) \zeta_j \in H_1(S \setminus \Sigma).$$

Let $p \in Y$ with infinite backward and forward orbits. There is a unique point $p' \in I$ such that the orbit of p' under the linear flow goes to p before returning to I . For $T \geq 0$, we denote by $\gamma_T(p)$ the curve $\gamma_{T_n}(p')$ where $T_{n-1}(p') < T \leq T_n(p')$.

LEMMA 8. – *We can choose U in such a way that there exist constants K_1 and K_2 such that for all $Y \in U$*

1. *for $j = 1, \dots, d$, the length l_j and height h_j of the rectangle $R_j = R_j(Y)$ satisfy $K_1^{-1} < h_j < K_1$ and $K_1^{-1} < l_j < K_1$,*
2. *for every point $p \in I$, the decomposition of the geodesic $\gamma_{K_2}(p) = \sum c_j \zeta_j$ in $H_1(Y \setminus \Sigma; \mathbb{Z})$ where the sum for j between 1 and d is such that no c_j is zero. In other words, any geodesic longer than K_2 goes through all rectangles R_j .*

Proof. – We consider the interval exchange transformation on the segment I associated to surfaces Y in a neighborhood of X . We recall from [19], that doubling all points of I which are preimages of a discontinuity of the interval exchange gives a Cantor set. The interval exchange transformation has a prolongation to this Cantor set and is a homeomorphism which is semi-conjugated to the initial transformation on T .

Around X , the lengths ℓ_j and heights h_j are continuous functions of Y . Hence, to fulfill the first condition it is enough to consider a relatively compact open set U inside the chart given by the zippered rectangles of X .

We prove that it is possible to satisfy the second one. Because of the regularity condition, the linear flow of X is minimal (Keane's Theorem [19]). Let $I \subset X$ be the segment associated to X . For any $p \in I$ with infinite future orbit, there exists a time $T = T(p)$ such that the curve $\gamma_T(p)$ has visited all rectangles. We choose $T(p)$ to be the first return time of p in I with this property. The map $p \mapsto T(p)$ is locally constant on I and hence also on the associated Cantor set $\tilde{I}$. By minimality, $T(p)$ is uniformly bounded, otherwise there would exist a point p for which its future does not cross some rectangle. Hence, on X , any curve of length longer than $K = \max_{p \in I} T(p) < \infty$ goes through all rectangles. In a small neighborhood of X , the rectangles associated to the time K are still rectangles and their heights have been modified continuously with respect to the surface. By choosing U small enough we may ensure that all rectangles of length less than K in Y are still rectangles in $Y' \in U$ and their heights are uniformly bounded by $K_2 = K + \varepsilon$ with $\varepsilon > 0$. $\square$

By taking smaller U if necessary, we assume that it is “flow box” that contains X . Namely, U is identified with a transversal P to the Teichmüller flow containing X times an interval $]-\varepsilon; \varepsilon[$. For $Y \in P$, we consider the return times $t_n = t_n(Y)$ of the surface Y in P . By ergodicity of μ , this is well defined for μ -almost all Y in P . The segment I in $g_{t_n} Y$ becomes a segment of length $e^{-t_n} I$ in Y . We define the curves $\zeta_j^{(n)}$ as the first return time associated to $e^{-t_n} I$ in Y . In homology, we have $\zeta_j^{(n)} = (B^{(t_n)})^* (\zeta_j)$.

5.2. Upper bound

Let X be a regular surface and P a transversal to the Teichmüller flow containing X as in Section 5.1. In particular we assume that Lemma 8 holds for all surfaces Y in P .

Let $Y \in P$ be a surface which is recurrent for the Teichmüller flow and let $\zeta_j^{(n)}$, $j = 1, \dots, d$, $n = 0, 1, \dots$ be the curves defined at the end of the previous section as first return times into

$I^{(n)} = e^{-t_n} I \subset Y$. By linear algebra, we know that the pairings $\langle f, \zeta_j^{(n)} \rangle = \langle B^{(t_n)} f, \zeta_j \rangle$ is bounded by $\|B^{(t_n)} f\|$ times a constant. To prove that the bound still holds for a geodesic not necessarily of the form $\zeta_j^{(n)}$ we use the following lemma.

