

SEPARATION FOR DOT-DEPTH TWO

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ABSTRACT. The dot-depth hierarchy of Brzozowski and Cohen classifies the star-free languages of finite words. By a theorem of McNaughton and Papert, these are also the first-order definable languages. The dot-depth rose to prominence following the work of Thomas, who proved an exact correspondence with the quantifier alternation hierarchy of first-order logic: each level in the dot-depth hierarchy consists of all languages that can be defined with a prescribed number of quantifier blocks. One of the most famous open problems in automata theory is to settle whether the membership problem is decidable for each level: is it possible to decide whether an input regular language belongs to this level ?

Despite a significant research effort, membership by itself has only been solved for low levels. A recent breakthrough was achieved by replacing membership with a more general problem: *separation*. Given two input languages, one has to decide whether there exists a third language in the investigated level containing the first language and disjoint from the second. The motivation for looking at separation is threefold: (1) while more difficult, it is more rewarding, as solving it requires a better understanding; (2) being more general, it provides a more convenient framework, and (3) all recent membership algorithms are actually reductions to separation for lower levels.

We present a separation algorithm for dot-depth two. A key point is that while dot-depth two is our most prominent application, our theorem is more general. We consider a family of hierarchies which includes the dot-depth: the concatenation hierarchies. They are built via a generic construction process. One first chooses an initial class, the basis, which serves as the lowest level in the hierarchy. Then, further levels are built by applying generic operations. Our main theorem states that for any concatenation hierarchy whose basis contains finitely many languages, separation is decidable for level one. In the special case of the dot-depth, this can be lifted to level two using previously known results.

1. INTRODUCTION

Concatenation hierarchies. Many fundamental problems about regular languages raised in the 70s [Pin17b] led to considerable advances, not only in automata theory but also in logic and algebra, thanks to the discovery of deep connections between these areas that led to the problems' solutions. Even if some of these questions are now well understood, a few others

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remain wide open, despite a wealth of research work spanning several decades. This is the case for the fascinating dot-depth problem [Pin17a], which has two elementary formulations: a language-theoretic one and a logical one. The language-theoretic one is the older of the two. It takes its roots in a theorem of Schützenberger [Sch65] (see also [DKRH16, DG08]), which gives an algorithm to decide whether a regular language is star-free, *i.e.*, can be expressed using union, complement and concatenation, but without the Kleene star operator. This celebrated result was highly influential for three reasons:

- First, Schützenberger precisely formalized the objective of “understanding the expressive power of a formalism” through a decision problem called *membership*, which asks whether an input regular language belongs to a class under study.
- Next, he developed a methodology for tackling it, which he applied to membership for the class of star-free languages.
- Finally, McNaughton and Papert [MP71] established that star-free languages are exactly the first-order definable ones.

This work highlighted the robustness of the notion of regularity, underlining the ties between automata theory and logic, and revealing new links with algebra. It also established membership as the reference problem for investigating classes of languages.

Schützenberger’s theorem led Brzozowski and Cohen to define the dot-depth hierarchy [BC71], an infinite classification [BK78] of all star-free languages counting the number of alternations between concatenations and complements needed to define them. This definition is a particular instance of a generic construction process, which was formalized later and named *concatenation hierarchies*. Any such hierarchy has a single parameter: a “level 0 class” (its *basis*). Then, one uses two operations, polynomial and Boolean closure, to build two kinds of classes: half levels $1/2, 3/2, 5/2 \dots$ and full levels $1, 2, 3 \dots$. Given a class of languages \mathcal{C} , its *polynomial closure* $Pol(\mathcal{C})$ is the least class of languages containing \mathcal{C} and closed under union and marked concatenation ($K, L \mapsto KaL$, where a is a letter). Its *Boolean closure* $Bool(\mathcal{C})$ is the least class containing \mathcal{C} and closed under union and complement. For any full level n , the next half and full levels are built as follows:

- Level $n + \frac{1}{2}$ is the polynomial closure of level n .
- Level $n + 1$ is the Boolean closure of level $n + \frac{1}{2}$.

Thus, a concatenation hierarchy is fully determined by its basis. In the paper, we are interested in hierarchies with a *finite* basis.

The most prominent hierarchies of this kind in the literature are the dot-depth and the Straubing-Thérien hierarchy [Str81, Th81]. They acquired this status when it was discovered [Tho82, PP86] that each of them coincides with the quantifier alternation hierarchy within an appropriate variant of first-order logic. These two variants have the same overall expressiveness but slightly different signatures (which impacts the properties that one can define at a given level of their quantifier alternation hierarchies).

These correspondences motivated a research program to solve membership for all levels of both hierarchies, thus also characterizing the alternation hierarchies of first-order logic. However, progress has been slow. Until recently, the classes that were solved for both variants are only level $1/2$ [Arf87, PW97], level 1 [Sim75, Kna83] and level $3/2$ [Arf87, PW97, GS00]. See [DGK08] for a survey. Following these results, membership for level 2 remained open for a long time and was named the “dot-depth two problem”.

Separation. Recently [PZ14, Pla15, Pla18], solutions were found for levels 2, 5/2 and 7/2. The key ingredient is a new problem stronger than membership: *separation*. Rather than asking whether an input language belongs to the class \mathcal{C} under investigation, the \mathcal{C} -separation problem takes as input *two* languages, and asks whether there exists a third one *from* \mathcal{C} containing the first and disjoint from the second. While the interest in separation is recent, it has quickly replaced membership as the central question. A first practical reason is that separation proved itself to be a key ingredient in obtaining all recent membership results. See [PZ15b, PZ18b] for an overview. A striking example is provided by a crucial theorem of [PZ14]. It establishes a generic reduction from $Pol(\mathcal{C})$ -separation to \mathcal{C} -membership which holds for any class \mathcal{C} . Combined with a separation algorithm for level 3/2 and a little extra work, this yields a membership algorithm for level 5/2.

