RELATIVELY FREE PROFINITE MONOIDS: AN INTRODUCTION AND EXAMPLES*

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Introduction

One of the main strongpoints of the concept of variety for the study of classes of algebras, say, of monoids, is the dual viewpoint which it allows: algebraic, or semantic, on one hand — a class of monoids closed under taking submonoids, homomorphic images and (arbitrary) direct products —; syntactic on the other hand — a class defined by a set of identities. Connecting the two points of view is the construction of the free object of a variety $\mathcal{V}$ over a set A as a subdirect product of all the A-generated elements of $\mathcal{V}$, or as the quotient of the free monoid A^* by the fully invariant congruence generated by the identities of $\mathcal{V}$ in $|A|$ variables.

When one wants to study pseudovarieties of finite monoids (algebraic characterization, solution of the membership problem, combinatorial description of the associated classes of recognizable languages), the symmetry of this double viewpoint is somewhat modified. Of course, as an intermediate case, if a pseudovariety of finite monoids $\mathbf{V}$ admits a free object over a finite set A, the understanding of this free object gives considerable information on the A-generated elements of $\mathbf{V}$: syntactic information on the identities satisfied by these monoids, and semantic-combinatorial information on the languages of A^* they recognize. In effect, the free object of $\mathbf{V}$ over A is a finite monoid which contains all the information on all A-generated elements of $\mathbf{V}$. However the existence of such free objects is far from being the general case, since it occurs if and only if $\mathbf{V}$ has only finitely many A-generated elements

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(up to isomorphism).

The theory of implicit operations and free profinite monoids presented here aims at dealing with the general case by constructing a possibly infinite topological monoid containing the desired information on the infinitely many A-generated elements of $\mathbf{V}$. This monoid, denoted $\overline{\Omega}_A\mathbf{V}$ and called the free pro-$\mathbf{V}$ monoid over A, is constructed by taking the projective limit of the A-generated elements of $\mathbf{V}$. It is always compact, so that it inherits many important properties of finite monoids (e.g. Green's relations $\mathcal{J}$ and $\mathcal{D}$ coincide). Moreover, the submonoid of $\overline{\Omega}_A\mathbf{V}$ generated by A is dense in it. We will see that the elements of $\mathbf{V}$ are exactly the continuous homomorphic images of the $\overline{\Omega}_A\mathbf{V}$, that the collection of the $\overline{\Omega}_A\mathbf{V}$ (A finite) characterizes the pseudovariety $\mathbf{V}$ (Reiterman's theorem), and that the languages of A^* which are recognized by monoids of $\mathbf{V}$ can be characterized in terms of the topology of $\overline{\Omega}_A\mathbf{V}$. These fundamental definitions and properties are covered in section 1.

So the program for understanding a given pseudovariety $\mathbf{V}$ can be viewed as computing the free pro-$\mathbf{V}$ monoids $\overline{\Omega}_A\mathbf{V}$, or at least learning something about their structure. Not surprisingly, this study is quite difficult in general. We will however discuss two classes of examples, namely the sub-pseudovarieties of $\mathbf{DS}$ (section 2) and the pseudovarieties of groups (section 3), for which some interesting results are known.

Recall that $\mathbf{DS}$ is the pseudovariety of all monoids in which each regular $\mathcal{D}$-class is a subsemigroup (or equivalently, in which each regular element is a group element). The key fact regarding the sub-pseudovarieties $\mathbf{V}$ of $\mathbf{DS}$ is that the elements of free pro-$\mathbf{V}$ monoids can be factored as products of elements of A and of regular elements. Moreover, one can give interesting combinatorial characterizations of these regular elements in terms of multiplicity of subwords. We will see that, for certain sub-pseudovarieties of $\mathbf{DS}$, more can be said on the regular elements (e.g. sufficient conditions for equality), and in favorable cases, one can give unique factorization theorems for the elements of $\overline{\Omega}_A\mathbf{V}$.

The case of groups is quite different. Profinite groups have been studied a lot, notably in the framework of number theory and algebraic geometry, and we can benefit from this work. The results presented here are of a geometrical nature. They deal with certain properties of the Cayley graphs of free profinite groups and with their consequences for the so-called reduced factorizations of elements of free profinite groups.

The paper closes with a case study which uses simultaneously properties of sub-pseudovarieties of $\mathbf{DS}$ and those of pseudovarieties of groups. We compute the free pro-$\mathbf{V}$ monoids when $\mathbf{V} = \mathbf{DG} \cap \mathbf{ECom}$ (the class of monoids in which the idempotents commute and the regular $\mathcal{D}$-classes are groups), and we provide a join decomposition for this pseudovariety.

Historically, the theory of free pro-$\mathbf{V}$ monoids and of implicit operations received its major impetus from Reiterman's theorem [20]. Much of the work thereafter was accomplished by Almeida ([3], [5], etc.) and, for what concerns the sub-pseudovarieties of $\mathbf{DS}$, by Azevedo [13] and by Almeida and Azevedo [8]. Almeida's book [6] offers a comprehensive treatment of the theory. The major connection with the classical study of free profinite groups, and in particular the introduction of the geometrical ideas involving the Cayley graphs of these groups, intervened through work of Ribes and Zalesskiĭ [23] who offered a direct proof of the topological version of the Rhodes type-II conjecture. (Independently and a year earlier, Ash [11] had proved the original statement of the Rhodes conjecture.) The results of sections 3 and 4, which make use of this geometric intuition, are due to Almeida and Weil [10].

In this respect, there is a difference between the first two sections of this paper and the last two. The first two offer material which is not so recent, which has been presented in conferences several times, has been published in journals, and now also partly appears in book form. The last two sections, on the contrary, deal with very recent results, which are presented here for the first time in a conference, and are still "to appear". This may account to some extent for stylistic differences between the two sets of sections, the first one being "slower" and assuming a more "pedagogical" tone. However, throughout the paper, we have tried to give a proof of (almost) all the statements, so as to keep it as self-contained as we could.

Finally, let us note that most statements in this paper are given in terms of monoids and pseudovarieties of monoids. Unless otherwise indicated, they also hold for semigroups and pseudovarieties of semigroups, up to obvious changes (e.g. replace free monoids by free semigroups). Examples will be drawn indistinctly from the realm of monoids as well as from that of semigroups.

Prerequisites

We assume the reader to be familiar with some basic notions of semigroup theory: idempotents (the set of idempotents of a semigroup S is denoted $E(S)$), regular elements, Green's relations.

We recall that a pseudovariety of monoids (resp. semigroups) is a class of finite monoids (resp. semigroups) which is closed under taking submonoids (resp. subsemigroups), homomorphic images and finitary direct products.

For certain sections, we also assume familiarity with the notion of languages recognized by a monoid or a semigroup, and with the notion of variety of languages. For details on these notions, the reader is referred to Pin's [19], or to Pin's contribution to this volume.

¿From the language setting, we borrow the following: an alphabet always means a finite set. If A is an alphabet, its elements are called letters. The elements of the free monoid A^* (or of the free semigroup A^+) over A can be seen as finite sequences of letters, and are called words. We will use freely this combinatorial terminology when dealing with free monoids.

1. Free profinite monoids and implicit operations

Projective limits constitute an essential tool for our purpose: to each pseudovariety $\mathbf{V}$ and to each finite set A, we associate the projective limit of the A-generated elements of $\mathbf{V}$, called the free pro-$\mathbf{V}$ monoid over A.

Projective limits can be defined for any type of algebras (e.g. groups, rings, modules) or partial algebras (e.g. graphs). In this section we will be interested only in monoids and semigroups but modules and graphs will play an important role in section 3. For this reason, section 1.1, which deals with the basic properties of projective limits, is written in terms of algebras in general.

Section 1.2 introduces relatively free profinite monoids, their topology and their elementary properties. Examples are computed in section 1.3. In section 1.4, we consider the monoids of implicit operations on a pseudovariety $\mathbf{V}$, and show that they are naturally isomorphic to the free pro-$\mathbf{V}$ monoids. Section 1.5 is devoted to the statement and proof of Reiterman's theorem on the definition of pseudovarieties by sets of pseudoidentities (formal equalities of implicit operations). Finally, section 1.6 deals with the characterization of the class of languages recognized by $\mathbf{V}$ in terms of the topology of the free pro-$\mathbf{V}$ monoids.

A general reference for the contents of this section is Almeida's book [6]. Other expository papers on the topic are Almeida's [2] and Weil's [26]. On the elementary properties of projective limits, we have used the first pages of Serre's [25] and of Fried and Jarden's [15]. In fact, the presentation given here, which introduces first the relatively free profinite monoids, and next the monoids of implicit operations, is in the reverse order of that adopted in [2], [6] and [26], but in keeping with that of Almeida's thorough [3]. Sections 1.1 to 1.5 are an expanded version of the first sections of a forthcoming paper by Almeida and Weil about implicit operations on semidirect products [10].

1.1. Projective limits

Recall that a partially ordered set (poset) I is *directed* if any two elements of I admit a common upper bound. A *directed system of algebras* $(S_i)_{i\in I}$ is a family of algebras indexed by a directed poset I such that, for each pair (i,j) of elements of I verifying $i \geq j$, there exists a homomorphism $\varphi_{i,j}: S_i \to S_j$, and such that, for all $i \geq j \geq k$ in I, we have $\varphi_{i,i} = \mathrm{id}_{S_i}$ and $\varphi_{j,k} \circ \varphi_{i,j} = \varphi_{i,k}$.

By definition, the *projective limit* of a directed system of algebras $(S_i)_{i\in I}$ is the following subalgebra of the direct product $\prod_{i\in I} S_i$:

$$\varprojlim (S_i)_{i\in I} = \big\{(x_i)_{i\in I} \in \prod_{i\in I} S_i : \forall i \geq j,\ \varphi_{i,j} x_i = x_j\big\}.$$

Example. Let $I = \mathbb{N}$ be the set of natural numbers. For each $n \geq 0$, let $S_n = \langle a; a^n = a^{n+1}\rangle = \{1, a, \ldots, a^n\}$ with

$$a^i a^j = \begin{cases} a^{i+j} & \text{if } i+j \leq n \\ a^n & \text{if } i+j > n. \end{cases}$$

For each $n, k \geq 0$, let $\varphi_{n+k,n}: S_{n+k} \to S_n$ be the morphism given by $\varphi_{n+k,n} a^i = a^{\min(i,n)}$. The morphisms $\varphi_{n+k,n}$ make $(S_n)_n$ into a directed system of monoids, and $\varprojlim (S_n)_n$ is isomorphic to the free monoid generated by a with a zero added, that is, $\varprojlim (S_n)_n$ is the set $\{1, a, a^2, \ldots, a^n, \ldots\} \cup \{0\}$.

$$\begin{array}{lllllll}
S_1 & 1 - a & & & & & \\
S_2 & 1 - a - a^2 & & & & & \\
\vdots & \quad\vdots & & & & & \\
S_n & 1 - a - a^2 & \cdots & - a^n & & & \\
\vdots & \quad\vdots & & & & & \\
\varprojlim (S_n)_n & 1 - a - a^2 & \cdots & - a^n & \cdots & & 0
\end{array}$$

The *canonical morphisms* $\pi_i \colon \varprojlim (S_i)_{i\in I} \to S_i$ are defined to be the restrictions of the coordinate projections to $\varprojlim (S_i)_i$. Note that these morphisms are not necessarily onto.

Example. Let $I = \mathbb{N}$. For each $n, k \geq 0$, let $S_n = \mathbb{Z}$, and let $\varphi_{n+k,n} \colon \mathbb{Z} \to \mathbb{Z}$ be given by $\varphi_{n+k,n} x = 2^k x$. These data define a projective system and $\varprojlim (S_n)_n = \{0\}$.

A *directed system of morphisms* defined on an algebra S is a family of morphisms $(\rho_i \colon S \to S_i)_{i\in I}$ indexed by a directed poset I such that:

- there exist morphisms $\varphi_{i,j} \colon S_i \to S_j$ for all pairs (i, j) of elements of I verifying $i \geq j$, and these morphisms make $(S_i)_i$ a directed system of algebras,
- for all $i \geq j$ in I, we have $\rho_j = \varphi_{i,j} \circ \rho_i$.

It is not difficult to verify the following universal property of projective limits.

Proposition 1.1 *Let $(\rho_i \colon S \to S_i)_{i\in I}$ be a directed system of morphisms defined on an algebra S. Then there exists a unique morphism $\rho \colon S \to \varprojlim (S_i)_i$ satisfying $\rho_i = \pi_i \circ \rho$ for all i. Moreover ρ is defined by $\rho s = (\rho_i s)_i$.*

This property defines $\varprojlim (S_i)_i$ up to an isomorphism.

We say that the morphism ρ described above is *induced* by the directed system $(\rho_i)_i$.

Let us now assume that the algebras S_i are topological algebras, and that the morphisms $\varphi_{i,j}$ are continuous. The direct product $\prod_i S_i$ is equipped with the product topology. By definition, a basis of open sets for this topology is given by the sets of the form

$$V_{i_1} \times \cdots \times V_{i_n} \times \prod_{\substack{j\in I \\ j\neq i_1,\ldots,i_n}} S_j ,$$

where $n \geq 1$, $i_1, \ldots, i_n$ are distinct elements of I and V_{i_h} is an open subset of S_{i_h}. Then one verifies that $\varprojlim (S_i)_i$ is a closed subalgebra of $\prod_i S_i$. Thus the canonical morphisms π_i are continuous. In particular, if the S_i are compact, then so is their direct product (by the classical Tikhonov's theorem), and hence so is their projective limit.

Note. Using the fact that the poset I is directed, one may verify that a basis of open subsets of $\varprojlim(S_i)_{i\in I}$ is given by the sets of the form $\pi_i^{-1}(V_i)$ where $i \in I$ and V_i is an open subset of S_i.

The case where the S_i are finite algebras is particularly important for our purpose. We consider each finite algebra to be equipped with the discrete topology: each of them is compact, and any morphism between them is continuous. In this situation the topological algebra $\varprojlim(S_i)_i$ is compact and totally disconnected.

We say that an algebra is *profinite* if it is the projective limit of a directed system of finite algebras. In fact, for the most usual kinds of algebras (like groups, rings, lattices, semigroups, monoids, sets; in general for those in which syntactic congruences are characterized by a finite number of terms), one can verify that an algebra is profinite if and only if it is compact and totally disconnected (cf. [4]). We do not know, for instance, if this property holds for groupoids.

More generally, for each class $\mathcal{C}$ of algebras, we say that an algebra is *pro-*$\mathcal{C}$ if it is a projective limit of algebras in $\mathcal{C}$.

Example. If we consider algebras with empty signature, that is, sets, it follows that profinite sets are compact totally disconnected topological spaces. Conversely, one can verify that any such topological space is a projective limit of finite sets, i.e., it is profinite.

Let us note the following important properties of projective limits.

Lemma 1.2 *Let $(\rho_i : S \to S_i)_{i\in I}$ be a directed system of morphisms defined on S. Let $\rho: S \to \varprojlim(S_i)_i$ be the induced morphism.*

a) ρ is one-to-one if and only if, for each $s \in S$, $\{s\} = \bigcap_{i\in I} \rho_i^{-1}(\rho_i s)$.

b) If S is compact and if, for each $i \in I$, ρ_i is onto, then ρ is onto.

Proof. (a) is immediate by definition of projective limits.

In order to prove (b), for each element $(s_i)_i$ of $\varprojlim(S_i)_i$, we consider the family of closed sets $\big(\rho_i^{-1}(s_i)\big)_i$. Since the system of morphisms $(\rho_i)_i$ is directed, each finite intersection of $\rho_i^{-1}(s_i)$ is non-empty. But S is compact, so $\bigcap_{i\in I}\rho_i^{-1}(s_i)$ is non-empty, which concludes the proof.□

The following lemma is an immediate but useful observation. It shows that, when we are given a pro-$\mathcal{C}$ algebra S where $\mathcal{C}$ is a class closed under taking closed subalgebras, we may always assume that S is the projective limit of a directed system $(S_i)_{i\in I}$ of algebras in $\mathcal{C}$ in such a way that the canonical morphisms $\pi_i : S \to S_i$ are onto.

Lemma 1.3 *Let $S = \varprojlim(S_i)_i$ and let $\pi_i : S \to S_i$ be the canonical morphisms. For each i, $\pi_i S$ is a subalgebra of S_i and $S = \varprojlim(\pi_i S)_i$.*□

The next lemma shows that the finite continuous images of a projective limit $\varprojlim(S_i)_i$ of compact algebras are exactly the images of the S_i.

Lemma 1.4 *Let $S = \varprojlim(S_i)_{i\in I}$ be a projective limit of compact algebras and let $\pi_i: S \to S_i$ $(i \in I)$ be the canonical morphisms, which we assume to be onto. Let T be a finite algebra and let $\varphi: S \to T$ be a continuous morphism. There exists $i \in I$ and a continuous morphism $\varphi_i: S_i \to T$ such that $\varphi = \varphi_i \circ \pi_i$.*

Proof. The $\pi_i^{-1}(V)$ ($i \in I$, V open subset of S_i) constitute a basis of open sets for the topology of S. Since φ is continuous, for each $s \in S$, there exists an element i_s of I and an open neighborhood V_s of $\pi_{i_s}s$ in S_{i_s} such that $s \in \pi_{i_s}^{-1}(V_s)$ and $\pi_{i_s}^{-1}(V_s) \subseteq \varphi^{-1}(\varphi s)$. Now S is compact and $S = \bigcup_s \pi_{i_s}^{-1}(V_s)$. So there exists an integer n, elements $s_1, \ldots, s_n \in S$, elements $i_1, \ldots, i_n \in I$ and open sets $V_1, \ldots, V_n$ respectively of $S_{i_1}, \ldots, S_{i_n}$ such that

$$S = \bigcup_{h=1}^{n} \pi_{i_h}^{-1}(V_h) \qquad \text{with } s_h \in \pi_{i_h}^{-1}(V_h) \text{ and } \pi_{i_h}^{-1}(V_h) \subseteq \varphi^{-1}(\varphi s_h) \text{ for all } h.$$

By hypothesis, I is a directed poset, so we may consider an element i of I which is greater than or equal to $i_1, \ldots, i_n$. For each $s \in S$, there exists $1 \le h \le n$ such that $s \in \pi_{i_h}^{-1}(V_h)$. Therefore $\pi_i^{-1}(\pi_i s) \subseteq \pi_{i_h}^{-1}(\pi_{i_h} s) \subseteq \pi_{i_h}^{-1}(V_h) \subseteq \varphi^{-1}(\varphi s_h)$. In particular $\varphi s_h = \varphi s$, so

$$\text{for each } s \in S, \quad \pi_i^{-1}(\pi_i s) \subseteq \varphi^{-1}(\varphi s).$$

Thus, letting $\varphi_i(\pi_i s) = \varphi s$ defines a mapping φ_i from S_i into T which is trivially a morphism and satisfies $\varphi = \varphi_i \circ \pi_i$.

There remains to see that φ_i is continuous. For this it suffices to verify that $\varphi_i^{-1}(t)$ is closed in S_i for each $t \in T$. By hypothesis, $\varphi^{-1}(t)$ is closed in S, and hence is compact. Since π_i is continuous, it follows that $\varphi_i^{-1}(t) = \pi_i(\varphi^{-1}(t))$ is compact, which concludes the proof.□

To conclude this section, we observe that compact monoids and semigroups, and among them profinite ones, have many useful properties in common with finite ones. In particular, we will use freely the following elementary properties of compact semigroups.

Proposition 1.5 *Let S be a compact semigroup. Then*

a) S contains an idempotent;

b) if $s \le_{\mathcal{J}} t$ in S and $t \le_{\mathcal{R}} s$ (resp. $t \le_{\mathcal{L}} s$, $t \le_{\mathcal{H}} s$), then $s \mathrel{\mathcal{R}} t$ (resp. $s \mathrel{\mathcal{L}} t$, $s \mathrel{\mathcal{H}} t$);

c) the Green relations $\mathcal{J}$ and $\mathcal{D}$ coincide.

Proof. Let $s \in S$ and let M_s be the closure of the subsemigroup generated by s. Note that M_s is compact and commutative. We claim that M_s admits a unique non-empty minimal closed ideal, namely the intersection of all its closed ideals. Indeed any finite intersection of closed ideals is non-empty, and the claim follows by compactness. Let I_s be this minimal ideal. If $x \in I_s$, then $xI_s = I_s$ by commutativity and minimality of I_s, so I_s is a group, and hence I_s contains an idempotent, thus proving (a).

Let now $s, t \in S$ be such that $s \leq_{\mathcal{J}} t$ and $t \leq_{\mathcal{R}} s$. Then $s = utv$ and $t = sx$ for some $u, v, x \in S$. So, for each $n \geq 0$, $s = u^n s(xv)^n$. By (1), there exists a sequence $\left((xv)^{k_n}\right)_n$ which converges to an idempotent e. Then $s = \lim_n u^{k_n} s(xv)^{2k_n}$. But, for each n, $u^{k_n} s(xv)^{2k_n} = sxv(xv)^{k_n - 1} = tv(xv)^{k_n-1} \leq_{\mathcal{R}} t$, so $s \leq_{\mathcal{R}} t$ by passing to the limit. This proves (b).

Finally, let $s, t \in S$ with $s \mathrel{\mathcal{J}} t$. Then $s = utv$ for some $u, v \in S$. We have $ut \leq_{\mathcal{J}} t \mathrel{\mathcal{J}} s$, so $ut \leq_{\mathcal{J}} s \leq_{\mathcal{R}} ut$. By (b), this implies $s \mathrel{\mathcal{R}} ut$. Now, $t \mathrel{\mathcal{J}} s \mathrel{\mathcal{J}} ut$ and $ut \leq_{\mathcal{L}} t$, so $t \mathrel{\mathcal{L}} ut$ by (b) again. Thus $s \mathrel{\mathcal{D}} t$. This completes the proof of the proposition.□

1.2. Relatively free profinite monoids and semigroups

Let A be a finite set and let $\mathbf{V}$ be a pseudovariety. An *A-generated profinite (*resp. *pro-$\mathbf{V}$) monoid* is a mapping $\mu: A \to M$ into a profinite (resp. pro-$\mathbf{V}$) monoid M such that the submonoid generated by μA is dense in M. By extension, we will say that the profinite (resp. pro-$\mathbf{V}$) monoid M is A-generated if there exists a mapping $\mu: A \to M$ as above.