LEMMA 9 ([12] Lemma 9.4, [35] Proposition 8). – *Let X be a regular surface and P a transversal containing X as above. Let $Y \in P$ be recurrent for the Teichmüller flow and $\zeta_j^{(n)}$ be the curves defined by first return times. Let $p \in I \subset Y$ be a point with infinite future orbit. For each $T \geq 0$ there exist an integer $n = n(T)$ and a decomposition*

$$\gamma_T(p) = \sum_{k=0}^n \sum_{j=1}^d m_j^{(k)} \zeta_j^{(k)} \quad \text{in } H_1(Y; \mathbb{Z}),$$

which satisfies

1. the $m_j^{(k)}$ are non negative integers for $k = 1, \dots, n$ and $j = 1, \dots, d$,
2. $\sum_j m_j^{(n)} \neq 0$ and $K_1^{-1} e^{t_n} < T$,
3. $K_1^{-1} e^{t_k} \leq \ell(\zeta_j^{(k)}) \leq K_1 e^{t_k}$ for $j = 1, \dots, d$, where ℓ denotes the length,
4. $\sum_j m_j^{(k)} \leq 2(K_1)^2 e^{t_{k+1} - t_k}$.

Proof. – Let $I = I^{(0)} \subset Y$ be the segment of the interval exchange transformation associated to the zippered rectangles decomposition of Y . For $k \geq 1$, let $I^{(k)}$ be the subintervals of Y which are the images of $I(g_{t_k} Y) \subset g_{t_k}(Y)$ under g_{-t_k} . We recall that this segment is the segment with the same left extremity as I but whose length is e^{-t_k} times the one of I .

We describe the so called *prefix-suffix* decomposition in symbolic dynamics for $\gamma_T(p)$. We assume that T is a return time of the linear flow in I and note $\gamma = \gamma_T(p)$. Let n be the largest k such that the closed curve γ crosses $I^{(k)}$ twice. We may decompose $\gamma = r_-^{(n)} \gamma^{(n)} r_+^{(n)}$ where

- $r_-^{(n)}$ starts from p and ends in $I^{(n)}$,
- $\gamma^{(n)}$ is a non empty concatenation of $\gamma_j^{(n)}$,
- $r_+^{(n)}$ starts from $I^{(n)}$ and ends at p .

We choose $r_-^{(n)}$ and $r_+^{(n)}$ to be minimal and hence their length are smaller than $K_1 e^{t_n}$ by Lemma 8, Property 1. On the other hand, by definition, each curve $\gamma_j^{(n)}$ is of length larger than $K_1^{-1} e^{t_n}$ and hence $T > K_1^{-1} e^{t_n}$. This proves Equation 2 and also 1 and 3 for the special case $k = n$.

We now proceed by induction and decompose $r_-^{(n)}$ and $r_+^{(n)}$ with respect to the other recurrence times $0 < t_k < t_n$. We assume that we built two curves $r_-^{(k)}$ and $r_+^{(k)}$ and two sequences $\gamma_-^{(k)}, \gamma_-^{(k+1)}, \dots, \gamma_-^{(n-1)}$ and $\gamma_+^{(k)}, \gamma_+^{(k+1)}, \dots, \gamma_+^{(n-1)}$ such that

$$\gamma = r_-^{(k)} \left(\gamma_-^{(k)} \gamma_-^{(k+1)} \dots \gamma_-^{(n-1)} \right) \gamma^{(n)} \left(\gamma_+^{(n-1)} \dots \gamma_+^{(k+1)} \gamma_+^{(k)} \right) r_+^{(k)}$$

with

- $\gamma_-^{(m)}$ starts from $I^{(m)}$, ends in $I^{(m+1)}$ and does not cross $I^{(m+1)}$ before its endpoint,
- $\gamma_+^{(m)}$ starts from $I^{(m+1)}$, ends in $I^{(m)}$ and does not cross $I^{(m+1)}$ after its start point,
- $r_-^{(k)}$ starts from $I^{(0)}$ and ends in $I^{(k)}$ and does not cross $I^{(k)}$ before its endpoint,

- $r_+^{(k)}$ starts from $I^{(k)}$ and ends in $I^{(0)}$ and does not cross $I^{(k)}$ after its start point.