However, the main reason is deeper. The primary motivation for considering such problems is to thoroughly understand the classes under investigation. In this respect, while harder, separation is also far more rewarding than membership. On one hand, a membership algorithm for a class \mathcal{C} only applies to languages of \mathcal{C} : it can detect them and build a description witnessing membership. On the other hand, a separation algorithm for \mathcal{C} is universal: it applies to *any* language. Indeed, one may view separation as an approximation problem: given an input pair (L_1, L_2) one wants to over-approximate L_1 by a language in \mathcal{C} , and L_2 serves to specify what a satisfying approximation is. This is why we look at separation: it yields a more robust understanding of the classes than membership.

The state of the art for separation is the following: it was shown to be decidable for levels 1/2, 1, 3/2 and 5/2 in the Straubing-Thérien hierarchy [CMM13, PvRZ13, PZ14, Pla15, Pla18]. These results can be lifted to dot-depth using a generic transfer theorem [PZ15a, PZ17a]. Notice the gap between levels 3/2 and 5/2: no algorithm is known for level 2. This is explained by the fact that obtaining separation algorithms presents very different challenges for half levels and for full levels. Indeed, it turns out that most separation algorithms rely heavily on closure under marked concatenation, which holds for half levels by definition, but not for full levels.

Contributions. Our main result is a separation algorithm for level 2 in the Straubing-Thérien hierarchy. Furthermore, by the aforementioned transfer theorem [PZ15a, PZ17a], this can be lifted to separation for dot-depth 2. A crucial point is that this separation result is actually an instance of a generic theorem, which applies to any finite class \mathcal{C} satisfying a few standard properties (namely closure under Boolean operations and quotients). It states that for such a class \mathcal{C} , $Bool(Pol(\mathcal{C}))$ has decidable separation. This has two important consequences,

- In *any* hierarchy whose basis is such a class, level 1 has decidable separation.
- In the specific case of the Straubing-Thérien hierarchy, this extends to level 2 since it is also level 1 of another finitely based concatenation hierarchy [PS85].

This generic result complements others recent results in a natural way. It has been shown in [Pla18] that $Pol(\mathcal{C})$ - and $Pol(Bool(Pol(\mathcal{C})))$ -separation are decidable for any finite class \mathcal{C} satisfying the aforementioned hypotheses. Combined with our results, this implies that for finitely based concatenation hierarchies, separation is decidable for all levels up to 3/2.

Remark 1.1. *Our proof argument exploits a theorem of [Pla18] about the simpler class $Pol(\mathcal{C})$. However, the techniques that we use here are very different from the ones used in [Pla18], which rely heavily on the fact that $Pol(\mathcal{C})$ and $Pol(Bool(Pol(\mathcal{C})))$ are closed under concatenation (again, this is not the case for $BPol(\mathcal{C})$).*

Being generic, our approach yields separation algorithms for a whole family of classes. Moreover, it serves to pinpoint the key hypotheses which are critical in order to solve separation for dot-depth 2. Let us also stress that we obtain new direct proofs that separation is decidable for level 1 in both the dot-depth and Straubing-Thérien hierarchies. This is of particular interest for dot-depth 1 since the previous solution was indirect, as it relied on a transfer result from [PZ15a, PZ17a].

Finally, a key point is that all our results are presented and proved using a general framework that was recently introduced in [PZ18a]. It is designed to handle the separation problem and present the solutions in an elegant manner. In fact, this framework considers a third decision problem which is even more general than separation: *covering*. Given a class \mathcal{C} , \mathcal{C} -covering takes two objects as input, a single regular language L and a finite set of regular languages \mathbf{L} . It asks whether there exists a \mathcal{C} -cover of L (*i.e.*, a finite set of languages in \mathcal{C} whose union includes L) such that no language in this \mathcal{C} -cover intersects all languages in \mathbf{L} . It is simple to show that separation is the special case of covering when the set \mathbf{L} is a singleton. Our main theorem actually states that $Bool(Pol(\mathcal{C}))$ -covering is decidable for any finite class \mathcal{C} satisfying mild hypotheses.

Organization. The paper is organized as follows. Section 2 gives preliminary definitions. In Section 3, we introduce concatenation hierarchies and state our generic theorem: $BPol(\mathcal{C})$ -covering is decidable for every finite quotient-closed Boolean algebra \mathcal{C} . The remainder of the paper is devoted to presenting the algorithm. In Section 4, we recall a framework which was designed in [PZ18a] to handle the covering problem. Then, we use this framework to formulate our algorithm for $Bool(Pol(\mathcal{C}))$ -covering in Section 5. The remaining sections are devoted to the correction proof of this algorithm.

This is the journal version of [PZ17b]. The main result (*i.e.*, that $BPol(\mathcal{C})$ -separation is decidable for every finite quotient-closed Boolean algebra \mathcal{C}) has been generalized to $BPol(\mathcal{C})$ -covering. Moreover, both its presentation and its proof have been completely reworked. Our results are now formulated using the general framework designed to handle separation and covering problems which was introduced in [PZ18a].

2. PRELIMINARIES

In this preliminary section, we provide standard definitions for the objects which are investigated in the paper. Moreover, we present the separation and covering problems.

2.1. Words, languages and classes. For the whole paper, we fix an arbitrary finite alphabet A . Recall that A^* denotes the set of all words over A , including the empty word ε . We let $A^+ = A^* \setminus \{\varepsilon\}$. If $u, v \in A^*$ are words, we write $u \cdot v$ or uv the word obtained by concatenating u and v .

A subset of A^* is called a *language*. We shall denote the singleton language $\{u\}$ by u . It is standard to generalize the concatenation operation to languages: given $K, L \subseteq A^*$, we write KL for the language $KL = \{uv \mid u \in K \text{ and } v \in L\}$. Moreover, we shall also consider *marked concatenation*, which is less standard. Given $K, L \subseteq A^*$, a marked concatenation of K with L is a language of the form KaL for some letter $a \in A$.

