

The planar geometry of first-order string transductions

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One possible motivation

Star-free languages are equivalently defined by:

- Star-free regexps: $E, E' ::= \emptyset \mid \{a\} \mid E \cup E' \mid E \cdot E' \mid E^c$ (complement)
e.g. $(ab)^* = (b\emptyset^c \cup \emptyset^c a \cup \emptyset^c aa\emptyset^c \cup \emptyset^c bb\emptyset^c)^c$ over the alphabet $\{a, b\}$
- $\varphi^{-1}(P)$ for some morphism φ to a finite and *aperiodic* monoid $M \supseteq P$
- counter-free automata (aperiodicity condition), first-order logic (FO), ...

Definition

A monoid M is *aperiodic* when $\forall x \in M, \exists n \in \mathbb{N} : x^n = x^{n+1}$.

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We characterize star-free languages (and FO transductions)
by “compositional/local” conditions on *behaviors of two-way automata*

Drawback: no counterpart to syntactic monoid / minimal DFA

Reminder: two-way automata (1)

Transitions: update finite state + move left/right depending on new state

Example: states $Q = \{q_1^{\rightarrow}, q_2^{\leftarrow}, q_3^{\leftarrow}\}$, initial state q_1^{\rightarrow}

$$q_1^{\rightarrow}, (a|b) \mapsto q_1^{\rightarrow} \quad q_1^{\rightarrow}, c \mapsto q_2^{\leftarrow} \quad q_2^{\leftarrow}, (a|b|c) \mapsto q_3^{\leftarrow} \quad q_3^{\leftarrow}, b \mapsto \text{accept}$$

Directed states are an old idea¹, more convenient

+ needed to define *reversible* 2DFAs (Dartois et al. ICALP'17)

¹cf. e.g. J.-C. Birget, *Concatenation of Inputs in a Two-Way Automaton* (1989)

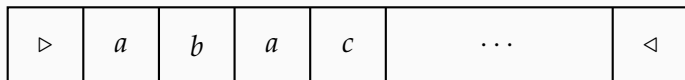
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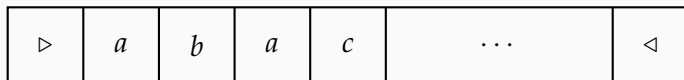
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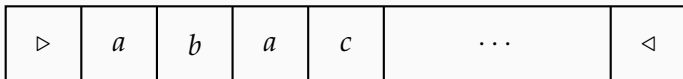
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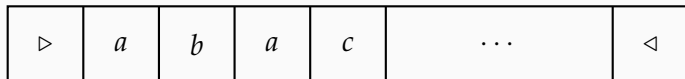
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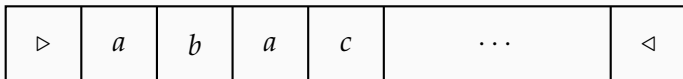
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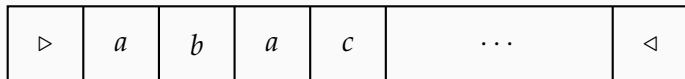
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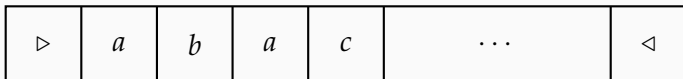
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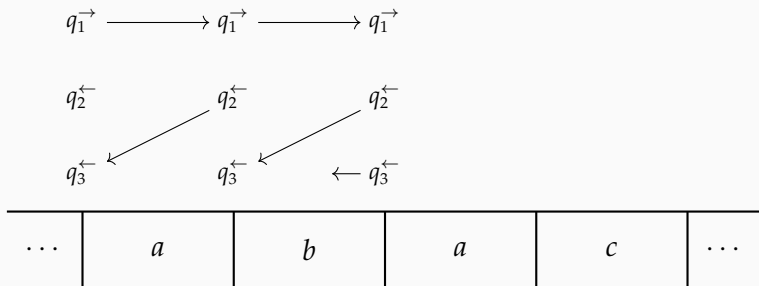
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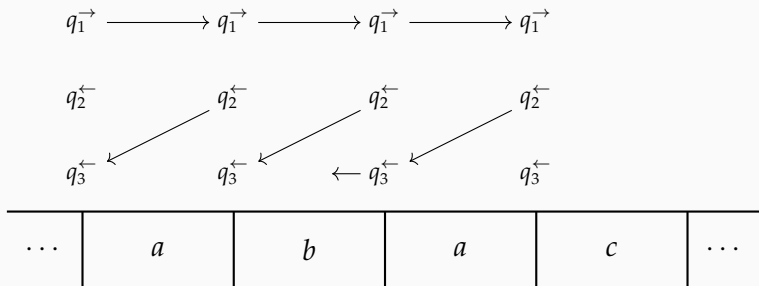
Graphical representation of transitions for each letter:



