Asymptotics of meanders with fixed number of bigons

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CNRS, LaBRI, Bordeaux

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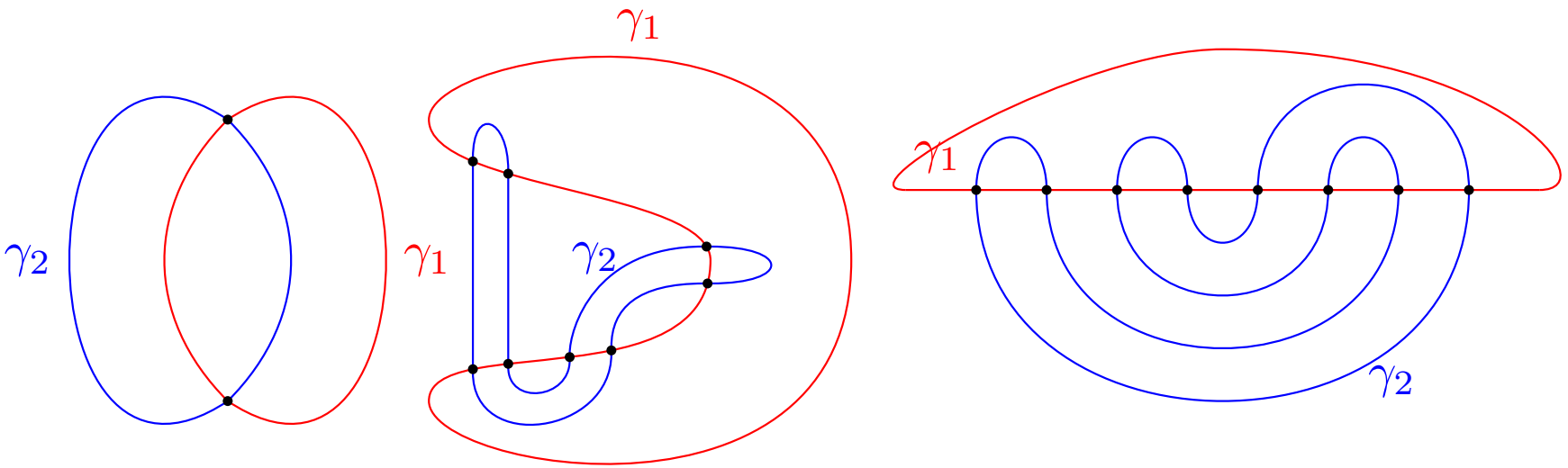
What is a meander

Definition

A meander is a homotopy class of pairs of transverse smooth curves (γ_1, γ_2) on the sphere.



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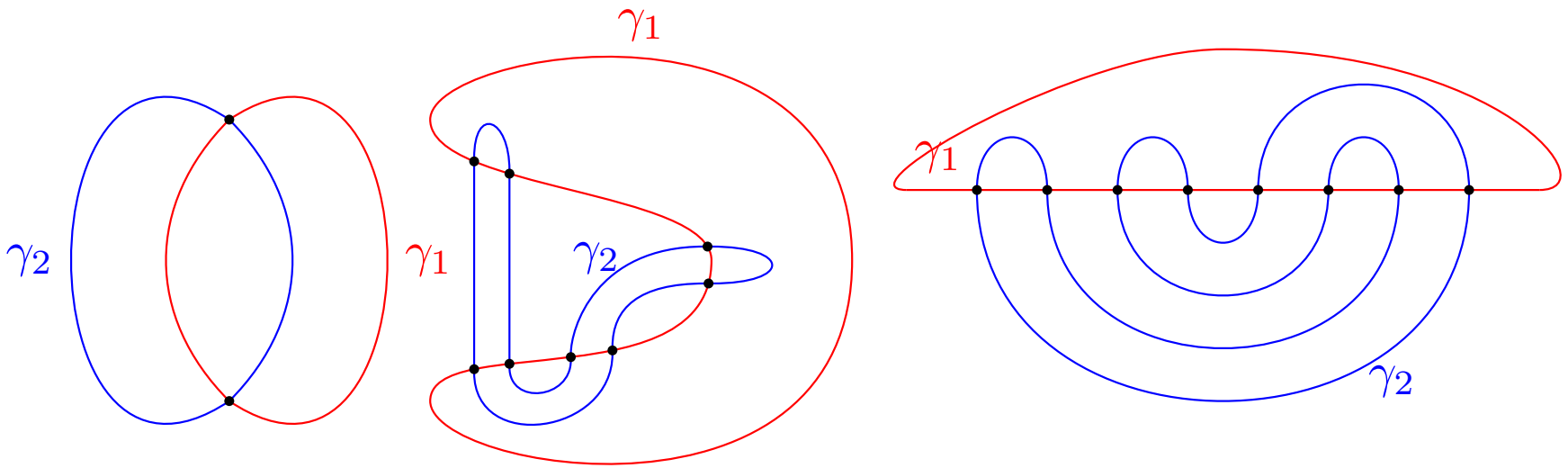
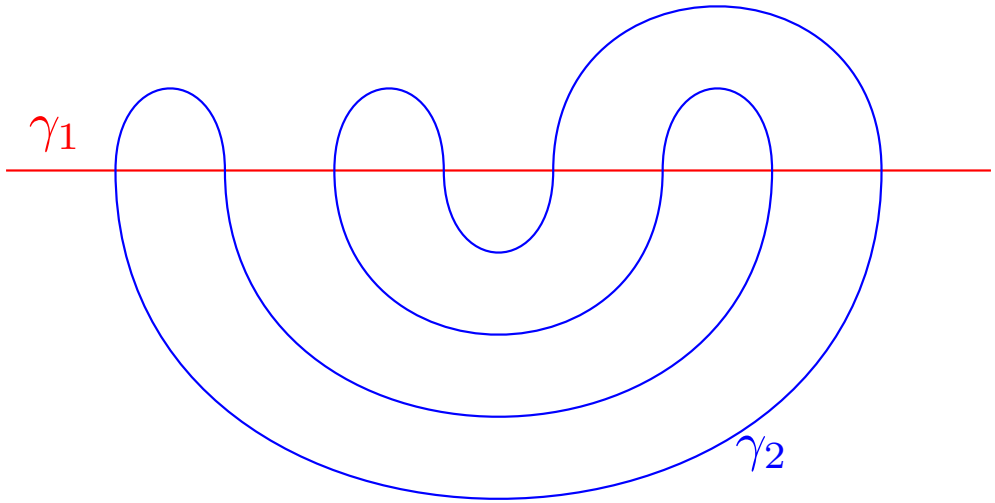


Following the literature, we will have a marked point on the blue curve so that intersections can be numbered (and avoid complications with automorphisms).

Faces and bigons

A meander cut the plane into faces with an even number of sides $2j$. We call $j-2$ the *degree*.



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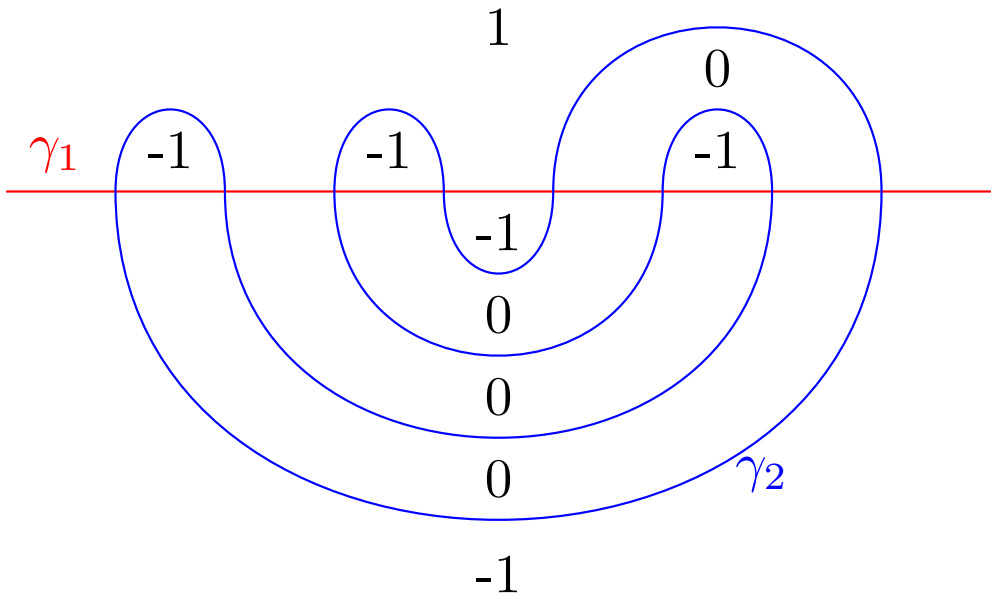