A-generated profinite monoids are the objects of a category whose morphisms are defined as follows: Let $\mu: A \to M$ and $\nu: A \to N$ be A-generated profinite monoids. A morphism of profinite monoids (that is, a monoid morphism which is continuous) $\varphi: M \to N$ is a *morphism of A-generated profinite monoids* from μ to ν if $\nu = \varphi \circ \mu$. In particular, φ must be onto.

Let $\mathbf{V}$ be a pseudovariety of finite monoids and let A be a finite set. The class $\mathbf{V}_A$ of all A-generated profinite monoids in $\mathbf{V}$, together with all the morphisms of A-generated profinite monoids between them, constitutes a category and a directed system. By definition, we let $\overline{\Omega}_A \mathbf{V}$ be its projective limit.

For each A-generated profinite monoid $\mu: A \to M$ with $M \in \mathbf{V}$, we may consider the natural extension of μ to a morphism $\mu: A^* \to M$. Since M is finite, we have $\mu A^* = M$. By the universal property of projective limits, these morphisms μ induce a morphism $\iota: A^* \to \overline{\Omega}_A \mathbf{V}$, which we call the *natural morphism* from A^* into $\overline{\Omega}_A \mathbf{V}$. We let ι_A be the restriction of ι to A. Note that ι need not be injective. However, if $\mathbf{V}$ is not trivial, then ι_A is one-to-one.

Example. Let $\mathbf{V} = \mathbf{Com}$ be the pseudovariety of all commutative monoids. Let A be a finite set, and let a, b be distinct elements of A. Since $\mu(ab) = \mu(ba)$ for each morphism $\mu: A^* \to M$ such that $M \in \mathbf{Com}$, we have $\iota(ab) = \iota(ba)$. So ι is not injective. In general, ι is injective if and only if $\mathbf{V}$ satisfies no nontrivial identity on $|A|$ variables.

Note. More precisely, one can verify that ιA^* is the free object of the variety generated by the pseudovariety $\mathbf{V}$. In the sequel, we will often identify the isomorphic monoids ιA^* and $A^*/\ker \iota$. In particular, when $\mathbf{V}$ satisfies no nontrivial identity, we will view A^* as a submonoid of $\overline{\Omega}_A \mathbf{V}$.

Proposition 1.6 *Let A be a finite set and let $\mathbf{V}$ be a pseudovariety of monoids. Then $\overline{\Omega}_A \mathbf{V}$ (together with the mapping ι_A) is the free pro-$\mathbf{V}$ monoid over the finite set A.*

Furthermore, an A-generated finite monoid is in $\mathbf{V}$ if and only if it is a continuous homomorphic image of $\overline{\Omega}_A \mathbf{V}$.

Proof. We first verify that ιA^* is dense in $\overline{\Omega}_A\mathbf{V}$. For each A-generated profinite monoid in $\mathbf{V}$, that is, for each onto morphism $\mu: A^* \to M$ with $M \in \mathbf{V}$, let $\pi_\mu: \overline{\Omega}_A\mathbf{V} \to M$ be the corresponding canonical morphism. Then $\pi_\mu \circ \iota = \mu$. By definition of the topology on $\overline{\Omega}_A\mathbf{V}$, a basis of neighborhoods of an element $x \in \overline{\Omega}_A\mathbf{V}$ consists of all $\pi_\mu^{-1}(\pi_\mu x)$. But $\mu A^* = M$ for each μ, so if $\pi_\mu x = \mu u$ for some $u \in A^*$, then $\iota u \in \pi_\mu^{-1}(\pi_\mu x)$. Thus ιA^* is dense in $\overline{\Omega}_A\mathbf{V}$.

Let now S be an A-generated pro-$\mathbf{V}$ monoid: there exists a continuous mapping $\sigma: A \to S$ such that the submonoid generated by σA is dense in S, and there exists a directed system $(S_i)_i$ of monoids in $\mathbf{V}$ such that $S = \varprojlim(S_i)_i$. Let $\pi_i: S \to S_i$ $(i \in I)$ be the corresponding canonical morphisms, which, by Lemma 1.3, can be assumed to be onto. By definition of projective limit, each element x of S is of the form $x = (\pi_i x)_i$. Let $\sigma_i = \pi_i \circ \sigma: A \to S_i$. Then the σ_i are A-generated profinite monoids in $\mathbf{V}$. Therefore, considering the canonical morphisms associated with the projective limit defining $\overline{\Omega}_A\mathbf{V}$, we find that there exist continuous morphisms $\varphi_i: \overline{\Omega}_A\mathbf{V} \to S_i$ such that $\varphi_i \circ \iota_A = \sigma_i = \pi_i \circ \sigma$. We use the universal property of projective limits once more to see that these morphisms induce an onto continuous morphism $\varphi: \overline{\Omega}_A\mathbf{V} \to S$ such that $\pi_i \circ \varphi = \varphi_i$. So $\pi_i \circ \varphi \circ \iota_A = \pi_i \circ \sigma$ for each i, and hence $\varphi \circ \iota_A = \sigma$.

The proof of the last statement of the proposition is an immediate consequence of the definition of $\overline{\Omega}_A\mathbf{V}$ and of Lemma 1.4.□

The following corollary of Proposition 1.6 will turn out to be rather useful.

Corollary 1.7 *Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of monoids such that $\mathbf{W} \subseteq \mathbf{V}$, let A and B be finite sets, let ι be the natural map from A^* into $\overline{\Omega}_A\mathbf{V}$, and let ι_A be the restriction of ι to A. If $\varphi: A \to \overline{\Omega}_B\mathbf{W}$ is a mapping, then φ induces a unique continuous morphism $\bar{\varphi}: \overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_B\mathbf{W}$ such that $\bar{\varphi} \circ \iota_A = \varphi$.*

In particular, the identity function of A induces an onto continuous morphism from $\overline{\Omega}_A\mathbf{V}$ onto $\overline{\Omega}_A\mathbf{W}$.□

The construction of the free pro-$\mathbf{V}$ monoid over a finite set A and of its topology can also be approached from a different angle. We first define an écart d on A^* as follows. For each pair of distinct words u and v of A^*, we let

$$\begin{aligned} r(u,v) &= \min\{|S| : S \in \mathbf{V}, \exists \mu: A^* \to S, \mu u \neq \mu v\} \\ d(u,v) &= 2^{-r(u,v)}. \end{aligned}$$

We also let $d(u,u) = 0$. Then we have

$$d(u,v) = d(v,u) \qquad \text{and} \qquad d(u,v) \le \max\big(d(u,w), d(w,v)\big)$$

for all $u, v, w \in A^*$. That is, d is an ultrametric écart and as such, d defines a uniform structure on A^*. The resulting topology is called the *pro-$\mathbf{V}$ topology* of A^*. One easily verifies that d is a distance (that is, $d(u,v) = 0$ if and only if $u = v$) if and only if $\mathbf{V}$ satisfies no nontrivial identity on $|A|$ variables, and hence if and only if the morphism $\iota: A^* \to \overline{\Omega}_A\mathbf{V}$ is injective.

More precisely, $d(u,v) = 0$ if and only if $\iota u = \iota v$. Thus d induces an ultrametric distance on the quotient set $A^*/\ker\iota$, called the *pro-$\mathbf{V}$ distance*. If one identifies the isomorphic monoids ιA^* and $A^*/\ker\iota$, one can also verify that the topology thus defined on $A^*/\ker\iota$ coincides with that induced by $\overline{\Omega}_A\mathbf{V}$ on ιA^*. For this reason and since $\overline{\Omega}_A\mathbf{V}$ is compact, $\overline{\Omega}_A\mathbf{V}$ is sometimes called the *pro-$\mathbf{V}$ completion* of A^*.

1.3. EXAMPLES

The situation is particularly simple when $\mathbf{V}$ admits a finite free object $F_A(\mathbf{V})$ over the set A. This is the case for instance when $\mathbf{V}$ is generated by a finite number of monoids. Then, the category $\mathbf{V}_A$ of A-generated elements of $\mathbf{V}$ admits only finitely many elements, up to isomorphism, and $F_A(\mathbf{V})$ is its initial object by definition. Therefore, we have:

Proposition 1.8 *Let A be a finite set and let $\mathbf{V}$ be a pseudovariety. If $\mathbf{V}$ admits a free object $F_A(\mathbf{V})$ over A, then $\overline{\Omega}_A\mathbf{V} = F_A(\mathbf{V})$. In particular, $\overline{\Omega}_A\mathbf{V}$ is finite and $\overline{\Omega}_A\mathbf{V}$ is an A-generated element of $\mathbf{V}$.*$\square$

A very important example of this situation is given by the pseudovariety $\mathbf{J}_1$ of finite semilattices, or equivalently, of finite idempotent and commutative monoids. It is well-known that $\mathbf{J}_1$ is generated by the 2-element monoid $\{0,1\}$, and that the free object of $\mathbf{J}_1$ over the set A is the set $\mathcal{P}(A)$ of all subsets of A, under union. Let $c\colon A^* \to \mathcal{P}(A)$ be the natural morphism from A^* into $\overline{\Omega}_A\mathbf{J}_1 = \mathcal{P}(A)$. This morphism is called the *content morphism*. For each word $u \in A^*$, the content of u is the set $c(u)$ of all letters of A which occur in u.

Let now $\mathbf{V}$ be a pseudovariety containing $\mathbf{J}_1$. By Corollary 1.7, the identity function of A induces an onto continuous morphism $c_{\mathbf{V}}\colon \overline{\Omega}_A\mathbf{V} \to \mathcal{P}(A)$. Let $\iota\colon A^* \to \overline{\Omega}_A\mathbf{V}$ be the natural morphism. Then the composed morphism $c_{\mathbf{V}} \circ \iota$ is equal to c. Thus, we have been able to extend the notion of content of a word to the elements of $\overline{\Omega}_A\mathbf{V}$. The morphism $c_{\mathbf{V}}$ is also called *content morphism*, and when there is no risk of ambiguity, we write c for $c_{\mathbf{V}}$.

Some information regarding the passage from the special case of a pseudovariety with free objects to the general one is given in the following proposition. Recall that each pseudovariety $\mathbf{V}$ is the union of an increasing sequence of pseudovarieties admitting free objects over each finite alphabet. Indeed, if we identify isomorphic monoids, then $\mathbf{V}$ is at most countable, so that we can enumerate its elements,

$$\mathbf{V} = \{S_0, S_1, \ldots, S_n, \ldots\}.$$

For each $k \geq 0$, let $\mathbf{V}_k$ be the pseudovariety generated by $S_0, \ldots, S_k$. Then $\mathbf{V}_k$ admits free objects over each finite set, and $\mathbf{V} = \bigcup_{k\geq 0}\mathbf{V}_k$.

Proposition 1.9 *Let A be a finite set and let $(\mathbf{V}_k)_{k\geq 0}$ be a sequence of pseudovarieties such that $\mathbf{V}_k \subseteq \mathbf{V}_{k+1}$ for all $k \geq 0$. Let $\mathbf{V} = \bigcup_k \mathbf{V}_k$. Then $\overline{\Omega}_A\mathbf{V} = \varprojlim(\overline{\Omega}_A\mathbf{V}_k)_k$.*

Proof. According to Corollary 1.7, we may consider the continuous morphisms π_k from $\overline{\Omega}_A\mathbf{V}$ onto $\overline{\Omega}_A\mathbf{V}_k$ induced by the identity mapping of A^*. These morphisms are easily seen to constitute a directed system.

By Lemma 1.2, it suffices to show that, for all $x, y \in \overline{\Omega}_A\mathbf{V}$, if $\pi_k x = \pi_k y$ for all k, then $x = y$. But, for each A-generated monoid M in $\mathbf{V}$, there exists $k \geq 0$ such that $M \in \mathbf{V}_k$. So the canonical morphism π_M from $\overline{\Omega}_A\mathbf{V}$ onto M factors through π_k, and hence $\pi_k x = \pi_k y$ implies $\pi_M x = \pi_M y$. Since this holds for each M in $\mathbf{V}$, and since $\overline{\Omega}_A\mathbf{V}$ is the projective limit of the A-generated elements of $\mathbf{V}$, this implies $x = y$.$\square$

We now consider a few simple, but nontrivial, examples, namely the pseudovarieties of semigroups **Nil**, **K**, $\mathbf{K}^{\mathrm{r}}$ and **LI**. These pseudovarieties are defined by

$$\begin{aligned} S \in \mathbf{Nil} &\iff \exists k,\ \forall s_1, \ldots, s_k, t \in S,\ s_1 \cdots s_k t = s_1 \cdots s_k = t s_1 \cdots s_k, \\ S \in \mathbf{K} &\iff \exists k,\ \forall s_1, \ldots, s_k, t \in S,\ s_1 \cdots s_k t = s_1 \cdots s_k, \\ S \in \mathbf{K}^{\mathrm{r}} &\iff \exists k,\ \forall s_1, \ldots, s_k, t \in S,\ t s_1 \cdots s_k = s_1 \cdots s_k, \\ S \in \mathbf{LI} &\iff \exists k,\ \forall s_1, \ldots, s_k, t \in S,\ s_1 \cdots s_k t s_1 \cdots s_k = s_1 \cdots s_k. \end{aligned}$$

The semigroups in **Nil** are called *nilpotent*, and those in **LI** are called *locally trivial.*

Let A be a fixed finite alphabet. For each $k \geq 1$, let I_k be the ideal of A^+ consisting of all words of length at least k. One can verify that each $S_k = A^+/I_k$ is a nilpotent semigroup. Moreover, each morphism from A^+ onto a nilpotent semigroup can be factored through some S_k. Therefore, $\overline{\Omega}_A\mathbf{Nil} = \varprojlim(S_k)_{k \geq 0}$. Note that the semigroup S_k is the set of all words of length less than k, with a zero added and with product given, for all words u and v of length less than k by

$$uv = \begin{cases} uv & \text{if } uv \text{ is of length less than } k \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this means that the natural morphism $\iota\colon A^+ \to \overline{\Omega}_A\mathbf{Nil}$ is injective.

As in the first example given in section 1.1, it is not difficult to verify that the semigroup $\overline{\Omega}_A\mathbf{Nil}$ is obtained from A^+ by adding a zero 0. A basis of clopen sets for its topology is the family of all $\{w\}$ $(w \in A^+)$ and the sets $I_k \cup \{0\}$ $(k \geq 1)$. So $\overline{\Omega}_A\mathbf{Nil}$ is the one-point compactification of A^+, where A^+ is endowed with the discrete topology. Moreover, one can verify that a sequence of words $(u_n)_n$ of A^+ converges in $\overline{\Omega}_A\mathbf{Nil}$ (if we identify A^+ and ιA^+) if and only if, either it is ultimately constant, or the lengths of the u_n tend towards infinity. In this last case, the sequence $(u_n)_n$ tends to 0.

We now turn to **K**. Again one can define a sequence of A-generated semigroups S_k in **K** such that $\overline{\Omega}_A\mathbf{K} = \varprojlim(S_k)_{k \geq 0}$, namely, by letting S_k be defined by the set of generators A and by the relation $y_1 \cdots y_k z = y_1 \cdots y_k$. Informally speaking, the projection of A^+ onto S_k tests prefixes of length k, and S_k can be identified with the set of all words of length less than or equal to k, with product

$$uv = \begin{cases} uv & \text{if } uv \text{ is of length less than or equal to } k \\ \text{prefix of length } k \text{ of } uv & \text{otherwise.} \end{cases}$$

Again, the natural morphism $\iota\colon A^+ \to \overline{\Omega}_A\mathbf{K}$ is injective. Moreover, one can check that a convergent sequence of words $(u_n)_n$ is either ultimately constant, or it satisfies

$$\forall k \quad \exists n_k \quad \forall n, m \geq n_k \qquad u_n \text{ and } u_m \text{ have identical prefixes of length } k.$$

So $\overline{\Omega}_A\mathbf{K}$ can be identified with the set $A^\infty = A^+ \cup A^\omega$, where A^ω denotes the set of (right) infinite words over A. In particular, $\overline{\Omega}_A\mathbf{K}$ is uncountable when $|A| \geq 2$. The product in A^∞ is extended from the product in A^+ by letting $ww' = w$ if w is an infinite word. A basis of clopen sets for the topology of $\overline{\Omega}_A\mathbf{K} = \varprojlim(S_k)_{k \geq 0}$ is the family of all $\{w\}$ and wA^∞ $(w \in A^+)$. Note that wA^∞ is the family of all finite and infinite words with prefix w. In particular, the topology on A^+ is discrete, while the topology on $A^\omega = A^{\mathbb{N}}$ is the product topology of the discrete topology on A, that is, A^ω is a Cantor set. Thus, $\overline{\Omega}_A\mathbf{K}$ can be sketchily described as a "nilpotent extension of a Cantor set".

Dually $\overline{\Omega}_A\mathbf{K}^{\mathrm{r}}$ can be identified with ${}^\infty A = A^+ \cup {}^\omega A$, where ${}^\omega A$ is the set of left infinite words over A.

Finally, similar considerations allow the computation of $\overline{\Omega}_A\mathbf{LI}$. One can show that $\overline{\Omega}_A\mathbf{LI}$ is a nilpotent extension of an uncountable rectangular band. More precisely, $\overline{\Omega}_A\mathbf{LI} = A^+ \cup (A^\omega \times {}^\omega A)$. A^+ has the discrete topology and $A^\omega \times {}^\omega A$ the product topology. Finally $A^\omega \times {}^\omega A$ is a rectangular band and if $w \in A^+$, $w_1 \in A^\omega$ and $w \in {}^\omega A$, then $w(w_1, w_2) = (ww_1, w_2)$ and $(w_1, w_2)w = (w_1, w_2w)$. A basis of clopen sets for this topology is the family of all $\{w\}$ and $(wA^\omega \times {}^\omega Aw') \cup wA^+w'$, where w and w' are finite words.

We will refer in section 2 another example of complete computation of a free pro-$\mathbf{V}$ monoid, namely the free pro-$\mathbf{J}$ monoid ($\mathbf{J}$ is the pseudovariety of $\mathcal{J}$-trivial monoids). The description of the free pro-$\mathbf{J}$ monoid will be obtained by a rather different approach.

1.4. Implicit operations and pro-V monoids

The theory of implicit operations constitutes yet another way to approach the profinite completions of the finitely generated free monoids induced by a pseudovariety $\mathbf{V}$. Informally speaking, an implicit operation on $\mathbf{V}$ is any operation that makes sense on each element of $\mathbf{V}$ and is preserved under morphisms between elements of $\mathbf{V}$. More precisely, if A is a finite set, an *A-ary implicit operation π on* $\mathbf{V}$ is a family $\pi = (\pi_S)$ indexed by the elements S of $\mathbf{V}$, such that π_S is a mapping from S^A into S, and such that, for each morphism $\varphi: S \to T$ between elements of $\mathbf{V}$, the following diagram is commutative,

$$\begin{tikzcd} S^A \arrow[r, "\pi_S"] \arrow[d, "\varphi^A"] & S \arrow[d, "\varphi"] \\ T^A \arrow[r, "\pi_T"] & T \end{tikzcd}$$

that is, such that $\varphi\pi_S\big((s_a)_a\big) = \pi_T\big((\varphi s_a)_a\big)$ for all $(s_a)_{a\in A}$ in S^A.

An unsophisticated, yet fundamental, class of examples is given by the words on some fixed countable alphabet $\{x_1, \ldots, x_n, \ldots\}$. For instance, the word $u = x_1x_2^2x_3x_1$ defines a 3-ary operation u_S on any monoid S, by letting, for all s_1, s_2 and s_3 in S, $u_S(s_1, s_2, s_3) = s_1s_2^2s_3s_1$. It is not difficult to verify that, for any pseudovariety $\mathbf{V}$, $u = (u_S)_{S\in\mathbf{V}}$ is an implicit operation. In general, a word on the alphabet A defines an A-ary operation. Such implicit operations, which are defined by words, are called *explicit operations*.

In general, there are implicit operations which are not explicit. The easiest to describe is the ω-power. Let S be a finite monoid. For each $s \in S$, let s^ω be the unique idempotent in the subsemigroup of S generated by s. It is not difficult to verify that, for any morphism $\varphi: S \to T$ and any $s \in S$, we have $\varphi(s^\omega) = (\varphi s)^\omega$. So, for any pseudovariety $\mathbf{V}$, $(\ )^\omega$ is a unary implicit operation on $\mathbf{V}$. For $\mathbf{V} = \mathbf{M}$, the pseudovariety of all finite monoids, or for $\mathbf{V} = \mathbf{Com}$, this operation is not explicit.

Of course, in other pseudovarieties, it may be explicit. For instance in any nontrivial pseudovariety of finite idempotent monoids, it is also defined by the word x_1. In the pseudovariety $(\mathbb{Z}/2\mathbb{Z})$ generated by the cyclic group of order 2, it is defined by x_1^2.

For each finite set A, pointwise multiplication makes the set of implicit operations a monoid: If π and π' are A-ary implicit operations on $\mathbf{V}$, the implicit operation $\pi\pi'$ is defined, for each $S \in \mathbf{V}$ and each $(s_a)_{a\in A} \in S^A$, by $(\pi\pi')_S\big((s_a)_a\big) = \pi_S\big((s_a)_a\big)\ \pi'_S\big((s_a)_a\big)$. We denote this monoid by $I_A(\mathbf{V})$. Its unit element is the explicit operation induced by the empty word.

Proposition 1.10 *Let* $\mathbf{V}$ *be a pseudovariety and let* A *be a finite set. Then the monoids* $I_A(\mathbf{V})$ *and* $\overline{\Omega}_A\mathbf{V}$ *are naturally isomorphic.*

Proof. For each A-generated profinite monoid $\mu: A \to M$ with $M \in \mathbf{V}$, we define a mapping φ_μ, from $I_A(\mathbf{V})$ into M, by letting $\varphi_\mu(\pi) = \pi_M\big((\mu a)_{a\in A}\big)$. This mapping is easily seen to be a morphism, and the φ_μ constitute a directed system of morphisms. By Lemma 1.2, the induced morphism $\varphi: I_A(\mathbf{V}) \to \overline{\Omega}_A\mathbf{V}$ is one-to-one.