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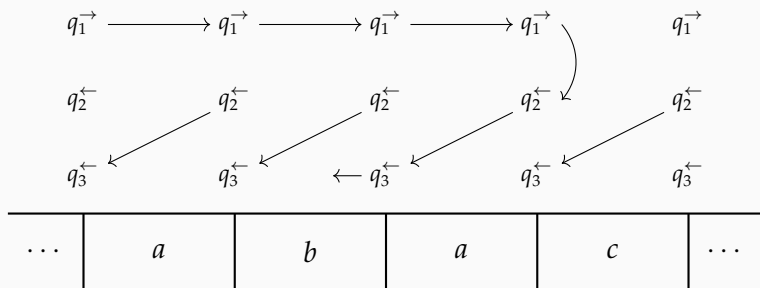
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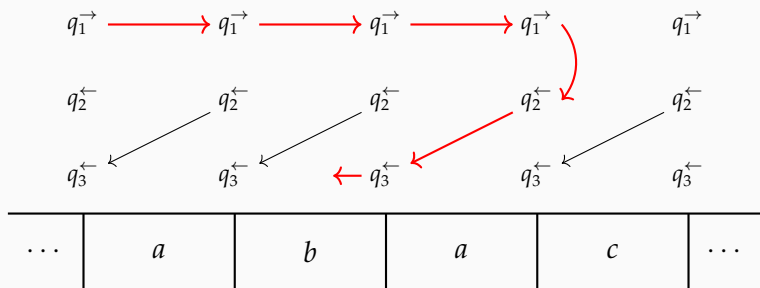
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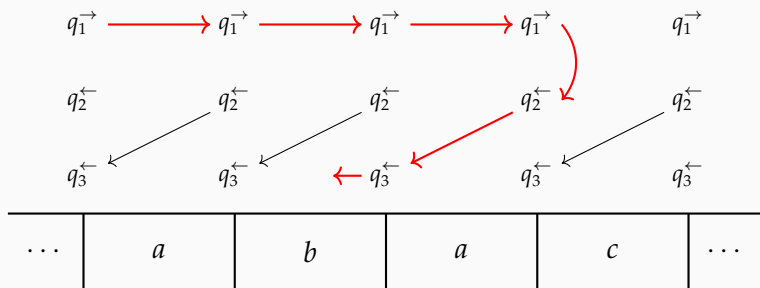
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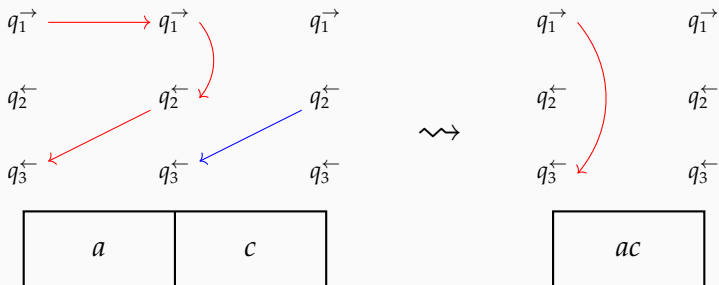


