

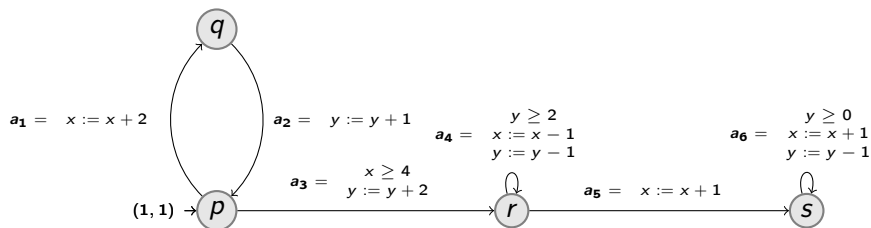
The Context-Freeness Problem is coNP-complete for Flat Counter Systems

Jérôme Leroux, Vincent Penelle, Grégoire Sutre

ATVA, November 5th, 2014

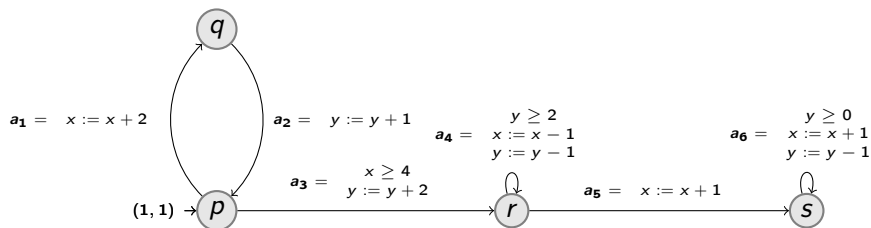
Introduction

Configuration : (s, x_1, \dots, x_k) , s a state, x_1, \dots, x_k positive counters.



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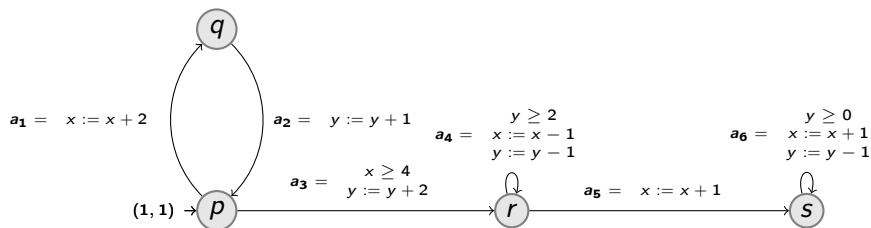
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$$L \subseteq \sigma_1^* \sigma_2^* \sigma_3^* \sigma_4^* \sigma_5^*$$

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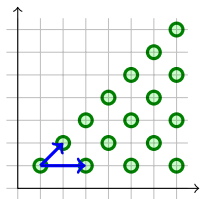
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Semilinear Sets

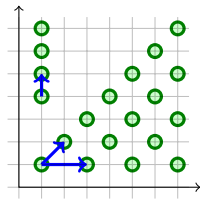
Definition (Ginsburg & Spanier '66)

Linear set : $\mathbf{b} + \mathbb{N}\mathbf{p}_1 + \dots + \mathbb{N}\mathbf{p}_m$ with $\mathbf{b}, \mathbf{p}_1, \dots, \mathbf{p}_m \in \mathbb{N}^d$.

Semilinear set : finite union of linear sets.



$$(1, 1) + \mathbb{N}(1, 1) + \mathbb{N}(2, 0)$$



$$(1, 1) + \mathbb{N}(1, 1) + \mathbb{N}(2, 0) \\ \cup (1, 4) + \mathbb{N}(0, 1)$$

Definition

A set of vectors P is stratified if:

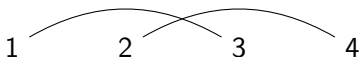
- Each $p \in P$ has at most two non-0 components
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$\{(a_1, 0, a_3, 0), (0, b_2, 0, b_4)\}$ is not stratified



Stratified Sets

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$\{(a_1, 0, a_3, 0), (b_1, 0, 0, b_4), (0, 0, c_3, c_4)\}$ is stratified



Stratifiable Semilinear Sets

Definition

A semilinear set is stratifiable if and only if it is the union of linear sets with a stratified set of periods

$$((1, 1, 1) + \mathbb{N}(1, 1, 1)) \cup ((2, 3, 2) + \mathbb{N}(1, 0, 1) + \mathbb{N}(0, 1, 0))$$

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Theorem (Ginsburg, 1966)

A language $L \subseteq \sigma_1^* \cdots \sigma_d^*$ is context-free if and only if

$$\{(n_1, \dots, n_d) \mid \sigma_1^{n_1} \cdots \sigma_d^{n_d} \in L\}$$

is a stratifiable semilinear set.