Repeating the same procedure, we end with the decomposition

$$\gamma = \gamma_-^{(0)} \gamma_-^{(1)} \dots \gamma_-^{(n-1)} \gamma_-^{(n)} \gamma_+^{(n-1)} \dots \gamma_+^{(1)} \gamma_+^{(0)}.$$

From the construction, we know that both $\gamma_-^{(k)}$ and $\gamma_+^{(k)}$ may be written as concatenations of curves $\zeta_j^{(k)}$ with positive coefficients. Let $\gamma_-^{(k)} + \gamma_+^{(k)} = \sum m_j^{(k)} \zeta_j^{(k)}$ in $H_1(Y; \mathbb{Z})$. The Property 1 is clearly satisfied. By maximality $\gamma_-^{(k)}$ satisfies $\ell(\gamma_-^{(k)}) < K_1 e^{t_{k+1}}$ and the same is true for $\gamma_+^{(k)}$. On the other hand each $\zeta_j^{(k)}$ is of length at least $K_1^{-1} e^{t_k}$ by Property 1 of Lemma 8 and hence Property 3 is satisfied. From the latter inequalities, we obtain

$$K_1^{-1} e^{t_k} \sum m_j^{(k)} < \sum m_j^{(k)} \ell(\zeta_j^{(k)}) < 2K_1 e^{t_{k+1}}.$$

which implies that $\sum m_j^{(k)} < 2(K_1)^2 e^{t_{k+1}-t_k}$ which is Property 4. $\square$

Now, we prove the upper bound in Theorem 2. We restrict to the Case 1 relative to one of the unstable subspace F_i^u of the Oseledets flag. The same proof works for the other cases. We follow mainly Section 9 of [12] (see also Section 6 of [35] and Section 4.9 of [36]). In what follows K_i for $i = 3, 4, \dots$ denotes constants which do not depend on the time T or the number $n = n(T)$.

We fix $Y \in P$ which is generically recurrent and Oseledets generic. We also fix p a point in Y with infinite forward orbit for the linear flow. By Lemma 9 and Oseledets' theorem (see Section 1), for any $\varepsilon > 0$, there exists a constant K_3 such that for T big enough the following estimation holds

$$(3) \quad |\langle f, \gamma_T(p) \rangle| \leq K_3 \sum_{k=1}^n e^{t_{k+1}-t_k} e^{(\nu_i + \varepsilon)t_k}.$$

Moreover, if Y is generically recurrent for the Teichmüller flow, one can ensure that

$$\lim_{k \rightarrow \infty} \frac{t_k}{k} = M$$

where M is the inverse of the transverse μ -measure of P . As Y is generically recurrent, for any δ the following estimation holds for k big enough

$$(4) \quad (M - \delta)k \leq t_k \leq (M + \delta)k.$$

Using (3) and (4) we get that for T big enough we get

$$\begin{aligned} |\langle f, \gamma_T(p) \rangle| &\leq K_4 \sum_{k=1}^n \exp((M + \delta)(k + 1) - (M - \delta)k) \exp(\nu_i + \varepsilon)(M + \delta)k \\ &\leq K_5 \sum_{k=1}^n \exp((\nu_i + \varepsilon)(M + \delta) + 2\delta)k \\ &\leq K_6 \exp((\nu_i + \varepsilon)(M + \delta) + 2\delta)n. \end{aligned}$$

Now, by equation (4) and the choice of $n = n(T)$ in the estimate 2 in Lemma 9 we have for T big enough

$$\exp((M - \delta)n) \leq \exp(t_n) \leq K_1 T.$$

Hence we get that for T big enough

$$\frac{\log |\langle f, \gamma_T(p) \rangle|}{\log T} \leq \frac{(\nu_i + \varepsilon)(M + \delta) + 2\delta}{M - \delta}.$$

As δ and ε can be chosen arbitrarily small we get the upper bound.

5.3. Lower bound

We now prove the lower bound in Theorem 2. The only non trivial case is the one of a cocycle in unstable part F_i^u or the central part F^c of the Oseledets flag (2). The proof is identical for both of them.

Let X be a regular surface and P be a transversal containing X as in Section 5.1. Let $Y \in P$ be recurrent for the Teichmüller flow and Oseledets generic. We denote by t_n the sequence of return times in P . We fix a point $p \in Y$ with infinite future orbit and a cocycle $f \in F_i^u \setminus F_{i+1}^u$ in the unstable part of $H^1(X; \mathbb{Z})$.