This two-way automaton is *deterministic*: outdegree ≤ 1

reversible: deterministic + indegree ≤ 1

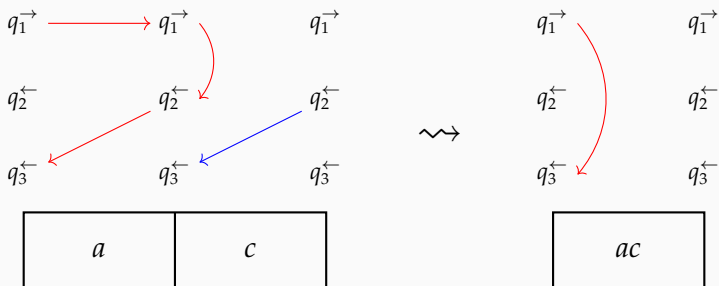
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Behaviors (or crossing types) form a monoid:



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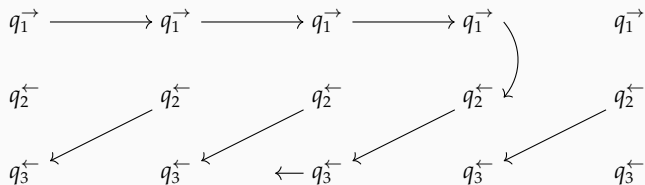
Behaviors (or crossing types) form a monoid:



This monoid is finite, therefore 2DFA recognize regular languages
(modern account of Shepherdson's construction (1959))

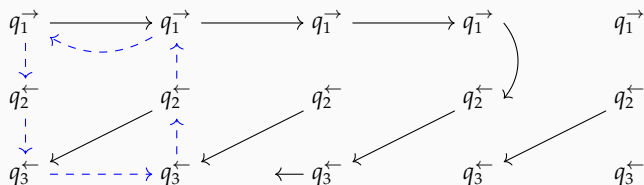
Reversible behaviors are closed under product, and
reversible 2DFA can recognize all regular languages (Dartois et al. ICALP'17)

Combinatorial planarity



This drawing is *planar*, i.e. without crossed edges.

Combinatorial planarity

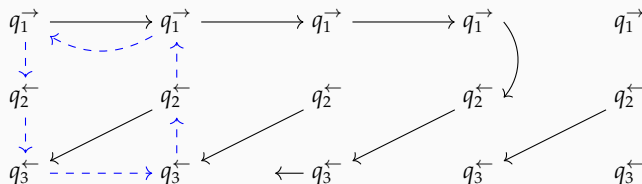


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Formally: for each of these 4 behaviors, the cyclic order

$$q_1^{\text{left}} \prec q_2^{\text{left}} \prec q_3^{\text{left}} \prec q_3^{\text{right}} \prec q_2^{\text{right}} \prec q_1^{\text{right}} \prec q_1^{\text{left}}$$

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does not contain any sub-cyclic-order $x \prec y \prec z \prec w \prec x$ such that

- x and z are connected by an edge (either $x \rightarrow z$ or $z \rightarrow x$)
- and y and w are also connected by an edge

\rightarrow depends on the choice of total order $q_1 < q_2 < q_3$

(More like planar combinatorial maps than planar graphs...)

The main theorem

Theorem

Let $L \subseteq \Sigma^*$. The following are equivalent:

- L is a star-free language.
- L is recognized by some planar 2DFA.
- L is recognized by some planar reversible 2DFA.

Our example of planar 2DFA recognizes $(\emptyset^c c \emptyset^c)^c b (a \cup b) c \emptyset^c$

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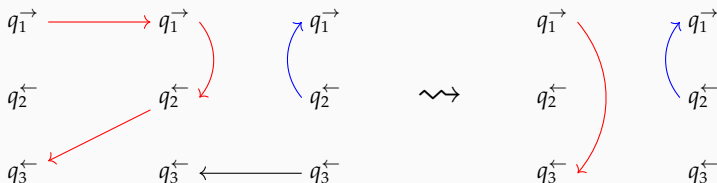
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Compositionality/locality: planar behaviors are closed under product



Planar *one-way* automata = *monotone* transitions, not powerful enough

A stronger theorem on transducers

Two-way deterministic *transducers* (2DFT) = 2DFA with output
(each transition can append a suffix to the output log)

2DFTs compute *regular functions* a.k.a. *MSO transductions*,
a well-behaved class of functions with many different characterizations
example: $w \mapsto w \cdot \text{reverse}(w)$

