

Dynamics of a linear maps

In this worksheet we study the dynamics of a very simple linear map

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Action on \mathbb{R}^2

Create a variable `A` initialized to be the above matrix (check the command `matrix`):

```
sage: # edit here
```

Create a variable `v` that contains the vector $(1, 1)$ (check the command `vector`):

```
sage: # edit here
```

To compute the image of a vector under a matrix you need to use the multiplication `*`. Compute the image `A * v`:

```
sage: # edit here
```

Write a function `draw_orbit(A, v0, n)` that draw the n -th first iterate of the orbit of v_0 under A . That is the sequence of vectors $v_0, Av_0, A^2v_0, \dots, A^{n-1}v_0$ (you can use the graphics primitives `point2d` or `line2d`):

```
sage: # edit here
```

On the same graphics, draw several of these orbits:

```
sage: # edit here
```

What is happening to them?

The object `AA` in Sage stands for "real algebraic numbers". It is one way to consider exact numbers beyond integers and rationals.:

```
sage: AA
```

In the following cell, we create the algebraic number $\sqrt{5}$ and the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$:

```
sage: a = AA(5).sqrt()
sage: phi = (1 + a) / 2
```

Check with Sage that the vectors $u_+ = (1, -\phi)$ and $u_- = (1, \phi - 1)$ are eigenvectors of the matrix A :

```
sage: up = vector([1, -phi])
sage: um = vector([1, phi-1])
sage: # edit here
```

Create a graphics with several orbits together with the lines $\mathbb{R}u_+$ and $\mathbb{R}u_-$:

```
sage: # edit here
```

Action on $\mathbb{R}^2/\mathbb{Z}^2$

Show that the linear action of A on \mathbb{R}^2 gives a well defined map \bar{A} on the quotient $\mathbb{R}^2/\mathbb{Z}^2$. Show that it is a bijection.

Let us consider the fundamental domain $T = [0, 1] \times [0, 1]$. Write a function `toral_map(v)` that applies the map \bar{A} to a vector v in T :

```
sage: # edit here
```

Let $v = (1/2, 0)$. Show that $\bar{A}^3 v = v$ but $\bar{A}^n v \neq v$ for $n = 1, 2$:

```
sage: # edit here
```

We say that $v = (1/2, 0)$ is a *periodic point of period 3*.

Show more generally that any vector in T with rational entries is a *periodic point*, that is, there exists an integer n so that $\bar{A}^n v = v$. Given a periodic point v the smallest positive integer n so that $\bar{A}^n v = v$ is called the *period*.

Write a function `compute_period(v)` that given a rational vector in T compute its period.:

```
sage: # edit here
```

Let

$$T_q = \left\{ \left(\frac{a}{q}, \frac{b}{q} \right) : a, b \in \{0, 1, \dots, q-1\} \text{ and } \gcd(a, b, q) = 1 \right\}.$$

Show that T_q is preserved by \bar{A} . What is the cardinality of T_q ?

Write a function `classify_periods(q)` that given a positive integer q compute all possible periods of points in T_q and their multiplicities:

```
sage: # edit here
```

By looking at small values of q , find the number of points of periods 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Can you make a conjecture about the number of periodic points of period n ?

Could you write a function `periodic_points(n)` that return the set of points whose period is n ?:

```
sage: # edit here
```