The reciprocal morphism ψ, from $\overline{\Omega}_A\mathbf{V}$ into $I_A(\mathbf{V})$, is constructed as follows. For each $S \in \mathbf{V}$ and for each A-tuple $s = (s_a)_a \in S^A$, we let μ_s be the mapping from A into S which maps each element a of A to s_a. Let $\hat{\mu}_s: \overline{\Omega}_A\mathbf{V} \to S$ be the corresponding canonical morphism. In particular, $\hat{\mu}_s \circ \iota_A = \mu_s$. For each element x of $\overline{\Omega}_A\mathbf{V}$, we define $(\psi x)_S$ by

$$\begin{aligned} (\psi x)_S: S^A &\longrightarrow S \\ s = (s_a)_a &\longmapsto \hat{\mu}_s x \end{aligned}$$

It is not difficult to verify that $\psi x = (\psi x)_S$ defines an implicit operation on $\mathbf{V}$ and that ψ is the reciprocal morphism of φ.$\square$

In the sequel, we drop the notation $I_A(\mathbf{V})$, and we identify the monoid of A-ary implicit operations on $\mathbf{V}$ with $\overline{\Omega}_A\mathbf{V}$. It will sometimes be useful to view the elements of the free pro-$\mathbf{V}$ monoid as implicit operations on $\mathbf{V}$. In this context, the results of section 1.2 translate as follows.

- The A-ary explicit operations on $\mathbf{V}$ form a dense submonoid of $\overline{\Omega}_A\mathbf{V}$. In the isomorphism between the monoid of implicit operations and $\overline{\Omega}_A\mathbf{V}$, this submonoid is mapped onto ιA^*, where ι is the natural morphism from A^* into $\overline{\Omega}_A\mathbf{V}$.
- A sequence $(\pi_n)_{n\geq 0}$ of A-ary implicit operations on $\mathbf{V}$ converges to an A-ary implicit operation π if, for each $S \in \mathbf{V}$, $\pi_S = (\pi_n)_S$ for all n large enough. That is, if and only if the sequence $(\pi_n)_n$ eventually coincides with π on each element S of $\mathbf{V}$.
- The distance on $A^*/\ker\iota$, that is, on the monoid of A-ary explicit operations on $\mathbf{V}$, which was defined at the end of section 1.2 can be extended in a natural way to a distance on the monoid of all A-ary implicit operations on $\mathbf{V}$ by letting, for each pair (π, π') of distinct A-ary implicit operations

$$\begin{aligned} r(\pi,\pi') &= \min\big\{|S| : S \in \mathbf{V}, \pi_S \neq \pi'_S\big\} \\ d(\pi,\pi') &= 2^{-r(\pi,\pi')}, \end{aligned}$$

and by letting $d(\pi,\pi) = 0$. This distance defines the topology of $\overline{\Omega}_A\mathbf{V}$.

Example. The ω-power $(\ )^\omega$ is the limit of the sequence of explicit operations $(a^{n!})_n$. Indeed, for each monoid S and each element s of S, the idempotent s^ω is equal to $s^{n!}$ for all $n \geq |S|$.

1.5. Reiterman's theorem

Let us recall the following important consequence of Corollary 1.7: if $\mathbf{V}$ and $\mathbf{W}$ are pseudovarieties such that $\mathbf{W} \subseteq \mathbf{V}$, and if A is a finite set, then id_{A^*} induces a continuous morphism from $\overline{\Omega}_A\mathbf{V}$ onto $\overline{\Omega}_A\mathbf{W}$, called the *natural projection* of $\overline{\Omega}_A\mathbf{V}$ onto $\overline{\Omega}_A\mathbf{W}$.

We will see that the natural projections of $\overline{\Omega}_A\mathbf{V}$ onto $\overline{\Omega}_A\mathbf{W}$ are injective for all finite sets A if and only if $\mathbf{V} = \mathbf{W}$. This very useful result will in fact appear as a consequence of a more precise statement, due to Reiterman, which is stated in terms of pseudoidentities.

Let $\mathbf{V}$ be a pseudovariety and let A be a finite set. A *pseudoidentity* on $\mathbf{V}$ on the set A (or, in $|A|$ variables) is a pair of elements (u, v) of the free pro-$\mathbf{V}$ monoid $\overline{\Omega}_A\mathbf{V}$ (we also say *the pseudoidentity* $u = v$). It is said to be nontrivial if the elements u and v of $\overline{\Omega}_A\mathbf{V}$ are distinct. Recall that an *identity* in $|A|$ variables is a pair (u, v) of elements of the free monoid A^* for some finite set A. Since each element of $\overline{\Omega}_A\mathbf{V}$ is the limit of a sequence of words, or rather of a sequence of explicit operations, we may view the pseudoidentity (u, v) as the limit of a sequence of identities.

Let S be a pro-$\mathbf{V}$ monoid. Recall that each mapping $\mu: A \to S$ induces a continuous morphism $\mu: \overline{\Omega}_A\mathbf{V} \to S$. We say that S *satisfies the pseudoidentity* (u, v), and we write $S \models u = v$ if, for each map $\mu: A \to S$, we have $\mu u = \mu v$. This generalizes the notion of S satisfying an identity. Note that the set of pseudoidentities of $\mathbf{V}$ on the set A satisfied by a given pro-$\mathbf{V}$ monoid S is a closed subset of $\overline{\Omega}_A\mathbf{V} \times \overline{\Omega}_A\mathbf{V}$.

Example. If $S = \varprojlim(S_i)_{i\in I}$, then S satisfies all pseudoidentities which are satisfied by the S_i. In particular, the free pro-$\mathbf{V}$ monoid $\overline{\Omega}_A\mathbf{V}$ satisfies all pseudoidentities which are satisfied by the A-generated elements of $\mathbf{V}$.

Let now Σ be a set of pseudoidentities on $\mathbf{V}$ (not necessarily involving a bounded number of variables). We let $[\![\Sigma]\!]_{\mathbf{V}}$ be the class of all elements of $\mathbf{V}$ satisfying all pseudoidentities in Σ. For instance, if $\mathbf{S}$, $\mathbf{M}$, $\mathbf{G}$ and $\mathbf{Ab}$ denote respectively the pseudovarieties of all semigroups, monoids, groups and abelian groups, then

$$\begin{aligned}
\mathbf{J}_1 &= [\![xy = yx, x^2 = x]\!]_{\mathbf{M}},\\
\mathbf{LI} &= [\![x^\omega y x^\omega = x^\omega]\!]_{\mathbf{S}},\\
\mathbf{K} &= [\![x^\omega y = x^\omega]\!]_{\mathbf{S}},\\
\mathbf{Nil} &= [\![x^\omega y = x^\omega, yx^\omega = x^\omega]\!]_{\mathbf{S}} = [\![yx^\omega = x^\omega]\!]_{\mathbf{K}},\\
\mathbf{G} &= [\![x^\omega y = y, yx^\omega = y]\!]_{\mathbf{M}},\\
\mathbf{Ab} &= [\![xy = yx]\!]_{\mathbf{G}}.
\end{aligned}$$

We can now state Reiterman's theorem.

Theorem 1.11 *Let* $\mathbf{V}$ *be a pseudovariety and let* $\mathbf{W} \subseteq \mathbf{V}$ *be a subclass of* $\mathbf{V}$. *Then* $\mathbf{W}$ *is a sub-pseudovariety if and only if there exists a class* Σ *of pseudoidentities on* $\mathbf{V}$ *such that* $\mathbf{W} = [\![\Sigma]\!]_{\mathbf{V}}$.

Proof. It is not difficult to verify that, if Σ is a class of pseudoidentities on $\mathbf{V}$, then $[\![\Sigma]\!]_{\mathbf{V}}$ is closed under taking submonoids, homomorphic images and finitary direct products. That is, $[\![\Sigma]\!]_{\mathbf{V}}$ is a sub-pseudovariety of $\mathbf{V}$.

We now consider a sub-pseudovariety $\mathbf{W}$ of $\mathbf{V}$, and the class Σ of all pseudoidentities on $\mathbf{V}$ which are satisfied by all elements of $\mathbf{W}$. Let S be a finite monoid in $\mathbf{V}$, satisfying all pseudoidentities in Σ. We need to show that $S \in \mathbf{W}$.

Since S is finite, there exists a finite set A and an onto continuous morphism $\sigma: \overline{\Omega}_A\mathbf{V} \to S$. Let also π be the natural projection of $\overline{\Omega}_A\mathbf{V}$ onto $\overline{\Omega}_A\mathbf{W}$. By definition, $\ker\pi$ is exactly the subset of Σ consisting of all pseudoidentities on $\mathbf{V}$ on the set A which are satisfied by the elements of $\mathbf{W}$. Thus, the hypothesis made on S implies that $\ker\pi \subseteq \ker\sigma$. Therefore σ factors through π, that is, there exists an onto morphism $\tau: \overline{\Omega}_A\mathbf{W} \to S$ such that $\sigma = \tau \circ \pi$. Note that, for each $s \in S$, $\sigma^{-1}(s)$ is closed and hence compact, so that, by continuity of π, $\tau^{-1}(s) = \pi(\sigma^{-1}(s))$ is closed. Therefore, τ is continuous. By Proposition 1.6, this implies $S \in \mathbf{W}$.□

We will often use the following corollary of Reiterman's theorem.

Corollary 1.12 *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties such that* $\mathbf{W} \subseteq \mathbf{V}$. *The natural projections of* $\overline{\Omega}_A\mathbf{V}$ *onto* $\overline{\Omega}_A\mathbf{W}$ *are injective for all finite sets* A *if and only if* $\mathbf{V} = \mathbf{W}$.

Proof. It suffices to notice that the projection of $\overline{\Omega}_A\mathbf{V}$ onto $\overline{\Omega}_A\mathbf{W}$ is injective if and only if $\mathbf{W}$ satisfies no nontrivial pseudoidentity of $\mathbf{V}$ on the set A.□

Note. Recall that we say that a monoid S ultimately satisfies the sequence of identities $(u_n = v_n)_n$ if it satisfies the identities $u_n = v_n$ for all large enough n. A class of finite monoids is ultimately defined by a sequence of identities $(u_n = v_n)_n$ if it consists exactly of all monoids which ultimately satisfy this sequence. Eilenberg-Schützenberger's theorem states that a class of finite monoids is a pseudovariety if and only if it is ultimately defined by a sequence of identities (see for instance Pin's contribution to this volume). Using the fact that a pseudoidentity can be viewed as the limit of a sequence of identities, it is not difficult to verify that Reiterman's theorem above implies Eilenberg-Schützenberger's theorem. On the other hand, Higgins gave a proof of Reiterman's theorem from Eilenberg-Schützenberger's theorem [17]. Unlike the proof given here, Higgins's proof explicitly avoids the use of the axiom of choice. This axiom intervenes in our treatment in the form of Tikhonov's theorem.

1.6. Languages recognized by a pseudovariety $\mathbf{V}$

There is a very interesting connection between the topology of $\overline{\Omega}_A\mathbf{V}$ and the rational languages of A^* which are recognized by the elements of $\mathbf{V}$, which we call $\mathbf{V}$-*recognizable*. We refer the reader to [19] and Pin's contribution to this volume for background on recognizable languages and varieties of recognizable languages.

In this section, we consider only pseudovarieties $\mathbf{V}$ which satisfy no nontrivial identity, that is, for which the natural morphism $\iota: A^* \to \overline{\Omega}_A\mathbf{V}$ is injective for each finite set A. In particular, we identify the free monoid A^* with a submonoid of the free pro-$\mathbf{V}$ monoid $\overline{\Omega}_A\mathbf{V}$. Furthermore, under this hypothesis, each morphism from A^* into a monoid S of $\mathbf{V}$ can be extended to a uniquely defined continuous morphism from $\overline{\Omega}_A\mathbf{V}$ into S.

The main observation in this section, is that the $\mathbf{V}$-recognizable languages over the alphabet A can be characterized in terms of the clopen subsets of $\overline{\Omega}_A\mathbf{V}$.

First we notice that, since A^* is dense in $\overline{\Omega}_A\mathbf{V}$, we have $K = \overline{K \cap A^*}$ for each clopen subset K of $\overline{\Omega}_A\mathbf{V}$. Next, one can verify that the following are equivalent for each language L in A^*.

i) L is $\mathbf{V}$-recognizable.
ii) $L = K \cap A^*$ for some clopen subset K of $\overline{\Omega}_A\mathbf{V}$.
iii) The closure $\overline{L}$ of L in $\overline{\Omega}_A\mathbf{V}$ is clopen and $L = \overline{L} \cap A^*$.

As a consequence, the closures of the $\mathbf{V}$-recognizable languages in A^* constitute exactly the set of clopen sets of $\overline{\Omega}_A\mathbf{V}$.

It is often an important question to characterize the $\mathbf{V}$-recognizable languages for a given peudovariety $\mathbf{V}$. These languages form a variety of languages, and in particular, on each finite alphabet, they form a Boolean algebra. In this context, we look for simple generating sets of this Boolean algebra, and the following observation turns out to be very useful.

We say that a family $\mathcal{X}$ of subsets of $\overline{\Omega}_A\mathbf{V}$ *separates* the points of $\overline{\Omega}_A\mathbf{V}$ if, for each pair of distinct elements x and y in $\overline{\Omega}_A\mathbf{V}$, there exists an element X of $\mathcal{X}$ such that either $x \in X$ and $y \notin X$, or $x \notin X$ and $y \in X$.

Proposition 1.13 *Let A be a finite alphabet, let $\mathbf{V}$ be a pseudovariety satisfying no nontrivial identity, and let $A^*\mathcal{V}$ be the Boolean algebra of languages over A recognized by $\mathbf{V}$. Let $\mathcal{L}$ be a subset of $A^*\mathcal{V}$ and let $\overline{\mathcal{L}}$ be the set of the topological closures in $\overline{\Omega}_A\mathbf{V}$ of the elements of $\mathcal{L}$.*

The Boolean algebra $A^*\mathcal{V}$ is generated by $\mathcal{L}$ if and only if the points of $\overline{\Omega}_A\mathbf{V}$ are separated by $\overline{\mathcal{L}}$.

Proof. Let $\mathcal{B}$ be the Boolean algebra generated by $\overline{\mathcal{L}}$. By the above remarks, $\mathcal{B}$ consists only of clopen subsets of $\overline{\Omega}_A\mathbf{V}$. Furthermore, the Boolean algebra $A^*\mathcal{V}$ is generated by $\mathcal{L}$ if and only if $\mathcal{B}$ is the class of all clopen subsets of $\overline{\Omega}_A\mathbf{V}$.

Let us first assume that $\overline{\mathcal{L}}$ separates the points of $\overline{\Omega}_A\mathbf{V}$ and let $\mathcal{L}^c$ be the class of all complements in A^* of the languages of $\mathcal{L}$. Let K be a clopen subset of $\overline{\Omega}_A\mathbf{V}$. For each $x \in \overline{\Omega}_A\mathbf{V} \setminus K$ and for each $y \in K$, there exists a language $L_{x,y}$ in $\mathcal{L}$ or $\mathcal{L}^c$, such that $\overline{L_{x,y}}$ contains y and not x. By compactness, one can select a finite collection of points of K, say, $y_1, \ldots, y_n$ such that $x \notin \bigcup_{i=1}^n \overline{L_{x,y_i}}$ and $K \subseteq \bigcup_{i=1}^n \overline{L_{x,y_i}}$. Therefore, for each $x \in \overline{\Omega}_A\mathbf{V} \setminus K$, there exists a language L_x such that $x \notin \overline{L_x}$, $K \subseteq \overline{L_x}$ and $\overline{L_x} \in \mathcal{B}$. In particular, $K = \bigcap_{x \notin K} \overline{L_x}$, and by compactness again, there exists a finite collection of points, say $x_1, \ldots, x_m$, such that $K = \bigcap_{j=1}^m \overline{L_{x_j}}$. Thus, $K \in \mathcal{B}$.

In order to prove the converse, we assume that $\mathcal{B}$ is the class of all clopen subsets of $\overline{\Omega}_A\mathbf{V}$, and we consider two distinct elements x and y of $\overline{\Omega}_A\mathbf{V}$. The topology of $\overline{\Omega}_A\mathbf{V}$ is compact and totally disconnected, so the clopen sets form a basis for this topology. Therefore, there exists $X \in \mathcal{B}$ such that $x \in X$ and $y \notin X$. But X can be expressed as a finite union of finite intersections of closures of languages in $\mathcal{L}$ and their complements. Note that the complement of a $\mathbf{V}$-recognizable language L is $\mathbf{V}$-recognizable, and that $\overline{A^* \setminus L} = \overline{\Omega}_A\mathbf{V} \setminus \overline{L}$. So there exist finitely many $\mathbf{V}$-recognizable languages $L_1, \ldots, L_k$ in $\mathcal{L}$ or $\mathcal{L}^c$ such that $x \in \bigcap_{i=1}^k \overline{L_i}$ and $y \notin \bigcap_{i=1}^k \overline{L_i}$.

Therefore, $x \in \overline{L_i}$ and $y \notin \overline{L_i}$ for some $1 \leq i \leq k$. This shows that $\overline{\mathcal{L}}$ separates the points of $\overline{\Omega}_A\mathbf{V}$, which concludes the proof.□

Example. Using the examples developed in section 1.3, we recover the well-known characterizations of the varieties of languages associated with **Nil**, **K**, $\mathbf{K}^{\mathrm{r}}$ and **LI**. For each finite alphabet A, the following hold.

- The Boolean algebra of all **Nil**-recognizable languages of A^+ is generated by the singleton languages.
- The Boolean algebra of all **K**- (resp. $\mathbf{K}^{\mathrm{r}}$-) recognizable languages of A^+ is generated by the languages of the form wA^* (resp. A^*w), where $w \in A^+$.
- The Boolean algebra of all **LI**-recognizable languages of A^+ is generated by the languages of the form wA^* and A^*w, where $w \in A^+$. This last result provides a proof of the equality $\mathbf{LI} = \mathbf{K} \vee \mathbf{K}^{\mathrm{r}}$.

2. Multiplicity of subwords and implicit operations on DS

This section is devoted to the study of implicit operations on sub-pseudovarieties of **DS**, the class of all monoids in which each regular $\mathcal{D}$-class is a subsemigroup. This is a rather large class of pseudovarieties, which contains notably the pseudovariety **J** of $\mathcal{J}$-trivial monoids, and the pseudovariety **G** of all groups.

As we mentioned in the introduction of this paper, the implicit operations on **DS** can be factored as products of explicit and regular operations, and this fact provides sufficient insight to obtain more detailed statements for certain sub-pseudovarieties of **DS**.

Sections 2.1, 2.2 and 2.3 lay the background for this factorization theorem, which is proved in section 2.4. Section 2.5 presents some further results on **DO** and **DG**, the classes of monoids in which the regular $\mathcal{D}$-classes are respectively orthodox semigroups and groups. These results will be used in section 4. Finally, section 2.6 summarizes some results concerning implicit operations on $\mathcal{J}$-trivial monoids. In this case, the monoids of implicit operations are countable, and are completely described.

Apart from section 2.6, this section is self-contained, and again we have not included the bibliographical references in the text. Proposition 2.1 is due to Schützenberger [24]. The results of sections 2.2, 2.3, 2.4, and 2.5 are due to Almeida and Azevedo, and can be found in Azevedo's [13], Almeida and Azevedo's [8], and of course in Almeida's book [6]. Finally, the results of section 2.6 are due to Almeida and can be found in [5] and in [6].

2.1. A few words about DS

If **V** is a pseudovariety of semigroups, we let **DV** be the class of all monoids in which each regular $\mathcal{D}$-class is a subsemigroup in **V**. It is not difficult to verify that **DV** is a pseudovariety.

In particular, **DS** is the class of all monoids in which each regular $\mathcal{D}$-class is a subsemigroup, or equivalently, in which each regular element is a group element.

$*$	$*$	$\cdots$	$*$
$*$	$*$	$\cdots$	$*$
$\vdots$			
$*$	$*$		$*$

A regular $\mathcal{D}$-class of a monoid in **DS**

This can be expressed by the following equalities.

$$\begin{aligned}\mathbf{DS} &= [\![\big((xy)^\omega (yx)^\omega (xy)^\omega\big)^\omega = (xy)^\omega]\!]_{\mathbf{M}} \\ &= [\![\big((xy)^\omega x\big)^{\omega+1} = (xy)^\omega x]\!]_{\mathbf{M}}\end{aligned}$$

Before we mention another important characterization of **DS**, let us recall that **LG** is the pseudovariety of all semigroups S such that eSe is a group for each idempotent e of S, or equivalently, the class of all semigroups in which the regular elements form the minimal ideal:

$$\mathbf{LG} = [\![(x^\omega y x^\omega)^\omega = x^\omega]\!]_{\mathbf{S}}.$$

Proposition 2.1 *Let S be a finite monoid. Then the following are equivalent:*
i) $S \in \mathbf{DS}$,
ii) For each $e \in E(S)$, $\{s \in S \mid e \leq_{\mathcal{J}} s\}$ is a submonoid of S,
*iii) S is a semilattice of semigroups in **LG**, that is, there exists a morphism $\varphi: S \to T$ with $T \in \mathbf{J}_1$ such that $\varphi^{-1}(t) \in \mathbf{LG}$ for each $t \in T$.*

Proof. Let us first assume that $S \in \mathbf{DS}$, and let T be the power set $T = \mathcal{P}\big(E(S)\big)$ under intersection. Then T is a monoid in $\mathbf{J}_1$. Let also $\varphi: S \to T$ be defined by

$$\varphi s = \{e \in E(S) : e \leq_{\mathcal{J}} s\}.$$

Let s and s' be elements of S. We want to show that $\varphi(ss') = \varphi s \cap \varphi s'$. If $e \leq_{\mathcal{J}} ss'$, then clearly $e \leq_{\mathcal{J}} s$ and $e \leq_{\mathcal{J}} s'$. Conversely, if $e \leq_{\mathcal{J}} s$ and $e \leq_{\mathcal{J}} s'$, there exists elements u, v, x, y of S such that $e = eusve = exs'ye$. In particular, e, eus and $s'ye$ are $\mathcal{D}$-equivalent regular elements of S. But $S \in \mathbf{DS}$, so $euss'ye$ is also $\mathcal{D}$-equivalent to e, and hence $e \leq_{\mathcal{J}} ss'$. Thus φ is a morphism.

Let now X be an element of T, that is, $X \subseteq E(S)$. Then

$$\varphi^{-1}(X) = \big\{s \in S : \forall e \in E(S),\ e \leq_{\mathcal{J}} s \Longleftrightarrow e \in X\big\}.$$

In particular, if $\varphi^{-1}(X) \neq \emptyset$, then X is saturated under the restriction of $\mathcal{J}$ to $E(S)$, and the idempotents of $\varphi^{-1}(X)$ are exactly the $\mathcal{J}$-maximal elements of X. Thus these idempotents are all $\mathcal{D}$-equivalent, which implies $\varphi^{-1}(X) \in \mathbf{LG}$. This proves that (i) implies (iii).

