

On the join of varieties of monoids with LI  
GT ALGA

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# Two fundamental problems in automata theory

## Characterisation

Class of regular languages  $\overset{?}{\leftrightarrow}$  Class of finite objects

## Decidability

Class of regular languages  $\overset{?}{\leftrightarrow}$  Algorithm for testing membership

First problem often linked to and motivated by the second problem.

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## Obvious approach

For class of regular languages, look at all **minimal finite automata**

- ▶ try to find characterising properties;
- ▶ ask whether they can be checked by an algorithm.

## A fruitful approach: the algebraic approach

Replace automaton  $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$  by monoid morphism  $\varphi: \Sigma^* \rightarrow M$  with  $\mathcal{L}(\mathcal{A}) = \varphi^{-1}(P)$  for some  $P \subseteq M$ .  $\rightsquigarrow \mathcal{L}(\mathcal{A})$  is recognised by  $\varphi$ .

### Miminal recognising morphism

- ▶ To each  $L \subseteq \Sigma^*$ , we can associate its **syntactic congruence**:  
 $\forall u, v \in \Sigma^*, u \sim_L v$  whenever  
 $\forall x, y \in \Sigma^*, xuy \in L \Leftrightarrow xvy \in L$ .
- ▶  $\Sigma^*/\sim_L$  is the **syntactic monoid** and

$$\eta_L: \Sigma^* \rightarrow \Sigma^*/\sim_L$$
$$u \mapsto [u]_{\sim_L}$$

the **syntactic morphism**.

- ▶ **Minimal** under some notion of division.

### Fundamental result

Language is regular iff its syntactic monoid is finite.

# The algebraic approach

## Characterisation

Class of regular languages  $\overset{?}{\longleftrightarrow}$  Class of finite syntactic monoids

## Example of decidable characterisations

▶ Star-free languages  $\longleftrightarrow$  Finite aperiodic syntactic monoids

▶ Disjoint unions of right-unambiguous monomials:

$A_0^* a_1 A_1^* \cdots a_k A_k^*$  where  $k \in \mathbb{N}$ ,  $a_1, \dots, a_k \in \Sigma$ ,

$A_0, A_1, \dots, A_k \subseteq \Sigma$  and  $a_i \notin A_{i-1}$  for all  $i \in [k]$

$\longleftrightarrow$  Finite  $\mathfrak{A}$ -trivial syntactic monoids

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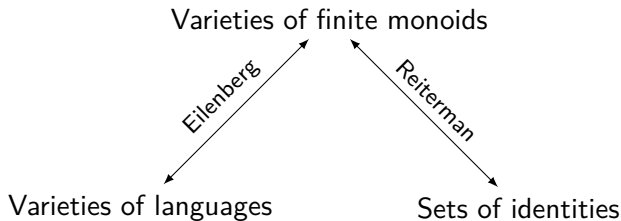
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## General situation



# The algebraic approach

## Identities

### What is an identity?

- ▶ Need profinite topology to define it formally.
- ▶ Informally and sufficiently for our examples, an equality of words  $u = v$  made up using letters of some alphabet and  $\omega$  powers.
- ▶  $M$  verifies  $u = v$  whenever for any substitution of the letters in  $u$  and  $v$  by elements of  $M$ , interpreting  $m^\omega$  as the idempotent power of  $m$  in  $M$ , we have equality.

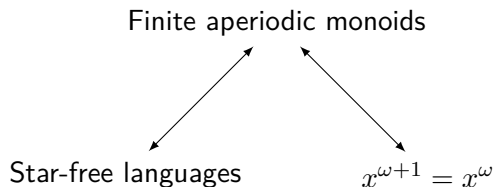
### Examples of identities

- ▶  $x^\omega = x^{\omega+1} \rightsquigarrow$  Finite aperiodic monoids
- ▶  $xy = yx \rightsquigarrow$  Finite commutative monoids
- ▶  $x^\omega = 1 \rightsquigarrow$  Finite groups

