Flat surfaces: computational problems

Summer school on Teichmüller dynamics, mapping class groups and applications

Investigate these problems in small groups. Some are delicate and involve a mix of thinking and computing. They are roughly ordered by increasing difficulty. Suggested softwares and libraries: SageMath, flipper, snappy, surface_dynamics, flatsurf.

Most articles in the reference are available for download at http://www.labri.fr/perso/vdelecro/grenoble_school/refs/

1 Where are my commutators?

Recall from Carlos Matheus’s lectures that an origami is given by a pair of permutations \((r, u) \in S_n \times S_n\) that together generate a transitive group, and tell, for each square on the surface, which square you find moving right \((r)\) and moving up \((u)\), see figure 1. The stratum \(H(\mu)\) of the origami given by \((r, u)\) can be deduced from the cycle lengths of the commutator \([r, u] = rur^{-1}u^{-1}\).

![Figure 1: Gluing pattern of the origami given by permutations r and u.](image)

Picking \((r, u)\) uniformly at random in \(S_n \times S_n\), and discarding pairs that generate a non transitive group, how is the genus \(g\) of the associated origami distributed? For given \(n\) and \(g\), how do these origamis distribute among the different strata?

2 Lyapunov exponents of a periodic windtree model

The periodic windtree model \(W_{a,b}\) is the billiard in the plane \(\mathbb{R}^2\) with size \(a \times b\) rectangular obstacles at all \(\mathbb{Z}^2\) lattice points. The associated flat surface is a \(\mathbb{Z}^2\)-cover of a genus 5 translation surface called \(X(a,b)\) (see [DHL14, §3] and Figure 2).

Prove that \(X(1/2,1/2)\) is a 12-square origami. Construct it with the package surface_dynamics. Compute approximations of its Lyapunov exponents. Do they match [DHL14, Theorem 7, page 1099]?

3 Arnoux-Yoccoz pseudo-Anosov example

The Arnoux-Yoccoz pseudo-Anosov homeomorphism \(f\) of a genus 3 surface is defined in [AYS1, Bo13].

1. Show that \(f\) fixes each singularity but does permute the separatrices.
2. Can \(f\) be obtained from Rauzy induction?
3. Can you decompose \(f\) as a product of Dehn twist?

Hint: you might want to use flipper to build the map \(f\) from a sequence of triangle flips.

4 World cup special (W. Veech)

A soccer ball is usually sowed from twenty regular hexagons and twelve regular pentagons. The corresponding polyhedron is a buckyball or truncated icosahedron, one of the 13 archimedean solids.

Any polytope in \(\mathbb{R}^3\) gives rise to a flat conical metric on the sphere (ie the metric is flat except at the vertices). When the angles of the polytope are rational multiples of \(\pi\) then there is an appropriate
ramified cover that is a translation surface. In the case of the buckyball, what are the properties of the translation surface $S$ obtained in this way? In particular...
1. What is its genus? its stratum?
2. What is its group of translation automorphisms?
3. Is it a Veech surface?
4. What can be said about its affine group?

Hint: you might want to use flatsurf, which allows dealing with conical metrics and taking covers.

5 Lost in triangles

There are several famous conjectures about billiards in triangles. Namely
1. Does every triangular billiard admit a periodic trajectory? (in the rational case, it is known that there are infinitely many of them [Ma86] and the answer is positive for all triangles for which all angles are smaller than 100 degrees [Sc09])
2. How does behave the complexity of a billiard sequence? (in the rational case, this is known to be linear [Hu95] while for irrational ones some polynomial upper bounds are known [GT95])
3. Can you find open sets of triangles that admit a periodic orbit? ([HS] for examples)

Investigate these two open problems using a computer program. For example
• Given a triangular billiard, find all possible codings of length $n$. What is the triangle that gives the largest complexity?
• Given a coding such as $ABACCABA$ find the set of angles for which this can be realized as a coding of a piece of trajectory.
• What about periodic trajectories?

References