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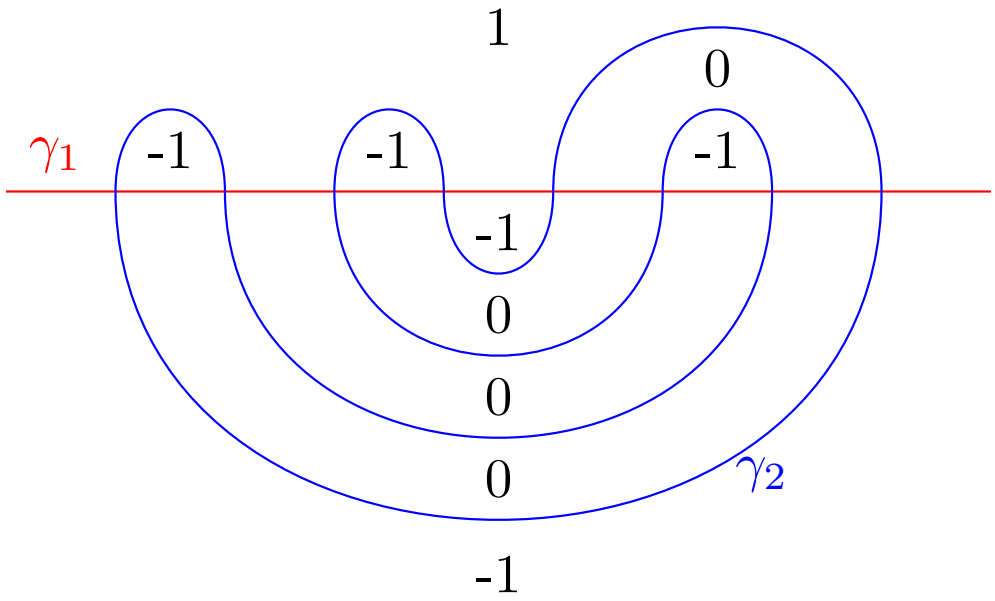

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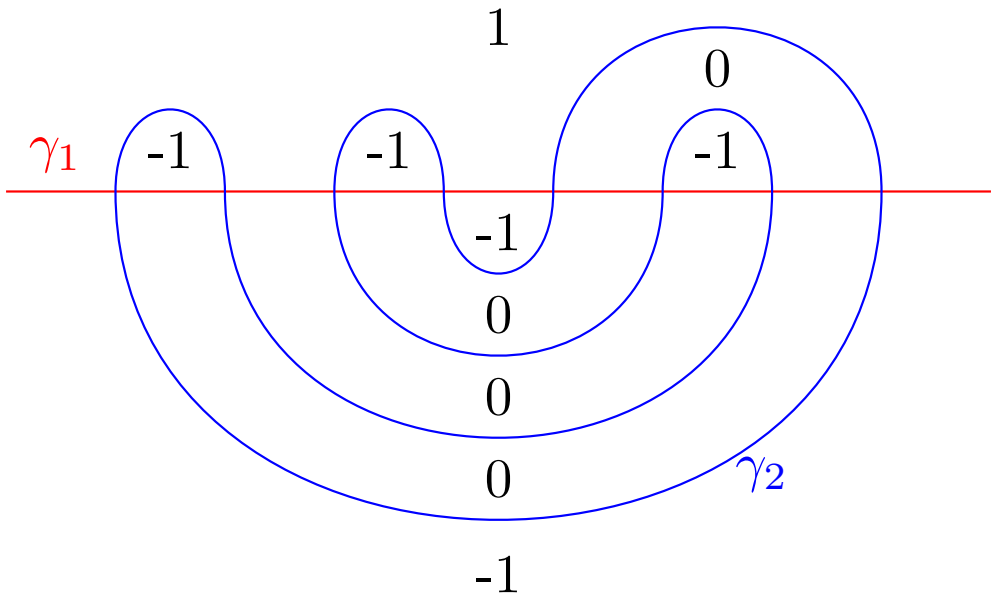

Sum of degrees = -4. The *bigons* are the faces of degree -1.

Some numbers

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n	$M_{n,4}$	$M_{n,5}$	$M_{n,6}$	$M_{n,7}$	$M_{n,8}$	$M_{n,9}$	$M_{n,10}$	M_n
2	2	0	0	0	0	0	0	2
3	6	0	2	0	0	0	0	8
4	8	16	16	0	2	0	0	42
5	20	40	110	60	30	0	2	262
6	12	168	416	576	462	144	48	1828
7	42	280	1470	3276	4228	2884	1288	13820
8	32	544	4128	13632	26424	31200	22096	110954
9	54	1152	9102	45468	130410	220488	243990	933458

Conjectural asymptotics for M_n

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Conjecture (Di Francesco-Golinelli-Guitter (2000))

$$M_n \sim C\ R^n\ n^{-\alpha}$$

where $C > 0$, $R \simeq 12.2628$ *and* $\alpha = \dfrac{29 + \sqrt{145}}{12}$.

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Extensive numerical experiments and extrapolations by Jensen and Guttman (2000) tend to confirm the conjecture.

Average order of $M_{n,k}$

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Theorem (Delecroix-Goujard-Zograf-Zorich)

For all k we have the following asymptotics

$$\sum_{n \leq x} M_{n,k} \sim \frac{2}{k!(k-4)!} \left(\frac{2}{\pi^2}\right)^{k-3} \binom{2k-4}{k-2}^2 \frac{x^{2k-5}}{2k-5}.$$

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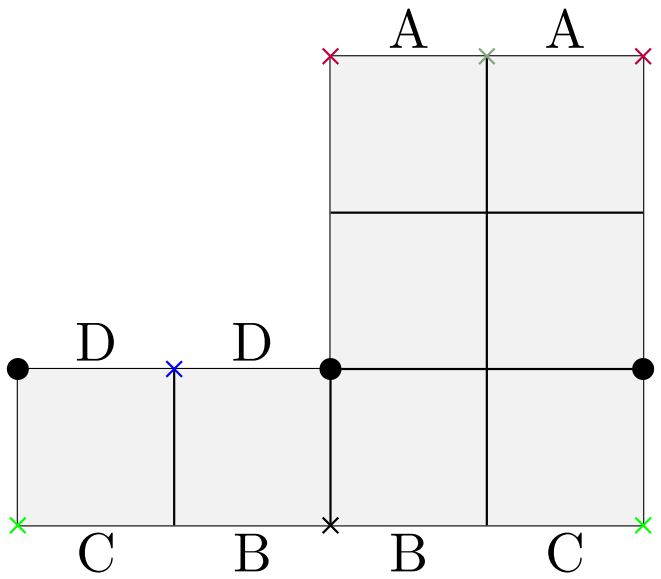
Examples

$$\sum_{n \leq x} M_{n,4} \sim \frac{6}{\pi^2} \frac{x^3}{3} \qquad \sum_{n \leq x} M_{n,5} \sim \frac{80}{3\pi^4} \frac{x^5}{5} \qquad \sum_{n \leq x} M_{n,6} \sim \frac{490}{9\pi^6} \frac{x^7}{7}$$

Meanders as square tiled surfaces of genus 0



A surface in $\mathcal{Q}(1, -1^5)$.

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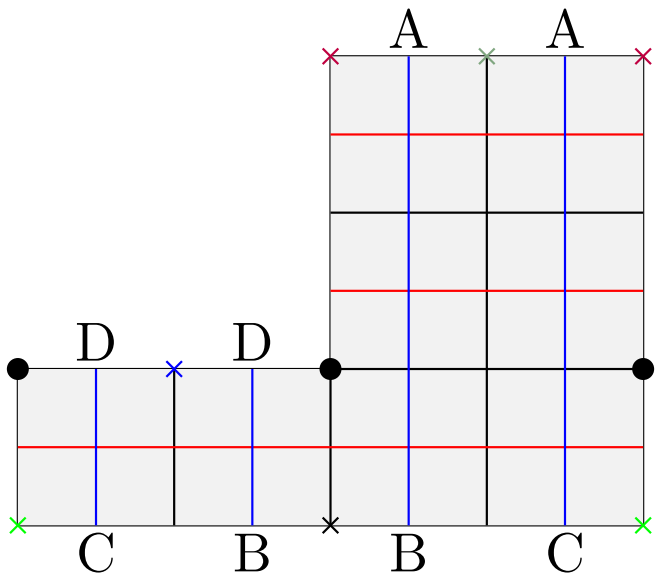

A surface in $\mathcal{Q}(1,-1^5)$. The horizontal curves and vertical curves dual to the squares gives a pair of transverse multicurve (γ_1, γ_2) on the sphere.