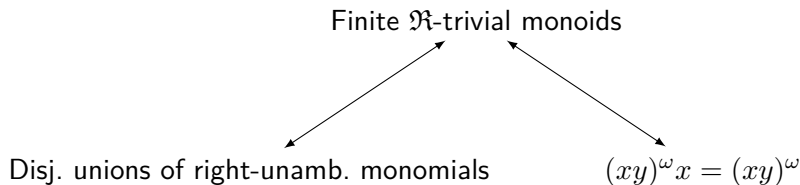
# The algebraic approach

## Examples of characterisations

### First example



### Second example





# The join

## Definition

$\mathbf{V}$  and  $\mathbf{W}$  varieties of monoids/semigroups,  $\mathbf{V} \vee \mathbf{W}$  is smallest variety of monoids/semigroups containing  $\mathbf{V}$  and  $\mathbf{W}$ .

## The languages in $\mathcal{L}(\mathbf{V} \vee \mathbf{W})$

- ▶ Parallel composition of automata for  $\mathcal{L}(\mathbf{V})$  and  $\mathcal{L}(\mathbf{W})$ .
- ▶ Boolean closure of  $\mathcal{L}(\mathbf{V}) \cup \mathcal{L}(\mathbf{W})$ .

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## Challenges

- ▶ Does not furnish decidable characterisation of  $\mathcal{L}(\mathbf{V} \vee \mathbf{W})$ .
- ▶ Finding set of identities defining  $\mathbf{V} \vee \mathbf{W}$  is difficult.

# Finite locally trivial semigroups

## The variety **LI**

- ▶ Defined by  $x^\omega y x^\omega = x^\omega$ .
- ▶  $\mathcal{L}(\mathbf{LI})$  contains all languages of the form  $U\Sigma^*V \cup W$  with  $U, V, W \subseteq \Sigma^*$  finite.

## $\mathbf{V} \vee \mathbf{LI}$ for $\mathbf{V}$ variety of finite monoids

- ▶  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ : automata for  $\mathcal{L}(\mathbf{V})$  with ability to check, in parallel, bounded-length prefixes and suffixes.
- ▶ Has been studied quite a lot, relying heavily on profinite topology (Azevedo'1990, Zeitoun'1995, Costa'2001).

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## My contribution

General **method** to find a set of identities defining  $\mathbf{V} \vee \mathbf{LI}$  when one has one for  $\mathbf{V}$  and  $\mathbf{V}$  verifies some criterion.

↪ Simple method drawing on algebraic and language-theoretic techniques.

# Essentially- $\mathbf{V}$ stamps

## Stamp

Surjective morphism  $\varphi: \Sigma^* \rightarrow M$  with  $M$  finite.

## Stability index (Chaubard-Pin-Straubing'2006)

For stamp  $\varphi: \Sigma^* \rightarrow M$ , smallest  $s \in \mathbb{N}_{>0}$  such that  $\varphi(\Sigma^{2s}) = \varphi(\Sigma^s)$ .

## Essentially- $\mathbf{V}$ stamps (G.-McKenzie-Segoufin'2021)

Stamp  $\varphi: \Sigma^* \rightarrow M$  with stability index  $s$  is **essentially- $\mathbf{V}$**  whenever there exists a stamp  $\mu: \Sigma^* \rightarrow N$  with  $N \in \mathbf{V}$  such that for all  $u, v \in \Sigma^*$ , we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

We will denote by **EV** the class of all essentially- $\mathbf{V}$  stamps.

## Essentially- $\mathbf{V}$ stamps

Intuitively, a stamp is essentially- $\mathbf{V}$  when it behaves like a stamp into a monoid of  $\mathbf{V}$  as soon as a sufficiently long beginning and ending of the input word has been fixed.

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## Trivial examples

- ▶ Let  $\varphi: \{a, b\}^* \rightarrow M$  be syntactic morphism of  $a(a + b)^*$ . It has stability index 1 because  $\varphi(aw) = \varphi(bw)$  for all  $w \in \{a, b\}^*$ . Thus  $\mu: \{a, b\}^* \rightarrow \{1\}$  verifies that for all  $u, v \in \Sigma^*$ , we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

$\varphi \in \mathbf{EI}$  ( $\mathbf{I}$ : trivial monoids).