We now verify that (iii) implies (ii). Let $\varphi: S \to T$ be a morphism with $T \in \mathbf{J}_1$, and such that $\varphi^{-1}(t) \in \mathbf{LG}$ for each $t \in T$. Let $e \in E(S)$ and let $s, s' \in S$ such that $e \leq_{\mathcal{J}} s, s'$. Then $\varphi e \leq_{\mathcal{J}} \varphi s, \varphi s'$. But $T \in \mathbf{J}_1$, so we have $\varphi e = \varphi(ess')$. Since $\varphi^{-1}(\varphi e) \in \mathbf{LG}$, $e \in E(S)$ and $ess' \leq_{\mathcal{J}} e$, this implies $e \mathrel{\mathcal{D}} ess'$, and hence $e \leq_{\mathcal{J}} ss'$.

Finally, we verify that (ii) implies (i). Let D be a regular $\mathcal{D}$-class of S, let $s, s' \in D$ and let e be an idempotent in D. Then $e \leq_{\mathcal{J}} s, s'$, and by hypothesis, this implies $e \leq_{\mathcal{J}} ss'$. But $ss' \leq_{\mathcal{J}} s \mathrel{\mathcal{J}} e$, so we have $e \mathrel{\mathcal{D}} ss'$, that is, $ss' \in D$: D is a subsemigroup of S. This completes the proof.□

We now list a few important sub-pseudovarieties of **DS**.

- **DO** is the pseudovariety of all monoids in which each regular $\mathcal{D}$-class is an orthodox subsemigroup, that is, a subsemigroup in which a product of idempotents is idempotent.
- **DG** is the pseudovariety of all monoids in which each regular $\mathcal{D}$-class is a group.
- **J** is the variety of $\mathcal{J}$-trivial monoids, that is, the class of aperiodic elements of **DG**, or the class of all the monoids in which each regular $\mathcal{D}$-class is a singleton.

Then we have

$$\mathbf{J}_1 \subset \mathbf{J} \subset \mathbf{DG} \subset \mathbf{DO} \subset \mathbf{DS}$$

and

$$\begin{aligned}
\mathbf{DO} &= [\![(xy)^\omega(yx)^\omega(xy)^\omega = (xy)^\omega]\!]_{\mathbf{M}}\\
\mathbf{DG} &= [\![(xy)^\omega = (yx)^\omega]\!]_{\mathbf{M}}\\
\mathbf{J} &= [\![x^{\omega+1} = x^\omega]\!]_{\mathbf{DG}}\\
&= [\![(xy)^\omega = (yx)^\omega,\ x^{\omega+1} = x^\omega]\!]_{\mathbf{M}}\\
&= [\![(xy)^\omega x = (xy)^\omega,\ y(xy)^\omega = (xy)^\omega]\!]_{\mathbf{M}}
\end{aligned}$$

2.2. Implicit operations on DS

Since the semigroups of **Nil** are $\mathcal{J}$-trivial and since **Nil** satisfies no nontrivial identity (see section 1.3), it follows that the same holds for **J**. In particular, this means that, for each finite set A, the free monoid A^* is a dense submonoid of $\overline{\Omega}_A\mathbf{J}$, and more generally of $\overline{\Omega}_A\mathbf{V}$ for each pseudovariety $\mathbf{V}$ containing **J**.

The characterization of **DS** given in the previous section has interesting consequences on the free pro-**DS** monoids $\overline{\Omega}_A\mathbf{DS}$. Indeed, by allowing us to consider the content of elements, the passage to these projective limits makes it possible to rewrite the third statement of Proposition 2.1 as follows. Recall that $c: \overline{\Omega}_A\mathbf{DS} \to \mathcal{P}(A) = \overline{\Omega}_A\mathbf{J}_1$ denotes the content morphism.

Proposition 2.2 *Let A be a finite set and let $x, y \in \overline{\Omega}_A\mathbf{DS}$. Then*

$$c(y) \subseteq c(x) \quad\Longrightarrow\quad (x^\omega y x^\omega)^\omega = x^\omega.$$

In particular, the content morphism c is such that, for each $t \in \mathcal{P}(A)$, the semigroup $c^{-1}(t)$ satisfies $(x^\omega y x^\omega)^\omega = x^\omega$: its regular elements form a $\mathcal{J}$-class which is its minimal ideal.

Proof. It suffices to show that, for each morphism $\mu: A^* \to S$ with $S \in \mathbf{DS}$ and for each $x, y \in \overline{\Omega}_A\mathbf{DS}$ such that $c(y) \subseteq c(x)$, we have $\pi(x^\omega y x^\omega)^\omega = \pi x^\omega$. Here π is the canonical morphism $\pi: \overline{\Omega}_A\mathbf{DS} \to S$ extending μ.

By Proposition 2.1, there exists a morphism $\varphi: S \to T$ into a monoid $T \in \mathbf{J}_1$ such that $\varphi^{-1}(t) \in \mathbf{LG}$ for each $t \in T$. Since $T \in \mathbf{J}_1$, the morphism $\varphi \circ \pi$ factors through the content morphism c, that is, there exists a morphism $\psi: \overline{\Omega}_A\mathbf{J}_1 \to T$ such that the following diagram commutes.

$$\begin{tikzcd} \overline{\Omega}_A\mathbf{DS} \arrow[r, "\pi"] \arrow[d, "c"'] & S \arrow[d, "\varphi"] \\ \overline{\Omega}_A\mathbf{J}_1 \arrow[r, "\psi"] & T \end{tikzcd}$$

By hypothesis, $c(x) = c(x^\omega y x^\omega)$, so $\psi\big(c(x)\big) = \psi\big(c(x^\omega y x^\omega)\big)$ and hence $\varphi\big(\pi(x)\big) = \varphi\big(\pi(x^\omega y x^\omega)\big)$. But $\varphi^{-1}\big(\varphi(\pi x)\big) \in \mathbf{LG}$, so $\big((\pi x)^\omega \pi(x^\omega y x^\omega)(\pi x)^\omega\big)^\omega = (\pi x)^\omega$. Now, $(\pi z)^\omega = \pi z^\omega$ for all $z \in \overline{\Omega}_A\mathbf{DS}$, so we have $\pi(x^\omega y x^\omega)^\omega = \pi x^\omega$, thus concluding the proof.$\square$

This result yields a complete description of the relation $\mathcal{J}$ for regular elements.

Corollary 2.3 *Let A be a finite set and let $\mathbf{V}$ be a pseudovariety containing $\mathbf{J}_1$. Then $\mathbf{V} \subseteq \mathbf{DS}$ if and only if*

$$\begin{array}{c} \text{for all } x, y \in \overline{\Omega}_A\mathbf{V}, \text{ if } x \text{ and } y \text{ are regular, then} \\ x \mathrel{\mathcal{J}} y \iff c(x) = c(y) \end{array} \tag{1}$$

For such a pseudovariety, $\overline{\Omega}_A\mathbf{V}$ has $2^{|A|}$ regular $\mathcal{D}$-classes.

Proof. Let us first assume that $\mathbf{V} \subseteq \mathbf{DS}$. Then $\overline{\Omega}_A\mathbf{V}$ is a continuous homomorphic image of $\overline{\Omega}_A\mathbf{DS}$, so the implication of Proposition 2.2 holds in $\overline{\Omega}_A\mathbf{V}$ as well. Let x and y be regular elements of $\overline{\Omega}_A\mathbf{V}$. If $x \mathrel{\mathcal{J}} y$, then $c(x) \mathrel{\mathcal{J}} c(y)$ and hence $c(x) = c(y)$.

Conversely, if $c(x) = c(y)$, then $(x^\omega y x^\omega)^\omega = x^\omega$ by Proposition 2.2. Moreover, by regularity, x can be written as $x = (xz)^\omega x$ for some z. But $\mathbf{DS}$ satisfies $\big((uv)^\omega u\big)^{\omega+1} = (uv)^\omega u$, so $x^{\omega+1} = x$. Thus $x \leq_{\mathcal{J}} y$ and, by symmetry, $x \mathrel{\mathcal{J}} y$.

Let us now assume that $\mathbf{V}$ satisfies (1), and let $x, y \in \overline{\Omega}_A\mathbf{V}$. Then $(xy)^\omega$ and $\big((xy)^\omega(yx)^\omega(xy)^\omega\big)^\omega$ are idempotents with equal content, and hence they are $\mathcal{J}$-equivalent. But $\big((xy)^\omega(yx)^\omega(xy)^\omega\big)^\omega \leq_{\mathcal{H}} (xy)^\omega$, so

$$\mathbf{V} \models \big((xy)^\omega(yx)^\omega(xy)^\omega\big)^\omega = (xy)^\omega$$

and hence $\mathbf{V} \subseteq \mathbf{DS}$.$\square$

2.3. Multiplicity of subwords and regular elements of $\overline{\Omega}_A\mathbf{DS}$

Let us now pay closer attention to the regular elements of $\overline{\Omega}_A\mathbf{DS}$. We first notice the following general property of regular elements of relatively free profinite monoids.

Lemma 2.4 *Let $\mathbf{V}$ be a pseudovariety, let A be a finite set, and let $x \in \overline{\Omega}_A\mathbf{V}$. Then x is regular if and only if, for each A-generated monoid $\mu: A \to S$ in $\mathbf{V}_A$, πx is a regular element of S, where $\pi: \overline{\Omega}_A\mathbf{V} \to S$ is the corresponding canonical morphism.*

Proof. If x is regular, then $x = xyx$ for some $y \in \overline{\Omega}_A\mathbf{V}$. Therefore, if $\pi: \overline{\Omega}_A\mathbf{V} \to S$ is a morphism, then $\pi x = \pi x \; \pi y \; \pi x$, so πx is regular.

We now prove the converse implication. Up to the identification of isomorphic elements, $\mathbf{V}_A$ has only countably many elements $\mu_n: A \to S_n$ $(n \geq 0)$. For each $n \geq 0$, we let $\mu'_n: A \to S_1 \times \cdots \times S_n$ be the direct product of $\mu_1, \ldots, \mu_n$, we let $\pi_n: \overline{\Omega}_A\mathbf{V} \to S_1 \times \cdots \times S_n$ be the corresponding canonical morphism, and we pick $y_n \in \overline{\Omega}_A\mathbf{V}$ such that $\pi_n(xy_nx) = \pi_n x$. By compactness, the sequence $(y_n)_{n\geq 0}$ admits a convergent subsequence $(y_{k_n})_n$. Let $y = \lim_n y_{k_n}$. Then, for each $n \geq 0$, $\pi_{k_n} y = \pi_{k_n} y_m$ for all m large enough, and hence $\pi_{k_n}(xyx) = \pi_{k_n} x$. But the definition of the π_n then implies that, for each $n \geq 0$, $\pi_n(xyx) = \pi_n x$. Therefore $xyx = x$ and x is regular.□

Recall that a word $u = a_1 \cdots a_r \in A^+$ $(a_i \in A)$ is a subword of a word $v \in A^*$ if v can be factored as $v = v_0a_1v_1 \cdots a_rv_r$ $(v_i \in A^*)$. Thus, if we consider words as finite sequences of letters, the subwords of a word v are given by its subsequences.

If $u, v \in A^*$ and $u \neq 1$, we let

$$\begin{bmatrix} v \\ u \end{bmatrix} = \max\{r \geq 0 : u^r \text{ is a subword of } v\}.$$

Example. $\begin{bmatrix} abaab \\ a \end{bmatrix} = 3$ and $\begin{bmatrix} abaab \\ ab \end{bmatrix} = 2$.

For each word $u = a_1 \cdots a_k \in A^*$ $(a_i \in A)$, we let $L(u)$ be the language $L(u) = A^*a_1A^* \cdots a_kA^*$ of all words admitting u for a subword. By definition of $\begin{bmatrix} v \\ u \end{bmatrix}$, we have

$$\begin{bmatrix} v \\ u \end{bmatrix} \geq r \Longleftrightarrow v \in L(u^r).$$

We will use the following observation.

Lemma 2.5 *For each finite set A and for each word $u \in A^*$, the language $L(u)$ is $\mathbf{J}$-recognizable.*

Proof. Let $u \in A^+$ and let k be the length of u. Let also $\sim$ be the syntactic congruence of $L(u)$ and let $x, y \in A^*$. To prove that the syntactic monoid of $L(u)$ is in $\mathbf{J}$, it suffices to prove that $x^{k+1} \sim x^k$ and $(xy)^k \sim (yx)^k$.

Let $r, s \in A^*$. Since rx^ks is a subword of $rx^{k+1}s$, $rx^ks \in L(u)$ implies $rx^{k+1}s \in L(u)$. Conversely, if $rx^{k+1}s \in L(u)$, then we can view u as a subsequence of $rx^{k+1}s$ using at most one letter of each of the $k+1$ factors x, and leaving one unused. Therefore, u is also a subword of rx^ks, and $rx^ks \in L(u)$. Thus, $x^{k+1} \sim x^k$.

Similarly, if $r(xy)^ks \in L(u)$, we can view u as a subsequence of $r(xy)^ks$ using at most one letter in each factor xy. This letter can then be found in yx as well, and hence u is a factor of $r(yx)^ks$ too. By symmetry, this shows that $r(xy)^ks \in L(u)$ if and only if $r(yx)^ks \in L(u)$, and hence $(xy)^k \sim (yx)^k$.□

The computation of multiplicity of subwords is directly relevant for the determination of the regular elements of monoids in $\mathbf{DS}$.

Proposition 2.6 *Let A be a finite set and let $u, v \in A^+$ be words with equal content. If $S \in \mathbf{DS}$ and $\begin{bmatrix} v \\ u \end{bmatrix} > |S|$, then, for each morphism $\mu: A^* \to S$, μv is regular. That is, $S \models v^{\omega+1} = v$.*

Proof. It is possible to factor v as $v = v_1 \cdots v_k$ with $k = \left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} v_i \\ u \end{smallmatrix}\right] = 1$ for each $1 \leq i \leq k$. For each morphism $\mu \colon A^* \to S$, there exist $0 < m < n \leq k$ such that $\mu(v_1 \cdots v_m) = \mu(v_1 \cdots v_n)$. Therefore $\mu(v_1 \cdots v_m) = \mu(v_1 \cdots v_m)\, \mu(v_{m+1} \cdots v_n)^\ell$ for all $\ell \geq 0$, and hence

$$\begin{aligned} \mu(v_1 \cdots v_m) &= \mu(v_1 \cdots v_m)\, \mu(v_{m+1} \cdots v_n)^{\omega+1} \\ \mu v &= \mu(v_1 \cdots v_m)\, \mu(v_{m+1} \cdots v_n)^{\omega+1}\, \mu(v_{n+1} \cdots v_k). \end{aligned}$$

But the v_i all have the same content, equal to $c(u)$, and $\mu(v_{m+1} \cdots v_n)^{\omega+1}$ is regular. Since $\mu v \leq_{\mathcal{J}} \mu(v_{m+1} \cdots v_n)^{\omega+1}$, it follows from Proposition 2.2 that μv is regular.□

Moreover, for each non-empty word u, it is possible to extend the definition of $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ to all implicit operations x on $\mathbf{M}$.

Proposition 2.7 *Let $\mathbf{V}$ be a pseudovariety containing $\mathbf{J}$. For each finite set A and for each $u \in A^+$, the mapping $v \mapsto \left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right]$ is uniformly continuous for the pro-$\mathbf{V}$ topology of A^*, and hence can be extended in a unique fashion to a continuous mapping*

$$\begin{aligned} \overline{\Omega}_A \mathbf{V} &\longrightarrow \mathbb{N} \cup \{\infty\} \\ x &\longmapsto \begin{bmatrix} x \\ u \end{bmatrix} \end{aligned}$$

Proof. It suffices to show that, for each $r \geq 0$, there exists $\varepsilon > 0$ such that, for all $v, v' \in A^*$,

$$d(v, v') < \varepsilon \quad \Longrightarrow \quad \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} v' \\ u \end{bmatrix} \text{ or } \begin{bmatrix} v \\ u \end{bmatrix}, \begin{bmatrix} v' \\ u \end{bmatrix} \geq r,$$

where d denotes the pro-$\mathbf{V}$ distance on A^* (see section 1.2).

For each $r \geq 0$, let n_r be the cardinality of the syntactic monoid of $L(u^r)$. Recall that this monoid is in $\mathbf{J}$. Therefore, if $d(v, v') < 2^{n_r}$, then v and v' are syntactically equivalent vis-à-vis $L(u^r)$, and hence $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ u \end{smallmatrix}\right] \geq r$ or $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} v' \\ u \end{smallmatrix}\right] < r$. It is now easy to conclude the proof.□

Note. Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties such that $\mathbf{J} \subseteq \mathbf{V} \subset \mathbf{W}$, and let π be the natural projection from $\overline{\Omega}_A \mathbf{W}$ onto $\overline{\Omega}_A \mathbf{V}$. Let $u \in A^+$. Both mappings $x \mapsto \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$, defined respectively on $\overline{\Omega}_A \mathbf{V}$ and $\overline{\Omega}_A \mathbf{W}$, are continuous extensions of the same mapping defined on A^*. So for all $x \in \overline{\Omega}_A \mathbf{W}$, $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} \pi x \\ u \end{smallmatrix}\right]$.

This leads to the following important characterization of the regular elements of $\overline{\Omega}_A \mathbf{J}$ and $\overline{\Omega}_A \mathbf{DS}$. We say that a word in A^* is *linear* if no letter occurs more than once in it. There are only finitely many linear words in A^*.

Corollary 2.8 *Let $\mathbf{V}$ be a pseudovariety containing $\mathbf{J}$ and contained in $\mathbf{DS}$. Let A be a finite set and let $x \in \overline{\Omega}_A \mathbf{V}$. Then the following are equivalent:*

i) x is regular;

ii) for each word $u \in A^+$, $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \{0, \infty\}$;

iii) for each linear word $u \in A^+$, $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \{0, \infty\}$.

Furthermore, if x is regular and $u \in A^+$, then $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \infty$ if and only if $c(u) \subseteq c(x)$.

Proof. Let x be a regular element of $\overline{\Omega}_A\mathbf{V}$. Then $x = xyx$ for some y, so that $x = (xy)^k x$ for all $k \geq 0$. Let $u \in A^+$ and let $(u_n)_n$ and $(v_n)_n$ be sequences of words converging respectively to x and y in $\overline{\Omega}_A\mathbf{V}$. By continuity, we have

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} (xy)^k x \\ u \end{bmatrix} = \lim_n \begin{bmatrix} (u_n v_n)^k u_n \\ u \end{bmatrix} \quad \text{for all } k.$$

But it is not difficult to see that $\left[\begin{smallmatrix} (u_n v_n)^k u_n \\ u \end{smallmatrix}\right] \geq (k+1)\left[\begin{smallmatrix} u_n \\ u \end{smallmatrix}\right]$, so $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \geq k\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ for all k, and hence either $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = 0$ or $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \infty$. This proves that (i) implies (ii). Similarly, if $c(u) \subseteq c(x)$ and u is of length k, then $\left[\begin{smallmatrix} (u_n v_n)^{k-1} u_n \\ u \end{smallmatrix}\right] \geq 1$ for all sufficiently large n, and hence $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \infty$.

It is clear that (ii) implies (iii). Let us now assume that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \{0, \infty\}$ for each linear word u in A^+, and let $(v_n)_{n \geq 0}$ be a sequence of words such that $x = \lim_n v_n$. We may assume the v_n to have all the same content as x. Then, up to extracting a subsequence of $(v_n)_n$, we can find a linear word u with the same content as x which is a subword of all the v_n. Indeed, there are only finitely many such linear words, so one of them is a subword of infinitely many v_n. For this word u, $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \lim_n \left[\begin{smallmatrix} v_n \\ u \end{smallmatrix}\right] \geq 1$, so $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \infty$.

Let now $\mu: A^* \to S$ be a morphism with $S \in \mathbf{V}$. Since $\mathbf{J} \subseteq \mathbf{V}$, A^* is a dense submonoid of $\overline{\Omega}_A\mathbf{V}$. We write also μ for the induced morphism $\mu: \overline{\Omega}_A\mathbf{V} \to S$. For all n large enough, $\left[\begin{smallmatrix} v_n \\ u \end{smallmatrix}\right] > |S|$ and $\mu v_n = \mu x$, so μx is regular by Proposition 2.6. This implies that x is regular in $\mathbf{DS}$ by Lemma 2.4.□

2.4. Factorization of implicit operations on DS

We are now ready to prove that each implicit operation on $\mathbf{DS}$ is a product of explicit operations and of regular implicit operations: that is, for each finite set A, the monoid $\overline{\Omega}_A\mathbf{DS}$ is generated by its regular elements and by the letters of A. It is immediate that this property is inherited by the monoids $\overline{\Omega}_A\mathbf{V}$ with $\mathbf{V} \subset \mathbf{DS}$, since these monoids are homomorphic images of $\overline{\Omega}_A\mathbf{DS}$.

Let A be a fixed alphabet and let π be the natural projection of $\overline{\Omega}_A\mathbf{M}$ onto $\overline{\Omega}_A\mathbf{DS}$. In terms of implicit operations, if $x \in \overline{\Omega}_A\mathbf{M}$, then πx is the restriction of x to $\mathbf{DS}$. For each $x \in \overline{\Omega}_A\mathbf{M}$, we let

$$\varphi x = \mathrm{Card}\Big\{u \in A^* : u \text{ linear}, \begin{bmatrix} x \\ u \end{bmatrix} \neq 0\Big\}.$$

First we notice that, for each $a \in c(x)$, $\left[\begin{smallmatrix} x \\ a \end{smallmatrix}\right] \geq 1$. Indeed, by continuity of the content morphism c, there exists a sequence $(v_n)_n$ of words converging to x in $\overline{\Omega}_A\mathbf{M}$ and such that $c(v_n) = c(x)$ for all n. For each n, $\left[\begin{smallmatrix} v_n \\ a \end{smallmatrix}\right] \geq 1$, and by continuity, $\left[\begin{smallmatrix} x \\ a \end{smallmatrix}\right] \geq 1$. Therefore, $\varphi x = 0$ if and only if $c(x) = \emptyset$, that is, if and only if $x = 1$, in which case x is regular.

Let us now consider $n \geq 1$, let us assume that if $x \in \overline{\Omega}_A\mathbf{M}$ and $\varphi x < n$, then x is a product of letters and of implicit operations whose restriction to $\mathbf{DS}$ is regular, and let us consider $x \in \overline{\Omega}_A\mathbf{M}$ such that $\varphi x = n$. Two cases arise. If $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \infty$ for each linear word u such that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \neq 0$, then πx is regular by Corollary 2.8.