We use a similar decomposition as in the proof of Lemma 9. For all n , we consider the sequence of return times of p into $I^{(n)}$. For the m -th return time T in $I^{(n)}$ we have a decomposition

$$\gamma_T(p) = \gamma_-^{(1)} \gamma_-^{(2)} \dots \gamma_-^{(n-1)} \zeta_{j_1}^{(n)} \zeta_{j_2}^{(n)} \dots \zeta_{j_m}^{(n)},$$

where the sequence $(j_k) = (j_k(n))$ does only depend on p and n and the sequence $\gamma_-^{(k)}$ only on p . The length of the initial segment $\gamma_-^{(1)} \gamma_-^{(2)} \dots \gamma_-^{(n-1)}$ corresponds to the first hitting time T_n^- of $I^{(n)}$ starting from p .

Let $\varepsilon > 0$; we want to prove that the following holds

$$\limsup_{T \rightarrow \infty} \frac{\log |\langle f, \gamma_T(p) \rangle|}{\log T} \geq \nu_i - \varepsilon.$$

We are done if the above equality holds for infinitely many prefixes $\gamma_-^{(1)} \dots \gamma_-^{(n-1)}$ of $\gamma_T(p)$. We assume that the latter condition does not hold and prove that the above inequality still holds. From that hypothesis, we get

$$(5) \quad \log \left| \left\langle f, \gamma_{T_n^-}(p) \right\rangle \right| = \log \left| \left\langle f, \gamma_-^{(1)} \gamma_-^{(2)} \dots \gamma_-^{(n-1)} \right\rangle \right| \leq (\nu_i - \varepsilon/2) \log T_n^-.$$

LEMMA 10. – *Let K_1 and K_2 be the constants of Lemma 8. There exist an index $\ell \in \{1, \dots, \lfloor K_1 K_2 \rfloor\}$, a constant $C > 0$ and an infinite subset $N \subset \mathbb{N}$ such that for all $n \in N$*

$$\forall \ell' \in \{1, \dots, \ell - 1\}, \quad \frac{|\langle B^{(t_n)} f, \zeta_{j_{\ell'}(n)} \rangle|}{\|B^{(t_n)} f\|} \leq \frac{C}{K_1 K_2} \quad \text{and} \quad \frac{|\langle B^{(t_n)} f, \zeta_{j_\ell(n)} \rangle|}{\|B^{(t_n)} f\|} \geq C.$$

Proof. – From equivalence of norms on the finite dimensional vector space $H^1(X; \mathbb{R})$, there exists a constant $C' > 0$ such that

$$\forall v \in H^1(X; \mathbb{Z}), \quad \max_{j=1, \dots, d} |\langle v, \zeta_j \rangle| \geq C' \|v\|.$$

Now, since the length of each curve ζ_j is at least K_1^{-1} and that after time K_2 all curves ζ_j appear (see Lemma 8), we know that before $\lfloor K_1 K_2 \rfloor$ return times in I any geodesic $\gamma_T(p)$ passes through all rectangles. In particular for at least one of the curves $\zeta_{j_1(n)}, \dots, \zeta_{j_{K_1 K_2}(n)}$ we have a uniform lower bound on the pairing with f .

Now, consider the sequence of pieces in first position $\zeta_{j_1(n)}$ as $n \in \mathbb{N}$. If

$$\limsup_{n \rightarrow \infty} \frac{|\langle B^{(t_n)} f, \zeta_{j_1(n)} \rangle|}{\|B^{(t_n)} f\|} > 0$$

then we are done by choosing C to be the half of the lim sup above. If not, we consider the sequence of pieces in second position $\zeta_{j_2(n)}$ and repeat the dichotomy. We know from the first part of the proof, that this process stops before the $(K_1 K_2)$ -th position. We get a position ℓ , a constant C , and a subsequence $N \subset \mathbb{N}$ that satisfy the right inequality of the statement of the lemma. By starting the subsequence far enough (i.e., considering $N \cap \{m, m+1, \dots\}$ for m big enough), by our construction since each of the lim inf for $l' = 1, \dots, k-1$ is 0, we may ensure by our construction that the $\ell-1$ inequalities on the left hold. $\square$