Ask for the monoid of behaviors to be *aperiodic*,
and you get *first-order transductions* (Carton & Dartois, CSL'15)

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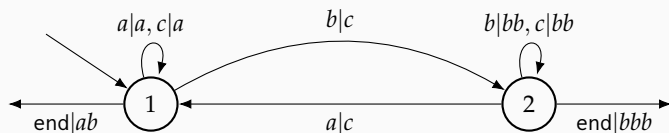
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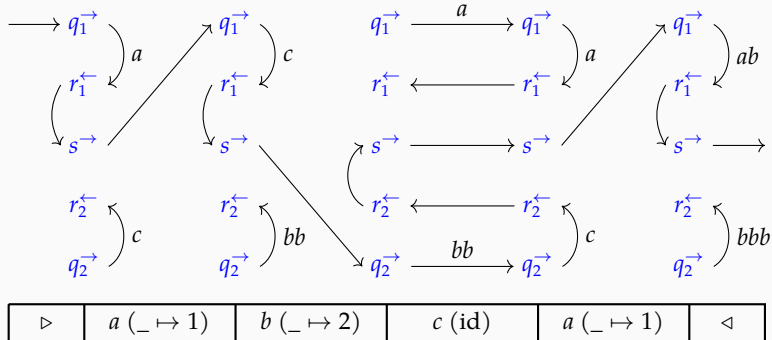
Let $f : \Gamma^* \rightarrow \Sigma^*$. The following are equivalent:

- f is a first-order transduction.
- f is computed by some planar 2DFT.
- f is computed by some planar reversible 2DFT.

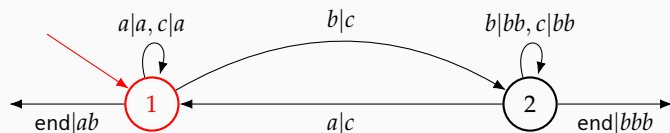
Example: simulating a sequential transducer



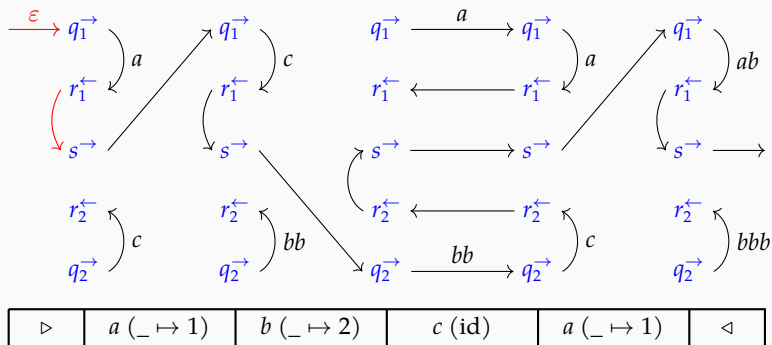
Let's look at a run on the input $abca$



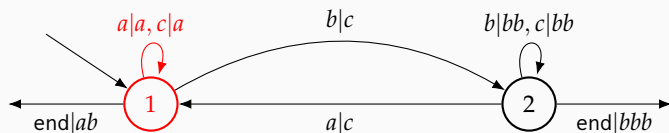
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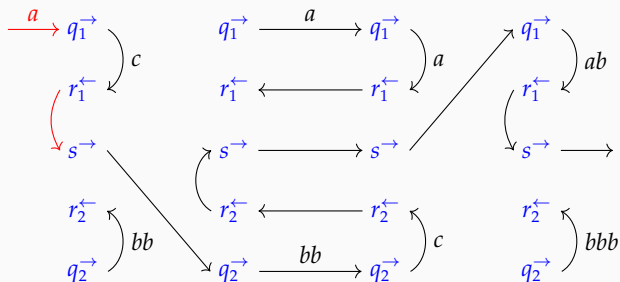
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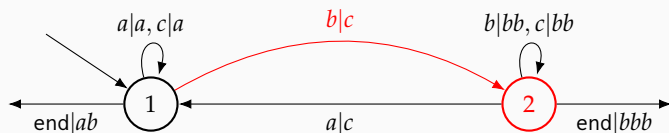