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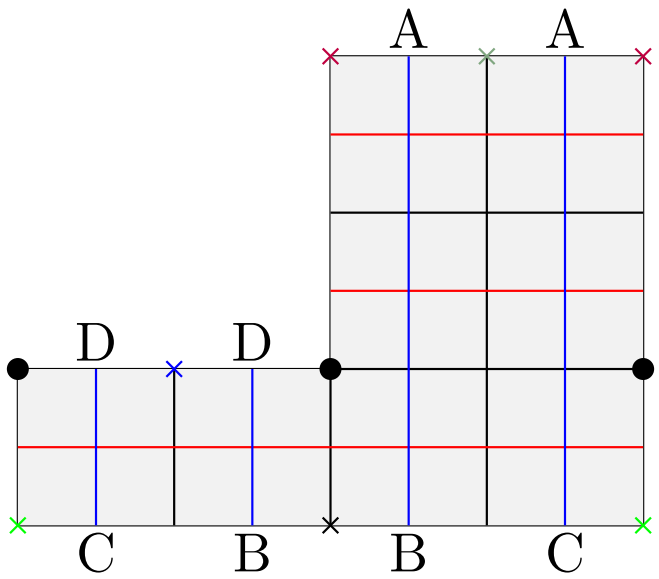

A surface in $\mathcal{Q}(1,-1^5)$. The horizontal curves and vertical curves dual to the squares gives a pair of transverse multicurve (γ_1, γ_2) on the sphere. In the example: γ_1 has three components while γ_2 has one.

Meanders as square tiled surfaces of genus 0

Let $\mathcal{Q}_{0,k}$ denote the moduli space of quadratic differentials on the sphere with k poles. We have

$$\mathcal{Q}_{0,4} = \mathcal{Q}(-1^4) \quad \mathcal{Q}_{0,5} = \mathcal{Q}(1,-1^5) \quad \mathcal{Q}_{0,6} = \mathcal{Q}(1^2,-1^6) \cup \mathcal{Q}(2,-1^6)$$

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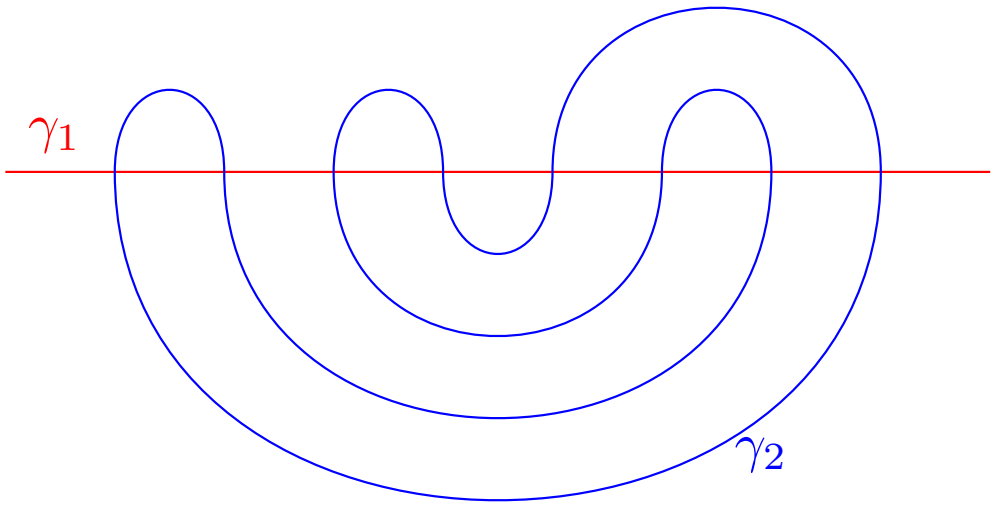
Lemma

We have

$$M_{n,k} = n \sum_S \frac{1}{|\operatorname{Aut}(S)|}$$

where the sum is over the square tiled surfaces in $\mathcal{Q}_{0,k}$ whose horizontal and vertical directions are made of a single cylinder of height 1.

Meanders as square tiled surfaces of genus 0 (proof)



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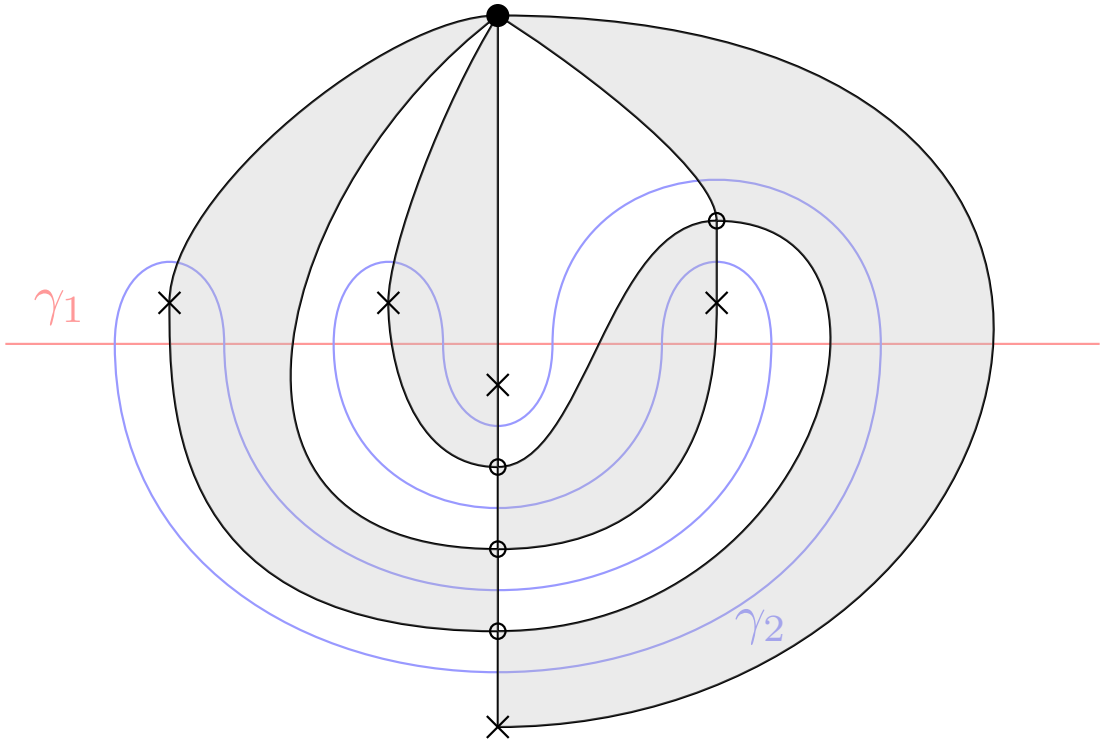

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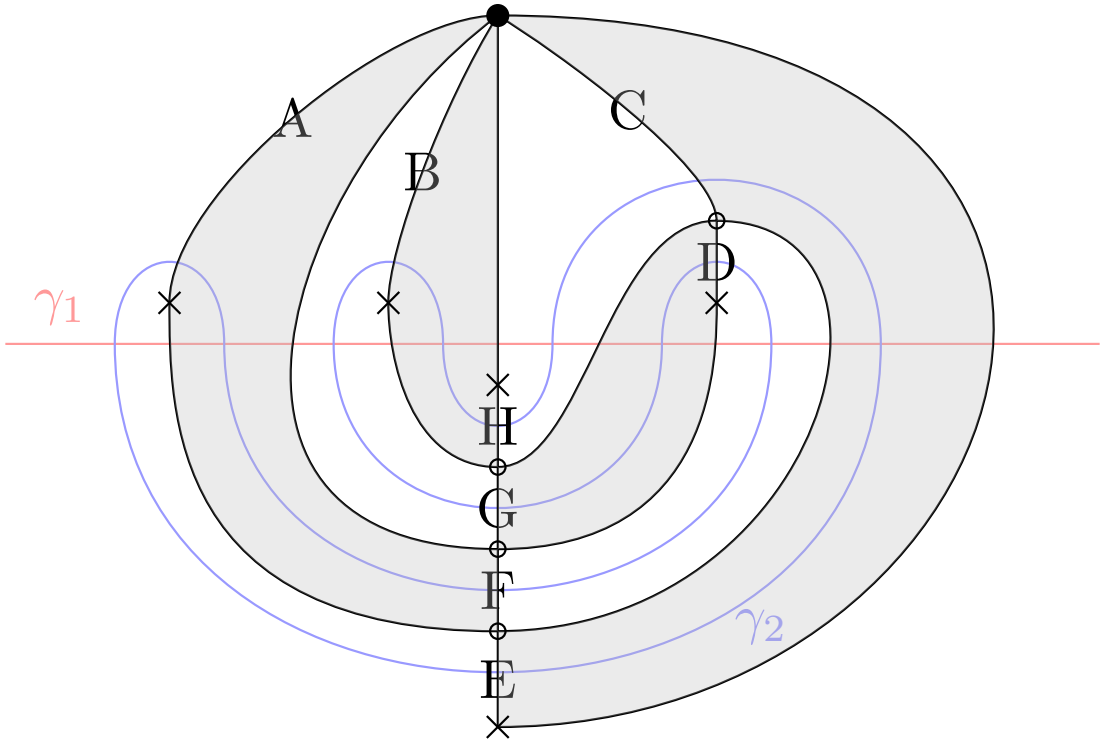