A *class of languages* \mathcal{C} is simply a set of languages. In the paper, we work with robust classes which satisfy standard closure properties:

- A *lattice* is a class \mathcal{C} closed under finite union and finite intersection, and such that $\emptyset \in \mathcal{C}$ and $A^* \in \mathcal{C}$.
- A *Boolean algebra* is a lattice closed under complement.
- Finally, a class \mathcal{C} is *quotient-closed* when for all $L \in \mathcal{C}$ and all $w \in A^*$, the following two languages belong to \mathcal{C} ,

$$w^{-1}L \stackrel{\text{def}}{=} \{u \in A^* \mid wu \in L\} \quad \text{and} \quad Lw^{-1} \stackrel{\text{def}}{=} \{u \in A^* \mid uw \in L\}$$

All classes considered in the paper will be (at least) quotient-closed lattices. Moreover, they are all included in the class of *regular languages*. These are the languages that can be equivalently defined by monadic second-order logic, finite automata or finite monoids. Let us briefly recall the definition based on monoids, which we shall use.

A *monoid* is a set M equipped with an associative multiplication (usually denoted by “ \cdot ”) which has a neutral element (usually denoted by “ 1_M ”). Recall that an idempotent within a semigroup S is an element $e \in S$ such that $ee = e$. Observe that A^* is a monoid when equipped with word concatenation as the multiplication (the neutral element is ε). Hence, given a monoid M , we may consider morphisms $\alpha : A^* \rightarrow M$. Given such a morphism, we say that a language $L \subseteq A^*$ is *recognized* by α when there exists $F \subseteq M$ such that $L = \alpha^{-1}(F)$. It is well-known that the regular languages are exactly those which can be recognized by a morphism $\alpha : A^* \rightarrow M$ where M is a **finite** monoid.

Remark 2.1. *Whenever we consider a morphism $\alpha : A^* \rightarrow M$ in the paper, the monoid M will be finite. For the sake of avoiding clutter, this will be implicit from now on.*

2.2. Separation and covering. In the paper, we investigate specific classes of languages which are part of concatenation hierarchies (introduced in Section 3). We do so by relying on two decision problems: separation and covering. The former is standard while the latter was introduced in [PZ18a]. Both of them are parametrized by an arbitrary class of languages \mathcal{C} . Let us start with the definition of separation.

Separation. Given three languages K, L_1, L_2 , we say that K *separates* L_1 from L_2 if $L_1 \subseteq K$ and $L_2 \cap K = \emptyset$. Given a class of languages \mathcal{C} , we say that L_1 is \mathcal{C} -*separable* from L_2 if some language in \mathcal{C} separates L_1 from L_2 . Observe that when \mathcal{C} is not closed under complement, the definition is not symmetrical: L_1 could be \mathcal{C} -separable from L_2 while L_2 is not \mathcal{C} -separable from L_1 . The separation problem associated to a given class \mathcal{C} is as follows:

INPUT: Two regular languages L_1 and L_2 .
OUTPUT: Is L_1 \mathcal{C} -separable from L_2 ?

Separation is meant to be used as a mathematical tool in order to investigate classes of languages. Intuitively, obtaining a \mathcal{C} -separation algorithm requires a solid understanding of \mathcal{C} .

Remark 2.2. *The \mathcal{C} -separation problem generalizes another classical decision problem: \mathcal{C} -membership which asks whether a single regular language L belongs to \mathcal{C} . Indeed, $L \in \mathcal{C}$ if and only if L is \mathcal{C} -separable from $A^* \setminus L$.*

Covering. Our second problem is more general and was introduced in [PZ18a]. Given a language L , a *cover* of L is a **finite** set of languages \mathbf{K} such that,

$$L \subseteq \bigcup_{K \in \mathbf{K}} K.$$

We speak of *universal cover* to mean a cover of A^* . Moreover, if \mathcal{C} is a class, a \mathcal{C} -cover of L is a cover \mathbf{K} of L such that all $K \in \mathbf{K}$ belong to \mathcal{C} .

Covering takes as input a language L_1 and a *finite set of languages* \mathbf{L}_2 . A *separating cover* for the pair (L_1, \mathbf{L}_2) is a cover \mathbf{K} of L_1 such that for every $K \in \mathbf{K}$, there exists $L \in \mathbf{L}_2$ which satisfies $K \cap L = \emptyset$. Finally, given a class \mathcal{C} , we say that the pair (L_1, \mathbf{L}_2) is \mathcal{C} -coverable when there exists a separating \mathcal{C} -cover. The \mathcal{C} -covering problem is now defined as follows:

INPUT: A regular language L_1 and a finite set of regular languages \mathbf{L}_2 .
OUTPUT: Is (L_1, \mathbf{L}_2) \mathcal{C} -coverable?

It is straightforward to prove that covering generalizes separation (provided that the class \mathcal{C} is a lattice) as stated in the following lemma (see Theorem 3.5 in [PZ18a] for the proof).

Lemma 2.3. *Let \mathcal{C} be a lattice and L_1, L_2 two languages. Then L_1 is \mathcal{C} -separable from L_2 , if and only if $(L_1, \{L_2\})$ is \mathcal{C} -coverable.*

In the paper, we shall not work with covering directly. Instead, we use a framework which is designed to handle these problems and was introduced in [PZ18a]. We recall it in Section 4.

2.3. Finite lattices. In the paper, we work with classes built from an arbitrary finite lattice (*i.e.*, one that contains finitely many languages) using generic operations (polynomial closure and Boolean closure, see Section 3). Here, we present standard results about such classes that will be important later.

Canonical preorder relations. Consider a finite lattice \mathcal{C} . It is classical to associate a *canonical preorder relation over A^* to \mathcal{C}* . Given $w, w' \in A^*$, we write $w \leq_{\mathcal{C}} w'$ if and only if the following holds:

$$\text{For all } L \in \mathcal{C}, \quad w \in L \Rightarrow w' \in L.$$

It is immediate from the definition that $\leq_{\mathcal{C}}$ is transitive and reflexive, making it a preorder.