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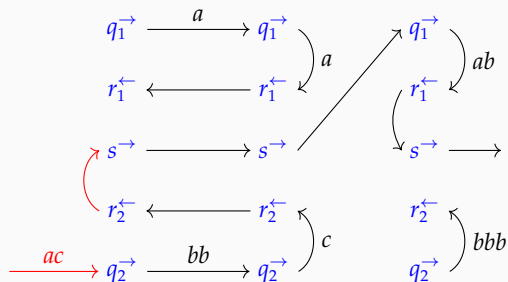


▷	a ($_ \mapsto 1$)	b ($_ \mapsto 2$)	c (id)	a ($_ \mapsto 1$)	◁
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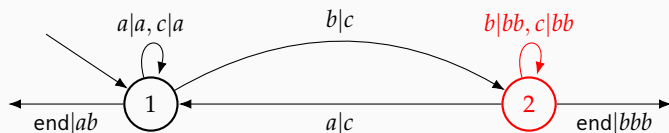


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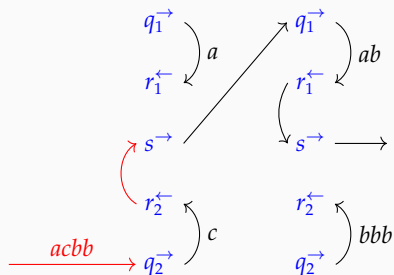


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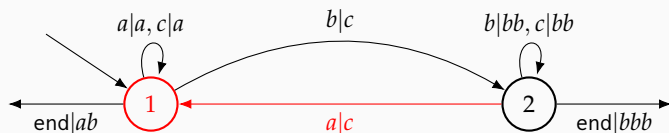


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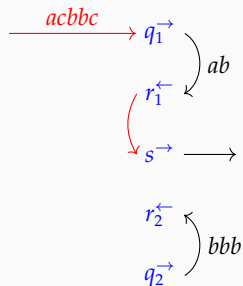


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- Hines, *A categorical framework for finite state machines* (2003)
Relates monoid of 2DFA behaviors to *geometry of interaction* (GoI),
a family of semantics for linear λ -calculi (as in *linear logic*)²
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Reversible planar DFA were considered by Hines in a talk
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(Papers on GoI are often named “The geometry of X” \longrightarrow this talk’s title)

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Conclusion

We introduced a notion of *planarity* of two-way transducers, based on the graphical representation of their behavior, and showed:

Main theorem

star-free language	\iff	planar 2DFA	\iff	planar reversible 2DFA
FO transduction	\iff	planar 2DFT	\iff	planar reversible 2DFT

- Planar behaviors form an aperiodic submonoid of all behaviors
→ unlike aperiodicity, planarity is compositional
- Expressivity established via factorization theorems (Krohn–Rhodes + extension to FO transductions)
 - Details in backup slides
- Inspiration from other areas of “logic in computer science”
 - Also motivated the study of *comparison-free polyregular functions*, a new transduction class → talk in Warsaw seminar **this afternoon**

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 - Details in backup slides
- Inspiration from other areas of “logic in computer science”
 - Also motivated the study of *comparison-free polyregular functions*, a new transduction class → talk in Warsaw seminar **this afternoon**

Thanks for your attention! Any questions?

Aperiodicity of planar behaviors (1)

To show that planar 2DFA can recognize *only* star-free languages, we use:

Lemma

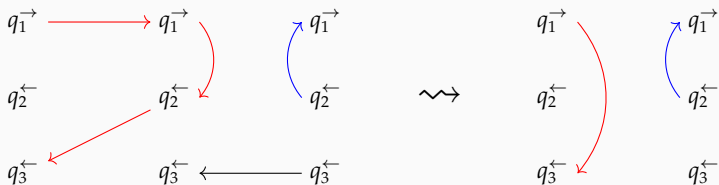
Let Q be a finite set of directed states. The finite monoid \mathfrak{P}_Q of all possible planar behaviors over Q is aperiodic: $\forall x \in \mathfrak{P}_Q, \exists n \in \mathbb{N} : x^n = x^{n+1}$.

