Dynamic Membership for Regular Languages

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Problem: dynamic membership for regular languages

- Fix a regular language L
  - $\rightarrow$ E.g., $L = (ab)^*$
- Read an input word w with $n := |w|$
  - $\rightarrow$ E.g., $w = abbbab$

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  - $\rightarrow$ E.g., we have $w \notin L$
- Maintain the membership of w to L under substitution updates
  - $\rightarrow$ E.g., replace character at position 3 with a: we now have $w \in L$

Design choices

- Model: RAM model
  - Cell size in $\Theta(\log(n))$
  - Unit-cost arithmetics
- Updates: only substitutions (so n never changes)
  - Otherwise, already tricky to maintain the current state of the word
- Memory usage: always polynomial in n by definition of the model
  - Our upper bounds only need $O(n)$ space
  - The lower bounds apply without this assumption
- Preprocessing:
  - The upper bounds only need $O(n)$ preprocessing
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A general-purpose algorithm in $O(\log n)$

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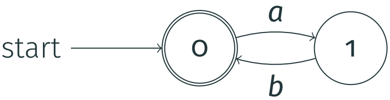

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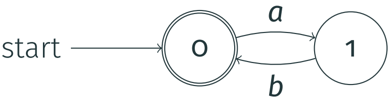
Fix the language $L = (ab)^*$: 

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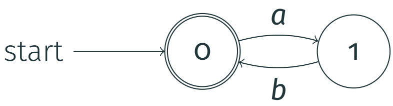

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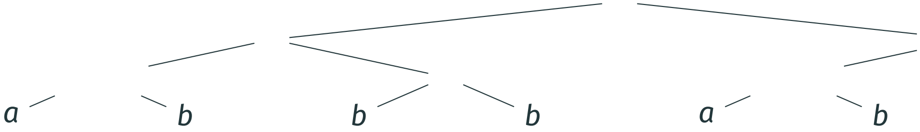

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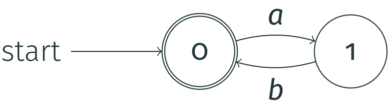

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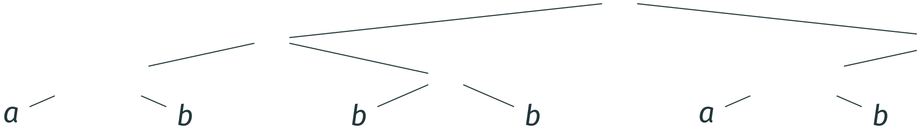

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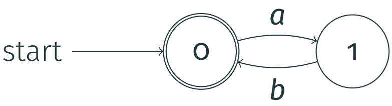
Fix the language $L = (ab)^*$: 

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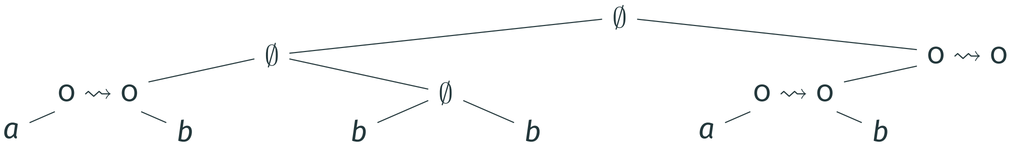

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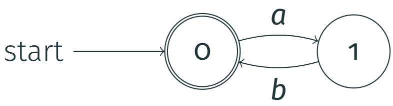
Fix the language $L = (ab)^*$: 

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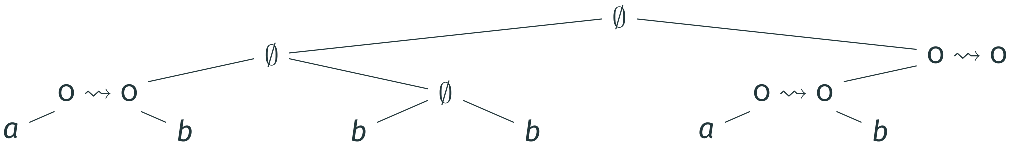

- The tree root describes if $w \in L$
- We can update the tree for each substitution in $O(\log n)$
- Can be improved to $O(\log n / \log \log n)$ with a log-ary tree

Can we do better than $O(\log n)$?

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- Check that n is even
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Question: what is the complexity of dynamic membership, depending on the fixed regular language L?

