

Flat surfaces: computational problems

Summer school on Teichmüller dynamics, mapping class groups and applications
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Investigate these problems in small groups. Some are delicate and involve a mix of thinking and computing. They are roughly ordered by increasing difficulty. Suggested softwares and libraries: SageMath¹, flipper², snappy³, surface_dynamics⁴, flatsurf⁵.

Most articles in the reference are available for download at http://www.labri.fr/perso/vdelecro/grenoble_school/refs/

1 Where are my commutators?

Recall from Carlos Matheus’s lectures that an origami is given by a pair of permutations $(r, u) \in S_n \times S_n$ that together generate a transitive group, and tell, for each square on the surface, which square you find moving right (r) and moving up (u), see figure 1. The stratum $\mathcal{H}(\mu)$ of the origami given by (r, u) can be deduced from the cycle lengths of the commutator $[r, u] = r u r^{-1} u^{-1}$.

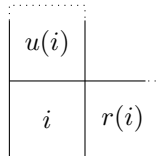


Figure 1: Gluing pattern of the origami given by permutations r and u .

Picking (r, u) uniformly at random in $S_n \times S_n$, and discarding pairs that generate a non transitive group, how is the genus g of the associated origami distributed? For given n and g , how do these origamis distribute among the different strata?

2 Lyapunov exponents of a periodic windtree model

The periodic windtree model $W_{a,b}$ is the billiard in the plane \mathbb{R}^2 with size $a \times b$ rectangular obstacles at all \mathbb{Z}^2 lattice points. The associated flat surface is a \mathbb{Z}^2 -cover of a genus 5 translation surface called $X(a, b)$ (see [DHL14, §3] and Figure 2)).

Prove that $X(1/2, 1/2)$ is a 12-square origami. Construct it with the package `surface_dynamics`. Compute approximations of its Lyapunov exponents. Do they match [DHL14, Theorem 7, page 1099]?

3 Arnoux-Yoccoz pseudo-Anosov example

The Arnoux-Yoccoz pseudo-Anosov homeomorphism f of a genus 3 surface is defined in [AY81, Bo13].

1. Show that f fixes each singularity but does permute the separatrices.
2. Can f be obtained from Rauzy induction?
3. Can you decompose f as a product of Dehn twist?

Hint: you might want to use `flipper` to build the map f from a sequence of triangle flips.

4 World cup special (W. Veech)

A soccer ball is usually sowed from twenty regular hexagons and twelve regular pentagons. The corresponding polyhedron is a buckyball or truncated icosahedron, one of the 13 archimedean solids.

Any polytope in \mathbb{R}^3 gives rise to a flat conical metric on the sphere (ie the metric is flat except at the vertices). When the angles of the polytope are rational multiples of π then there is an appropriate

¹More information on SageMath at <https://www.sagemath.org>

²More information on flipper at <http://flipper.readthedocs.io>

³More information on snappy at <https://www.math.uic.edu/t3m/SnapPy/>

⁴More information on surface_dynamics at http://www.labri.fr/perso/vdelecro/flatsurf_sage.html

⁵More information on flatsurf at <https://github.com/videlec/sage-flatsurf>

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