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The **blue** edge is conserved: right-right edges of $y \subseteq$ right-right edges of xy (right-right edges of x^n) $_{n \in \mathbb{N}}$ monotone, hence eventually constant

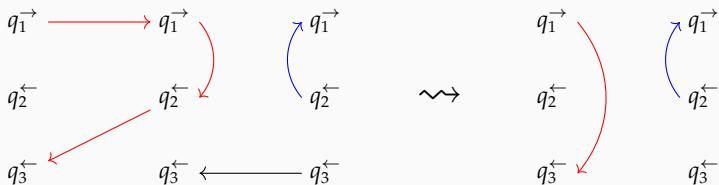
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Aperiodicity of planar behaviors (2)

Left-right edges are entirely determined by *degrees*:

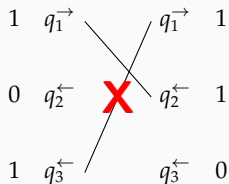
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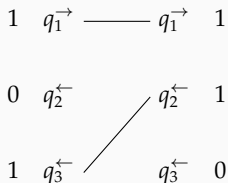
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\rightarrow (degrees of x^n) $_{n \in \mathbb{N}}$ non-increasing, hence eventually constant

Combine with previous slide: $\exists n \in \mathbb{N} : x^n = x^{n+1}$!

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More conceptual POV: Green's relations on the monoid \mathfrak{P}_Q

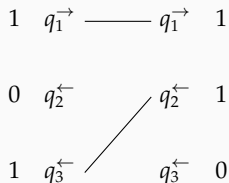
behavior \nearrow for $\preceq_{\mathcal{L}} \implies$ right degrees \nearrow and right-right edges \searrow

\rightarrow \mathcal{L} -class invariant + \mathcal{R} -class invariant determine element of \mathfrak{P}_Q

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Next: the converse direction of the main theorem

Expressiveness of reversible planar 2DFTs (1)

Theorem (Part of the main theorem on transducers)

Any first-order transduction can be computed by a reversible planar 2DFT.

Let's start with *aperiodic sequential functions* (\subsetneq FO transductions)

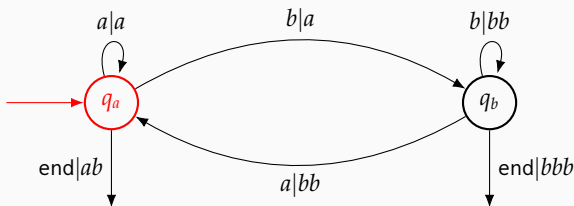
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Sequential transducers (see below) with aperiodic transition monoids