Let ℓ , C and $N \subset \mathbb{N}$ that satisfy the conclusion of Lemma 10. Let $n \in N$ and $p_n \in I^{(n)}$ be the endpoint of the prefix $\gamma_-^{(1)} \dots \gamma_-^{(n-1)}$. Then

$$\begin{aligned} \langle f, \gamma_{T_n}(p) \rangle &= \langle f, \gamma_{T_n^-}(p) \rangle + \langle f, \gamma_{T_n - T_n^-}(p_n) \rangle \\ &= \langle f, \gamma_-^{(1)} \dots \gamma_-^{(n-1)} \rangle + \langle f, \zeta_{j_1(n)} \rangle + \dots + \langle f, \zeta_{j_{\ell-1}(n)} \rangle + \langle f, \zeta_{j_\ell(n)} \rangle \\ &= \langle f, \gamma_-^{(1)} \dots \gamma_-^{(n-1)} \rangle + \langle B^{(t_n)} f, \zeta_{j_1(n)} \rangle + \dots \\ &\quad + \langle B^{(t_n)} f, \zeta_{j_{\ell-1}(n)} \rangle + \langle B^{(t_n)} f, \zeta_{j_\ell(n)} \rangle. \end{aligned}$$

Using triangular inequality, we get

$$\begin{aligned} |\langle f, \gamma_{T_n}(p) \rangle| &\geq \left| \langle B^{(t_n)} f, \zeta_{j_\ell(n)} \rangle \right| - \sum_{k=1}^{\ell-1} \left| \langle B^{(t_n)} f, \zeta_{j_k(n)} \rangle \right| - \left| \langle f, \gamma_{T_n^-}(p) \rangle \right| \\ &\geq C \|B^{(t_n)} f\| - \frac{(\ell-1)C}{K_1 K_2} \|B^{(t_n)} f\| - \left| \langle f, \gamma_{T_n^-}(p) \rangle \right| \\ &\geq \frac{C}{K_1 K_2} \|B^{(t_n)} f\| - \left| \langle f, \gamma_{T_n^-}(p) \rangle \right|. \end{aligned}$$

We use twice Lemma 8 to prove that T_n grows like e^{t_n} . First of all, as T_n is a time for which the orbit of p under the linear flow has reached at least twice the interval $I^{(n)}$ we have $T_n > K_1^{-1} e^{t_n}$. Moreover, by construction, $T_n < K_1(l+1)e^{t_n} \leq K_1(K_1 K_2 + 1)e^{t_n}$. Hence

$$(6) \quad \lim_{n \rightarrow \infty} \frac{\log T_n}{t_n} = 1.$$

From our assumption (5), for $n \in N$ big enough, the term $|\langle f, \gamma_{T_n^-}(p) \rangle|$ is exponentially smaller than $\|B^{(t_n)} f\|$. We hence get that

$$\begin{aligned} \limsup_{T \rightarrow \infty} \frac{|\langle f, \gamma_T(p) \rangle|}{\log T} &\geq \limsup_{n \in N} \frac{\log \left(C / (K_1 K_2) \|B^{(t_n)} f\| - \left| \langle f, \gamma_{T_n^-}(p) \rangle \right| \right)}{t_n} \\ &= \limsup_{n \in N} \frac{\log \|B^{(t_n)} f\|}{\log t_n} = \nu_i, \end{aligned}$$

where the last inequality holds, as we assume Y to be Osseledets generic. In other words, under assumption (5) we exhibit a subsequence on which the lim sup is achieved.

5.4. Theorem 6 for the whole locus $\mathcal{G}$

We now prove our main theorem about the wind-tree model in the following form

LEMMA 11. – *Let $0 < a < 1$ and $0 < b < 1$. Let $f \in H^1(X(a, b); \mathbb{Z}^2)$ be the cocycle that defines the wind-tree model. For $Y \in U$, let $Y_\theta = e^{-i\theta}Y$. Then for all $Y \in U$, Lebesgue almost every $\theta \in S^1$, every point $p \in Y_\theta$ with infinite forward orbit for the linear flow*

$$\limsup_{T \rightarrow \infty} \frac{\log |\langle f, \gamma_T(p) \rangle|}{\log(T)} = \frac{2}{3}.$$

Proof. – By Chaika-Eskin result [5], generic recurrence and Oseledets hold for almost every θ . We only need to analyze to which components of the Oseledets splitting f belongs.