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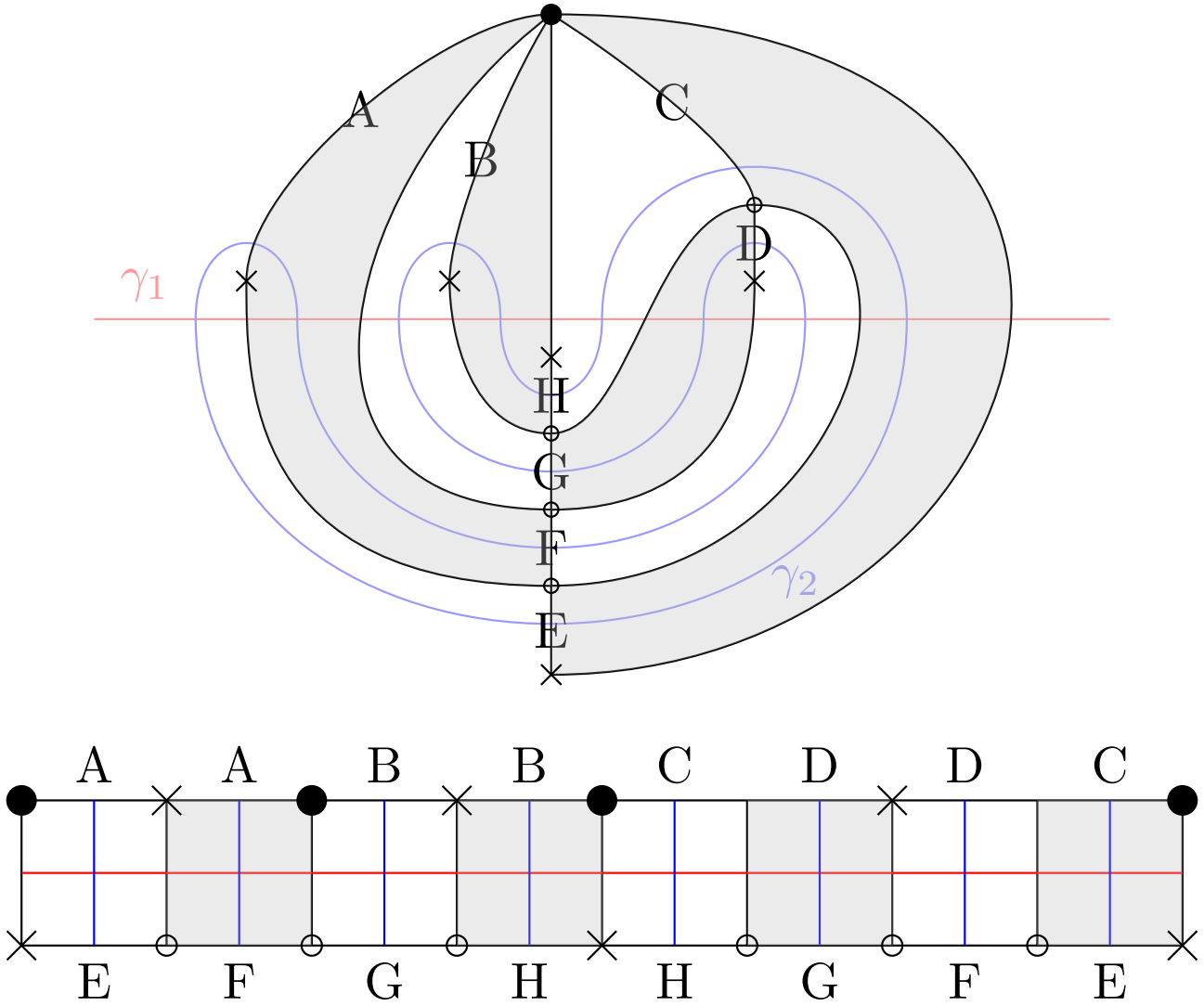

Reformulation of the asymptotics theorem

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Theorem (Delecroix-Goujard-Zograf-Zorich)

Let $\mathcal{C}^h_{\mathbb{Z}}$ (respectively $\mathcal{C}^v_{\mathbb{Z}}$) denote the set of square tiled surfaces in $\mathcal{C} = \mathcal{Q}(1^{k-4}, -1^k)$ that are made of a single horizontal cylinder (resp. vertical cylinder) of height one. Then

$$\#\{S \in \mathcal{C}^h_{\mathbb{Z}} \cap \mathcal{C}^v_{\mathbb{Z}} : \operatorname{Area}(S) \leq x\} \sim \frac{2}{k!(k-4)!} \left(\frac{2}{\pi^2}\right)^{k-3} \binom{2k-4}{k-2}^2 \frac{x^{2k-6}}{2k-6}.$$

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1. $$c(\mathcal{C}_{\mathbb{Z}}^h \cap \mathcal{C}_{\mathbb{Z}}^v) = \frac{c(\mathcal{C}_{\mathbb{Z}}^h)\ c(\mathcal{C}_{\mathbb{Z}}^v)}{c(\mathcal{C}_{\mathbb{Z}})}$$

   where $c(\mathcal{D})$ stands for the constant so that $\#\{S \in \mathcal{D} : \text{Area}(S) \leq x\} \sim c(\mathcal{D}) \frac{x^{2k-6}}{2k-6}$.

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2. compute the terms in rhs: $c(\mathcal{C}_{\mathbb{Z}}^h)$, $c(\mathcal{C}_{\mathbb{Z}}^v)$ and $c(\mathcal{C}_{\mathbb{Z}})$.

The case $k = 4$

For $\mathcal{C} = \mathcal{Q}(-1^4)$ we have

$$\#\{S \in \mathcal{C}_{\mathbb{Z}} : \text{Area}(S) = n\} = \sigma(n)$$
$$\#\{S \in \mathcal{C}^h_{\mathbb{Z}} : \text{Area}(S) = n\} = n$$
$$\#\{S \in \mathcal{C}^h_{\mathbb{Z}} \cap \mathcal{C}^v_{\mathbb{Z}} : \text{Area}(S) = n\} = \frac{M_{n,4}}{n} = \phi(n)$$

where ϕ is the Euler totient function and $\sigma(n) = \sum_{d|n} d$.