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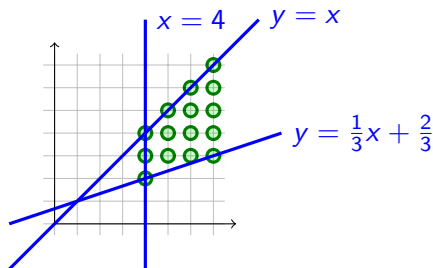
The stratifiability problem for semilinear sets is still open

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Integral Polyhedra

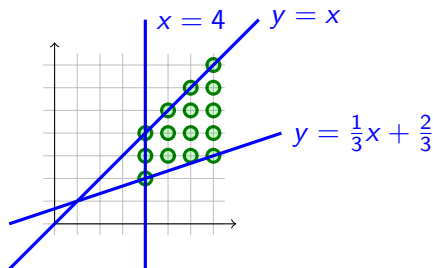
An integral polyhedron is a set of the form $\{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ where $\mathbf{A} \in \mathbb{Z}^{n \times d}$ and $\mathbf{b} \in \mathbb{Z}^n$



$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 3 \\ 1 & 0 \end{pmatrix}, \mathbf{b} = (0, 2, 4)$$

Integral Polyhedra

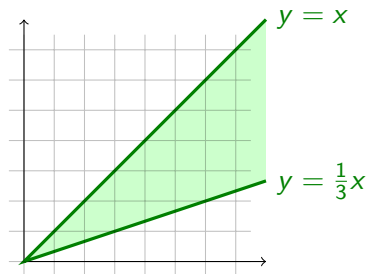
An integral polyhedron is a set of the form $\{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{Ax} \geq \mathbf{b}\}$ where $\mathbf{A} \in \mathbb{Z}^{n \times d}$ and $\mathbf{b} \in \mathbb{Z}^n$



$$\begin{aligned} & (4, 2) + \mathbb{N}(1, 1) + \mathbb{N}(3, 1) \\ \cup & (4, 3) + \mathbb{N}(1, 1) + \mathbb{N}(3, 1) \\ \cup & (4, 4) + \mathbb{N}(1, 1) + \mathbb{N}(3, 1) \\ \cup & (6, 3) + \mathbb{N}(1, 1) + \mathbb{N}(3, 1) \end{aligned}$$

Cones

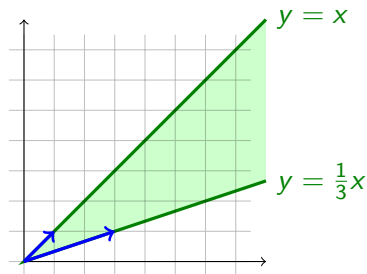
- A cone is a set of the form $\{\mathbf{x} \in \mathbb{Q}_{\geq 0}^d \mid \mathbf{A}\mathbf{x} \geq \mathbf{0}\}$



$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ -1 & 3 \\ -1 & 0 \end{pmatrix}$$

Cones

- A cone is a set of the form $\{\mathbf{x} \in \mathbb{Q}_{\geq 0}^d \mid \mathbf{A}\mathbf{x} \geq \mathbf{0}\}$
- \mathbf{X} is a cone iff $\mathbf{X} = \mathbb{Q}_{\geq 0}\mathbf{p}_1 + \cdots + \mathbb{Q}_{\geq 0}\mathbf{p}_m$, with $\mathbf{p}_1, \dots, \mathbf{p}_m \in \mathbb{N}^d$



$$\mathbb{Q}_{\geq 0}(3, 1) + \mathbb{Q}_{\geq 0}(1, 1)$$

Characterisation of Stratifiability of Integral Polyhedra

Definition

A cone \mathbf{X} is nested if every $\mathbf{x} \in \mathbf{X}$ can be decomposed into:

$$\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_m$$

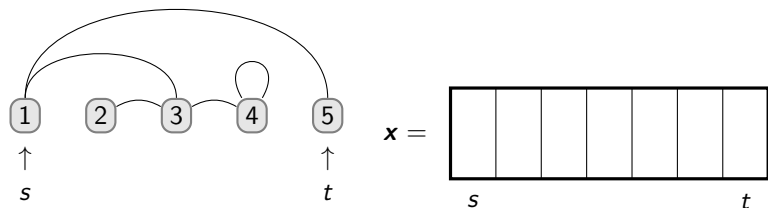
where $\{\mathbf{x}_1, \dots, \mathbf{x}_m\} \subseteq \mathbf{X}$ is stratified.