Example 2.4. *Consider the class AT which contains all Boolean combinations of languages A^*aA^* , for some $a \in A$ (“AT” stands for “alphabet testable”: $L \in \text{AT}$ if and only if membership of a word w in L depends only on the letters occurring in w). Clearly, AT is a finite Boolean algebra. In that case, \leq_{AT} is an equivalence relation which we denote by \sim_{AT} : $w \sim_{\text{AT}} w'$ if and only if w and w' have the same alphabet (*i.e.*, contain the same set of letters).*

We shall use several results about the relation $\leq_{\mathcal{C}}$. We omit the proofs which are simple and available in [Pla18].

The first lemma is where we use the hypothesis that \mathcal{C} is finite. We say that a language $L \subseteq A^*$ is an *upper set* (for $\leq_{\mathcal{C}}$) when for any two words $u, v \in A^*$, if $u \in L$ and $u \leq_{\mathcal{C}} v$, then $v \in L$. Furthermore, given $u \in A^*$, we let $\uparrow_{\mathcal{C}} u \subseteq A^*$ be the least upper set containing u : $\uparrow_{\mathcal{C}} u = \{v \in A^* \mid u \leq_{\mathcal{C}} v\}$.

Lemma 2.5. *Let \mathcal{C} be a finite lattice. Then, for any $L \subseteq A^*$, we have $L \in \mathcal{C}$ if and only if L is an upper set for $\leq_{\mathcal{C}}$. In particular, $\leq_{\mathcal{C}}$ has finitely many upper sets.*

Let us complete these definitions with a few additional useful results. First, as we observed for AT in Example 2.4, when the finite lattice \mathcal{C} is actually a Boolean algebra, it turns out that $\leq_{\mathcal{C}}$ is an equivalence relation, which we shall denote by $\sim_{\mathcal{C}}$.

Lemma 2.6. *Let \mathcal{C} be a finite Boolean algebra. Then for any alphabet A , the canonical preorder $\leq_{\mathcal{C}}$ is an equivalence relation $\sim_{\mathcal{C}}$ which admits the following direct definition:*

$$w \sim_{\mathcal{C}} w' \text{ if and only if for all } L \in \mathcal{C}, \quad w \in L \Leftrightarrow w' \in L$$

Thus, for any $L \subseteq A^$, we have $L \in \mathcal{C}$ if and only if L is a union of $\sim_{\mathcal{C}}$ -classes. In particular, $\sim_{\mathcal{C}}$ has finite index.*

Another important and useful property is that when \mathcal{C} is quotient-closed, the canonical preorder $\leq_{\mathcal{C}}$ is compatible with word concatenation.

Lemma 2.7. *A finite lattice \mathcal{C} is quotient-closed if and only if its associated canonical preorder $\leq_{\mathcal{C}}$ is compatible with word concatenation. That is, for any words u, v, u', v' ,*

$$u \leq_{\mathcal{C}} u' \quad \text{and} \quad v \leq_{\mathcal{C}} v' \quad \Rightarrow \quad uv \leq_{\mathcal{C}} u'v'.$$

\mathcal{C} -compatible morphisms. We use these notions to define special monoid morphisms. We fix a finite quotient-closed Boolean algebra \mathcal{C} for the definition.

By Lemma 2.6, $\sim_{\mathcal{C}}$ has finite index and the languages in \mathcal{C} are exactly the unions of $\sim_{\mathcal{C}}$ -classes. Moreover, since \mathcal{C} is quotient-closed, we know from Lemma 2.7 that $\sim_{\mathcal{C}}$ is a congruence for word concatenation. It follows that the quotient set $A^*/\sim_{\mathcal{C}}$ is a finite monoid and the map $w \mapsto [w]_{\mathcal{C}}$ is a morphism from A^* to $A^*/\sim_{\mathcal{C}}$. Consider an arbitrary morphism $\alpha : A^* \rightarrow M$. We say that α is \mathcal{C} -compatible when, for every $s \in M$, there exists a $\sim_{\mathcal{C}}$ -class denoted by $[s]_{\mathcal{C}}$ such that $\alpha^{-1}(s) \subseteq [s]_{\mathcal{C}}$ (i.e., $[w]_{\mathcal{C}} = [s]_{\mathcal{C}}$ for every $w \in A^*$ such that $\alpha(w) = s$).

Remark 2.8. *Given $s \in M$, the $\sim_{\mathcal{C}}$ -class $[s]_{\mathcal{C}}$ is determined by α when $\alpha^{-1}(s) \neq \emptyset$ ($[s]_{\mathcal{C}} = [w]_{\mathcal{C}}$ for any $w \in \alpha^{-1}(s)$). If $\alpha^{-1}(s) = \emptyset$, we may choose any $\sim_{\mathcal{C}}$ -class as $[s]_{\mathcal{C}}$. When we consider a \mathcal{C} -compatible morphism, we implicitly assume that the map $s \mapsto [s]_{\mathcal{C}}$ is fixed.*

We prove that we may assume without generality that all our morphisms are \mathcal{C} -compatible: any regular language L is recognized by a \mathcal{C} -compatible morphism which recognizes L .

Lemma 2.9. *Let \mathcal{C} be a finite quotient-closed Boolean algebra. Given a regular language $L \subseteq A^*$, one may compute a \mathcal{C} -compatible morphism $\alpha : A^* \rightarrow M$ recognizing L .*

Proof. Since L is regular, we may compute a finite monoid N and a morphism $\beta : A^* \rightarrow N$ (not necessarily \mathcal{C} -compatible) recognizing L . Since we know that the quotient set $A^*/\sim_{\mathcal{C}}$ is a finite monoid, the Cartesian product $M = N \times (A^*/\sim_{\mathcal{C}})$ is a finite monoid for the componentwise multiplication. It now suffices to define the morphism $\alpha : A^* \rightarrow M$ by $\alpha(w) = (\beta(w), [w]_{\mathcal{C}})$ for any $w \in A^*$. Clearly, α is a morphism which recognizes L and one may verify that it is \mathcal{C} -compatible. \square

3. CLOSURE OPERATIONS AND MAIN THEOREM

In this section, we define the family of classes that we investigate in the paper and present a few results about them. Then, we state our main theorem and discuss its consequences.