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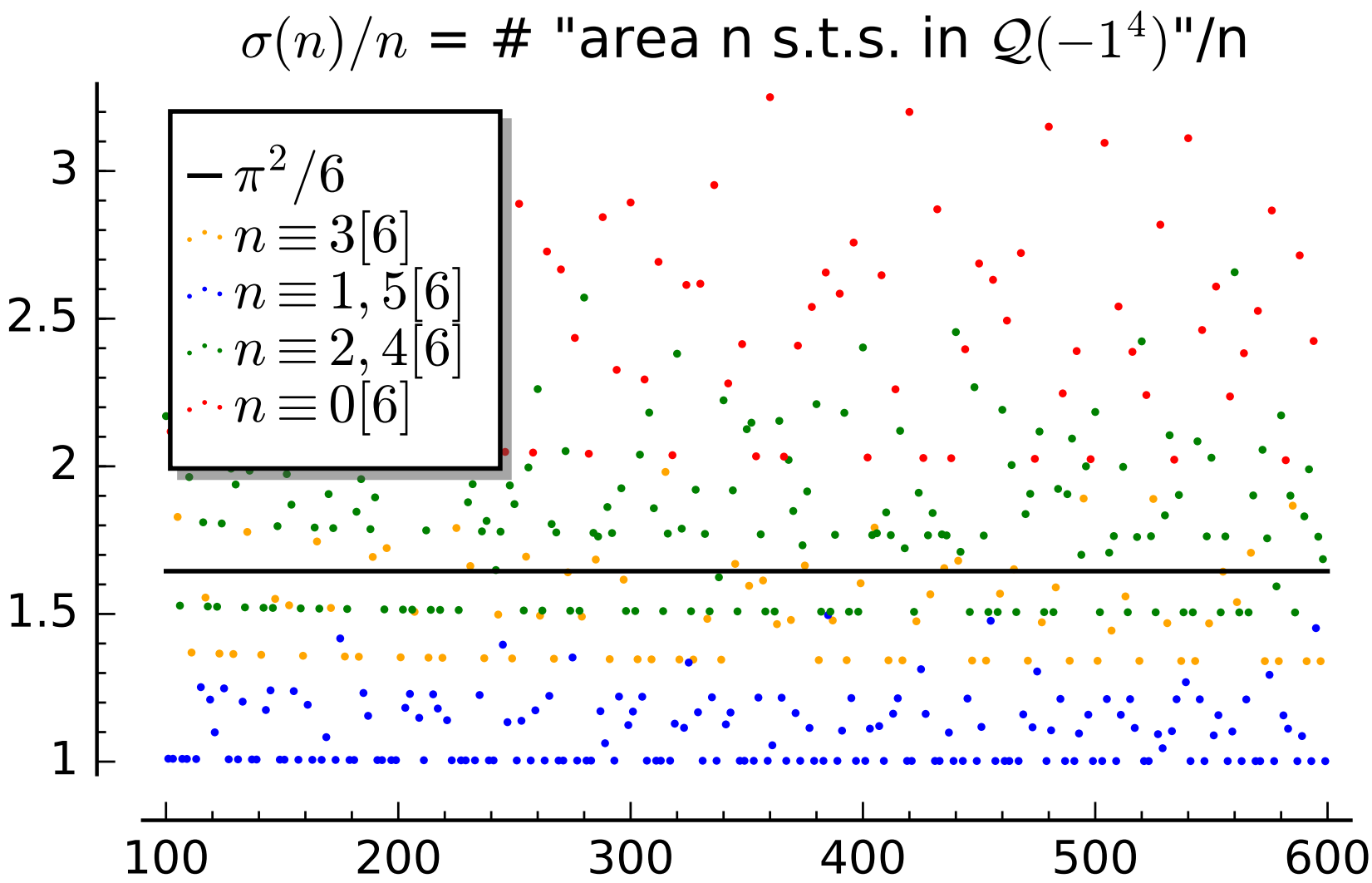
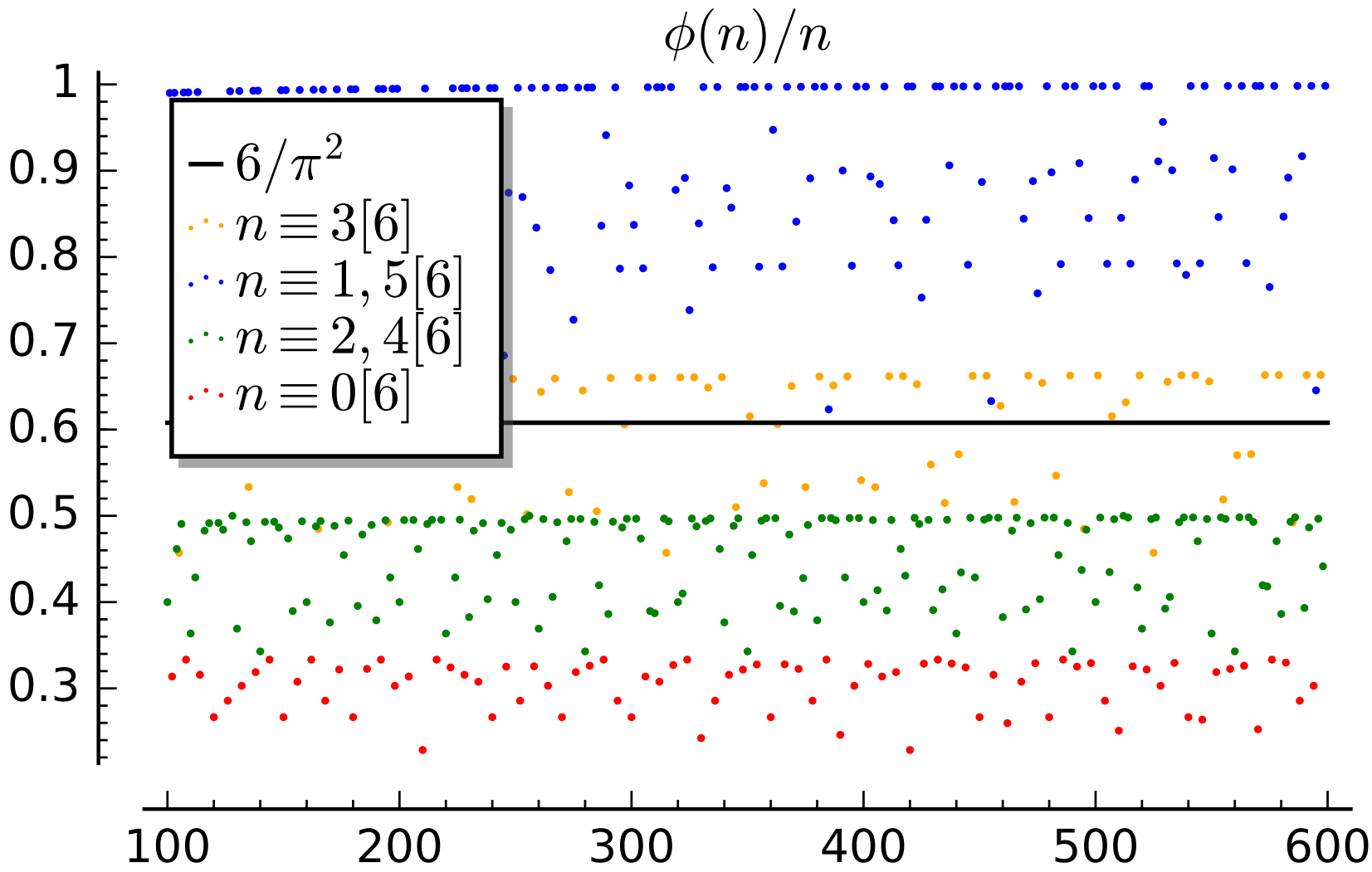
where ϕ is the Euler totient function and $\sigma(n) = \sum_{d|n} d$. We have

$$\sum_{n \leq x} \phi(n) \sim \frac{6}{\pi^2} x^2 \qquad \text{and} \qquad \sum_{n \leq x} \sigma(n) \sim \frac{\pi^2}{6} x^2.$$

$\sigma(n)/n$


$\sigma(n)/n$ = # "area n s.t.s. in $\mathcal{Q}(-1^4)$"/n
$-\pi^2/6$
$n \equiv 3[6]$
$n \equiv 1,5[6]$
$n \equiv 2,4[6]$
$n \equiv 0[6]$
3
2.5
2
1.5
1
100
200
300
400
500
600

$\phi(n)/n$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$-6/\pi^2$
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1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
100
200
300
400
500
600

The case $k = 5$

For $\mathcal{C} = \mathcal{Q}(1, -1^5)$ we have

$$\#\{S \in \mathcal{C}_{\mathbb{Z}} : \text{Area}(S) = n\} = \frac{3}{8}\left(\sigma_3(n) - (2n-1)\sigma(n)\right)$$

$$\#\{S \in \mathcal{C}^h_{\mathbb{Z}} : \text{Area}(S) = n\} = \frac{n(n-1)(n-2)}{6}$$

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where $\sigma_3(n) = \sum_{d|n} d^3$.

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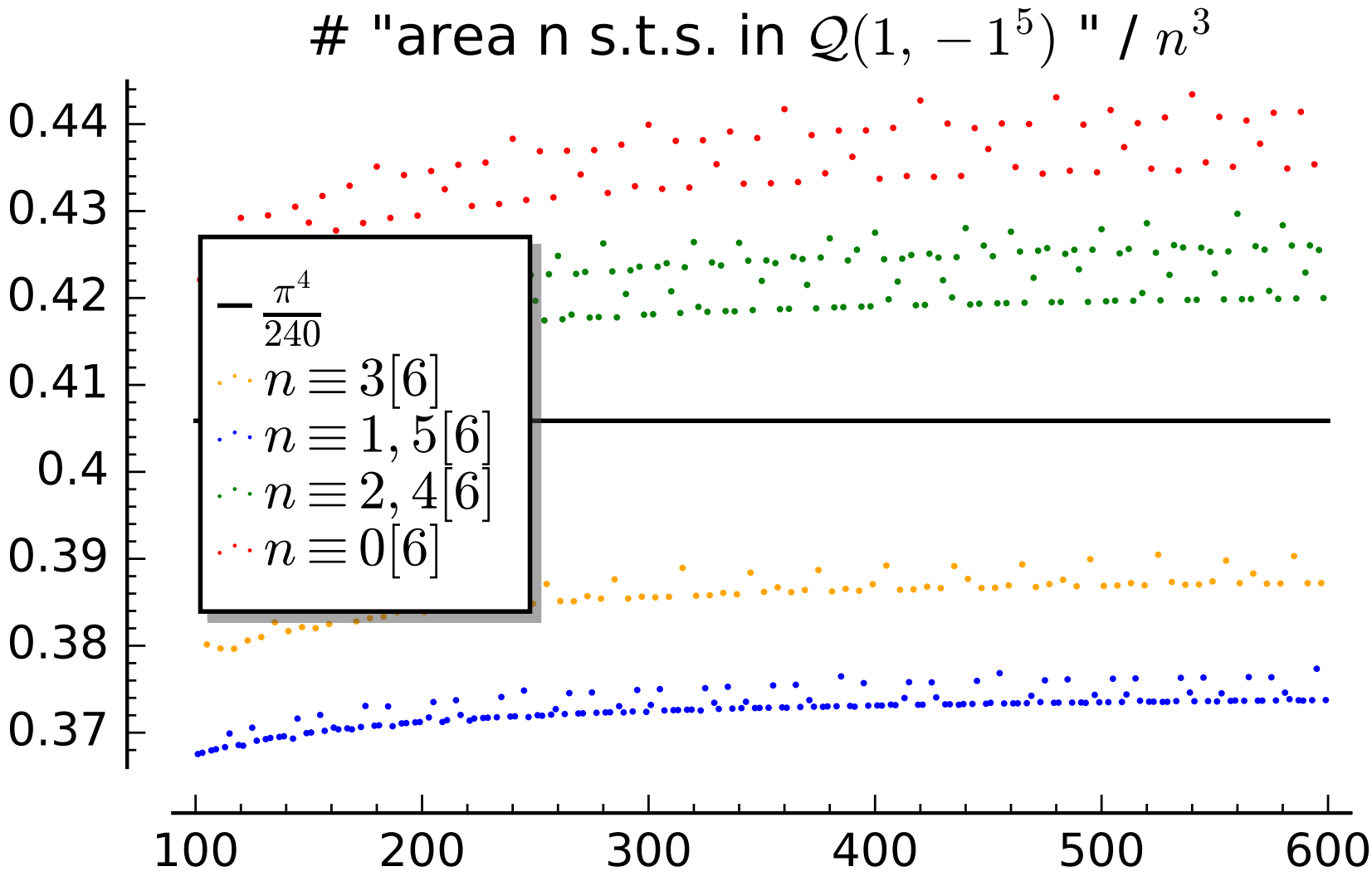
where $\sigma_3(n) = \sum_{d|n} d^3$. For the general $\mathcal{C} = \mathcal{Q}(1^{k-4}, -1^k)$ the counting functions

