

# Flat surfaces: computational problems

Summer school on Teichmüller dynamics, mapping class groups and applications  
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Investigate these problems in small groups. Some are delicate and involve a mix of thinking and computing. They are roughly ordered by increasing difficulty. Suggested softwares and libraries: SageMath<sup>1</sup>, flipper<sup>2</sup>, snappy<sup>3</sup>, surface\_dynamics<sup>4</sup>, flatsurf<sup>5</sup>.

Most articles in the reference are available for download at [http://www.labri.fr/perso/vdelecro/grenoble\\_school/refs/](http://www.labri.fr/perso/vdelecro/grenoble_school/refs/)

## 1 Where are my commutators?

Recall from Carlos Matheus’s lectures that an origami is given by a pair of permutations  $(r, u) \in S_n \times S_n$  that together generate a transitive group, and tell, for each square on the surface, which square you find moving right ( $r$ ) and moving up ( $u$ ), see figure 1. The stratum  $\mathcal{H}(\mu)$  of the origami given by  $(r, u)$  can be deduced from the cycle lengths of the commutator  $[r, u] = rur^{-1}u^{-1}$ .

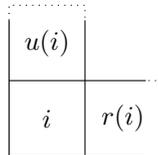


Figure 1: Gluing pattern of the origami given by permutations  $r$  and  $u$ .

Picking  $(r, u)$  uniformly at random in  $S_n \times S_n$ , and discarding pairs that generate a non transitive group, how is the genus  $g$  of the associated origami distributed? For given  $n$  and  $g$ , how do these origamis distribute among the different strata?

## 2 Lyapunov exponents of a periodic windtree model

The periodic windtree model  $W_{a,b}$  is the billiard in the plane  $\mathbb{R}^2$  with size  $a \times b$  rectangular obstacles at all  $\mathbb{Z}^2$  lattice points. The associated flat surface is a  $\mathbb{Z}^2$ -cover of a genus 5 translation surface called  $X(a, b)$  (see [DHL14, §3] and Figure 2)).

Prove that  $X(1/2, 1/2)$  is a 12-square origami. Construct it with the package `surface_dynamics`. Compute approximations of its Lyapunov exponents. Do they match [DHL14, Theorem 7, page 1099]?

## 3 Arnoux-Yoccoz pseudo-Anosov example

The Arnoux-Yoccoz pseudo-Anosov homeomorphism  $f$  of a genus 3 surface is defined in [AY81, Bo13].

1. Show that  $f$  fixes each singularity but does permute the separatrices.
2. Can  $f$  be obtained from Rauzy induction?
3. Can you decompose  $f$  as a product of Dehn twist?

*Hint:* you might want to use `flipper` to build the map  $f$  from a sequence of triangle flips.

## 4 World cup special (W. Veech)

A soccer ball is usually sowed from twenty regular hexagons and twelve regular pentagons. The corresponding polyhedron is a buckyball or truncated icosahedron, one of the 13 archimedean solids.

Any polytope in  $\mathbb{R}^3$  gives rise to a flat conical metric on the sphere (ie the metric is flat except at the vertices). When the angles of the polytope are rational multiples of  $\pi$  then there is an appropriate

<sup>1</sup>More information on SageMath at <https://www.sagemath.org>

<sup>2</sup>More information on flipper at <http://flipper.readthedocs.io>

<sup>3</sup>More information on snappy at <https://www.math.uic.edu/t3m/SnapPy/>

<sup>4</sup>More information on surface\_dynamics at [http://www.labri.fr/perso/vdelecro/flatsurf\\_sage.html](http://www.labri.fr/perso/vdelecro/flatsurf_sage.html)

<sup>5</sup>More information on flatsurf at <https://github.com/videlec/sage-flatsurf>



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