Otherwise, there exists a linear word u such that $r = \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \notin \{0, \infty\}$. By continuity of the mapping $x \mapsto \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$, there exists a sequence of words $(v_n)_{n \geq 0}$ converging to x in $\overline{\Omega}_A\mathbf{M}$, and such that $\left[\begin{smallmatrix} v_n \\ u \end{smallmatrix}\right] = r$ for all $n \geq 0$. Thus, if $u^r = a_1 a_2 \cdots a_k$ $(a_i \in A)$, each v_n can be factored as

$$v_n = v_{n,0} a_1 v_{n,1} \cdots a_k v_{n,k} \qquad (v_{n,i} \in A^*).$$

Furthermore, this factorization can be chosen to show the leftmost occurrence of $a_1 \cdots a_k$ as a subword of v_n, that is, we can assume that, for each $1 \leq i \leq k$, the letter a_i does not occur in $v_{n,i-1}$. Now, by compactness and up to extracting a subsequence of $(v_n)_n$, we may assume that each sequence $(v_{n,i})_n$ converges in $\overline{\Omega}_A\mathbf{M}$: let x_i be its limit. Then

$$x = \lim_n v_n = x_0 a_1 x_1 \cdots a_k x_k.$$

By construction, the word u is not a subword of any of the $v_{n,i}$ $(0 \leq i < k)$ since one of its letters does not occur in $v_{n,i}$. It is not a subword of $v_{n,k}$ either since $\left[\begin{smallmatrix} v_n \\ u \end{smallmatrix}\right] = r$. Therefore, for each $0 \leq i \leq k$, $\left[\begin{smallmatrix} x_i \\ u \end{smallmatrix}\right] = 0$. On the other hand, let u' be a linear word such that $\left[\begin{smallmatrix} x_i \\ u' \end{smallmatrix}\right] \neq 0$. Then u' is a subword of almost all the $v_{n,i}$, and hence of almost all the v_n. Therefore $\left[\begin{smallmatrix} x \\ u' \end{smallmatrix}\right] \neq 0$. Thus, for each $0 \leq i \leq k$, $\varphi x_i < \varphi x$.

So, using induction on φx, we can show that x is a product of letters and of implicit operations whose restriction to $\mathbf{DS}$ is regular.

We can make this statement more precise by noticing the following. Let y be a regular element of $\overline{\Omega}_A\mathbf{DS}$ and let z be an element of $\overline{\Omega}_A\mathbf{DS}$ whose content is contained in that of y. Then yz, zy and y have the same content and, by Proposition 2.2, yz and zy are regular as well.

Theorem 2.9 *Let A be a finite set. Then each element of $\overline{\Omega}_A\mathbf{M}$ admits a factorization of the form*

$$x = u_0 x_1 u_1 \cdots x_r u_r$$

where

- *for each $0 \leq i \leq r$, u_i is in A^*,*
- *for each $1 \leq i \leq r$, the restriction of x_i to $\mathbf{DS}$ is regular,*
- *for each $1 \leq i \leq r$ (resp. $0 \leq i < r$) such that u_i is not the empty word, the first (resp. last) letter of u_i does not lie in the content of x_i (resp. u_{i+1}),*
- *for each $1 \leq i \leq r$ such that $u_i = 1$, the contents of u_{i-1} and u_i are $\subseteq$-incomparable.*□

Corollary 2.10 *Let $\mathbf{V}$ be a pseudovariety contained in $\mathbf{DS}$ and let A be a finite set. Then $\overline{\Omega}_A\mathbf{V}$ is generated by its regular elements and by the letters of A.*□

2.5. Implicit operations on DO

In the case of sub-pseudovarieties of $\mathbf{DO}$, we can go further in the description of the monoids of implicit operations, and in particular of their regular elements. We show that these elements are determined by their idempotent power and by their restriction to groups.

Lemma 2.11 *Let $S \in \mathbf{DO}$, let $e \in E(S)$, and let $s, t \in S$ such that $e \leq_{\mathcal{J}} s, t$. Then $este = esete$.*

Proof. By Proposition 2.1, es and te are group elements in the $\mathcal{J}$-class of e. So $este = es(es)^{\omega}(te)^{\omega}te$. But $(es)^{\omega}(te)^{\omega}$ is idempotent since $S \in \mathbf{DO}$ and $(es)^{\omega}(te)^{\omega} \ \mathcal{H}\ e$. So $(es)^{\omega}(te)^{\omega} = e$, and $este = esete$.□

Proposition 2.12 *Let $\mathbf{V}$ be a pseudovariety contained in $\mathbf{DO}$ and let $\mathbf{H} = \mathbf{V} \cap \mathbf{G}$. Let A be a finite set and let $x, y \in \overline{\Omega}_A\mathbf{V}$. If x and y are regular, then $x = y$ if and only if $x^{\omega} = y^{\omega}$ and $\mathbf{H} \models x = y$.*

Proof. If $\mathbf{V} \subseteq \mathbf{G}$, then $\mathbf{V} = \mathbf{H}$ and the proposition is trivial. We now suppose that $\mathbf{V}$ is not a pseudovariety of groups, and hence, that $\mathbf{J}_1 \subseteq \mathbf{V}$. Let $x \in \overline{\Omega}_A\mathbf{V}$ and let $S \in \mathbf{V}$. There exists a word $u \in A^*$ with content $c(u) = c(x)$, and such that x coincides with the explicit operation u on S, that is, $x_S = u_S$.

Let $s = (s_a)_{a\in A} \in S^A$. Then $x_S(s) = u_S(s)$ is a regular element of S: let $e = x_S^{\omega}(s)$ and let H be its $\mathcal{H}$-class. In particular, $x_S(s) = e\, x_S(s)\, e = e\, u_S(s)\, e \in H$. Moreover, e is equal to a certain product of the s_a ($a \in c(x)$), so $e \leq_{\mathcal{J}} s_a$ for each $a \in c(x)$.

Let $s' = (s'_a)_{a\in A} \in S^A$ be given by $s'_a = es_ae$ if $a \in c(x)$ and $s'_a = e$ otherwise. Using the fact that u is an explicit operation of content $c(x)$, Lemma 2.11 implies $e\, u_S(s)\, e = u_S(s')$, and hence $x_S(s) = x_S(s')$. Now we observe that $es_ae \in H$ for each $a \in c(x)$, so that $x_S(s') = x_H(s')$. Thus, x_S is determined by x_S^{ω} and by the restrictions of x to the subgroups of S.□

In section 4, we will be interested in certain sub-pseudovarieties of $\mathbf{DG}$. For these pseudovarieties, the above proposition yields the following.

Corollary 2.13 *Let $\mathbf{V}$ be a pseudovariety containing $\mathbf{J}_1$ and contained in $\mathbf{DG}$, and let $\mathbf{H} = \mathbf{V} \cap \mathbf{G}$. Let A be a finite set and let $x, y \in \overline{\Omega}_A\mathbf{V}$. If x and y are regular, then $x = y$ if and only if $c(x) = c(y)$ and $\mathbf{H} \models x = y$.*

Proof. This is immediate in view of Proposition 2.12 above, since $x^{\omega} = y^{\omega}$ if and only if $c(x) = c(y)$ (Corollary 2.3).□

2.6. Implicit operations on J

In the case of aperiodic elements of $\mathbf{DG}$, that is, of $\mathcal{J}$–trivial monoids, the regular elements are all idempotent, so Corollary 2.13 (or, alternately, Corollary 2.3) implies that the idempotents are determined by their contents. In particular, each idempotent is the ω-power of a linear word (in fact, of any word with the appropriate content). Moreover, since each element of $\overline{\Omega}_A\mathbf{J}$ is a product of letters and of idempotents by Theorem 2.9, it follows that $\overline{\Omega}_A\mathbf{J}$ is countable.

In fact, we can state a unique factorization result for the free pro-$\mathbf{J}$ monoids. Its proof is too long to be included here, and we refer the reader to Almeida's [6] or [5].

Theorem 2.14 *Let A be a finite set.*

a) If $x, y \in \overline{\Omega}_A\mathbf{J}$, then $x = y$ if and only if $\binom{x}{u} = \binom{y}{u}$ for each word $u \in A^+$.

b) If $x \in \overline{\Omega}_A\mathbf{M}$ and if $x = u_0x_1u_1 \cdots x_ru_r$ and $x = v_0y_1v_1 \cdots y_sv_s$ are factorizations as in Theorem 2.9 (with respect to $\mathbf{J}$), then $r = s$, $u_i = v_i$ for all $1 \le i \le r$, and $\mathbf{J} \models x_i = y_i$, that is, $c(x_i) = c(y_i)$ for all $0 \le i \le r$.

c) $\overline{\Omega}_A\mathbf{J}$ is the free monoid with a unary operation $x \mapsto x^\omega$ over A in the variety defined by the identities $(xy)^\omega = (yx)^\omega = (x^\omega y^\omega)^\omega$ and $x^\omega x = xx^\omega = (x^\omega)^\omega = x^\omega$. Moreover, the word problem in $\overline{\Omega}_A\mathbf{J}$ (viewed as such a (2,1)-algebra) is decidable.

As a consequence of this remarkable result, we obtain Simon's theorem on the combinatorial description of $\mathbf{J}$-recognizable languages. Recall that, for a word $u = a_1 \cdots a_n \in A^*$, the language $L(u) = A^*a_1A^* \cdots a_nA^*$ is $\mathbf{J}$-recognizable (Lemma 2.5).

Theorem 2.15 *Let A be a finite set. The Boolean algebra of all $\mathbf{J}$-recognizable languages in A^* is generated by the $L(u)$ ($u \in A^+$).*

Proof. Recall that a word $v \in A^*$ lies in $L(u)$ ($u \in A^+$) if and only if $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right] \ge 1$. By continuity of the mapping $x \mapsto \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ on $\overline{\Omega}_A\mathbf{J}$, it follows that an element x of $\overline{\Omega}_A\mathbf{J}$ lies in $\overline{L(u)}$ if and only if $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \ge 1$. Therefore statement (1) of Theorem 2.14 above implies that the points of $\overline{\Omega}_A\mathbf{J}$ are separated by the closures of the languages $L(u)$ ($u \in A^+$). By Proposition 1.13, this suffices to prove the theorem.$\square$

3. Relatively free profinite groups

Cayley graphs provide a convenient tool for working with relatively free profinite groups such as the groups $\overline{\Omega}_A\mathbf{H}$ of implicit operations on a pseudovariety $\mathbf{H}$ of groups. Since not only the algebraic structure of $\overline{\Omega}_A\mathbf{H}$ is important, we will need to deal with some additional structure besides the usual combinatorial Cayley graphs. We therefore start by reviewing some basic notions and results of the theory of profinite graphs and then pursue to study specifically profinite Cayley graphs of profinite groups.

Section 3.1 introduces profinite graphs in general. Lemmas 3.3 and 3.4 are taken from Zalesskiĭ and Mel'nikov [27] and Ribes and Zalesskiĭ [23], respectively, where (homologically defined) tree-like properties of profinite graphs were investigated. Cayley graphs of some $\mathbf{H}$-free profinite groups were shown by Gildenhuys and Ribes [16] (see also Ribes [22]) to be $\mathbb{A}$-trees for a naturally associated profinite ring (of homological coefficients). The complete characterization of the pseudovarieties of groups $\mathbf{H}$ with this property, due to Almeida and Weil [9], is the theme of section 3.2. Such pseudovarieties of groups, called arborescent, turn out to be particularly amenable to syntactical arguments since certain "reduced factorizations" have a kind of uniqueness property in this case, a topic which is considered in detail in section 3.3.

The main result of section 3.3 (Theorem 3.14) and its applications (Propositions 3.15 and 3.16) were first proved in Almeida and Weil [9]; the proof presented here, in terms of cuts, which is developed in sections 3.1 and 3.3, reformulates in a more geometrical language — and somewhat simplifies — the original proof.

3.1. Profinite graphs

By a *graph* we mean a pair $G = (V, E)$ of disjoint sets endowed with two mappings $\alpha, \omega : E \to V$. The elements of V are called *vertices* and the elements of E are called *edges*. We often write $G = V \cup E$ when the context makes clear which elements are vertices and which are edges. For $e \in E$, αe and ωe are called, respectively, the *beginning* and the *end* vertices of the edge e and we often draw the picture

$$\alpha e \cdot \xrightarrow{\ e\ } \cdot\, \omega e$$

to represent this situation. The vertices αe and ωe are said to be the *extremities* of e and to be *joined* by e. Gluing all these pictures via the identification of vertices with the same name, we obtain the usual graphical representation of G.

A *morphism of graphs* $\varphi : (V, E) \to (V', E')$ is given by two functions $\varphi_1 : V \to V'$ and $\varphi_2 : E \to E'$ which respect the operations in the following sense: for each $e \in E$, $\chi\varphi_2 e = \varphi_1\chi e$ ($\chi \in \{\alpha, \omega\}$). A *quotient* of a graph G is a morphic image of G. Just as in section 1.1, where total algebras were considered, profinite graphs are defined as projective limits of finite graphs. In particular, in a profinite graph, the vertices and the edges each constitute topological spaces which are compact and zero-dimensional and the mappings α and ω are continuous. Whenever we consider graph (quotient) morphisms defined on profinite graphs, we will be referring to continuous graph morphisms.

Let $\mathbb{A}$ be a profinite commutative ring with 1. Viewing $\mathbb{A}$-modules as (universal) algebras of a suitable type, we may also consider profinite $\mathbb{A}$-modules, i.e., projective limits of finite $\mathbb{A}$-modules, as just another case of profinite (universal) algebras. We denote by $\mathbb{A}[S]$ the free profinite $\mathbb{A}$-module over the profinite set S. In particular, if S is finite, then $\mathbb{A}[S]$ is a direct sum of $|S|$ copies of $\mathbb{A}$. In general, if $S = \varprojlim S_i$, then $\mathbb{A}[S] = \varprojlim \mathbb{A}[S_i]$ and the submodule of $\mathbb{A}[S]$ generated by S, which is the free $\mathbb{A}$-module freely generated by S, is only a dense submodule of $\mathbb{A}[S]$. Thus, the elements of $\mathbb{A}[S]$ are limits of (Cauchy) sequences of formal linear combinations of elements of S with coefficients in $\mathbb{A}$.

Let now $G = (V, E)$ be a profinite graph. We define two morphisms of profinite $\mathbb{A}$-modules by

$$\begin{array}{rcl} \partial : \mathbb{A}[E] & \longrightarrow & \mathbb{A}[V] \\ e \in E & \longmapsto & \omega e - \alpha e \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \varepsilon : \mathbb{A}[V] & \longrightarrow & \mathbb{A} \\ v \in V & \longmapsto & 1 \end{array}$$

and we consider the sequence

$$\mathbb{A}[E] \xrightarrow{\ \partial\ } \mathbb{A}[V] \xrightarrow{\ \varepsilon\ } \mathbb{A}$$

where we note that $\operatorname{Im}\partial \subseteq \operatorname{Ker}\varepsilon$. The corresponding homology $\mathbb{A}$-modules are

$$H_0(G, \mathbb{A}) = \operatorname{Ker}\varepsilon / \operatorname{Im}\partial \quad \text{and} \quad H_1(G, \mathbb{A}) = \operatorname{Ker}\partial.$$

If $\varphi : G \to G'$ is a morphism of profinite graphs then there are associated morphisms of profinite $\mathbb{A}$-modules $\mathbb{A}[E] \to \mathbb{A}[E']$ and $\mathbb{A}[V] \to \mathbb{A}[V']$ and the following diagram commutes

$$\begin{array}{ccccc} \mathbb{A}[E] & \xrightarrow{\partial_G} & \mathbb{A}[V] & \xrightarrow{\varepsilon_G} & \mathbb{A} \\ \downarrow & & \downarrow & & \| \\ \mathbb{A}[E'] & \xrightarrow{\partial_{G'}} & \mathbb{A}[V'] & \xrightarrow{\varepsilon_{G'}} & \mathbb{A} \end{array}$$

It is then just an easy diagram chasing exercise to show that there are also induced morphisms $H_0(G,\mathbb{A}) \to H_0(G',\mathbb{A})$ and $H_1(G,\mathbb{A}) \to H_1(G',\mathbb{A})$, that the first one is onto if φ is onto, that the second one is one-to-one if φ is one-to-one, and that the functors $H_0(\_,\mathbb{A})$ and $H_1(\_,\mathbb{A})$ preserve projective limits.

Now, for a finite graph G, we claim that $H_0(G,\mathbb{A}) = 0$ if and only if G is connected in the usual (undirected) sense. Indeed, if G is not connected, then G has a quotient graph G' with two vertices and no edges except possibly loops; whence, for G', $\operatorname{Im}\partial = 0 \neq \operatorname{Ker}\varepsilon$ and, since the associated mapping $H_0(G,\mathbb{A}) \to H_0(G',\mathbb{A})$ is onto, we deduce that $H_0(G,\mathbb{A}) \neq 0$. Conversely, if G is connected, fixing a vertex v_0, we may define a distance function $d(v)$ on V as the length of the shortest undirected path from v_0 to v. If $\sum_{v\in V}\lambda_v = 0$, i.e., $\sum_{v\in V}\lambda_v v \in \operatorname{Ker}\varepsilon$, and $v_1 \in V$ is such that $d(v_1)$ is maximal among the vertices v for which $\lambda_v \neq 0$, then there is a vertex v_1' at smaller distance from v_0 and an edge e_1 with $\{\alpha e_1, \omega e_1\} = \{v_1, v_1'\}$. Then $\pm\partial(\lambda_{v_1}e_1) = \lambda_{v_1}v_1 - \lambda_{v_1}v_1'$ and so $\sum_{v\in V}\lambda_v v - \big(\pm\partial(\lambda_{v_1}e_1)\big)$ is an element of $\operatorname{Ker}\varepsilon$ in which the coefficient of v_1 is 0. An easy inductive scheme then shows that $\sum_{v\in V}\lambda_v v \in \operatorname{Im}\partial$ and proves the above claim. Since the functor $H_0(\_,\mathbb{A})$ transforms onto morphisms into onto morphisms, it follows that, for a profinite graph G, $H_0(G,\mathbb{A}) = 0$ if and only if all finite quotients of G are connected. We then say that G is *profinitely connected*.

On the other hand, for an arbitrary graph G, if $(e_1,\dots,e_n)$ is an undirected path from v_1 to v_2, then $\partial(\sum_{i=1}^n \pm e_i) = v_2 - v_1$ where an edge is affected of the minus sign if it appears in the wrong direction in the path. So, if G has (finite) cycles, then $H_1(G,\mathbb{A}) \neq 0$. Conversely, if $\sum_{e\in E}\lambda_e e \in \operatorname{Ker}\partial \setminus \{0\}$ and $\lambda_{e_1} \neq 0$, then, as $\sum_{e\in E}\lambda_e(\omega e - \alpha e) = 0$, there is some $e_2 \in E\setminus\{e_1\}$ such that $\lambda_{e_2} \neq 0$ and e_2 has one of its ends equal to, say, ωe_1. Again an appropriate induction scheme allows us to construct arbitrarily long undirected paths in G. Thus, in case G is finite, it follows that G has cycles. Hence a finite graph G is acyclic if and only if $H_1(G,\mathbb{A}) = 0$. Since $H_1(\_,\mathbb{A})$ preserves projective limits, if G is a projective limit of acyclic finite graphs, then $H_1(G,\mathbb{A}) = 0$. However, the converse is not true as the examples in the next section will show. Basically, this is because a quotient of an acyclic graph is not necessarily acyclic.

We say that a profinite graph G is $\mathbb{A}$-*acyclic* if $H_1(G,\mathbb{A}) = 0$ and that G is an $\mathbb{A}$-*tree* if G is profinitely connected and $\mathbb{A}$-acyclic. Since $H_1(\_,\mathbb{A})$ transforms one-to-one morphisms into one-to-one morphisms, a closed subgraph of an $\mathbb{A}$-acyclic graph is also $\mathbb{A}$-acyclic.

The *degree* of a vertex v of a graph G is the number of edges of G with an extremity at v. The *(vertex) degree* of a graph G is the supremum of the degrees of its vertices in case all these degrees are finite, and is ∞ otherwise. We next use more or less standard compactness arguments to show that profinitely connected graphs of finite degree must have long paths.

Lemma 3.1 *Let* $G = \varprojlim G_n$ *be the projective limit of a projective sequence* $G_1 \leftarrow G_2 \leftarrow \cdots$ *of finite graphs in which the projection morphisms are onto. Then the following conditions are equivalent:*

i) G *has finite degree;*

ii) all the vertices of G *have finite degree;*

iii) there is a bound for the degrees of the G_n $(n \geq 1)$.

Proof. Clearly (i)$\Longrightarrow$(ii) by definition of degree of a graph.

If G has a vertex of degree n, then some G_k must also have a vertex of degree n since the edges of G are separated by the morphisms onto the G_k. Hence (iii)$\Longrightarrow$(i).

Finally, suppose there is no bound for the degrees of the vertices of the G_n $(n \geq 1)$. Then we may assume that, for each $n \geq 1$, G_n has some vertex v_n of degree $\geq n$. Let $\psi_n : G \to G_n$ be the canonical projection and let $\varphi_{mn} : G_m \to G_n$ $(m > n)$ be the onto morphism associated with the projective sequence, so that $\psi_n = \varphi_{mn} \circ \psi_m$. Then, for each $n \geq 1$, there are n distinct edges $e_{n1}, \ldots, e_{nn}$ of G such that, for $i = 1, \ldots, n$, $\psi_n e_{ni}$ has v_n as one of its extremities.

The sequence $(e_{n1})_{n\geq 1}$ of edges of the profinite graph G has some convergent subsequence $(e_{n_k 1})_{k\geq 1}$, say with limit e_1. In turn, the sequence $(e_{n_k 2})_{k\geq 2}$ has some convergent subsequence $(e_{n_{k_l} 2})_{l\geq 1}$, say with limit e_2. Inductively, for each positive integer p, there is a strictly increasing sequence $(n_r)_{r\geq 1}$ of indices such that each of the sequences $(e_{n_r i})_{r\geq i}$ $(i = 1, \ldots, p)$ converges, say to the respective edges e_i. Moreover, this choice can be made so that, at each step, the already chosen edges $e_1, \ldots, e_{p-1}$ are preserved and so we obtain an infinite sequence $e_1, e_2, \ldots$ of edges.