1. for $\mathcal{C}_{\mathbb{Z}}$ is expressible in terms of σ ("quasimodularity" by Eskin, Okounkov, Pandharipande, Engels)
2. for $\mathcal{C}^h_{\mathbb{Z}}$ is a rational multiple of $\binom{n-k+4}{k-3}$
3. is not known for the meanders $(\mathcal{C}^h_{\mathbb{Z}} \cap \mathcal{C}^v_{\mathbb{Z}})$!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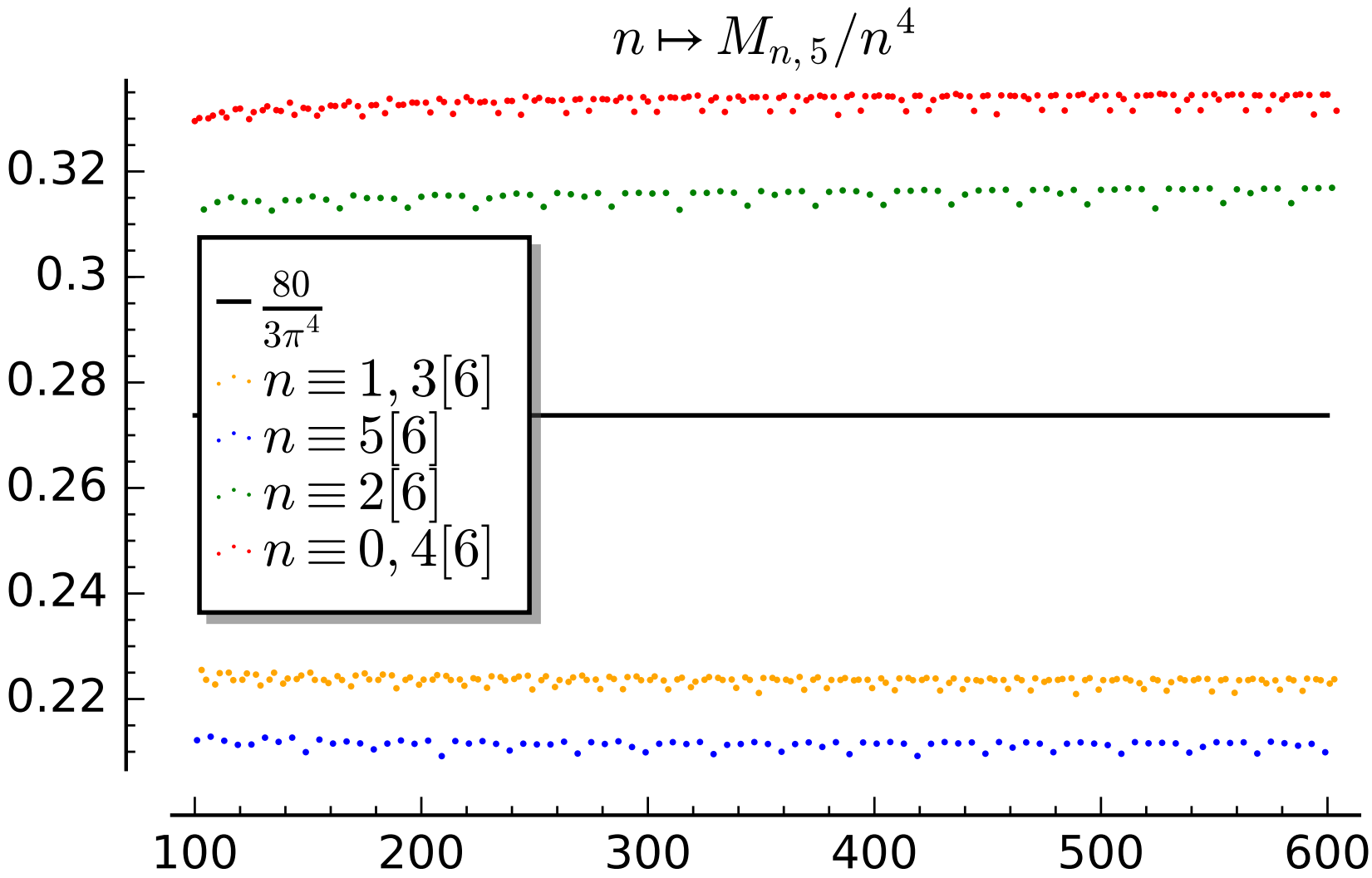
$n \mapsto M_{n,5}/n^4$
0.32
0.3
0.28
0.26
0.24
0.22
$-\frac{80}{3\pi^4}$
$n \equiv 1, 3[6]$
$n \equiv 5[6]$
$n \equiv 2[6]$
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100
200
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600

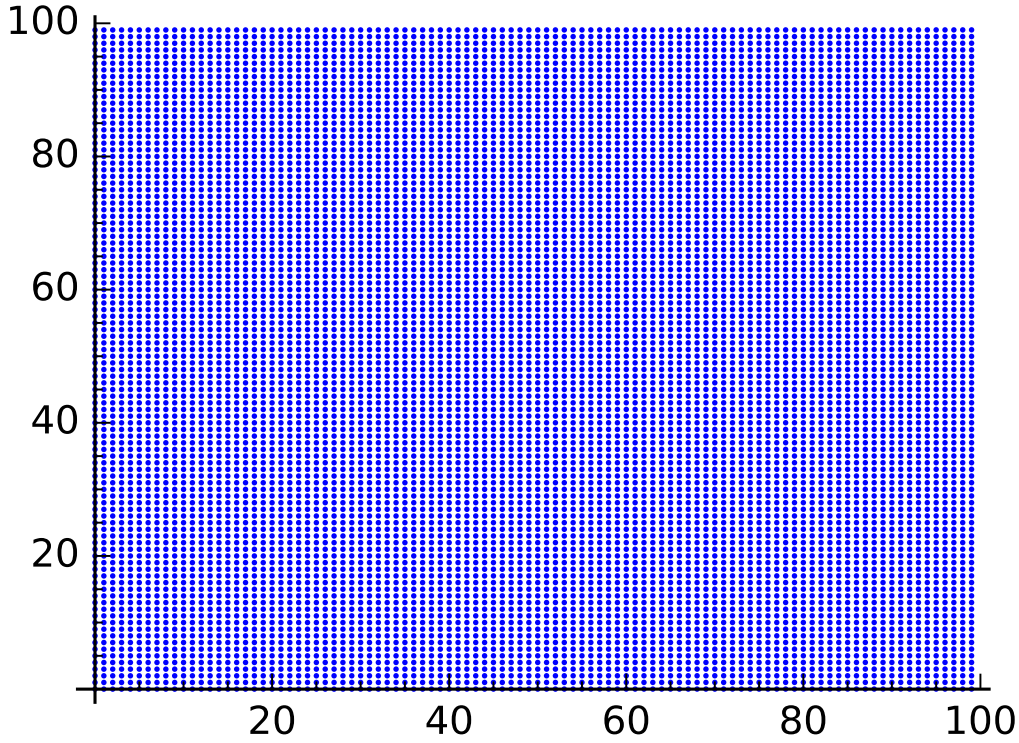
Existence of densities

Given a connected component of stratum $\mathcal{C}$, our aim is now to prove that $c(\mathcal{C}_{\mathbb{Z}})$, $c(\mathcal{C}^h_{\mathbb{Z}})$, $c(\mathcal{C}^v_{\mathbb{Z}})$ and $c(\mathcal{C}^h_{\mathbb{Z}} \cap \mathcal{C}^v_{\mathbb{Z}})$ exist and satisfy

$$c(\mathcal{C}^h_{\mathbb{Z}} \cap \mathcal{C}^v_{\mathbb{Z}}) = \frac{c(\mathcal{C}^h_{\mathbb{Z}})\ c(\mathcal{C}^v_{\mathbb{Z}})}{c(\mathcal{C}_{\mathbb{Z}})}.$$

Equidistribution in $\mathbb{Z}^2$

Let $\Lambda = \mathbb{Z}^2$ be the set of integral vectors.