From Lemma 3 and Lemma 4, we know that the cocycle $f \in H^1(X(a, b); \mathbb{Z}^2)$ decomposes into two pieces $f^{+-} \in E^{+-}(\mathbb{Q})$ and $f^{-+} \in E^{-+}(\mathbb{Q})$ where each of $E^{+-}(\mathbb{R})$ and $E^{-+}(\mathbb{R})$ are rank 2 subbundles stable under the Kontsevich-Zorich cocycle. From Theorem 7 and Corollary 1, the Lyapunov exponents of the KZ cocycle for $e^{i\theta}Y$ exist and are $2/3$ and $-2/3$ in both of $E^{+-}(\mathbb{R})$ and $E^{-+}(\mathbb{R})$. The only thing to prove in Lemma 11 is that f^{+-} (resp. f^{-+}) does not belong to the stable subspace of $E^{+-}(\mathbb{R})$ (resp. $E^{-+}(\mathbb{R})$) associated to $-2/3$. If f belongs to the stable subspace, then $\|B^{(t)}f\|$ goes to zero as t tends to infinity. But recall that the Kontsevich-Zorich cocycle takes values in the set of integer matrices of determinant 1. In particular it preserves the set $E^{+-}(\mathbb{Z})$. On the other hand, f is a rational vector and there exists N such that $Nf \in E^{+-}(\mathbb{Z})$. In particular, the quantity $\|B^{(t)}f\|$ is bounded below by

$$\frac{1}{N} \min_{v \in E^{+-}(\mathbb{Z}) \setminus \{0\}} \|v\|$$

which is different from 0 as the integer vectors $E^{+-}(\mathbb{Z})$ form a lattice in $E^{+-}(\mathbb{R})$. $\square$

BIBLIOGRAPHY

- [1] M. BAINBRIDGE, Euler characteristics of Teichmüller curves in genus two, *Geom. Topol.* **11** (2007), 1887–2073.
- [2] I. I. BOUW, M. MÖLLER, Teichmüller curves, triangle groups, and Lyapunov exponents, *Ann. of Math.* **172** (2010), 139–185.
- [3] A. I. BUFETOV, Limit theorems for translation flows, *Ann. of Math.* **179** (2014), 431–499.
- [4] K. CALTA, Veech surfaces and complete periodicity in genus two, *J. Amer. Math. Soc.* **17** (2004), 871–908.
- [5] J. CHAIKA, A. ESKIN, Every flat surface is Birkhoff generic and Oseledets generic in almost every direction, preprint arXiv:13051104.
- [6] J.-P. CONZE, E. GUTKIN, On recurrence and ergodicity for geodesic flows on non-compact periodic polygonal surfaces, *Ergodic Theory Dynam. Systems* **32** (2012), 491–515.