Theorem

$\{\mathbf{x} \in \mathbb{N}^d \mid \mathbf{Ax} \geq \mathbf{b}\}$ is stratifiable
 \iff
it is empty or $\{\mathbf{x} \in \mathbb{Q}_{\geq 0}^d \mid \mathbf{Ax} \geq \mathbf{0}\}$ is nested

Characterisation of nestedness

Is $x \in X$ nested ?

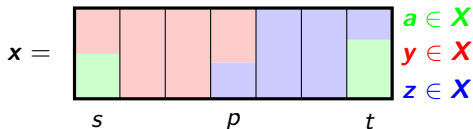
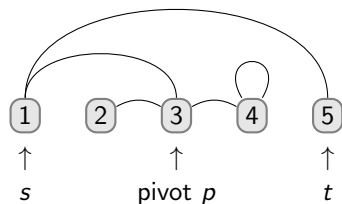


Theorem

A cone is nested if and only if it contains no irreducible vector

Characterisation of nestedness

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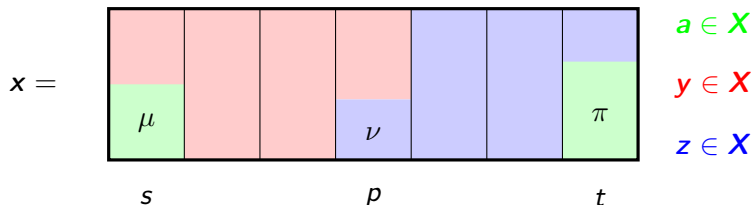
$$x = a + y + z$$
$$a, y, z \in X$$

Theorem

A cone is nested if and only if it contains no irreducible vector

CoNP-complete

x is reducible if and only if $\forall s,p,t \exists \mu \exists \nu \exists \pi, \psi_{s,p,t}(\mu, \nu, \pi, x)$



Theorem

The cone nestedness is solvable in coNP

Corollary

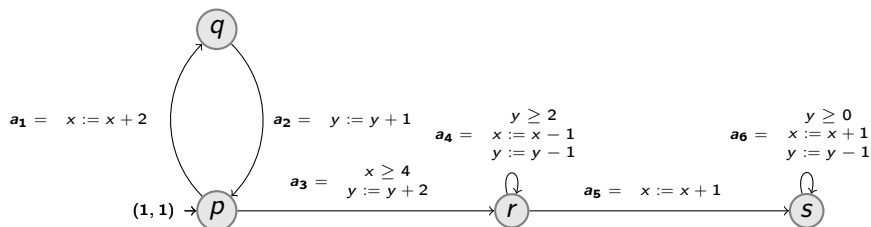
The stratifiability problem for integral polyhedra is coNP-complete

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Flat Counter System

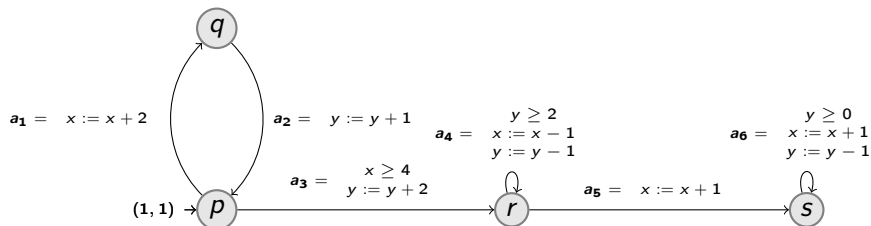
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$$L = \text{Pref}(\{(a_1 a_2)^n a_3 a_4^m a_5 a_6^p \mid n \geq 2, m \leq 2n + 1, p \leq n - r + 3\})$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix}, \mathbf{b} = (2, -1, -3)$$

The Context-Freeness Problem for Flat Counter System

Input: A flat counter system \mathcal{S}

Output: Whether the trace of \mathcal{S} is context-free or not

Theorem

The context-freeness problem for flat counter systems is coNP-complete

The Context-Freeness Problem for Flat Counter System

Input: A flat counter system \mathcal{S}

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Theorem

The context-freeness problem for flat counter systems is coNP-complete

Is the trace of \mathcal{S} context-free ? \Leftrightarrow Is an integral polyhedron stratifiable ? \Leftrightarrow Is a cone nested ?

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Conclusion

- We solved the stratifiability problem for integral polyhedra and have shown it to be coNP-complete
- It can be used to solve the context-freeness problem for flat counter systems
- This generalises our result over VAS

	Flat	General
VAS	coNP-complete	EXPSpace-complete ¹²
Counter Systems	coNP-complete	undecidable

- The general stratifiability problem is still open

¹Leroux, Penelle and Sutre in LICS 2013

²Leroux, Praveen and Sutre in CONCUR 2013