3.1. Closure operations. Consider a class \mathcal{C} . The *Boolean closure* of \mathcal{C} , denoted by $Bool(\mathcal{C})$ is defined as the least Boolean algebra containing \mathcal{C} . We have the following lemma which can be obtained from the definitions (this amounts to verifying that quotients commute with Boolean operations).

Lemma 3.1. *Let \mathcal{C} be a quotient-closed lattice. Then $Bool(\mathcal{C})$ is a quotient-closed Boolean algebra.*

The second operation that we shall consider is slightly more involved. Given a class \mathcal{C} , the *polynomial closure* of \mathcal{C} , denoted by $Pol(\mathcal{C})$, is the least class containing \mathcal{C} which is closed under both union and marked concatenation:

$$\text{for all } K, L \in Pol(\mathcal{C}) \text{ and } a \in A, \quad K \cup L \in Pol(\mathcal{C}) \text{ and } KaL \in Pol(\mathcal{C}).$$

While this is not obvious from the definition, when the input class \mathcal{C} is a quotient-closed lattice, its polynomial closure $Pol(\mathcal{C})$ is a quotient-closed lattice as well (the difficulty is to prove that $Pol(\mathcal{C})$ is closed under intersection). This was originally proved by Arfi [Arf87] (see also [Pin13] or [PZ18b] for recent proofs).

Theorem 3.2 (Arfi [Arf87]). *Let \mathcal{C} be a quotient-closed lattice. Then, $Pol(\mathcal{C})$ is a quotient-closed lattice closed under concatenation and marked concatenation.*

In the paper, we consider classes of the form $Bool(Pol(\mathcal{C}))$ built by applying polynomial closure and Boolean closure successively to some arbitrary quotient-closed Boolean algebra \mathcal{C} . For the sake of avoiding clutter, we shall write $BPol(\mathcal{C})$ for $Bool(Pol(\mathcal{C}))$. Note that by Lemma 3.1 and Theorem 3.2, we have the following corollary.

Corollary 3.3. *Let \mathcal{C} be a quotient-closed lattice. Then, $BPol(\mathcal{C})$ is a quotient-closed Boolean algebra.*

A key remark is that classes built with Boolean closure (such as $BPol(\mathcal{C})$) are *not* closed under concatenation. This contrasts with polynomial closure in which closure under (marked) concatenation holds by definition. This is an issue, as most of our techniques designed for handling separation and covering rely heavily on concatenation. We cope with this problem by using the following weak concatenation principle which holds for any class that is the Boolean closure of another class which is itself closed under concatenation.

Lemma 3.4. *Let \mathcal{G} be a lattice closed under concatenation. Consider $L, L' \in \mathcal{G}$ and let \mathbf{K}, \mathbf{K}' be $Bool(\mathcal{G})$ -covers of L and L' respectively. There exists a $Bool(\mathcal{G})$ -cover \mathbf{H} of LL' such that for every $H \in \mathbf{H}$, we have $H \subseteq KK'$ for some $K \in \mathbf{K}$ and $K' \in \mathbf{K}'$.*

Proof. Every language in $\mathbf{K} \cup \mathbf{K}'$ is a Boolean combination of languages in \mathcal{G} and $L, L' \in \mathcal{G}$. Therefore, there exists a finite lattice $\mathcal{C} \subseteq \mathcal{G}$ which satisfies the two following properties:

- (1) $L, L' \in \mathcal{C}$ and,
- (2) every language $K \in \mathbf{K} \cup \mathbf{K}'$ belongs to $Bool(\mathcal{C})$.

We define \mathcal{F} as the least lattice such that $HH' \in \mathcal{F}$ for every $H, H' \in \mathcal{C}$. Since $\mathcal{C} \subseteq \mathcal{G}$ and \mathcal{G} is a lattice closed under concatenation, we know that \mathcal{F} is a finite lattice such that $\mathcal{F} \subseteq \mathcal{G}$. It follows that $Bool(\mathcal{F})$ is a finite Boolean algebra such that $Bool(\mathcal{F}) \subseteq Bool(\mathcal{G})$.

Consider the canonical equivalence $\sim_{Bool(\mathcal{F})}$ associated to $Bool(\mathcal{F})$ (it compares words belonging to the same languages of $Bool(\mathcal{F})$, see Section 2). Since $L, L' \in \mathcal{C}$, it is immediate by definition of \mathcal{F} that we have $LL' \in \mathcal{F} \subseteq Bool(\mathcal{F})$. Therefore, Lemma 2.6 implies that LL' is a union of $\sim_{Bool(\mathcal{F})}$ -classes. We let \mathbf{H} be the set consisting of all these $\sim_{Bool(\mathcal{F})}$ -classes.

By definition, \mathbf{H} is a $Bool(\mathcal{F})$ -cover of LL' and therefore a $Bool(\mathcal{G})$ -cover as well since $Bool(\mathcal{F}) \subseteq Bool(\mathcal{G})$. It remains to prove that for every $H \in \mathbf{H}$, we have $H \subseteq KK'$ for some $K \in \mathbf{K}$ and $K' \in \mathbf{K}'$. We fix $H \in \mathbf{H}$ for the proof. We use the following fact.