Dynamic word problem for monoids

To answer the question, we study the dynamic word problem for monoids:

- Problem definition:
  - Fix a monoid M (set with associative law and neutral element)
  - Input: word w of elements of M
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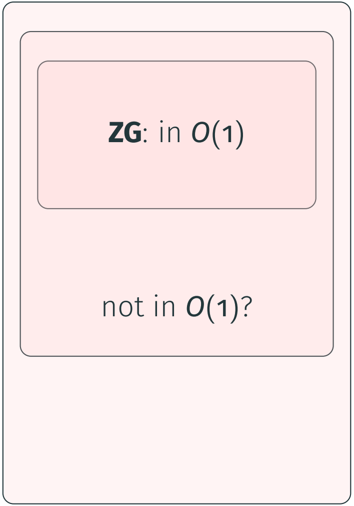
  - **Input:** word w of elements of M
  - Maintain the **product** of the elements under substitution updates
- This is a **special case** of dynamic membership for regular languages
  - e.g., it assumes that there is a **neutral element**
- This problem was studied by [Skovbjerg Frandsen et al., 1997]:

  → in $O(1)$ for **commutative monoids**

  → in $O(\log \log n)$ for **group-free monoids**

  → in $\Theta(\log n / \log \log n)$ for a certain class of monoids

Our results on the dynamic word problem for monoids



- We identify the class **ZG** satisfying $x^{\omega+1}y = yx^{\omega+1}$:
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- We identify the class **SG** satisfying $x^{\omega+1}yx^{\omega} = x^{\omega}yx^{\omega+1}$
  - for any monoid **in SG**, the problem is **in $O(\log \log n)$**
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- The problem is always in $O(\log n / \log\log n)$

Results on the dynamic membership problem for regular languages

QLZG: in $O(1)$

QSG: in $O(\log \log n)$
not in $O(1)$?

All: in $\Theta(\log n / \log \log n)$

Our results extend to regular language classes called **QLZG** and **QSG**

$\rightarrow$ We define them in the sequel

Results on monoids

$O(1)$ upper bound for monoids

Theorem

The dynamic word problem for commutative monoids is in $O(1)$

Algorithm:

- Count the number n_m of occurrences of each element m of M in w
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Lemma (Closure under monoid variety operations)

The submonoids, direct products, quotients of tractable monoids are also tractable

$O(1)$ upper bound for monoids (cont'd)

Theorem

The monoids S^1 where we add an identity to a nilpotent semigroup S are in $O(1)$

Idea of the proof: consider $e^*ae^*be^*$

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This technique applies to monoids where we intuitively need to track a constant number of non-neutral elements

$O(1)$ upper bound for monoids (end)

Call **ZG** the variety of monoids satisfying $x^{\omega+1}y = yx^{\omega+1}$ for all x, y

$\rightarrow$ Elements of the form $x^{\omega+1}$ are those belonging to a subgroup of the monoid

$\rightarrow$ This includes in particular all idempotents ($xx = x$)

$\rightarrow$ The $x^{\omega+1}$ are central: they commute with all other elements

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Lemma

ZG *is exactly the monoids obtainable from commutative monoids and monoids of the form S^1 for a nilpotent semigroup S via the monoid variety operators*

Theorem

The dynamic word problem for monoids in ***ZG*** *is in $O(1)$*

$O(\log\log n)$ upper bound for monoids

Call **SG** the variety of monoids satisfying $x^{\omega+1}yx^{\omega} = x^{\omega}yx^{\omega+1}$ for all x, y

→ Intuition: we can swap the elements of any given subgroup of the monoid

Examples:

- All **ZG** monoids (where elements $x^{\omega+1}$ commute with everything)
- All group-free monoids (where subgroups are trivial)
- Products of **ZG** monoids and group-free monoids

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Tools: induction on $\mathcal{J}$-classes, Rees-Sushkevich theorem, Van Emde Boas trees

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Specifically:

- Prefix-$\mathbb{Z}_d$: for $\Sigma = \{0, \ldots, d-1\}$, does the input prefix sum to 0 modulo d?
  - $\rightarrow$ Known lower bound of $\Omega(\log n / \log \log n)$
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Theorem (Lower bounds on a monoid M)

- *If M is not in* **SG***, then for some $d \in \mathbb{N}$ the Prefix-$\mathbb{Z}_d$ problem reduces to the dynamic word problem for M*
- *If M is in* **SG** $\setminus$ **ZG***, then Prefix-U_1 reduces to the dynamic word problem for M*

Results on languages
(via semigroups)

From monoids to semigroups

- **Semigroup:** like a monoid but possibly without a neutral element
- **Dynamic word problem for semigroups:** defined like for monoids

What is the difference?

