Algorithmic of translation surfaces

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What and why
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What is a translation surface?

**definition:** edge to edge gluings by translation.

![Diagram of translation surface]

A translation structure is $\in \mathbb{R}^2$ such that for each edge $e$: $v(-e) = -v(e)$, for each triangle $e_1, e_2, e_3$: $v(e_1) + v(e_2) = -v(e_3)$ and $\det(e_1, e_2) > 0$. 
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convenient encoding: $T$ a triangulation, $E(T)$ = oriented edges. A translation structure is $e \in E(T) \mapsto v(e) \in \mathbb{R}^2$ such that

- for each edge $e$: $v(-e) = -v(e)$,
- for each triangle $e_1, e_2, e_3$: $v(e_1) + v(e_2) = -v(e_3)$ and $\det(e_1, e_2) > 0$. 
Why do we study translation surfaces?

- simpler than hyperbolic geometry
- same flavour than embedded graphs geometry
- (algebraic geometry) a translation surface is a Riemann surface endowed with a nonzero holomorphic one form
- (dynamics) polygonal billiards
Goal of the talk

- algorithmic of translation surfaces and implementation in https://flatsurf.github.io/
- four open questions and two conjectures
Geometry
Saddle connections and flat triangulations

**Definition**

*saddle connection*: straight line segment joining two conical singularities.

*flat triangulation*: triangulation of the translation surface whose edges are saddle connections.
The space of triangulations of $M$

Theorem (folklore?)

Any set of saddle connections in $M$ with disjoint interiors can be completed into a flat triangulation.
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**Theorem (folklore?)**

*Any set of saddle connections in $M$ with disjoint interiors can be completed into a flat triangulation.*

**Theorem (Masur)**

*Any two flat triangulations of $M$ can be joined by a sequence of edge flips.*
Theorem (Veech?)

One can compute the Delaunay triangulation (with respect to the set of conical singularities) via edge flipping.
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Consequence: one can

- compute the shortest saddle connection
- decide isometry between two translation surfaces

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conjecture 1: Given $\left( T, \{v(e)\}_{e \in E(T)} \right)$, one can compute the Delaunay triangulation in $O\left(\log \max_{e \in E(T)} \|v(e)\| \right)$. 
Enumerating saddle connections

Theorem (Masur, Vorobets)

*The number of saddle connections of length at most $L$ on $M$ is $\Theta(L^2)$.*
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conjecture 2: There is a $O(L^2)$-algorithm (aka optimal) based on flips in triangulation.
Tightening geodesics

Theorem (folklore)

A curve in a translation surface is a geodesic if and only if it is a concatenation of straight line segments meeting at conical singularities with angles $\geq \pi$.

Given a path which is a concatenation of saddle connections, there is a tightening procedure to homotope the path to a geodesic.

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open questions 1 and 2:
- complexity of tightening (possible combinatorial explosion)
- how hard it is to approximate the volume entropy?
Translation flow ($\mathbb{R}$-action on $M$)
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Theorem (Katok, Keane)

Let $M$ be a translation surface. Let $\phi_t: M \to M$ be its translation flow. Let $x \in M$ be a point with an infinite orbit. Then \( \{\phi_t(M)(x) : t \geq 0\} \) is either a circle (iff $x$ is a periodic point) or a subsurface bounded by finitely many saddle connections.
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consequence: the surface decomposes into finitely many

- cylinders,
- minimal components
- saddle connections.
A partial result

Theorem (VD + Julian Rüth)

There exists a semi-algorithm that compute and certify the decomposition of the surface into cylinders + minimal components + saddle connections.
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open question 3: is the following decidable

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OUTPUT: whether $y \in \{\phi_{M}^{t}(x) : t \geq 0\}$
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GL$_2$(\(\mathbb{R}\)))-action on the moduli space of translation surfaces
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**Theorem (Eskin-Mirzakhani-Mohammadi)**

*For any \(M\), \(\text{GL}_2(\mathbb{R}) \cdot M\) is nice.*
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**partial result:** there exists a semi-algorithm to compute \( \text{GL}_2(\mathbb{R}) \cdot M \) (available in flatsurf).