Fact 3.5. *Consider a finite language $G \subseteq H$. Then, there exists $K \in \mathbf{K}$ and $K' \in \mathbf{K}'$ such that $G \subseteq KK'$.*

Let us first admit Fact 3.5 and apply it to finish the main proof. For every $n \in \mathbb{N}$, we let $G_n \subseteq H$ be the finite language containing all words of length at most n in H . Clearly, we have,

$$H = \bigcup_{n \in \mathbb{N}} G_n \quad \text{and} \quad G_n \subseteq G_{n+1} \quad \text{for all } n \in \mathbb{N}$$

For every $n \in \mathbb{N}$, Fact 3.5 yields $K_n \in \mathbf{K}$ and $K'_n \in \mathbf{K}'$ such that $G_n \subseteq K_n K'_n$. Since \mathbf{K} and \mathbf{K}' are finite sets, there exist $K \in \mathbf{K}$ and $K' \in \mathbf{K}'$ such that $K_n = K$ and $K'_n = K'$ for infinitely many n . Since $G_n \subseteq G_{n+1}$ for all n , it follows that $G_n \subseteq KK'$ for every $n \in \mathbb{N}$. Finally, since $H = \bigcup_{n \in \mathbb{N}} G_n$, this implies $H \subseteq KK'$, finishing the proof.

It remains to prove Fact 3.5. Consider a finite language $G \subseteq H$ and let $G = \{w_1, \dots, w_n\}$. We exhibit $K \in \mathbf{K}$ and $K' \in \mathbf{K}'$ such that $G \subseteq KK'$.

By definition, H is a $\sim_{Bool(\mathcal{F})}$ -class included in LL' . This implies that $w_1, \dots, w_n \in LL'$ and $w_1 \sim_{Bool(\mathcal{F})} \dots \sim_{Bool(\mathcal{F})} w_n$. Using these equivalences, we first prove the following claim which involves the canonical preorder \leq_C associated to the finite lattice \mathcal{C} .

Claim. *For every $u, v \in A^*$ such that $w_n = uv$, there exist $u_1, \dots, u_n, v_1, \dots, v_n \in A^*$ such that $w_i = u_i v_i$ for every $i \leq n$, $u \leq_C u_1 \leq_C \dots \leq_C u_n$ and $v \leq_C v_1 \leq_C \dots \leq_C v_n$.*

Proof. We prove the existence of $u_1, v_1 \in A^*$ such that $w_1 = u_1 v_1$, $u \leq_C u_1$ and $v \leq_C v_1$ using the hypothesis that $w_n = uv$ and $w_n \sim_{Bool(\mathcal{F})} w_1$. One may then iterate the argument to build $u_2, \dots, u_n \in A^*$ and $v_2, \dots, v_n \in A^*$ since $w_1 \sim_{Bool(\mathcal{F})} \dots \sim_{Bool(\mathcal{F})} w_n$.

Consider the languages $U = \uparrow_C u$ and $V = \uparrow_C v$ (the upper sets of u and v for \leq_C). By Lemma 2.5, we have $U, V \in \mathcal{C}$. Therefore, we have $UV \in \mathcal{F}$ by definition of \mathcal{F} . Clearly, $w_n = uv \in UV$. Therefore, $w_n \sim_{Bool(\mathcal{F})} w_1$, implies that $w_1 \in UV$. This yields a decomposition $w_1 = u_1 v_1$ with $u_1 \in U$ and $v_1 \in V$. By definition of U, V , this implies $u \leq_C u_1$ and $v \leq_C v_1$, finishing the proof. \square

Since $w_n \in LL'$, it admits at least one decomposition $w_n = uv$ with $u \in L$ and $v \in L'$. Moreover, since w_1 is a finite word, it admits finitely many decompositions $w_1 = u_1 v_1$ with $u_1, v_1 \in A^*$. Therefore, a repeated application of the claim together with the pigeon-hole principle yield $u_1, \dots, u_n, v_1, \dots, v_n \in A^*$ such that $w_i = u_i v_i$ for every $i \leq n$ and,

$$u \leq_C u_1 \leq_C \dots \leq_C u_n \leq_C u_1 \quad v \leq_C v_1 \leq_C \dots \leq_C v_n \leq_C v_1.$$

Since $u \in L$, $v \in L$ and $L, L' \in \mathcal{C}$ by definition of \mathcal{C} , we get that $u_1 \in L$ and $v_1 \in L'$ by definition of \leq_C . Therefore, since \mathbf{K} and \mathbf{K}' are covers of L and L' respectively, there exist $K \in \mathbf{K}$ and $K' \in \mathbf{K}'$ such that $u_1 \in K$ and $v_1 \in K'$.

Moreover, one may verify from the definitions that for every $x, y \in A^*$, $x \leq_C y$ and $y \leq_C x$ imply that $x \sim_{Bool(\mathcal{C})} y$. Therefore, the above implies that,

$$u_1 \sim_{Bool(\mathcal{C})} \dots \sim_{Bool(\mathcal{C})} u_n \quad v_1 \sim_{Bool(\mathcal{C})} \dots \sim_{Bool(\mathcal{C})} v_n$$

Therefore, since $K, K' \in Bool(\mathcal{C})$ by definition of \mathcal{C} , it is immediate that $u_1, \dots, u_n \in K$ and $v_1, \dots, v_n \in K'$. Altogether, we obtain that $G = \{u_1 v_1, \dots, u_n v_n\} \subseteq KK'$, finishing the proof. \square

Finally, we shall need the following variant of Lemma 3.4 which considers marked concatenation instead of standard concatenation. The proof is identical to the one of Lemma 3.4 and left to the reader.

Lemma 3.6. *Let \mathcal{G} be a lattice closed under marked concatenation. Consider $L, L' \in \mathcal{G}$ and $a \in A$ and let \mathbf{K}, \mathbf{K}' be $\text{Bool}(\mathcal{G})$ -covers of L and L' respectively. There exists a $\text{Bool}(\mathcal{G})$ -cover \mathbf{H} of LaL' such that for every $H \in \mathbf{H}$, we have $H \subseteq KaK'$ for some $K \in \mathbf{K}$ and $K' \in \mathbf{K}'$.*

3.2. Main theorem. We may now state the main theorem of the paper: whenever \mathcal{C} is a finite quotient-closed Boolean algebra, $B\text{Pol}(\mathcal{C})$ -separation and $B\text{Pol}(\mathcal{C})$ -covering are both decidable.