- [7] P. EHRENFEST, T. EHRENFEST, Begriffliche Grundlagen der statistischen Auffassung in der Mechanik, in *Encykl. d. Math. Wissensch.* IV 2 II, Heft 6, 1912 (in German, translated in:) *The conceptual foundations of the statistical approach in mechanics*, 10-13 Cornell University Press, Ithaca NY (1959).
- [8] A. ESKIN, M. KONTSEVICH, A. ZORICH, Lyapunov spectrum of square-tiled cyclic covers, *J. Mod. Dyn.* **5** (2011), 319–353.
- [9] A. ESKIN, M. KONTSEVICH, A. ZORICH, Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow, *Publ. Math. I.H.É.S.* **120** (2014), 207–333.
- [10] A. ESKIN, M. MIRZAKHANI, Invariant and stationary measures for the $SL_2(\mathbb{R})$ action on moduli space, preprint arXiv:1302.3320.
- [11] A. ESKIN, M. MIRZAKHANI, M. MOHAMADI, Isolation, equidistribution, and orbit closures for the $SL_2(\mathbb{R})$ action on moduli space, preprint arXiv:1305.3015.
- [12] G. FORNI, Deviation of ergodic averages for area-preserving flows on surfaces of higher genus, *Ann. of Math.* **155** (2002), 1–103.
- [13] G. FORNI, C. MATHEUS, A. ZORICH, Square-tiled cyclic covers, *J. Mod. Dyn.* **5** (2011), 285–318.
- [14] J. HARDY, J. WEBER, Diffusion in a periodic wind-tree model, *J. Math. Phys.* **21** (1980), 1802–1808.
- [15] W. P. HOOPER, P. HUBERT, B. WEISS, Dynamics on the infinite stair case surface, preprint arXiv:0905.3736, to appear in *Dis. Cont. Dyn.*
- [16] W. P. HOOPER, B. WEISS, Generalized staircases: recurrence and symmetry, *Ann. Inst. Fourier (Grenoble)* **62** (2012), 1581–1600.
- [17] P. HUBERT, S. LELIÈVRE, S. TROUBETZKOY, The Ehrenfest wind-tree model: periodic directions, recurrence, diffusion, *J. reine angew. Math.* **656** (2011), 223–244.
- [18] P. HUBERT, G. SCHMITHÜSEN, Infinite translation surfaces with infinitely generated Veech groups, *J. Mod. Dyn.* **4** (2010), 715–732.
- [19] M. KEANE, Interval exchange transformations, *Math. Z.* **141** (1975), 25–31.
- [20] M. KONTSEVICH, A. ZORICH, Connected components of the moduli spaces of Abelian differentials with prescribed singularities, *Invent. Math.* **153** (2003), 631–678.
- [21] M. KONTSEVICH, A. ZORICH, Lyapunov exponents and Hodge theory, preprint arXiv:hep-th/9701164.
- [22] S. LELIÈVRE, R. SILHOL, Multi-geodesic tessellations, fractional Dehn twists and uniformization, preprint arXiv:math/0702374.
- [23] H. MASUR, Interval exchange transformations and measured foliations, *Ann. of Math.* **115** (1982), 169–200.
- [24] C. T. McMULLEN, Billiards and Teichmüller curves on Hilbert modular surfaces, *J. Amer. Math. Soc.* **16** (2003), 857–885.
- [25] C. T. McMULLEN, Teichmüller curves in genus two: discriminant and spin, *Math. Ann.* **333** (2005), 87–130.
- [26] C. T. McMULLEN, Dynamics of $SL_2(\mathbb{R})$ over moduli space in genus two, *Ann. of Math.* **165** (2007), 397–456.

- [27] P. PRZYTICKI, G. SCHMITHÜSEN, F. VALDEZ, Veech groups of Loch Ness monsters, *Ann. Inst. Fourier (Grenoble)* **61** (2011), 673–687.
- [28] G. SCHMITHÜSEN, Examples of origamis, in *The Geometry of Riemann Surfaces and Abelian Varieties*, Contemp. Math. **397**, 2006, 193–206.
- [29] S. TABACHNIKOV, *Billiards*, Panor. Synth. **1**, Soc. Math., France, 1995.
- [30] F. VALDEZ, Veech groups, irrational billiards and stable abelian differentials, *Discrete Contin. Dyn. Syst.* **32** (2012), 1055–1063.
- [31] W. A. VEECH, Gauss measures for transformations on the space of interval exchange maps, *Ann. of Math.* **115** (1982), 201–242.
- [32] M. VIANA, Dynamics of interval exchange maps and Teichmüller flows, preprint <http://w3.impa.br/~viana/out/ietf.pdf>.
- [33] J.-C. YOCOZ, Continued fraction algorithms for interval exchange maps: an introduction, in *Frontiers in number theory, physics, and geometry. I*, Springer, Berlin, 2006, 401–435.
- [34] A. ZORICH, Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents, *Ann. Inst. Fourier (Grenoble)* **46** (1996), 325–370.
- [35] A. ZORICH, Deviation for interval exchange transformations, *Ergodic Theory Dynam. Systems* **17** (1997), 1477–1499.
- [36] A. ZORICH, How do the leaves of a closed 1-form wind around a surface?, in *Pseudoperiodic topology*, Amer. Math. Soc. Transl. Ser. 2 **197**, Amer. Math. Soc., Providence, RI, 1999, 135–178.
- [37] A. ZORICH, Flat surfaces, in *Frontiers in number theory, physics, and geometry. I*, Springer, Berlin, 2006, 437–583.

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