- ▶ In general, the syntactic morphism of  $x\Sigma^*y$  for  $x, y \in \Sigma^*$  belongs to  $\mathbf{EI}$ .

## Essentially- $\mathbf{V}$ stamps

### One more sophisticated example

Let  $\varphi: \{a, b\}^* \rightarrow M$  be syntactic morphism of  $a(a+b)^*b(a+b)^*a$ . It has stability index 3 because

$$\varphi(xwy) = \begin{cases} \varphi(xby) & \text{if } w \in (a+b)^*b(a+b)^* \\ \varphi(xay) & \text{otherwise} \end{cases}$$

for all  $x, y \in \{a, b\}$  and  $w \in \{a, b\}^+$ . Thus the syntactic morphism  $\mu: \{a, b\}^* \rightarrow N$  of  $(a+b)^*b(a+b)^*$  verifies that for all  $u, v \in \Sigma^*$ , we have

$$\mu(u) = \mu(v) \Rightarrow (\varphi(xuy) = \varphi(xvy) \quad \forall x, y \in \Sigma^s) .$$

### Proposition

Let  $\mathbf{V}$  be a variety of finite monoids. A language over  $\Sigma$  has its syntactic morphism in  $\mathbf{EV}$  iff it is a Boolean combination of languages of the form  $xLy$  for  $L \subseteq \Sigma^*$  in  $\mathcal{L}(\mathbf{V})$  and  $x, y \in \Sigma^*$ .



# Essentially- $\mathbf{V}$ stamps and $\mathbf{V} \vee \mathbf{LI}$

Let  $\mathbf{V}$  be a variety of finite monoids.

The class of finite semigroups  $\varphi(\Sigma^+)$  for  $\varphi: \Sigma^* \rightarrow M$  stamp in  $\mathbf{EV}$  is a variety of finite semigroups.

$\rightsquigarrow$  We identify it with  $\mathbf{EV}$ .

**Lemma**

$\mathbf{V} \vee \mathbf{LI} \subseteq \mathbf{EV}$ .

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## Lemma

$\mathbf{V} \vee \mathbf{LI} \subseteq \mathbf{EV}$ .

## Definition

$\mathbf{V}$  verifies criterion (A) whenever for any  $L \subseteq \Sigma^*$  in  $\mathcal{L}(\mathbf{V})$  and  $x, y \in \Sigma^*$ , we have  $xLy \in \mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ .

## Lemma

$\mathbf{EV} \subseteq \mathbf{V} \vee \mathbf{LI}$  iff  $\mathbf{V}$  verifies criterion (A).

# Characterisation and decidability of $\mathbf{EV}$

## Decidability

For any variety of finite monoids  $\mathbf{V}$ , if  $\mathbf{V}$  is decidable, then so is  $\mathbf{EV}$ .

## Proposition

*Let  $\mathbf{V}$  be a variety of finite monoids and let  $E$  be a set of identities that defines it. Then  $\mathbf{EV}$  is defined by*

$$\{x^\omega yuzt^\omega = x^\omega yvzt^\omega \mid u = v \in E, x, y, z, t \text{ with not in } uv\} .$$

## Remark

Rediscovery of Costa'2001 who

- ▶ for set  $E$  of identities, defines  $U(E)$  to be above set;
- ▶ gives cancellation property for variety of finite semigroups  $\mathbf{V}$  so that  $U(E)$  defines  $\mathbf{V} \vee \mathbf{LI}$  for  $E$  defining  $\mathbf{V}$ .

# Method for getting set of identities for $\mathbf{V} \vee \mathbf{LI}$

## The method

For a variety of finite monoids  $\mathbf{V}$

1. show that  $\mathbf{V}$  verifies criterion (A);
2. from a set of identities  $E$  defining  $\mathbf{V}$ , deduce that

$$\{x^\omega yuzt^\omega = x^\omega yvzt^\omega \mid u = v \in E, \text{ with } x, y, z, t \text{ not in } uv\}$$

defines  $\mathbf{V} \vee \mathbf{LI}$ .