For a fixed p, the convergence of the above sequences implies that, for every k, there is r such that, for each $i \in \{1, \ldots, p\}$, $\psi_k e_{n_r i} = \psi_k e_i$, whence $\psi_k e_i = \psi_k e_{n_r i} = \varphi_{n_r k} \psi_{n_r} e_{n_r i}$. Since the edges $\psi_{n_r} e_{n_r i}$ $(i = 1, \ldots, p)$ have the common extremity v_{n_r}, the edges $\psi_k e_i$ $(i = 1, \ldots, p)$ also have a common extremity. Being true for all $k \geq 1$, this forces $e_1, \ldots, e_p$ to have a common extremity. Since p is arbitrary, this shows that G has infinite degree, thereby proving (ii)$\Longrightarrow$(iii).□

The *diameter* of a graph is the supremum of the lengths of finite paths with no repeated vertices.

Proposition 3.2 *Let* G *be an infinite profinitely connected graph of finite degree. Then* G *contains some infinite path without repeated vertices.*

Proof. Let $G = \varprojlim G_n$, a projective limit of a projective sequence of finite connected graphs in which the projection morphisms are onto. Let ψ_n and φ_{mn} be morphisms as in the proof of Lemma 3.1. Note that, since G is infinite, the sizes of the graphs G_n $(n \geq 1)$ are not bounded. By Lemma 3.1, the G_n $(n \geq 1)$ have bounded degrees and, therefore, have unbounded diameters and so we may assume that each G_n has diameter $\geq n$. For each $n \geq 1$, choose edges $e_{n1}, \ldots, e_{nn}$ of G such that $(\psi_n e_{n1}, \ldots, \psi_n e_{nn})$ is a path without repeated vertices.

Proceeding as in the proof of Lemma 3.1, we conclude that, for each $p \geq 1$, there is a strictly increasing sequence $(n_k)_{k\geq 1}$ of indices such that each of the sequences

$(e_{n_r i})_{r\geq 1}$ $(i = 1, \ldots, p)$ converges to an edge e_i. Moreover, the argument is such that, having done so for p, the choice for $p+1$ preserves the choice of the first p edges $e_1, \ldots, e_p$, and so we obtain an infinite sequence $e_1, e_2, \ldots$ of edges. Again the argument of the proof of Lemma 3.1 shows that $(e_1, \ldots, e_p)$ is a path without repeated vertices. Hence $(e_1, e_2, \ldots)$ is an infinite path without repeated vertices.□

We proceed with a couple of useful properties of $\mathbb{A}$-trees.

Lemma 3.3 *Let G be a profinite graph.*

a) Let H_1 and H_2 be profinitely connected subgraphs of G. If $H_1 \cap H_2 \neq \emptyset$, then $H_1 \cup H_2$ is profinitely connected.

b) If G is an $\mathbb{A}$-tree and $(H_i)_i$ is a collection of profinitely connected subgraphs of G such that $\bigcap_i H_i \neq \emptyset$, then $\bigcap_i H_i$ is an $\mathbb{A}$-tree.

Proof. (a) In each finite quotient H' of $H = H_1 \cup H_2$, the images of H_1 and H_2 are two connected subgraphs with at least one vertex in common and so H' is connected. Hence H is profinitely connected.

(b) Let $H = \bigcap_i H_i$. Since H is a closed subgraph of G, it suffices to show that H is profinitely connected. Let $H_i = (V_i, E_i)$ for each i and let $H = (V, E)$. Then $\mathbb{A}[E] = \bigcap_i \mathbb{A}[E_i]$ and $\mathbb{A}[V] = \bigcap_i \mathbb{A}[V_i]$. Note that

$$\operatorname{Ker} \varepsilon_H = \mathbb{A}[V] \cap \operatorname{Ker} \varepsilon_G = \bigcap_i \operatorname{Ker} \varepsilon_{H_i} \quad \text{and} \quad \operatorname{Im} \partial_H \subseteq \bigcap_i \operatorname{Im} \partial_{H_i}.$$

Since G is an $\mathbb{A}$-tree, the mapping ∂_G is one-to-one and so the inclusion above must be an equality. Finally, since $\operatorname{Ker} \varepsilon_{H_i} = \operatorname{Im} \partial_{H_i}$ for all i, we conclude that H is profinitely connected.□

In view of Lemma 3.3, for any two vertices u and v of an $\mathbb{A}$-tree G, the intersection of all profinitely connected subgraphs of G which contain u and v is an $\mathbb{A}$-tree and the smallest profinitely connected subgraph containing u and v; it is called the *segment joining u and v* and is denoted by $[u, v]$.

Lemma 3.4 *Let G be an $\mathbb{A}$-tree, let u, v and w be vertices of G, and let H and K be two profinitely connected subgraphs of G. Then:*

a) $[u, v] \subseteq [u, w] \cup [w, v]$;

b) $[u, v] = [u, w] \cup [w, v]$ if and only if $w \in [u, v]$;

c) if $H \cap K \neq \emptyset$, $u \in H$ and $v \in K$, then $[u, v] \cap H \cap K \neq \emptyset$;

d) if $[u, w] \cap [w, v] = \{w\}$, then $[u, v] = [u, w] \cup [w, v]$.

Proof. (a) By Lemma 3.3(a), $[u, w] \cup [w, v]$ is a profinitely connected subgraph of G containing u and v and so it contains $[u, v]$.

(b) is immediate in view of (a).

(c) Again by Lemma 3.3(a), $H \cup K$ is a profinitely connected subgraph of G containing u and v and so it contains the segment $[u, v]$. Hence

$$[u, v] = ([u, v] \cap H) \cup ([u, v] \cap K)$$

is a decomposition of $[u, v]$ into closed subgraphs. If these two graphs are disjoint, then $[u, v]$ has a quotient which has only two vertices and no edges joining them, which contradicts profinite connectedness of $[u, v]$. Hence $[u, v] \cap H \cap K \neq \emptyset$.

(d) Take $H = [u, w]$, $K = [w, v]$ and apply (c) to conclude that $w \in [u, v]$. The result follows from (b).$\Box$

A vertex v is said to be a *cut* of an $\mathbb{A}$-tree G if G is the union of two nontrivial profinitely connected subgraphs with no edges in common and whose only vertex in common is v.

For the remainder of this section, G denotes an $\mathbb{A}$-tree.

Lemma 3.5 *Let u and v be two vertices of G. Then a vertex w is a cut of the segment $[u, v]$ if and only if $w \neq u, v$ and $[u, w] \cap [w, v] = \{w\}$.*

Proof. Suppose first that $[u, w] \cap [w, v] = \{w\}$, so that $[u, v] = [u, w] \cup [w, v]$ by Lemma 3.4(d). If $w \neq u, v$, then the segments $[u, w]$ and $[w, v]$ are nontrivial and so, being profinitely connected with only the vertex w in common and having union $[u, v]$, it follows that w is a cut of $[u, v]$.

Conversely, suppose that w is a cut of $[u, v]$ and let H and K be two nontrivial profinitely connected subgraphs with only the vertex w in common whose union is $[u, v]$. Without loss of generality, we may assume that $u \in H$ and $v \in K$ since $[u, v]$ cannot be entirely contained in one of H or K. Consider the segments $[u, w]$ and $[w, v]$. By definition of segment, since $\{u, w\} \subseteq H$ and $\{w, v\} \subseteq K$, we then have $[u, w] \cap [w, v] \subseteq H \cap K = \{w\}$.$\Box$

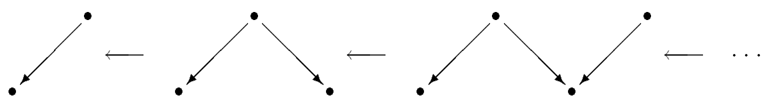
Lemma 3.6 *Let u and v be two vertices of G and let w_1 and w_2 be two distinct cuts of the segment $[u, v]$ such that $w_1 \in [u, w_2]$. Then w_1 is a cut of $[u, w_2]$ and w_2 is a cut of $[w_1, v]$.*

Proof. Since $w_1 \in [u, w_2]$, if $w_2 \in [u, w_1]$, then the two segments $[u, w_1]$ and $[u, w_2]$ coincide. Then, as $[u, v] = [u, w_1] \cup [w_1, v] = [u, w_2] \cup [w_2, v]$ and $[u, w_i] \cap [w_i, v] = \{w_i\}$ for $i = 1, 2$, we must also have $[w_1, v] = [w_2, v]$, whence $w_1 = w_2$, in contradiction with the hypothesis. Hence $w_2 \notin [u, w_1]$ and so $w_2 \in [w_1, v]$. To conclude the proof, by symmetry, it suffices to show that w_1 is a cut of $[u, w_2]$. Indeed, since $w_2 \in [w_1, v]$, $[u, w_1] \cap [w_1, w_2] \subseteq [u, w_1] \cap [w_1, v] = \{w_1\}$.$\Box$

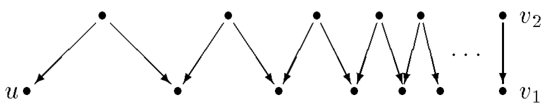
In the situation of Lemma 3.6, we write $w_1 \prec_{[u,v]} w_2$. More generally, for two cuts w_1 and w_2 of the segment $[u, v]$ (**from** u **to** v), we write $w_1 \preceq_{[u,v]} w_2$ if $w_1 \in [u, w_2]$ or, equivalently, if $w_2 \in [w_1, v]$.

Proposition 3.7 *The cuts of a segment $[u, v]$ of an $\mathbb{A}$-tree are linearly ordered by the relation $\preceq_{[u,v]}$.*$\Box$

Example. There may be infinitely many cuts in a segment. For example, consider the projective limit G of the sequence of graphs



in which each morphism identifies the two rightmost edges. Then G is a projective limit of trees and therefore, it is an $\mathbb{A}$-tree for any $\mathbb{A}$. The graph G looks as follows:



The segment $[u, v_1] = [u, v_2]$ is the whole graph and has infinitely many cuts, namely all vertices but u, v_1 and v_2 are cuts. This example also shows that the set of all cuts of a segment, together with the two extremities, may not be closed.

Lemma 3.8 *Let w_1 be a cut of $[u, v]$ and let w_2 be a cut of $[w_1, v]$. Then w_2 is a cut of $[u, v]$.*

Proof. Since $[u, w_2] \subseteq [u, w_1] \cup [w_1, w_2]$ by Lemma 3.4(a),

$$[u, w_2] \cap [w_2, v] \subseteq \big([u, w_1] \cap [w_2, v]\big) \cup \big([w_1, w_2] \cap [w_2, v]\big) = \{w_2\}$$

because $[u, w_1] \cap [w_2, v] \subseteq [u, w_1] \cap [w_1, v] = \{w_1\}$ and $w_1 \notin [w_2, v]$.$\square$

3.2. Profinite Cayley graphs

Let G be a profinite A-generated group, i.e., A is a profinite set and a continuous mapping $\varphi : A \to G$ is fixed such that the subgroup of G generated by φA is dense. Although we do not restrict φ to be one-to-one, we usually denote the image of $a \in A$ under φ by a. The *profinite Cayley graph of G (with respect to φ)* is the graph $\Gamma(G) = (G, G \times A)$ with $\alpha(g, a) = g$ and $\omega(g, a) = ga$. For an edge (g, a), we call the letter a its *label*. We will only be interested here in the case where A is finite. If G is finite, then $\Gamma(G)$ is the usual Cayley graph of G. Since $\Gamma(\_)$ preserves projective limits, in general $\Gamma(G)$ is the projective limit of the Cayley graphs of the finite quotients of G. In particular, $\Gamma(G)$ is profinitely connected. On the other hand, the Cayley graph of any nontrivial finite group has nontrivial cycles. Nevertheless, under certain conditions, the profinite Cayley graph of a profinite group may be $\mathbb{A}$-acyclic for suitable $\mathbb{A}$. We proceed to examine an important case in which this situation arises.

Fix a nontrivial pseudovariety $\mathbf{H}$ of groups. For a finite set A, the profinite Cayley graph of $\overline{\Omega}_A\mathbf{H}$ with respect to the natural mapping $A \to \overline{\Omega}_A\mathbf{H}$ is denoted by $\Gamma_{\mathbf{H}}(A)$. The finite quotients of the group $\overline{\Omega}_1\mathbf{H}$ have a natural ring structure which is respected by morphisms of 1-generated groups. Hence, we may view $\overline{\Omega}_1\mathbf{H}$ as a profinite ring. We say that $\Gamma_{\mathbf{H}}(A)$ is an $\mathbf{H}$-*tree* if $\Gamma_{\mathbf{H}}(A)$ is $\overline{\Omega}_1\mathbf{H}$-acyclic. The nontrivial pseudovariety $\mathbf{H}$ is said to be *arborescent* if $\Gamma_{\mathbf{H}}(A)$ is an $\mathbf{H}$-tree for every finite alphabet A.

For example, the pseudovariety $\mathbf{Ab}$ of all finite abelian groups is not arborescent since, if the alphabet A contains at least two distinct letters a and b, then the edges $(1, a)$, (a, b), (b, a) and $(1, b)$ constitute a cycle in $\Gamma_{\mathbf{Ab}}(A)$. On the other hand, $\Gamma_{\mathbf{Ab}}(\{a\}) = \Gamma_{\mathbf{G}}(\{a\})$ and so $\Gamma_{\mathbf{Ab}}(\{a\})$ is an $\mathbf{Ab}$-tree (i.e., a $\mathbf{G}$-tree) by Theorem 3.9 below, even though no finite nontrivial quotient of $\Gamma_{\mathbf{Ab}}(\{a\})$ is acyclic.

Let us try to reformulate the condition that $\Gamma_{\mathbf{H}}(A)$ is an $\mathbf{H}$-tree. Since $\Gamma_{\mathbf{H}}(A)$ is profinitely connected, i.e., $H_0(\Gamma_{\mathbf{H}}(A), \mathbb{A}) = 0$ for $\mathbb{A} = \overline{\Omega}_1\mathbf{H}$, $\Gamma_{\mathbf{H}}(A)$ is an $\mathbf{H}$-tree if and only if the mapping

$$\partial : \mathbb{A}[\overline{\Omega}_A\mathbf{H} \times A] \to \mathbb{A}[\overline{\Omega}_A\mathbf{H}]$$

defines an isomorphism between $\mathbb{A}[\overline{\Omega}_A\mathbf{H} \times A]$ and the profinite ring $I = \operatorname{Im}\partial = \operatorname{Ker}\varepsilon$, which is often called the *augmentation ideal*. Indeed, note that the $\mathbb{A}$-module $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$ is in fact an $\mathbb{A}$-algebra where the multiplication continuously and linearly extends the multiplication of $\overline{\Omega}_A\mathbf{H}$. (In case $\overline{\Omega}_A\mathbf{H}$ is finite, $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$ is just the usual group ring.) This ring $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$ acts naturally on $\mathbb{A}[\overline{\Omega}_A\mathbf{H} \times A]$ by the continuous linear extension of the action of $\overline{\Omega}_A\mathbf{H}$ on $\overline{\Omega}_A\mathbf{H} \times A$ given by $g(h,a) = (gh,a)$. It is then an easy exercise on universal properties to show that $\mathbb{A}[\overline{\Omega}_A\mathbf{H} \times A]$ is the $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$-free module over $\{1\} \times A$. Note also that ∂ respects the action of $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$. We therefore conclude that $\Gamma_{\mathbf{H}}(A)$ is an $\mathbf{H}$-tree if and only if the augmentation ideal I is freely generated by the $\partial(1,a) = a - 1$ ($a \in A$) as a profinite $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$-module.

We claim that the elements $a - 1$ ($a \in A$) always generate I as a profinite $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$-module, i.e., the $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$-module generated by these elements is dense in I. Indeed, the latter assertion is equivalent to the augmentation ideal I_G of $\mathbb{A}[G]$ being generated by the $a-1$ ($a \in A$) for each A-generated G in $\mathbf{H}$. By induction on the number of nonzero coefficients λ_g in an element $\sum_{g\in G}\lambda_g g$ of I_G, it is easy to show that the elements $g - 1$ ($g \in G$) generate I_G as an $\mathbb{A}$-module. But, if $g = a_1 \dots a_n$ with $a_i \in A$, then $g - 1 = \sum_{i=1}^n a_1 \dots a_{i-1}(a_i - 1)$ lies in the $\mathbb{A}[G]$-module generated by the $a - 1$ ($a \in A$), which proves the claim.

Recall that, for two semigroups S and T and a monoid homomorphism $T^1 \to \operatorname{End} S$ with the image of $s \in S$ under the endomorphism corresponding to $t \in T^1$ denoted by ts, the associated semidirect product is the semigroup $S*T$ with universe the cartesian product $S\times T$ and operation given by $(s_1,t_1)(s_2,t_2) = (s_1 + t_1 s_2, t_1 t_2)$ where additive notation is adopted for S. For two pseudovarieties $\mathbf{V}$ and $\mathbf{W}$, $\mathbf{V} * \mathbf{W}$ denotes the pseudovariety generated by all semidirect products $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$.

Consider now a profinite $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$-module M and a continuous mapping $r : A \to M$. The action of $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$ on M yields, in particular, a semidirect product $M*\overline{\Omega}_A\mathbf{H}$. Suppose that the mapping $r' : A \to M * \overline{\Omega}_A\mathbf{H}$ defined by $r'(a) = \big(r(a), a\big)$ extends to a continuous group homomorphism $\widehat{r'} : \overline{\Omega}_A\mathbf{H} \to M * \overline{\Omega}_A\mathbf{H}$. Composing with the projection on the first component, we obtain a continuous mapping $d : \overline{\Omega}_A\mathbf{H} \to M$ such that $\widehat{r'}(g) = \big(d(g), g\big)$ for all $g \in \overline{\Omega}_A\mathbf{H}$. Although d is not in general a group homomorphism, it is a derivation in the sense that

$$d(gh) = d(g) + g \cdot d(h) \quad \text{for all } g,h \in \overline{\Omega}_A\mathbf{H}. \tag{2}$$

We extend d to a morphism of profinite $\mathbb{A}$-modules $\widehat{d} : \mathbb{A}[\overline{\Omega}_A\mathbf{H}] \to M$ by using the universal property of $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$. By continuity and $\mathbb{A}$-linearity of $\widehat{d}$, from (2) we deduce that

$$\widehat{d}(xy) = \varepsilon(y) \cdot d(x) + x \cdot d(y) \quad \text{for all } x, y \in \mathbb{A}[\overline{\Omega}_A\mathbf{H}].$$

In particular, the restriction ρ of $\widehat{d}$ to the augmentation ideal I is a morphism of profinite $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$-modules satisfying, for each $a \in A$,

$$\rho(a-1) = \widehat{d}(a-1) = d(a) - d(1) = d(a) = r(a).$$

The various mappings involved in this argument are depicted in the following diagram:

$$\begin{tikzcd}
A \arrow[r, dashed, "r'"] \arrow[d] \arrow[dr, "r"] & M * \overline{\Omega}_A\mathbf{H} \arrow[d] \\
\overline{\Omega}_A\mathbf{H} \arrow[r, dashed, "d"] \arrow[d] \arrow[ur, dashed, "\widehat{r'}"] & M \\
\mathbb{A}[\overline{\Omega}_A\mathbf{H}] \arrow[ur, dashed, "\widehat{d}"] & I \arrow[l, hook'] \arrow[u, dashed, "\rho"']
\end{tikzcd}$$

Finally, note that, if the abelian group $M * \overline{\Omega}_A\mathbf{H}$ is pro-$\mathbf{H}$, then the mapping r' does extend to $\widehat{r'}$ as indicated above. We thus have established half of the following theorem.

Theorem 3.9 *A nontrivial pseudovariety* $\mathbf{H}$ *of groups is arborescent if and only if* $(\mathbf{H} \cap \mathbf{Ab}) * \mathbf{H} = \mathbf{H}$.

Proof. It remains to be shown that, if $\mathbf{H}$ is arborescent, S and T are groups in $\mathbf{H}$ with S abelian, and $S * T$ is a semidirect product, then $S * T \in \mathbf{H}$.

Let A be a finite set and let $a \mapsto (s_a, t_a)$ define a mapping $A \to S * T$ whose image generates $S * T$. Since $T \in \mathbf{H}$, the correspondence $a \mapsto t_a$ defines a unique continuous homomorphism $\psi : \overline{\Omega}_A\mathbf{H} \to T$. If $\varphi : T \to \operatorname{End} S$ defines the action of T on S, the composite $\varphi \circ \psi$ gives an action of $\overline{\Omega}_A\mathbf{H}$ on S and a semidirect product $S * \overline{\Omega}_A\mathbf{H}$. On the other hand, since $S \in \mathbf{H} \cap \mathbf{Ab}$, S is a direct product of cyclic groups in $\mathbf{H}$ on which $\mathbb{A} = \overline{\Omega}_1\mathbf{H}$ acts naturally. Hence, S is a (finite) $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$-module.

Consider the mapping $r : A \to S$ given by $a \mapsto s_a$. Since $\mathbf{H}$ is arborescent, there is a morphism of profinite $\mathbb{A}[\overline{\Omega}_A\mathbf{H}]$-modules $\rho : I \to S$ such that $\rho(a-1) = r(a)$ $(a \in A)$. Then $g \mapsto \rho(g-1)$ defines a continuous derivation $d : \overline{\Omega}_A\mathbf{H} \to S$ and so the mapping $\sigma : \overline{\Omega}_A\mathbf{H} \to S * \overline{\Omega}_A\mathbf{H}$ is a continuous homomorphism. Composing with the homomorphism $(s, g) \mapsto \big(s, \psi(g)\big)$ $(s \in S,\ g \in \overline{\Omega}_A\mathbf{H})$, we obtain a continuous homomorphism $\overline{\Omega}_A\mathbf{H} \to S * T$ which is onto since the $(s_a, t_a) = \big(d(a), \psi(a)\big)$ $(a \in A)$ generate $S * T$. Hence $S * T \in \mathbf{H}$ by Proposition 1.6.□

3.3. Reduced factorizations in relatively free profinite groups

It is well known that the Cayley graph of a free group with respect to a set A of free generators is a tree in the usual combinatorial sense. Reading the second components of the edges (g, a) from the shortest path from 1 to h, taking a^{-1} instead of a if the edge appears in the wrong direction, we obtain a so-called *reduced word* $b_1 \ldots b_n$ on the alphabet $A \cup \overline{A}$ where $\overline{A} = \{a^{-1} : a \in A\}$ in the sense that no factor of length two is of any of the forms aa^{-1} or $a^{-1}a$ with $a \in A$.