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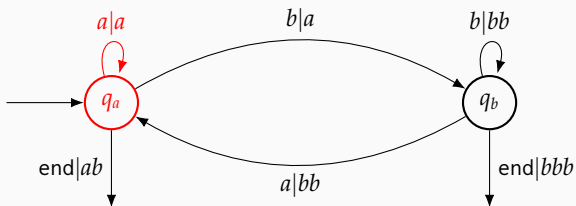
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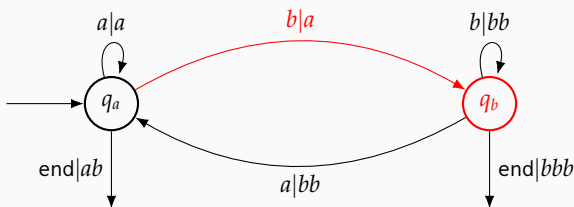
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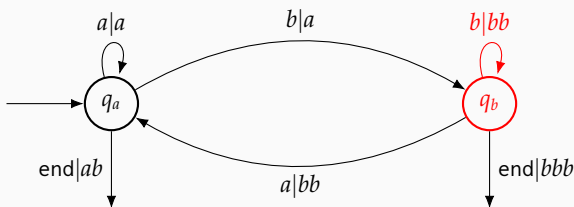
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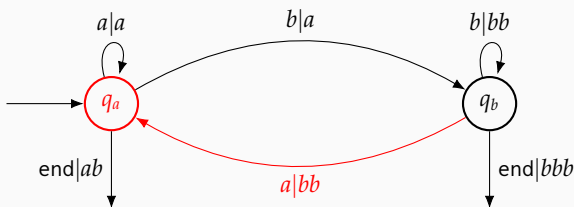
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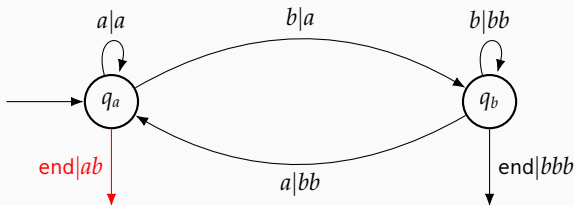
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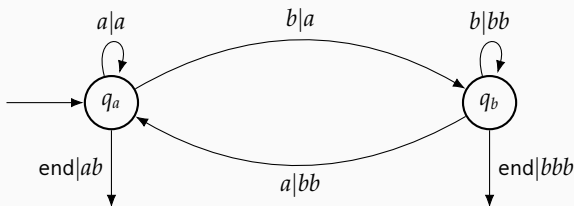
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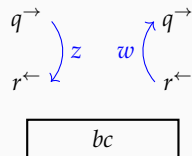
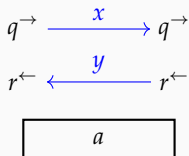
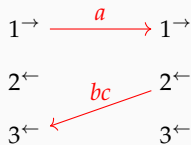
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Reminder: Krohn–Rhodes decomposition theorem

Aperiodic sequential functions are generated by aper. seq. transducers with 2 states (like the one above) + function composition.

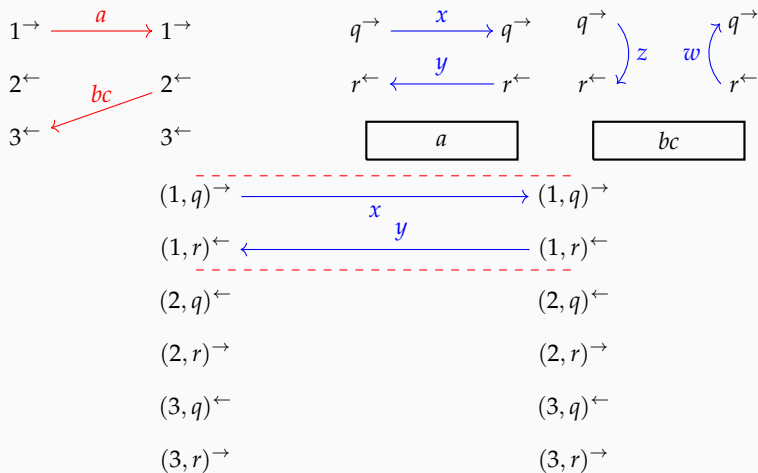
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Composition of reversible 2DFTs uses a wreath-product-like construction (do you see why reversibility is needed?) *preserving planarity*



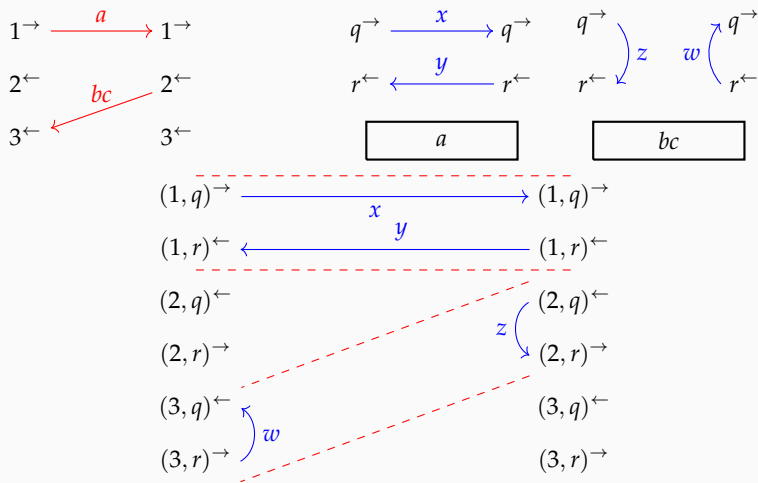
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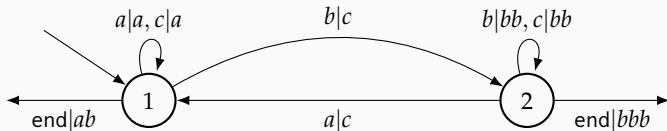
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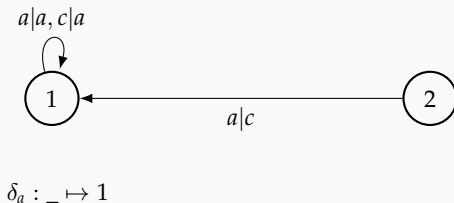
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The Krohn–Rhodes decomposition involves aperiodic sequential transducers *with 2 states*, such as:



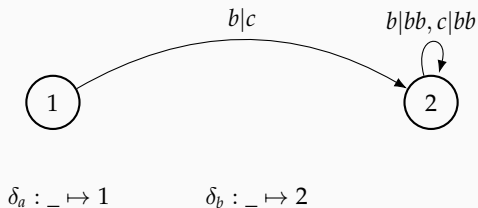
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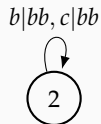


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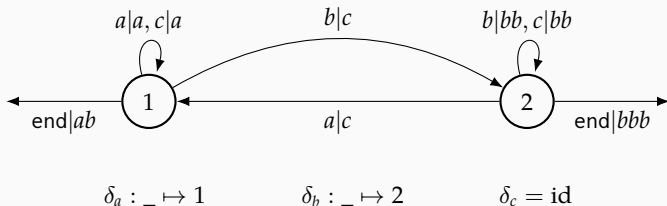


$$\delta_b : _ \mapsto 2$$

$$\delta_c = \text{id}$$

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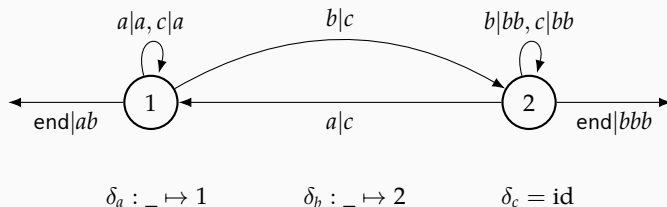
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 $\{\delta_a, \delta_b, \delta_c\}$ is the largest aperiodic submonoid of $Q \rightarrow Q$

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Translation into reversible planar 2DFT already illustrated

Expressiveness of reversible planar 2DFTs (5): FO transductions

We just proved that reversible planar 2DFTs are *closed under composition* and can simulate *two-state aperiodic sequential transducers*.

By Krohn–Rhodes, we get all aper. seq. functions.

Expressiveness of reversible planar 2DFTs (5): FO transductions

We just proved that reversible planar 2DFTs are *closed under composition* and can simulate *two-state aperiodic sequential transducers*.

By Krohn–Rhodes, we get all aper. seq. functions. To go further, we use:

Theorem (Bojańczyk et al.⁴)

Any first-order transduction can be obtained as a composition of:

- *aperiodic sequential functions*;
- $\text{mapReverse}_\Sigma, \text{mapDuplicate}_\Sigma : (\Sigma \cup \{\#\})^* \rightarrow (\Sigma \cup \{\#\})^*$ for $\# \notin \Sigma$

For $w_1, \dots, w_n \in \Sigma^*$, $\text{mapReverse}_\Sigma(w_1\# \dots \#w_n) = \text{rev}(w_1)\# \dots \#\text{rev}(w_n)$

$\text{mapDuplicate}_\Sigma(w_1\# \dots \#w_n) = w_1w_1\# \dots \#w_nw_n$

With this, we can conclude the proof of the main theorems.

⁴Not entirely explicit in the literature; variants can be found in:

Bojańczyk, Daviaud & Krishna, *Regular and first-order list functions* (2018)

Bojańczyk & Stefański, *Single-use automata and transducers for infinite alphabets* (2020)