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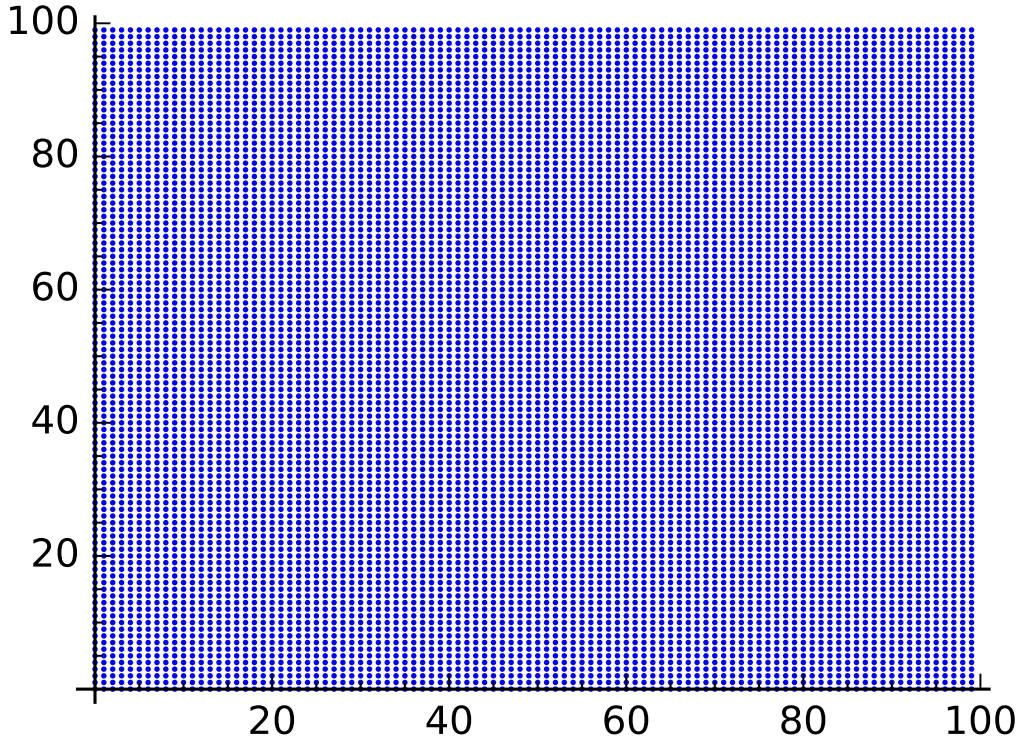


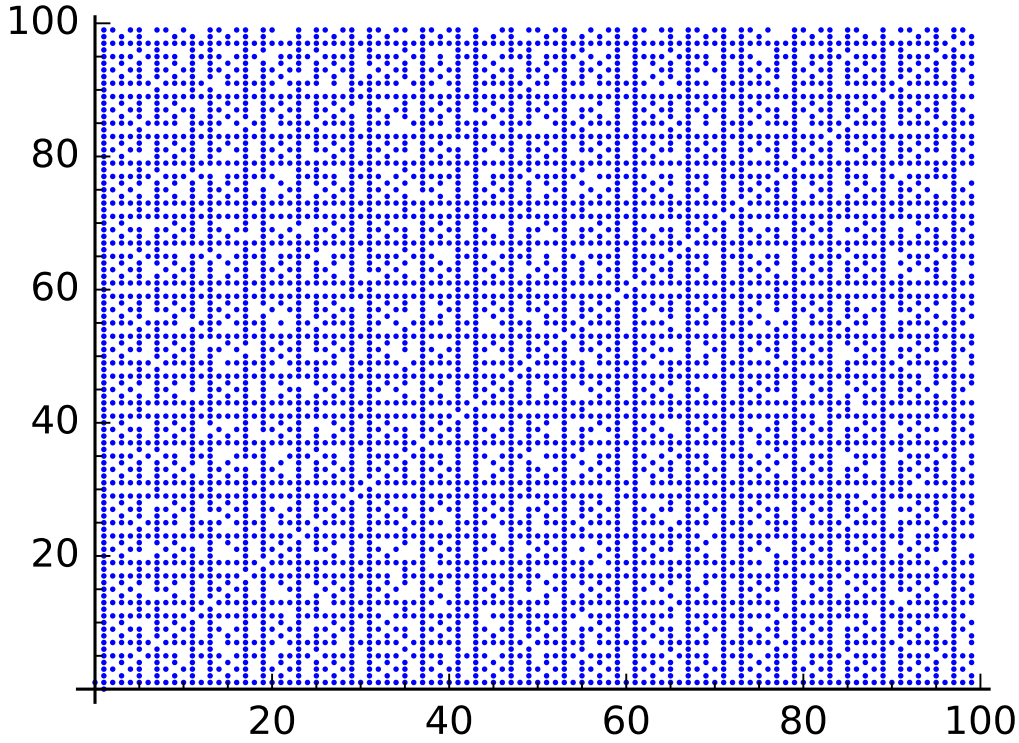
Theorem

For any relatively compact open set $U \subset \mathbb{R}^2$ we have as $\epsilon \to 0$ the following asymptotics

$$\#(U \cap (\epsilon\Lambda)) \sim \frac{\mathrm{Vol}(U)}{\epsilon^2}.$$

Equidistribution in $\mathbb{Z}^2$ (bis)

Let $\Lambda' \subset \mathbb{Z}^2$ be the set of integral vectors with relatively prime entries.



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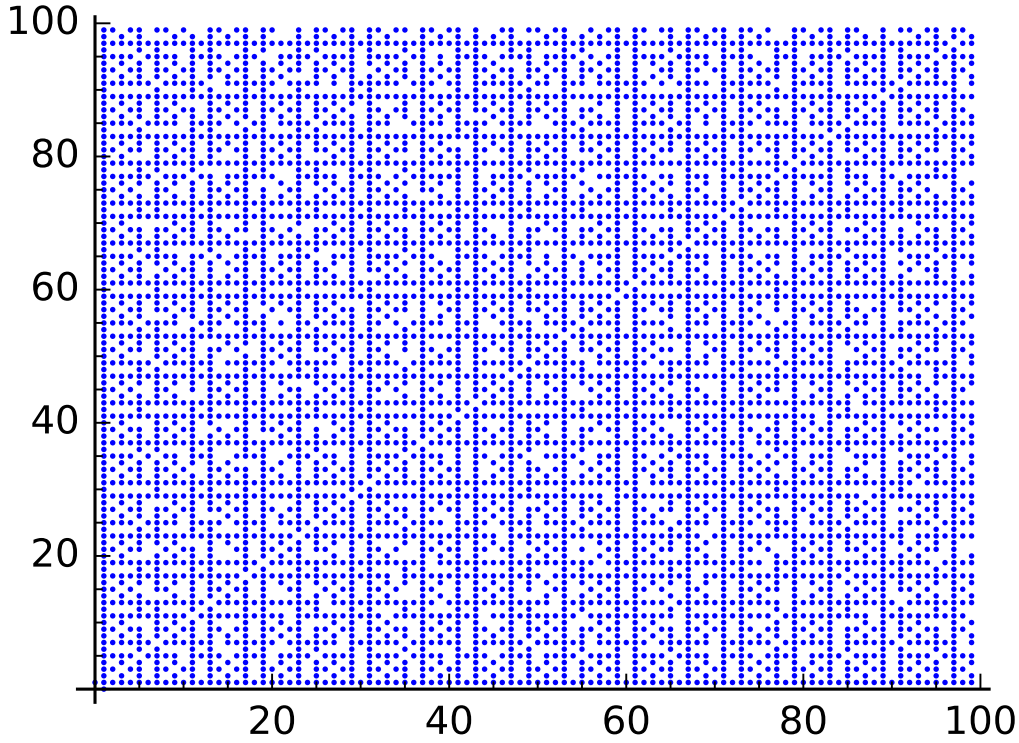

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$$\#(U \cap (\epsilon\Lambda')) \sim \frac{6}{\pi^2} \frac{\mathrm{Vol}(U)}{\epsilon^2}.$$

Equidistribution of square tiled surfaces

Let $\mathcal{D}$ be a subset of the set of square tiled surfaces in a component of stratum $\mathcal{C}$. We say that $\mathcal{D}$ has *uniform density in* $\mathcal{C}$ if there exists a number $\delta(\mathcal{D})$ so that for any open set $U \subset \mathcal{C}$ we have as $\epsilon \to 0$ the following equivalent

$$\#(U \cap (\epsilon\mathcal{D})) \sim \delta(\mathcal{D}) \frac{\mu_{MV}(U)}{\epsilon^d}$$

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Equidistribution of square tiled surfaces

Theorem (Delecroix-Goujard-Zograf-Zorich)

Let $\mathcal{C}$ be a component of stratum of quadratic differentials. Let $\mathcal{C}_{\mathbb{Z}}$ be the square tiled surfaces in $\mathcal{C}$ and $\mathcal{C}^h_{\mathbb{Z}}$ (respectively $\mathcal{C}^v_{\mathbb{Z}}$) be the s.t.s with a single horizontal cylinder with height 1 (resp. vertical cylinder with height 1). Then $\mathcal{C}^h_{\mathbb{Z}}$, $\mathcal{C}^v_{\mathbb{Z}}$ and $\mathcal{C}^h_{\mathbb{Z}} \cap \mathcal{C}^v_{\mathbb{Z}}$ have uniform densities and

$$\delta(\mathcal{C}_h \cap \mathcal{C}_v) = \delta(\mathcal{C}_h)\ \delta(\mathcal{C}_v) = \delta(\mathcal{C}_h)^2.$$

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$$\#\{S \in \mathcal{D} : \ \mathrm{Area}(S) \leq x\} \sim c(\mathcal{D}) \ \frac{x^d}{2d}.$$

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step 3: the intersection $\mathcal{C}^h_{\mathbb{Z}} \cap \mathcal{C}^v_{\mathbb{Z}}$ satisfies $\delta(\mathcal{C}^h_{\mathbb{Z}} \cap \mathcal{C}^v_{\mathbb{Z}}) = \delta(\mathcal{C}^h_{\mathbb{Z}}) \ \delta(\mathcal{C}^v_{\mathbb{Z}})$ because of the product structure of $\mathcal{C}$.