Now, suppose that $\mathbf{H}$ is an arborescent pseudovariety of groups and A is a finite alphabet. Then the profinite Cayley graph $\Gamma_{\mathbf{H}}(A)$ is an $\mathbf{H}$-tree so that, as was already observed in section 3.1, for each $g \in \overline{\Omega}_A\mathbf{H}$ we have a segment $[1, g]$. This segment is not necessarily a finite path and so we may not be able to apply the procedure described above for the free group. For example, if $g \in \overline{\Omega}_{\{a\}}\mathbf{H}$ is such that both g and g^{-1} are non-explicit (monoid) operations, then no finite path connects 1 with g in $\Gamma_{\mathbf{H}}(\{a\})$. In $\overline{\Omega}_{\{a\}}\mathbf{H}$, each vertex h has exactly one edge leading to it, namely (ha^{-1}, a), and one edge leaving from it, namely (h, a). By Proposition 3.2, it follows that, if $[1, g]$ is not finite, then it contains an infinite sequence $h, ha, ha^2, \ldots$ or an infinite sequence $h, ha^{-1}, ha^{-2}, \ldots$ and so all vertices of $\Gamma_{\mathbf{H}}(\{a\})$ (and, therefore, also all edges). These arguments, although formulated for the case when $\mathbf{H}$ is arborescent, prove the following general result.

Proposition 3.10 *For any pseudovariety* $\mathbf{H}$ *of groups, the only possibly infinite profinitely connected subgraph of* $\Gamma_{\mathbf{H}}(\{a\})$ *is* $\Gamma_{\mathbf{H}}(\{a\})$ *itself.*$\square$

Another way to look at reduced words $b_1 \ldots b_n$ in free groups is to note that the property of having no mutually inverse consecutive letters amounts to saying that the corresponding path repeats no edges. We use an appropriate abstraction of this property to define reduced factorizations in the profinite setting.

Throughout the remainder of this section, $\mathbf{H}$ denotes a fixed arborescent pseudovariety of groups.

For $\pi, \rho \in \overline{\Omega}_A\mathbf{H}\setminus\{1\}$, we say that the product $\pi\rho$ is *reduced* if $[1, \pi] \cap [\pi, \pi\rho] = \{\pi\}$. More generally, a product $\pi_1 \cdots \pi_n$ of $n \geq 0$ factors $\pi_1, \ldots, \pi_n \in \overline{\Omega}_A\mathbf{H}\setminus\{1\}$ is *reduced* if each product $\pi_i\pi_{i+1}$ $(1 \leq i < n)$ is reduced; we then also say that $\pi_1 \cdots \pi_n$ is a *reduced factorization* of $\pi = \pi_1 \cdots \pi_n$. For example, by Proposition 3.10, the only reduced products in $\overline{\Omega}_{\{a\}}\mathbf{H}$ of length greater than 1 are those of the forms $\underbrace{a \cdots a}_{n}$ and $\underbrace{a^{-1} \cdots a^{-1}}_{n}$ $(n > 1)$. In particular, since $\overline{\Omega}_{\{a\}}\mathbf{H}$ is uncountable, most of its elements do not admit nontrivial reduced factorizations. Sufficient conditions for nontrivial reduced factorizability are given below but they are obviously not necessary.

Note that the action of $\overline{\Omega}_A\mathbf{H}$ on the edges of $\Gamma_{\mathbf{H}}(A)$ considered in section 3.2 is in fact part of a graph action in the sense that each $g \in \overline{\Omega}_A\mathbf{H}$ gives a continuous endomorphism of the graph $\Gamma_{\mathbf{H}}(A)$ by acting on the edges as described in section 3.2 and on vertices by left multiplication. It follows that, for any $\pi, \rho, \sigma \in \overline{\Omega}_A\mathbf{H}$, $\pi[\rho, \sigma] = [\pi\rho, \pi\sigma]$.

Lemma 3.11 *Let* $\pi_1, \ldots, \pi_n \in \overline{\Omega}_A\mathbf{H} \setminus \{1\}$. *If the product* $\pi_1 \cdots \pi_n$ *is reduced then* $[1, \pi_1] \cap [\pi_1, \pi_1 \cdots \pi_n] = \{\pi_1\}$.

Proof. We proceed by induction on n. The case $n = 1$ is obvious and the case $n = 2$ follows directly from the definition of reduced product. So, assume that $n > 2$ and the result is valid for $n - 1$.

Since the product $\pi_2 \cdots \pi_n$ is reduced, by the induction hypothesis $[1, \pi_2] \cap [\pi_2, \pi_2 \cdots \pi_n] = \{\pi_2\}$ and so $[\pi_1, \pi_1\pi_2] \cap [\pi_1\pi_2, \pi_1 \cdots \pi_n] = \{\pi_1\pi_2\}$. By Lemma

3.4(d), it follows that

$$[\pi_1, \pi_1 \cdots \pi_n] = [\pi_1, \pi_1\pi_2] \cup [\pi_1\pi_2, \pi_1 \cdots \pi_n]. \tag{3}$$

Now, suppose that σ is a vertex different from π_1 in $[1, \pi_1] \cap [\pi_1, \pi_1 \cdots \pi_n]$. By (3), since $[1, \pi_1] \cap [\pi_1, \pi_1\pi_2] = \{\pi_1\}$ by Lemma 3.8, we must also have $\sigma \in [1, \pi_1] \cap [\pi_1\pi_2, \pi_1 \cdots \pi_n]$. Then

$$[\pi_1, \pi_1\pi_2] \subseteq [\pi_1, \sigma] \cup [\sigma, \pi_1\pi_2] \subseteq [1, \pi_1] \cup [\pi_1\pi_2, \pi_1 \cdots \pi_n]$$

and so

$$[\pi_1, \pi_1\pi_2] \subseteq \big([1, \pi_1] \cap [\pi_1, \pi_1\pi_2]\big) \cup \big([\pi_1\pi_2, \pi_1 \cdots \pi_n] \cap [\pi_1, \pi_1\pi_2]\big) = \{\pi_1, \pi_1\pi_2\}$$

yielding $[\pi_1, \pi_1\pi_2] = \{\pi_1, \pi_1\pi_2\}$, which is a contradiction since $\pi_2 \neq 1$ and so the finite graph $\{\pi_1, \pi_1\pi_2\}$ is not connected. This shows that there is no such vertex σ and completes the induction step.□

The next proposition gives a "geometrical" interpretation of the definition of reduced factorization.

Proposition 3.12 *Let $\pi_1, \ldots, \pi_n \in \overline{\Omega}_A\mathbf{H}$. Then the product $\pi_1 \cdots \pi_n$ is reduced if and only if the vertices $\pi_1, \pi_1\pi_2, \ldots, \pi_1 \cdots \pi_{n-1}$ constitute a strictly increasing chain of cuts of the segment $[1, \pi_1 \cdots \pi_n]$.*

Proof. Suppose first that the product $\pi_1 \cdots \pi_n$ is reduced. Proceeding by induction on n, we may assume that the result is valid for $n-1$. Then, by Lemma 3.11, π_1 is a cut of $[1, \pi_1 \cdots \pi_n]$ and, by the induction hypothesis, the vertices $\pi_1\pi_2, \ldots, \pi_1 \cdots \pi_{n-1}$ are cuts of the segment $[\pi_1, \pi_1 \cdots \pi_n]$, whence also of the segment $[1, \pi_1 \cdots \pi_n]$ by Lemma 3.8.

Conversely, suppose that $\pi_1, \pi_1\pi_2, \ldots, \pi_1 \cdots \pi_{n-1}$ constitute a strictly increasing chain of cuts of $[1, \pi_1 \cdots \pi_n]$. Then, for any $i \in \{1, \ldots, n-1\}$, denoting $\pi_1 \cdots \pi_j$ by σ_j, we have

$$[\sigma_{i-1}, \sigma_i] \cap [\sigma_i, \sigma_{i+1}] \subseteq [\sigma_1, \sigma_i] \cap [\sigma_i, \sigma_n]$$

by the definition of the ordering of cuts and so $[\sigma_{i-1}, \sigma_i] \cap [\sigma_i, \sigma_{i+1}] = \{\sigma_i\}$ because σ_i is a cut of $[1, \sigma_n]$. Hence $[1, \pi_i] \cap [\pi_1, \pi_{i+1}] = \{\pi_i\}$ and so the product $\pi_1 \cdots \pi_n$ is reduced.□

Given $\pi, \rho \in \overline{\Omega}_A\mathbf{H}$, we write $\pi \preceq \rho$ if $\pi = 1$, $\pi = \rho$ or π is a cut of the segment $[1, \rho]$ of $\Gamma_{\mathbf{H}}(A)$. This defines a partial order on $\overline{\Omega}_A\mathbf{H}$ which, by Proposition 3.7, is in fact a tree ordering, in the sense that elements with a common upper bound are comparable.

We say that a factorization $\pi = \pi_1 \cdots \pi_m$ *refines* a factorization $\pi = \rho_1 \cdots \rho_n$ of the same element if there are indices i_j $(j = 0, \ldots, n)$ such that $0 = i_0 < i_1 < \cdots < i_n = m$ and $\rho_j = \pi_{i_{j-1}+1} \ldots \pi_{i_j}$ $(j = 1, \ldots, n)$.

Lemma 3.13 *Let $\pi = \pi_1 \cdots \pi_m$ and $\pi = \rho_1 \cdots \rho_n$ be two reduced factorizations in $\overline{\Omega}_A\mathbf{H}$ of the same element. Then $\pi_1 \cdots \pi_m$ refines $\rho_1 \cdots \rho_n$ if and only if*

$$\{\pi_1, \pi_1\pi_2, \ldots, \pi_1 \cdots \pi_{m-1}\} \supseteq \{\rho_1, \rho_1\rho_2, \ldots, \rho_1 \cdots \rho_{n-1}\}.$$

Proof. Since necessity is obvious, it suffices to show that the inclusion in the statement is sufficient for $\pi_1 \cdots \pi_m$ to refine $\rho_1 \cdots \rho_n$. Indeed, for each $j \in \{1, \dots, n-1\}$, there is $i_j \in \{1, \dots, m-1\}$ such that $\rho_1 \cdots \rho_j = \pi_1 \cdots \pi_{i_j}$ and, since both sequences $(\pi_1, \pi_1\pi_2, \dots, \pi_1 \cdots \pi_{m-1})$ and $(\rho_1, \rho_1\rho_2, \dots, \rho_1 \cdots \rho_{n-1})$ are strictly increasing under $\preceq$, the sequence $0 = i_0, i_1, \dots, i_n = m$ is also strictly increasing and $\rho_j = \pi_{i_{j-1}+1} \cdots \pi_{i_j}$ for $j = 1, \dots, n$.□

We may now establish the confluence of refinement of reduced factorizations.

Theorem 3.14 *Any two reduced factorizations in $\overline{\Omega}_A\mathbf{H}$ of the same element have a common refinement.*

Proof. Let $\pi = \pi_1 \cdots \pi_m$ and $\pi = \rho_1 \cdots \rho_n$ be two reduced factorizations. Then, by Proposition 3.7, the union of the two chains $\pi_1 \prec \pi_1\pi_2 \prec \cdots \prec \pi_1 \cdots \pi_{m-1}$ and $\rho_1 \prec \rho_1\rho_2 \prec \cdots \prec \rho_1 \cdots \rho_{n-1}$ of cuts of $[1, \pi]$ is also a chain of cuts which we may write in the form $\xi_1 \prec \xi_1\xi_2 \prec \cdots \prec \xi_1 \cdots \xi_{r-1}$. Taking $\xi_r = (\xi_1 \cdots \xi_{r-1})^{-1}\pi$, we obtain a reduced factorization $\pi = \xi_1 \cdots \xi_r$ by Proposition 3.12 which refines the two given factorizations by Lemma 3.13.□

Note that the order $\preceq$ is not stable under multiplication. For example, if $a \in A$, then $a \preceq a^3$ but $a^{-1} = a^{-2}a \not\preceq a^{-2}a^3 = a$ since $[1, a] = \{1, a, (1, a)\}$. To our best knowledge, no example has yet been found of an element π of $\overline{\Omega}_A\mathbf{H}$ such that $[1, \pi]$ has infinitely many cuts. If no such example exists, then by Theorem 3.14 every element π of $\overline{\Omega}_A\mathbf{H}$ has a "canonical" reduced factorization, namely that which uses all cuts of $[1, \pi]$ according to Proposition 3.12. But, not even the weaker property that the set $\{\rho : \rho \preceq \pi\}$ is closed has yet been established, and perhaps both properties fail. (Compare with the example near the end of section 3.1.)

We conclude this section with two applications of Theorem 3.14 which will be useful in the next section. We sketch only the proof of the first and refer the reader to [9] for details.

Proposition 3.15 *Let $\mathbf{H}$ be an arborescent pseudovariety of groups and let A be an alphabet. For $1 \le i \le n$, let A_i and B_i be subsets of A, let $\pi_i \in \overline{\Omega}_{A_i}\mathbf{H} \setminus \{1\}$ and $\rho_i \in \overline{\Omega}_{B_i}\mathbf{H} \setminus \{1\}$, and let u_j ($1 \le j \le n-1$) be reduced words over $A \cup \overline{A}$ such that the first letter of u_j is not in $A_j \cup \overline{A}_j \cup B_j \cup \overline{B}_j$, while its last letter is not in $A_{j+1} \cup \overline{A}_{j+1} \cup B_{j+1} \cup \overline{B}_{j+1}$. Then the factorizations $\pi_1 u_1 \pi_2 \dots u_{n-1}\pi_n$ and $\rho_1 u_1 \rho_2 \dots u_{n-1}\rho_n$ are reduced.*

If $\pi_1 u_1 \pi_2 \dots u_{n-1}\pi_n = \rho_1 u_1 \rho_2 \dots u_{n-1}\rho_n$ in $\overline{\Omega}_A\mathbf{H}$, then $\pi_i = \rho_i$ for all $1 \le i \le n$.

Proof. The two factorizations $\pi_1 u_1 \pi_2 \dots u_{n-1}\pi_n$ and $\rho_1 u_1 \rho_2 \dots u_{n-1}\rho_n$ are reduced since the $\Gamma_{\mathbf{H}}(A_j)$ and $\Gamma_{\mathbf{H}}(B_j)$ are profinitely connected subgraphs of $\Gamma_{\mathbf{H}}(A)$ which do not include any edge whose label is either the last letter of u_{j-1} or the first letter of u_{j+1} (or their inverses). By Theorem 3.14, the two factorizations have a common refinement $\xi_1 \cdots \xi_r$. Let $\pi_1 = \xi_1 \cdots \xi_i$ and $\rho_1 = \xi_1 \cdots \xi_j$ and suppose that $i < j$. Arguing by induction, it suffices to show this leads to a contradiction. Indeed, we have $u_1 = \xi_{i+1} \cdots \xi_k$ for some $k > i$. Hence, if $a \in A$ is such that a or a^{-1} is the first letter of u_1, then the segment $[1, \xi_{i+1}]$ has an edge labeled a while $[1, \rho_1] \subseteq \Gamma_{\mathbf{H}}(B_1)$ includes no such edge.□

Proposition 3.16 *Let* $\mathbf{H}$ *be an arborescent pseudovariety of groups, let* A *be an alphabet and let* B *and* C *be disjoint non-empty subsets of* A. *Let* $\pi_1, \ldots, \pi_r, \rho_1, \ldots, \rho_s$ *be elements of* $\overline{\Omega}_B\mathbf{H} \setminus \{1\}$, *and let* $\sigma_0, \ldots, \sigma_r$ *and* $\tau_0, \ldots, \tau_s$ *be elements of* $\overline{\Omega}_C\mathbf{H}$ *such that* $\sigma_i \neq 1$ *if* $1 \leq i < r$ *and* $\tau_j \neq 1$ *if* $1 \leq j < s$. *Then the factorizations* $\sigma_0\pi_1\sigma_1 \cdots \pi_r\sigma_r$ *and* $\tau_0\rho_1\tau_1 \cdots \rho_s\tau_s$ *are reduced.*

If $\sigma_0\pi_1\sigma_1 \cdots \pi_r\sigma_r = \tau_0\rho_1\tau_1 \cdots \rho_s\tau_s$ *in* $\overline{\Omega}_A\mathbf{H}$, *then* $r = s$, $\sigma_i = \tau_i$ *for* $0 \leq i \leq r$ *and* $\pi_i = \rho_i$ *for* $1 \leq i \leq r$.

4. A case study: DS with commuting idempotents

Let **ECom** and **ZE** denote, respectively, the pseudovarieties of all finite monoids in which idempotents commute (among themselves) and in which idempotents are central (i.e., commute with every element). The aim of this section is to give a canonical factorization for implicit operations on the pseudovariety $\mathbf{DS} \cap \mathbf{ECom} = \mathbf{DG} \cap \mathbf{ECom}$ and to deduce some join decompositions for certain subpseudovarieties.

Section 4.1 presents the study of canonical factorizations on free profinite monoids over pseudovarieties of the forms $\mathbf{DH} \cap \mathbf{ECom}$ and $\mathbf{ZE} \cap \overline{\mathbf{H}}$ where $\mathbf{H}$ is a pseudovariety of groups. Such factorizations are obtained by simplifying the general factorizations over **DS** considered in section 2.4. To establish their uniqueness, suitable finite automata are considered. As an application of these unique factorization statements and the results of section 3.3, via Reiterman's Theorem, join decompositions are given for $\mathbf{DH} \cap \mathbf{Ecom}$ and $\mathbf{ZE} \cap \overline{\mathbf{H}}$ for the case of an arborescent pseudovariety $\mathbf{H}$ of groups. We also present examples to show that such join decompositions are not valid for arbitrary $\mathbf{H}$ and to indicate that the arborescence of $\mathbf{H}$ is not a necessary condition for them to hold. Finally, section 4.3 gives the counterpart of these results for varieties of languages via Eilenberg's correspondence between these entities and pseudovarieties of monoids [14].

All results of this section are taken from Almeida and Weil [9]. Theorems 4.6 and 4.8 extend the particular cases where $\mathbf{H}$ is the pseudovariety of all finite groups which had already been obtained by Almeida and Azevedo [7] and Almeida [1], respectively. Theorem 4.10 first appeared in Ash, Hall and Pin [12]. For related problems concerning join decompositions, see Rhodes [21] and Almeida [6].

4.1. Some canonical factorizations

We start by observing that the commutation of idempotents in a monoid implies that the regular elements constitute a submonoid. This allows us to simplify the factorization of implicit operations on subpseudovarieties of $\mathbf{DG} \cap \mathbf{ECom}$ given by Theorem 2.9.

Let $\mathbf{V} \subseteq \mathbf{DG} \cap \mathbf{ECom}$ and let $\pi \in \overline{\Omega}_A\mathbf{M}$. We say that a factorization of π of the form

$$\pi = u_0\pi_1 u_1 \ldots \pi_r u_r$$

is *normal* (*with respect to* $\mathbf{V}$) if

- $u_i \in A^*$ $(0 \leq i \leq r)$;
- $u_i \neq 1$ $(0 < i < r)$;
- the restriction of π_i to $\mathbf{V}$ is regular $(1 \leq i \leq r)$;

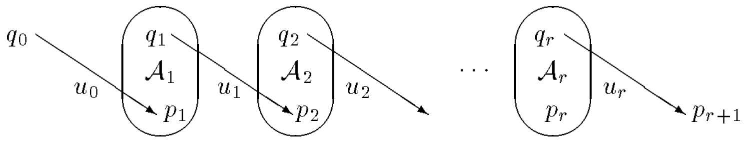
- for each $i < r$, the last letter of the word u_i (if it exists) does not belong to the content of π_{i+1};
- for each $i > 0$, the first letter of the word u_i (if it exists) does not belong to the content of π_i.

Note that, if $\mathbf{W} \subseteq \mathbf{V}$ and a factorization is normal with respect to $\mathbf{V}$, then it is also normal with respect to $\mathbf{W}$.

Proposition 4.1 *Let* $\mathbf{V}$ *be a subpseudovariety of* $\mathbf{DG} \cap \mathbf{ECom}$. *Then every* $\pi \in \overline{\Omega}_A\mathbf{M}$ *admits a normal factorization with respect to* $\mathbf{V}$.$\Box$

For our analysis of implicit operations on $\mathbf{DG} \cap \mathbf{ECom}$, we need some test monoids to separate distinct factorizations. They are constructed as transition monoids of suitable automata.

For $r \geq 0$, let $u_0, \ldots, u_r \in A^*$ and let $A_1, \ldots, A_r \subseteq A$ be such that $A_i \neq \emptyset$ $(1 \leq i \leq r)$, $u_i \neq 1$ $(0 < i < r)$, the last letter of each u_i does not belong to A_{i+1}, and the first letter of each u_i does not belong to A_i. For each $i \in \{1, \ldots, r\}$, let $\mathcal{A}_i$ be a permutation automaton on the alphabet A_i with set of states Q_i and let $p_i, q_i \in Q_i$. The automaton $\mathcal{A} = \mathcal{A}(u_0; \mathcal{A}_1, p_1, q_1; u_1; \ldots; \mathcal{A}_r, p_r, q_r; u_r)$ is the following

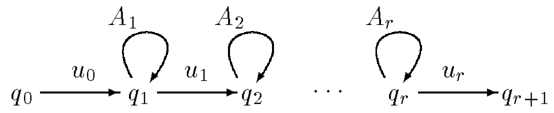


The condition on the extreme letters of the words u_i guarantees that $\mathcal{A}$ is a deterministic and co-deterministic automaton, meaning that each letter defines a partial one-to-one transformation of the set Q of states. Hence all words define partial one-to-one transformations of the set Q. In particular, the transition monoid $M_{\mathcal{A}}$ of the automaton $\mathcal{A}$ belongs to $\mathbf{ECom}$. A slightly more detailed analysis of the structure of $\mathcal{A}$ proves the following lemma.

Lemma 4.2 *The monoid* $M_{\mathcal{A}}$ *lies in* $\mathbf{DG} \cap \mathbf{ECom}$ *and its subgroups lie in the pseudovariety generated by the permutation groups* $M_{\mathcal{A}_i}$. *If* $A_1 = \cdots = A_r$, $A_1 \cap c(u_0 \ldots u_r) = \emptyset$, $u_0 = u_r = 1$, *and* $|u_i| = 1$ *(*$i = 1, \ldots, r-1$*), then* $M_{\mathcal{A}} \in \mathbf{ZE}$. *Furthermore, if* $w \in A^+$, $k > |u_0 u_1 \ldots u_r| + r$ *and* $p, q \in Q$ *are such that* $p\,w^k = q$, *then there is some* $i \in \{1, \ldots, r\}$ *such that* $p, q \in Q_i$ *and* $w \in A_i^+$.