Theorem 3.7. *Let \mathcal{C} be a finite quotient-closed Boolean algebra. Then, separation and covering are decidable for $B\text{Pol}(\mathcal{C})$.*

Before we detail the applications of Theorem 3.7, let us make an important observation. This result completes an earlier one presented in [Pla18] which applies to classes of the form $\text{Pol}(\mathcal{C})$ and $\text{Pol}(B\text{Pol}(\mathcal{C}))$ (when \mathcal{C} is a finite quotient-closed Boolean algebra). Let us recall this result.

Theorem 3.8 ([Pla18]). *Let \mathcal{C} be a finite quotient-closed Boolean algebra. Then, separation and covering are decidable for $\text{Pol}(\mathcal{C})$ and $\text{Pol}(B\text{Pol}(\mathcal{C}))$.*

An important point is that while $B\text{Pol}(\mathcal{C})$ is an intermediary class between $\text{Pol}(\mathcal{C})$ and $\text{Pol}(B\text{Pol}(\mathcal{C}))$, the proof of Theorem 3.7 involves ideas which are very different from those used in [Pla18] to prove Theorem 3.8. This is not surprising and we already mentioned the reason above: unlike $\text{Pol}(\mathcal{C})$ and $\text{Pol}(B\text{Pol}(\mathcal{C}))$, the class $B\text{Pol}(\mathcal{C})$ is not closed under concatenation in general. This difference is significant since most of the techniques we have for handling covering rely heavily on concatenation. In practice, this means that Boolean closure is harder to handle than polynomial closure, at least with such techniques.

However, we do reuse a result of [Pla18] to prove Theorem 3.7. More precisely, it turns out that the arguments for handling $B\text{Pol}(\mathcal{C})$ (in this paper) and $PB\text{Pol}(\mathcal{C})$ (in [Pla18]) both exploit the same sub-result for the simpler class $\text{Pol}(\mathcal{C})$ (albeit in very different ways). This sub-result is stronger than the decidability of $\text{Pol}(\mathcal{C})$ -covering and is proved in [Pla18]. We recall it in Section 7. However, it is important to keep in mind that from this preliminary result for $\text{Pol}(\mathcal{C})$, the arguments for $B\text{Pol}(\mathcal{C})$ and $\text{Pol}(B\text{Pol}(\mathcal{C}))$ build in orthogonal directions.

Remark 3.9. *This discussion might seem surprising. Indeed, by definition, $\text{Pol}(B\text{Pol}(\mathcal{C}))$ is built from $B\text{Pol}(\mathcal{C})$ using polynomial closure. Hence, intuition suggests that one needs some knowledge about the latter to handle the former. However, this is not the case as Boolean closure can be bypassed in the definition of $\text{Pol}(B\text{Pol}(\mathcal{C}))$. Specifically, one may prove that $\text{Pol}(B\text{Pol}(\mathcal{C})) = \text{Pol}(\text{co-Pol}(\mathcal{C}))$ where $\text{co-Pol}(\mathcal{C})$ is the class containing all complements of languages in $\text{Pol}(\mathcal{C})$. This is exactly how $\text{Pol}(B\text{Pol}(\mathcal{C}))$ is handled in [Pla18].*

The remaining sections of the paper are devoted to proving Theorem 3.7. We rely on a framework which was designed in [PZ18a] for the specific purpose of handling the covering problem. We recall it in Section 4. The algorithm for $B\text{Pol}(\mathcal{C})$ -covering is presented in Section 5. The remaining sections are then devoted to the correction proof of this algorithm. However, let us first conclude the current section by detailing the important applications of Theorem 3.7.

3.3. Applications of the main theorem. Classes of the form $BPol(\mathcal{C})$ are important: they are involved in natural hierarchies of classes of languages, called *concatenation hierarchies*. Let us briefly recall what they are (we refer the reader to [PZ18b] for a detailed presentation). A particular concatenation hierarchy depends on a single parameter: an arbitrary quotient-closed Boolean algebra of regular languages \mathcal{C} , called its *basis*. Once the basis is chosen, the construction is uniform. Languages are classified into levels of two kinds: full levels (denoted by $0, 1, 2, \dots$) and half levels (denoted by $1/2, 3/2, 5/2, \dots$):

- Level 0 is the basis (*i.e.*, our parameter class \mathcal{C}).
- Each *half level* $n + \frac{1}{2}$, for $n \in \mathbb{N}$, is the *polynomial closure* of the previous full level, *i.e.*, of level n .
- Each *full level* $n + 1$, for $n \in \mathbb{N}$, is the *Boolean closure* of the previous half level, *i.e.*, of level $n + \frac{1}{2}$.

The generic process is depicted in the following figure.

$$0 \xrightarrow{\text{Pol}} \frac{1}{2} \xrightarrow{\text{Bool}} 1 \xrightarrow{\text{Pol}} \frac{3}{2} \xrightarrow{\text{Bool}} 2 \xrightarrow{\text{Pol}} \frac{5}{2} \dots$$

Hence, a reformulation of Theorem 3.7 is that for any concatenation hierarchy whose basis is *finite*, separation is decidable for level one. There are two prominent examples of finitely based hierarchies:

- The *Straubing-Thérien* hierarchy [Str81, Thé81], whose basis is the class $\{\emptyset, A^*\}$.
- The *dot-depth* hierarchy of Brzozowski and Cohen [BC71], whose basis is the class $\{\emptyset, \{\varepsilon\}, A^+, A^*\}$.

Consequently, Theorem 3.7 implies that separation and covering are decidable for level one in these two hierarchies. These are not new results. By definition, level one in the Straubing-Thérien hierarchy is exactly the class of *piecewise testable languages*. For this class, separation has been solved in [CMM13, PvrZ13] and covering has been solved in [PZ18a]. For dot-depth one the decidability of covering and separation was originally obtained indirectly. Indeed, it is known [PZ17a] that separation and covering for any level in the dot-depth hierarchy reduce to the same problem for the corresponding level in the Straubing-Thérien hierarchy. Therefore, while the decidability of separation and covering for dot-depth one is not a new result, an advantage of Theorem 3.7 is that we obtain a new *direct* proof of this result.

