Dynamics of a linear maps

In this worksheet we study the dynamics of a very simple linear map

\[ A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \]

**Action on \( \mathbb{R}^2 \)**

Create a variable \( A \) initialized to be the above matrix (check the command `matrix`):

```sage
sage: # edit here
```

Create a variable \( v \) that contains the vector \((1,1)\) (check the command `vector`):

```sage
sage: # edit here
```

To compute the image of a vector under a matrix you need to use the multiplication \(*\). Compute the image \( A * v \):

```sage
sage: # edit here
```

Write a function `draw_orbit(A, v0, n)` that draw the \( n \)-th first iterate of the orbit of \( v0 \) under \( A \). That is the sequence of vectors \( v_0, A v_0, A^2 v_0, \ldots, A^{n-1} v_0 \) (you can use the graphics primitives `point2d` or `line2d`):

```sage
sage: # edit here
```

On the same graphics, draw several of these orbits:

```sage
sage: # edit here
```

What is happening to them?

The object `AA` in Sage stands for "real algebraic numbers". It is one way to consider exact numbers beyond integers and rationals.:

```sage
sage: AA
```

In the following cell, we create the algebraic number \( \sqrt{5} \) and the golden ratio \( \phi = \frac{1+\sqrt{5}}{2} \):

```sage
sage: a = AA(5).sqrt()
sage: phi = (1 + a) / 2
```

Check with Sage that the vectors \( u_+ = (1,-\phi) \) and \( u_- = (1,\phi-1) \) are eigenvectors of the matrix \( A \):

```sage
sage: up = vector([1, -phi])
sage: um = vector([1, phi-1])
sage: # edit here
```

Create a graphics with several orbits together with the lines \( R u_+ \) and \( R u_- \):

```sage
sage: # edit here
```
**Action on $\mathbb{R}^2/\mathbb{Z}^2$**

Show that the linear action of $A$ on $\mathbb{R}^2$ gives a well defined map $\overline{A}$ on the quotient $\mathbb{R}^2/\mathbb{Z}^2$. Show that it is a bijection.

Let us consider the fundamental domain $T = [0, 1] \times [0, 1]$. Write a function `toral_map(v)` that applies the map $\overline{A}$ to a vector $v$ in $T$:

```python
sage: # edit here
```

Let $v = (1/2, 0)$. Show that $\overline{A}^3 v = v$ but $\overline{A}^n v \neq v$ for $n = 1, 2$:

```python
sage: # edit here
```

We say that $v = (1/2, 0)$ is a periodic point of period 3.

Show more generally that any vector in $T$ with rational entries is a periodic point, that is, there exists an integer $n$ so that $\overline{A}^n v = v$. Given a periodic point $v$ the smallest positive integer $n$ so that $\overline{A}^n v = v$ is called the period.

Write a function `compute_period(v)` that given a rational vector in $T$ compute its period:

```python
sage: # edit here
```

Let

\[ T_q = \left\{ \left( \frac{a}{q}, \frac{b}{q} \right) : a, b \in \{0, 1, \ldots, q-1\} \text{ and } \gcd(a, b, q) = 1 \right\}. \]

Show that $T_q$ is preserved by $\overline{A}$. What is the cardinality of $T_q$?

Write a function `classify_periods(q)` that given a positive integer $q$ compute all possible periods of points in $T_q$ and their multiplicities:

```python
sage: # edit here
```

By looking at small values of $q$, find the number of points of periods 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Can you make a conjecture about the number of periodic points of period $n$?

Could you write a function `periodic_points(n)` that return the set of points whose period is $n$:

```python
sage: # edit here
```