- The language $\Sigma^*(ae^*a)\Sigma^*$ on $\Sigma = \{a, b, e\}$ has a **neutral letter** e that we intuitively need to "**jump over**"
- The language $\Sigma^*aa\Sigma^*$ on $\Sigma = \{a, b\}$ without e can be **maintained in $O(1)$** by counting the factors aa

Local monoids in semigroups

- A **local monoid** of a semigroup S is a subset of S that has a **neutral element**
  - $\rightarrow$ If S has a local monoid M then the dynamic word problem for M **reduces** to S
  - $\rightarrow$ **Lower bounds** on M thus apply to S

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- Hence, we define:
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  - **LZG**: all local monoids are in **ZG**
    - $\rightarrow$ We have **LZG** $\neq$ **ZG** and show bounds for semigroups in **LZG**

From semigroups to languages

We now move back to dynamic membership for regular languages

- Dynamic membership for a regular language L is like the dynamic word problem for its syntactic semigroup
  - → This is like the transition monoid but without the neutral element
- Difference: not all elements of the syntactic semigroup can be achieved as one letter

→ We use instead the stable semigroup, which intuitively groups letters together into blocks of a constant size

From semigroups to languages (cont'd)

Call **QLZG** and **QSG** the languages whose stable semigroup is in **ZG** and **SG**

Theorem

Our results on semigroups in **SG** *and* **LZG** *extend to regular languages in* **QSG** *and* **QLZG**

From semigroups to languages (cont'd)

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Our results on semigroups in **SG** *and* **LZG** *extend to regular languages in* **QSG** *and* **QLZG**

For any regular language L*:*

- *If* L is *in* **QLZG** *then dynamic membership is in* $O(1)$
- *If* L is not *in* **QSG** $\setminus$ **QLZG** *then dynamic membership is in* $O(\log \log n)$ *and has a reduction from prefix-*U_1
- *If* L is *not in* **QSG** *then dynamic membership is in* $\Theta(\log n / \log \log n)$

Conclusion and future work

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  - Yes with **randomization**: one language **in $\Theta(\log \log n)$** and one **in $O(\sqrt{\log \log n})$**
  - **Question:** can the intermediate classes be **characterized**?

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Thanks for your attention!

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$O(\log\log n)$ upper bound for monoids (proof sketch)

Example : $\Sigma^*(ae^*a)\Sigma^*$ on $\Sigma = \{a, b, e\}$

- **Idea:** maintain the count of factors ae^*a
- **Problem:** to do this, we need to "jump over" the e's

$\rightarrow$ **Van Emde Boas tree** data structure:

- maintain a subset of $\{1, \ldots, n\}$ under insertions/deletions
- jump to the prev/next element in $O(\log\log n)$

Full proof: induction on $\mathcal{J}$-classes and Rees-Sushkevich theorem

Extending SG to semigroups

We can show that, for semigroups:

Lemma

A semigroup satisfies the equation of **SG** *iff it is in* **LSG**

Hence, as the algorithm for **SG** works for semigroups as well as monoids:

Theorem

For any semigroup S*:*

- *If* S *is in* **SG***, then the dynamic word problem is in* $O(\log \log n)$
- *Otherwise, the dynamic word problem is in* $\Theta(\log n / \log \log n)$

Case of ZG

We have **ZG** $\neq$ **LZG**, but we can still show:

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Proof sketch: only need to show the upper bound:

- We show the $O(1)$ upper bound on the semidirect product $\mathbf{ZG} * \mathbf{D}$ of **ZG** with definite semigroups
- We show an independent locality result: $\mathbf{LZG} = \mathbf{ZG} * \mathbf{D}$
  - $\to$ Technical proof relying on finite categories and Straubing's delay theorem

Difference between the stable semigroup and syntactic semigroup

- Dynamic membership for $(aa)^*ba^*$ is in $O(1)$: count the b's at even and odd positions
- The dynamic word problem for its syntactic semigroup has a reduction from $\mathbb{Z}_2$