Explicit numbers in genus 0

Theorem

Let $\mathcal{C} = \mathcal{Q}(1^{k-4}, -1^k)$ and $\mathcal{C}^h_{\mathbb{Z}}$ be the set of square tiled surfaces in $\mathcal{C}$ with a single horizontal cylinder of height. Then

1. *(Athreya-Eskin-Zorich)*

$$c(\mathcal{C}_{\mathbb{Z}}) = \mu^{(1)}_{MV}(\mathcal{Q}(1^{k-4}, -1^k)) = \frac{\pi^{2k-6}}{2^{k-5}} = \frac{1}{4}\left(\frac{\pi^2}{2}\right)^{k-3}$$

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$$\sum_{n \leq x} M_{n,k} \sim \frac{2}{k!(k-4)!}\left(\frac{2}{\pi^2}\right)^{k-3}\binom{2k-4}{k-2}^2 \frac{x^{2k-5}}{2k-5}.$$

The computation of $\mathcal{C}_{\mathbb{Z}}$ (Athreya-Eskin-Zorich)

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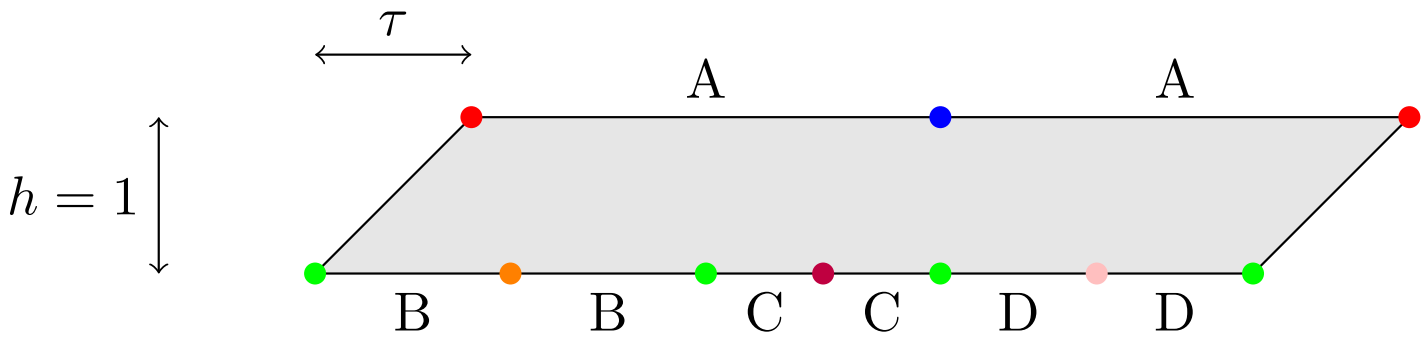
The proof of Athreya-Eskin-Zorich is quite convoluted and uses the formula for the sum of Lyapunov exponents Λ^+ (Eskin-Kontsevich-Zorich) and the fact that on the sphere $\Lambda^+ = 0$ (bundle has dimension 0).

The computation of $\mathcal{C}_{\mathbb{Z}}^{h}$ (Delecroix-Goujard-Zograf-Zorich)

Let us consider the case of $\mathcal{Q}(1,-1^5)$.



$$\{(\lambda_A,\lambda_B,\lambda_C,\lambda_D,\tau)\in\mathbb{Z}_{>0} : \lambda_A=\lambda_B+\lambda_C+\lambda_D \quad \text{and} \quad 0\leq\tau<\lambda_A\}$$

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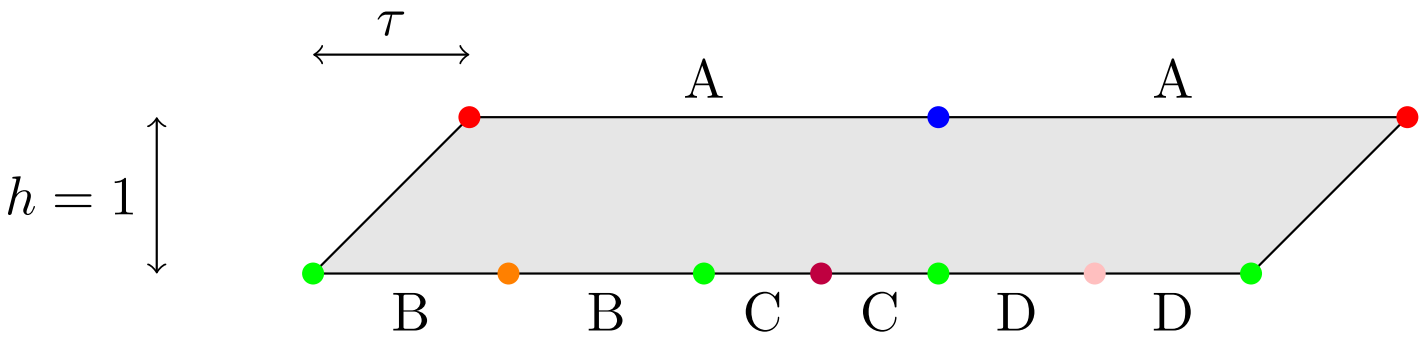

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For fixed area n we got $\frac{1}{3}\binom{n}{2}n$ such surfaces.

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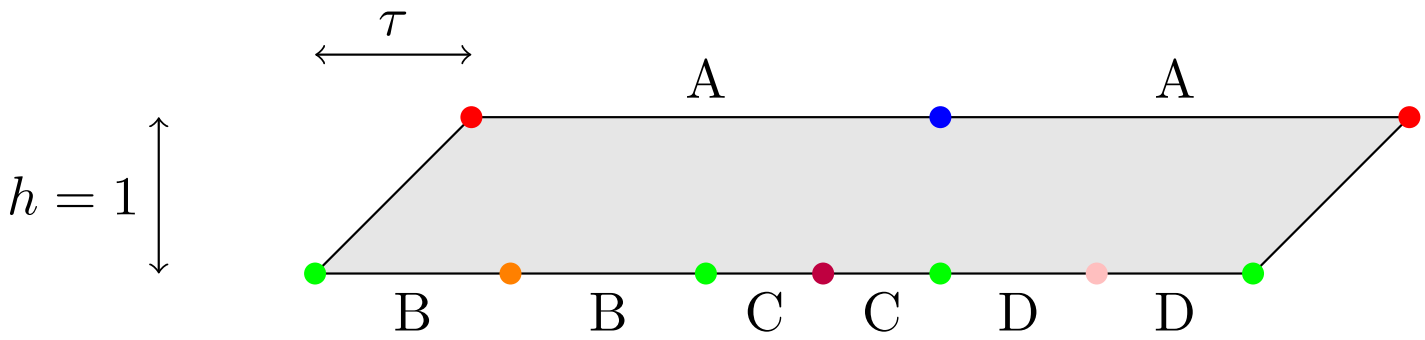

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In general we have to make a count over all separatrix diagrams that are counted by trees.

References

- S. K. Lando, A. K. Zvonkin *Plane and projective meanders* Theoret. Comput. Sci. 117 (1993).
- P. Di Francesco, O. Golinelli, E. Guitter
  - *Meanders: a direct enumeration approach* Nuclear Phys. B 482 (1996).
  - *Meander, folding, and arch statistics* Combinatorics and physics, Math. Comput. Modelling 26 (1997).
  - *Meanders and the Temperley-Lieb algebra* Comm. Math. Phys. 186 (1997).
  - *Meanders: exact asymptotics* Nuclear Phys. B 570 (2000).
- I. Jensen, Iwan *A transfer matrix approach to the enumeration of plane meanders* J. Phys. A 33 (2000).
- I. Jensen, A. J. Guttmann *Critical exponents of plane meanders* J. Phys. A 33 (2000).
- V. Delecroix, É. Goujard, P. Zograf A. Zorich *Enumeration of meanders and Masur-Veech volumes* `https://arxiv.org/abs/1705.05190`