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## Problems

- ▶ Are there any varieties verifying criterion (A)?
- ▶ To characterise  $\mathbf{V} \vee \mathbf{LI}$ , I need to prove that some languages belong to  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$ . Did I really made progress in view of my prior knowledge that  $\mathcal{L}(\mathbf{V} \vee \mathbf{LI})$  is the Boolean closure of  $\mathcal{L}(\mathbf{V}) \cup \mathcal{L}(\mathbf{LI})$ ?

# Applications

## Why it works

### Lemma

*Let  $\mathbf{V}$  be a variety of finite monoids. If for any  $L \subseteq \Sigma^*$  in  $\mathcal{L}(\mathbf{V})$  and  $x, y \in \Sigma^*$ , there exists some  $K \subseteq \Sigma^*$  in  $\mathcal{L}(\mathbf{V})$  such that  $L = x^{-1}Ky^{-1}$ , then  $\mathbf{V}$  verifies criterion (A).*

### Proof idea.

Because then for any  $L \subseteq \Sigma^*$  in  $\mathcal{L}(\mathbf{V})$  and  $x, y \in \Sigma^*$ , we have  $xLy = K \cap x\Sigma^*y$  for some  $K \subseteq \Sigma^*$  in  $\mathcal{L}(\mathbf{V})$ . □

### Remarks

- ▶ This quotient-expressibility condition for  $\mathbf{V}$  does only depend on  $\mathcal{L}(\mathbf{V})$ .
- ▶ It can be weakened to get equivalence with criterion (A).

# Applications

## First application

### Theorem

$\mathbf{H} \vee \mathbf{LI} = \mathbf{EH}$  for any variety of finite groups  $\mathbf{H}$ .

### Proof.

Let  $\mathbf{H}$  be a variety of finite groups. Let  $L \subseteq \Sigma^*$  in  $\mathcal{L}(\mathbf{H})$  and  $x, y \in \Sigma^*$ .

Take  $\eta: \Sigma^* \rightarrow M$  syntactic morphism of  $L$ . We have

$$L = x^{-1}\eta^{-1}(\eta(x)\eta(L)\eta(y))y^{-1}$$

because for all  $w \in \Sigma^*$  and  $w' \in L$ , it holds that

$$\eta(xwy) = \eta(x)\eta(w')\eta(y) \implies \eta(w) = \eta(w') .$$



# Applications

## Second application

### Theorem

$\mathbf{R} \vee \mathbf{LI} = \mathbf{ER}$ , so it is defined by  $x^\omega y(ab)^\omega azt^\omega = x^\omega y(ab)^\omega zt^\omega$ .

### Proof idea.

Take  $A_0^* a_1 A_1^* \cdots a_k A_k^*$  right-unambiguous monomial and  $x, y \in \Sigma^*$ . Observe that  $y$  can be uniquely written as  $y = zt$  where  $z \in A_k^*$  and  $t \in \{\varepsilon\} \cup (\Sigma \setminus A_k)\Sigma^*$ . We have

$$\begin{aligned} & A_0^* a_1 A_1^* \cdots a_k A_k^* \\ &= x^{-1} \left( x A_0^* a_1 A_1^* \cdots a_k A_k^* t \cap \bigcap_{v \in A_k^{<|z|}} (x A_0^* a_1 A_1^* \cdots a_k v t) \right) y^{-1} \end{aligned}$$

using the convention that  $x A_0^* a_1 A_1^* \cdots a_k v t = x v t$  for all  $v \in A_k^{<|z|}$  when  $k = 0$ . □



# Conclusion

## Contribution

- ▶ General method to give set of identities defining  $\mathbf{V} \vee \mathbf{LI}$ ; new proofs of such characterisations.
- ▶ Simple method using algebraic and language-theoretic techniques, reducing profinite topology to the minimum.
- ▶ But **ad hoc**.

## Open questions

- ▶ What are the varieties of finite monoids verifying criterion (A)?
- ▶ Is the strong quotient-expressibility condition I presented necessary for verifying criterion (A)?

Thank you for listening.