Theorem 4.3 *Let* $\mathbf{H}$ *be a pseudovariety of groups and let* $\mathbf{V} = \mathbf{DH} \cap \mathbf{ECom}$. *Let* $\pi, \rho \in \overline{\Omega}_A\mathbf{M}$ *and let* $\pi = u_0\pi_1 u_1 \ldots \pi_r u_r$ *and* $\rho = v_0 \rho_1 v_1 \ldots \rho_s v_s$ *be factorizations in normal form with respect to* $\mathbf{V}$. *Then* $\mathbf{V}$ *satisfies* $\pi = \rho$ *if and only if* $r = s$, $u_i = v_i$ *(*$0 \leq i \leq r$*) and* $\mathbf{V}$ *satisfies* $\pi_i = \rho_i$ *(*$1 \leq i \leq r$*) (or, equivalently,* $c(\pi_i) = c(\rho_i)$ *and* $\mathbf{H} = \mathbf{V} \cap \mathbf{G}$ *satisfies* $\pi_i = \rho_i$*).*

Proof. Let $A_i = c(\pi_i)$ $(1 \leq i \leq r)$ and let $B_j = c(\rho_j)$ $(1 \leq j \leq s)$. Consider the following automaton $\mathcal{A}$:



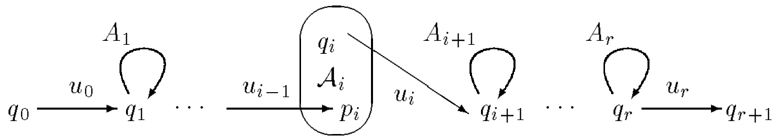
and its transition homomorphism $\mu : A^* \to M$. Let $\hat{\mu} : \overline{\Omega}_A\mathbf{M} \to M$ be the unique continuous homomorphic extension of μ. By Lemma 4.2, $M \in \mathbf{J} \cap \mathbf{ECom}$ and so $M \in \mathbf{V}$. Let $k > \max\{|u_0 \dots u_r| + r, |v_0 \dots v_s| + s\}$.

Since the factors π_i and ρ_j are regular on $\mathbf{V}$, their images in M, $\hat{\mu}(\pi_i)$ and $\hat{\mu}(\rho_j)$, are idempotents. By density of A^* in $\overline{\Omega}_A\mathbf{M}$, there are words w_i such that $c(w_i) = A_i$ and $\hat{\mu}(\pi_i) = \mu(w_i^k)$. By Lemma 4.2, $\mu(w_i^k)$ is a partial identity function whose domain consists of all q_j such that $c(w_i) = A_i \subseteq A_j$.

We claim that $\hat{\mu}(\pi) = \mu(u_0 w_1^k u_1 \dots w_r^k u_r)$ is the function $\{q_0\} \to \{q_{r+1}\}$. Clearly $q_0\,\hat{\mu}(\pi) = q_{r+1}$. Suppose that a state $q \neq q_0$ belongs to the domain of $\hat{\mu}(\pi)$. Let $p = q\,u_0 w_1^k u_1 \dots u_{i-1} w_i^k$. Since $\mu(w_i^k)$ is idempotent, $p\,w_i^k = p$ and so $p = q_{j(i)}$ for some $j(i)$ such that $c(w_i) = A_i \subseteq A_{j(i)}$. Then either $u_0 = 1$ and $q_0 = q_1$, and so $j(1) > 1$ because there is no way back from right to left in the automaton $\mathcal{A}$; or $u_0 \neq 1$ and so $j(1) = 1$ would imply that the last letter of u_0 belongs to $c(\pi_1) = A_1$, contrary to the assumption that the product $u_0 w_1^k u_1 \dots w_r^k u_r$ is in canonical form. This shows that $j(1) > 1$. Suppose next that i is such that $j(i) > i$. Then $j(j(i)) = j(i)$ implies $c(u_i w_{i+1}^k u_{i+1} \dots u_{j(i)-1} w_{j(i)}^k) \subseteq A_{j(i)}$ which again contradicts the assumption that we are dealing with a normal factorization. Hence $j(j(i)) > j(i)$. Moreover, this argument applies whenever $i < j(i) \leq r$. Whence $1, j(1), j(j(1)), \dots$ is a strictly increasing sequence of integers which is bounded above by r, which is absurd. Hence $q = q_0$ and the claim is established.

Consider now words w'_i $(1 \leq i \leq s)$ such that $c(w'_i) = B_i$ and $\hat{\mu}(\rho_i) = \mu({w'_i}^k)$. Again by Lemma 4.2, $\mu({w'_i}^k)$ is a partial identity with domain the set of all q_j such that $B_i \subseteq A_j$. Since $\mathbf{V} \models \pi = \rho$, we have $\hat{\mu}(\pi) = \hat{\mu}(\rho)$ and so $q_0 v_0 {w'_1}^k v_1 \dots {w'_s}^k v_s = q_{r+1}$. As above, $q_0\, v_0 {w'_1}^k = q_i$ for some $i \in \{1, \dots, r\}$. This implies that $q_0\, v_0 = q_i$ and so u_0 is a prefix of v_0. By symmetry, $v_0 = u_0$ so that $i = 1$ and $c(w'_1) \subseteq A_1 = c(w_1)$. Again by symmetry, it follows that $c(\pi_1) = c(\rho_1)$. Iterating the above argument, we deduce that $r = s$, $u_i = v_i$ $(0 \leq i \leq r)$ and $c(\pi_i) = c(\rho_i)$ $(1 \leq i \leq r)$.

Finally, to establish that $\mathbf{V} \models \pi_i = \rho_i$, since π_i and ρ_i are regular on $\mathbf{V}$ and have the same content, by Corollary 2.13, it suffices to show that $\mathbf{H} \models \pi_i = \rho_i$. Let G be an A_i-generated group in $\mathbf{H}$ and let $\mathcal{A}_i$ be its Cayley graph with respect to the set of generators A_i. Note that the transition monoid of $\mathcal{A}_i$ is precisely G. Let $A_i = \{a_{i1}, \dots, a_{in_i}\}$ and consider the vertices $p_i = 1$ and $q_i = (\pi_i)_G(a_{i1}, \dots, a_{in_i})$. We have an associated automaton $\mathcal{A}$ defined as follows:



(3)

We again denote by μ and $\hat{\mu}$ respectively the transition homomorphism of $\mathcal{A}_i$ and its continuous homomorphic extension to $\overline{\Omega}_A\mathbf{M}$. By the first part of the proof, the mappings $\hat{\mu}(u_0\rho_1u_1\dots\rho_{i-1}u_{i-1})$, $\hat{\mu}(u_i\rho_{i+1}u_{i+1}\dots\rho_ru_r)$ and $\hat{\mu}(u_0\pi_1u_1\dots\pi_ru_r)$ send, respectively, q_0 to p_i, q_i to q_{r+1}, and q_0 to q_{r+1}. Hence, since $\hat{\mu}(u_0\pi_1u_1\dots\pi_ru_r) = \hat{\mu}(u_0\rho_1u_1\dots\rho_ru_r)$ by Lemma 4.2 and we are dealing with partial one-to-one mappings, the equality $(\rho_i)_G(a_{i1},\dots,a_{in_i}) = (\pi_i)_G(a_{i1},\dots,a_{in_i})$ holds, which shows that $\mathbf{H} \models \pi_i = \rho_i$.□

The above proof shows that the transition monoids of the automata of the form (4.1) suffice to separate any two distinct normal products with respect to $\mathbf{DH} \cap \mathbf{ECom}$.

Corollary 4.4 *Let* $\mathbf{H}$ *be a pseudovariety of groups. Then the pseudovariety* $\mathbf{DH} \cap \mathbf{ECom}$ *is generated by the transition monoids of the automata of the form (4.1) with* $\mathcal{A}_i$ *a permutation automaton over the alphabet* A_i *such that* $M_{\mathcal{A}_i} \in \mathbf{H}$*, the last letter of each* u_i *not in* A_{i+1}*, and the first letter of each* u_i *not in* A_i*.*□

Corollary 4.5 *An implicit operation* π *on* M *is regular on* $\mathbf{DG} \cap \mathbf{ECom}$ *if and only if it is regular on* $\mathbf{J} \cap \mathbf{ECom}$*. In particular, a factorization is normal with respect to any pseudovariety in the interval* $[\mathbf{J} \cap \mathbf{ECom}, \mathbf{DG} \cap \mathbf{ECom}]$ *if and only if it is normal with respect to* $\mathbf{J} \cap \mathbf{Ecom}$*.*

Proof. Since $\overline{\Omega}_A\mathbf{J} \cap \mathbf{ECom}$ is a quotient of $\overline{\Omega}_A\mathbf{DG} \cap \mathbf{ECom}$, clearly regularity on $\mathbf{DG} \cap \mathbf{ECom}$ implies regularity on $\mathbf{J} \cap \mathbf{ECom}$. For the converse, just note that a transition function on an automaton $\mathcal{A}$ of the form (4.1) is regular in the transition monoid $M_{\mathcal{A}_i}$ if and only if the corresponding function on the quotient automaton obtained by identifying all states of $\mathcal{A}_i$ is regular.□

A similar but somewhat simpler situation arises when we consider the pseudovarieties $\mathbf{DH} \cap \mathbf{ZE} = \overline{\mathbf{H}} \cap \mathbf{ZE}$, where $\overline{\mathbf{H}}$ denotes the pseudovariety of all finite monoids whose subgroups lie in $\mathbf{H}$.

Theorem 4.6 *Let* $\mathbf{H}$ *be a pseudovariety of groups and let* $\mathbf{V} = \mathbf{ZE} \cap \overline{\mathbf{H}}$*. Each element* π *of* $\overline{\Omega}_A\mathbf{V}$ *admits a factorization of the form* $\pi = \pi_1u_1\pi_2\dots u_{r-1}\pi_r$*, where* $u_i \in A^+$ $(1 < i < r)$*,* π_i *is regular* $(1 \le i \le r)$*,* $\pi_i \ne \pi_i^\omega$ $(1 \le i < r)$*,* $\pi_r = \pi_r^\omega$ *implies* $\pi_1 = \pi_1^\omega$*,* $c(\pi_1) = c(\pi_2) = \dots = c(\pi_r)$ *and* $c(u_1\cdots u_{r-1}) \cap c(\pi_1) = \emptyset$*.*

Furthermore, if $\pi \in \overline{\Omega}_A\mathbf{V}$ *and* $\pi = \pi_1u_1\pi_2\dots u_{r-1}\pi_r$ *and* $\pi = \rho_1v_1\rho_2\dots v_{s-1}\rho_s$ *are two such factorizations, then* $r = s$*,* $u_i = v_i$ $(1 \le i < r)$*,* $c(\pi_1) = c(\rho_1)$ *and* $\mathbf{H}$ *satisfies* $\pi_i = \rho_i$ $(1 \le i \le r)$*.*

4.2. Some join decompositions

Putting together the results of section 4.1 and Propositions 3.15 and 3.16, we now give some join decompositions of pseudovarieties.

Theorem 4.7 *Let* $\mathbf{H}$ *be an arborescent pseudovariety of groups. Then*

$$(\mathbf{J} \cap \mathbf{ECom}) \vee \mathbf{H} = \mathbf{DH} \cap \mathbf{ECom}.$$

Proof. The inclusion from left to right is obvious. By Reiterman's Theorem, to prove the reverse inclusion, it suffices to show that, for any $\pi, \rho \in \overline{\Omega}_A\mathbf{M}$,

$$(\mathbf{J} \cap \mathbf{ECom}) \vee \mathbf{H} \models \pi = \rho \quad \Longrightarrow \quad \mathbf{DH} \cap \mathbf{ECom} \models \pi = \rho.$$

Let $\pi = u_0\pi_1u_1 \ldots \pi_ru_r$ and $\rho = v_0\rho_1v_1 \ldots \rho_sv_s$ be normal factorizations with respect to $\mathbf{J}\cap\mathbf{ECom}$ (and so also with respect to $\mathbf{DH}\cap\mathbf{ECom}$ by Corollary 4.5) and suppose that the pseudoidentity $\pi = \rho$ is valid in $\mathbf{J} \cap \mathbf{ECom}$ and in $\mathbf{H}$. Then, by Theorem 4.3, $r = s$, $u_i = v_i$ $(0 \leq i \leq r)$ and $c(\pi_i) = c(\rho_i)$ $(1 \leq i \leq r)$. On the other hand, by Proposition 3.15, $\mathbf{H} \models \pi_i = \rho_i$ $(i = 1, \ldots, r)$. Hence $\pi = \rho$ is valid in $\mathbf{DH} \cap \mathbf{ECom}$ by Corollary 2.13.□

Theorem 4.8 *Let* $\mathbf{H}$ *be an arborescent pseudovariety of groups. Then*

$$(\mathbf{A} \cap \mathbf{Com}) \vee \mathbf{H} = \mathbf{ZE} \cap \overline{\mathbf{H}}.$$

Proof. As in the proof of Theorem 4.7, suppose that both $\mathbf{A} \cap \mathbf{Com}$ and $\mathbf{H}$ satisfy the pseudoidentity $\pi = \rho$ and let $\pi = \pi_1u_1\pi_2 \ldots u_{r-1}\pi_r$ and $\rho = \rho_1v_1\rho_2 \ldots v_{s-1}\rho_s$ be factorizations according to Theorem 4.6. We start by observing that, by substituting 1 for every variable except for a given variable a, we obtain a pseudoidentity $|\pi|_a = |\rho|_a$ which is valid in all finite cyclic monoids (which generate $\mathbf{A}\cap\mathbf{Com}$) if and only if it is trivial. But $|\pi|_a$ is a word if and only if $a \notin c(\pi_1)$. Hence, from $\mathbf{A}\cap\mathbf{Com} \models \pi = \rho$ we deduce that $c(\pi_1) = c(\rho_1)$. Then a direct application of Proposition 3.16 shows that $r = s$, $u_i = v_i$ $(i = 1, \ldots, r-1)$ and $\mathbf{H} \models \pi_i = \rho_i$ $(i = 1, \ldots, r)$. Finally, by Corollary 2.13, $\pi = \rho$.□

Note that the above proof of Theorem 4.8 does not really use Theorem 4.6. We next present an example to show that the decompositions of Theorems 4.7 and 4.8 are not valid in general. For the second decomposition, we may already observe that any $\mathbf{H} \subseteq \mathbf{Ab}$ gives a counter-example since $\mathbf{ZE} \cap \overline{\mathbf{Ab}}$ contains non-commutative monoids, namely all monoids of the form S^1 where S is a nilpotent semigroup (in fact, these monoids generate $\mathbf{ZE} \cap \overline{\mathbf{I}} = \mathbf{ZE} \cap \mathbf{A}$).

Example. For a group G, let $G_0 = G$ and let G_{n+1} be the subgroup generated by the commutators $[g,h] = ghg^{-1}h^{-1}$ with $g \in G_n$ and $h \in G$. Recall that the group G is said to be *nilpotent of class* n if $G_n = \{e\}$. We denote by $\mathbf{G}_{nil,n}$ the pseudovariety of all finite nilpotent groups of class n.

Let $A_\infty = \{a_1, a_2, \ldots\}$ be an infinite alphabet. Define, recursively, the words $\alpha_0 = a_1$, $\beta_0 = 1$, $\alpha_{n+1} = \alpha_na_{n+1}\beta_n$ and $\beta_{n+1} = \beta_na_{n+1}\alpha_n$. It is well-known that $\mathbf{G}_{nil,n}$ is the subpseudovariety of $\mathbf{G}$ defined by the identity $\alpha_n = \beta_n$ and that $\mathbf{G}_{nil,n}$ is strictly contained in $\mathbf{G}_{nil,n+1}$ [18].

Consider the pseudoidentity

$$\alpha_n(\alpha_n\beta_n)^\omega a_{n+1}\beta_n(\alpha_n\beta_n)^\omega = \beta_n(\alpha_n\beta_n)^\omega a_{n+1}\alpha_n(\alpha_n\beta_n)^\omega.$$

It is valid in $\mathbf{J} \cap \mathbf{ECom}$ and in $\mathbf{G}_{nil,n+1}$. However, since both its sides are in normal form, it is not valid in $\mathbf{DG}_{nil,n+1}\cap\mathbf{ECom}$. Hence $(\mathbf{J}\cap\mathbf{ECom})\vee\mathbf{G}_{nil,n+1} \neq \mathbf{DG}_{nil,n+1} \cap \mathbf{ECom}$.

On the other hand, the pseudovariety $\mathbf{G}_{nil}$ of all finite nilpotent groups is generated by a union of arborescent pseudovarieties of groups, namely the pseudovarieties $\mathbf{G}_p$ of all finite p-groups, where p is a prime. This implies that $\mathbf{G}_{nil}$ satisfies a pseudoidentity $\pi = \rho$ if and only if $\pi = \rho$ is valid in all $\mathbf{G}_p$ (p prime). Therefore, adopting the same notation as in the proof of Theorem 4.7, by Proposition 3.15 we have

$$\mathbf{G}_{nil} \models u_0\pi_1 u_1 \dots \pi_r u_r = u_0 \rho_1 u_1 \dots \rho_r u_r \quad \Longrightarrow \quad \mathbf{G}_{nil} \models \pi_i = \rho_i \text{ for } 1 \leq i \leq r.$$

Hence $(\mathbf{J} \cap \mathbf{ECom}) \vee \mathbf{G}_{nil} = \mathbf{DG}_{nil} \cap \mathbf{ECom}$. Similarly, $(\mathbf{A} \cap \mathbf{Com}) \vee \mathbf{G}_{nil} = \mathbf{ZE} \cap \overline{\mathbf{G}_{nil}}$.

4.3. The corresponding varieties of languages

The results of the preceding sections may now be easily translated into corresponding results on varieties of languages according to Eilenberg's correspondence.

For a pseudovariety $\mathbf{H}$ of groups, denote by $\mathcal{L}(\mathbf{H})$ the class of all languages of the form $u_0 L_1 u_1 \dots L_r u_r$ such that $r \geq 0$, $u_i \in A^*$ and:

- $u_i \neq 1$ for $0 < i < r$;
- each L_i is a group language over the non-empty sub-alphabet A_i of A, whose syntactic monoid lies in $\mathbf{H}$ and whose minimal automaton has only one terminal state;
- the first letter of each u_i is not in A_i;
- the last letter of each u_i is not in A_{i+1}.

Note that, under the notation of section 4.1, if the minimal automaton of L_i is $\mathcal{A}_i$, then the minimal automaton of such a language $u_0 L_1 u_1 \dots L_r u_r$ is the automaton $\mathcal{A} = \mathcal{A}(u_0; \mathcal{A}_1, p_1, q_1; u_1; \dots; \mathcal{A}_r, p_r, q_r; u_r)$ where the initial and terminal states are, respectively, q_0 and p_{r+1}. Hence, by Lemma 4.2, the languages of $\mathcal{L}(\mathbf{H})$ are $\mathbf{DH} \cap \mathbf{ECom}$-recognizable. If $L \in \mathcal{L}(\mathbf{H})$, then the closure $\overline{L}$ of L in $\overline{\Omega}_A(\mathbf{DH} \cap \mathbf{ECom})$ consists precisely of those elements of $\mathbf{DH} \cap \mathbf{ECom}$ whose canonical images in the transition monoid of $\mathcal{A}$ map q_0 to p_{r+1}. Finally, note that, in case $\mathbf{H} = \mathbf{I}$, the languages L_i are of the form A_i^*.

Theorem 4.9 *For any pseudovariety* $\mathbf{H}$ *of groups, the class of languages in* A^* *which are recognized by monoids in* $\mathbf{DH} \cap \mathbf{ECom}$ *is the Boolean algebra generated by* $\mathcal{L}(\mathbf{H})$.

Proof. In view of the above remarks, by Proposition 1.13, it suffices to observe that the closures of the languages in $\mathcal{L}(\mathbf{H})$ separate the points of $\overline{\Omega}_A(\mathbf{DH} \cap \mathbf{ECom})$ and this was done in the proof of Theorem 4.3.□

Take for $\mathcal{L}'(\mathbf{H})$ the class of all languages of the form $L_1 a_1 L_2 \dots a_{r-1} L_r$ such that $r \geq 0$ and:

- each L_i is a group language over a fixed subset B of A, whose syntactic monoid (over B) is in $\mathbf{H}$ and whose minimal automaton has only one terminal state;
- each $a_i \in A \setminus B$.

Then, just as above, we have the following result.

Theorem 4.10 *For any pseudovariety* $\mathbf{H}$ *of groups, the class of languages in* A^* *which are recognized by monoids in* $\mathbf{ZE} \cap \overline{\mathbf{H}}$ *is the Boolean algebra generated by* $\mathcal{L}'(\mathbf{H})$.$\square$

Finally, in case $\mathbf{H}$ is arborescent, the results of section 4.2 allow us to restrict even further the generators needed to obtain the above varieties of languages.

Theorem 4.11 *Let* $\mathbf{H}$ *be an arborescent pseudovariety of groups. The class of languages in* A^* *which are recognized by monoids in* $\mathbf{DH} \cap \mathbf{ECom}$ *is the Boolean algebra generated by the group languages with syntactic monoid in* $\mathbf{H}$ *and the languages of the form* $u_0 A_1^* u_1 \cdots A_r^* u_r$, *where* $r \geq 0$, $\emptyset \neq A_i \subseteq A$, *each* u_i *is in* A^* *with* $|u_i| \geq 1$ *for* $0 < i < r$, *the first letter of each* u_i *is not in* A_i *and the last letter of each* u_i *is not in* A_{i+1}.$\square$

Theorem 4.12 *Let* $\mathbf{H}$ *be an arborescent pseudovariety of groups. The class of languages in* A^* *which are recognized by monoids in* $\mathbf{ZE} \cap \overline{\mathbf{H}}$ *is the Boolean algebra generated by the group languages with syntactic monoid in* $\mathbf{H}$ *and the languages of the form* $L(a,k,n) = \{u \in A^* : |u|_a = k\}$, *where* $0 \leq k < n$ *and* $a \in A$.$\square$

Of course, Theorems 4.11 and 4.12 fail exactly for those pseudovarieties for which Theorems 4.7 and 4.8, respectively, fail. So, in particular, they fail for each $\mathbf{H} = \mathbf{G}_{nil,n}$ with $n \geq 1$.

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