However, these are not the main applications of Theorem 3.7. It turns out that the theorem also applies to level two in the Straubing-Thérien hierarchy. Indeed, it is known that this level is also level one in another finitely based concatenation hierarchy.

Recall the class AT presented in Example 2.4: it contains all Boolean combinations of languages A^*aA^* , for some $a \in A$. It is straightforward to verify that AT is a finite quotient-closed Boolean algebra. The following theorem was shown in [PS85] (see also [PZ18a] for a recent proof).

Theorem 3.10 ([PS85]). *Level two in the Straubing-Thérien hierarchy is exactly the class $BPol(AT)$.*

In view of Theorem 3.10, we obtain the following immediate corollary of Theorem 3.7.

Corollary 3.11. *Separation and covering are decidable for level two in the Straubing-Thérien hierarchy.*

Finally, this result may be lifted to dot-depth two using again the generic transfer theorem proved in [PZ17a]. Hence, we obtain the following additional corollary.

Corollary 3.12. *Separation and covering are decidable for dot-depth two.*

Remark 3.13. *Logical characterizations of these two hierarchies are known (see [Tho82] for the dot-depth hierarchy and [PP86] for the Straubing-Thérien hierarchy). Each of them corresponds to a quantifier alternation hierarchy within a particular variant of first-order logic over words. The two variants differ by the set of predicates which are allowed in sentence (they have the same overall expressive power, but this changes the levels in their respective quantifier alternation hierarchies). We refer the reader to [PZ18b] for details and a recent proof of these results.*

In particular, level two in the Straubing-Thérien hierarchy corresponds to a logic denoted by $\mathcal{BS}_2(<)$ and dot-depth two corresponds to another logic denoted by $\mathcal{BS}_2(<, +1)$. Hence, our results also imply that covering and separation are decidable for these two logics.

4. FRAMEWORK: RATING MAPS AND OPTIMAL COVERS

In this section, we present the framework which we use to formulate our covering algorithm for $BPol(\mathcal{C})$ (when \mathcal{C} is a finite quotient-closed Boolean algebra) announced in Theorem 3.7. The framework itself was designed and applied to several specific classes in [PZ18a]. Moreover, it was also used in [Pla18] to formulate algorithms for $Pol(\mathcal{C})$ - and $Pol(BPol(\mathcal{C}))$ -covering. Here, we recall the part of this framework that we shall actually need in the paper. We refer the reader to [PZ18a] for a complete and detailed presentation.

Consider a lattice \mathcal{D} . In \mathcal{D} -covering, the input is a pair (L, \mathbf{L}) where L is a regular language and \mathbf{L} a finite set of regular languages: we have to decide whether there exists a \mathcal{D} -cover of L which is separating for \mathbf{L} . The main idea in the framework of [PZ18a] is to replace the set \mathbf{L} by a (more general) algebraic object called *rating map*. Intuitively, rating maps are designed to measure the quality of \mathcal{D} -covers. Given a rating map ρ and a language L , we use ρ to rank the existing \mathcal{D} -covers of L . This leads to the definition of “**optimal**” \mathcal{D} -cover of L . We are then able to reformulate \mathcal{D} -covering with these notions. Instead of deciding whether (L, \mathbf{L}) is \mathcal{D} -coverable, we compute an optimal \mathcal{D} -cover for L for a rating map ρ that we build from \mathbf{L} . An advantage of this approach is that it yields elegant formulations for the covering algorithms which are formulated with it. We refer the reader to [PZ18a] and [Pla18] for examples (in addition to $BPol(\mathcal{C})$ which is presented in this paper). Another important motivation for using this framework is that in order to handle $BPol(\mathcal{C})$, we require a result for the simpler class $Pol(\mathcal{C})$ which is stronger than the decidability of covering (this result is proved in [Pla18]). The framework of [PZ18a] is designed to formulate this result.

We start by defining rating maps. Then, we explain how they are used to measure the quality of a cover and define optimal covers. Finally, we connect these notions to the covering problem. Let us point out that several statements presented here are without proof. We refer the reader to [PZ18a] for these proofs.

4.1. Rating maps. Rating maps involve commutative and idempotent monoids. We shall write such monoids $(R, +)$: we call the binary operation “+” *addition* and denote the neutral element by 0_R . Being idempotent means that for all $r \in R$, we have $r + r = r$. Observe that for every commutative and idempotent monoid $(R, +)$, we may define a canonical ordering \leq over R :

$$\text{For all } r, s \in R, \quad r \leq s \text{ when } r + s = s.$$

It is straightforward to verify that \leq is a partial order which is compatible with addition. Moreover, we have the following fact which is immediate from the definitions.

Fact 4.1. *Let $(R, +)$ and $(Q, +)$ be two commutative and idempotent monoids. Moreover, let $\gamma : (R, +) \rightarrow (Q, +)$ be a morphism. Then, γ is increasing: for every $s, t \in R$ such that $s \leq t$, we have $\gamma(s) \leq \gamma(t)$.*

Example 4.2. *For any E , it is immediate that $(2^E, \cup)$ is an idempotent and commutative monoid. The neutral element is \emptyset . Moreover, the canonical ordering is set inclusion.*

When manipulating the subsets of a commutative and idempotent monoid $(R, +)$ we shall often need to apply a *downset operation*. Given $S \subseteq R$, we write $\downarrow_R S$ for the set,

$$\downarrow_R S = \{r \in R \mid r \leq s \text{ for some } s \in S\}.$$

We extend this notation to Cartesian products of arbitrary sets with R . Given some set X and a subset $S \subseteq X \times R$, we write $\downarrow_R S$ for the set,

$$\downarrow_R S = \{(x, r) \in X \times R \mid \text{there exists } s \in R \text{ such that } r \leq s \text{ and } (x, s) \in S\}.$$

We may now define rating maps. A rating map is a morphism $\rho : (2^{A^*}, \cup) \rightarrow (R, +)$ where $(R, +)$ is a *finite* idempotent and commutative monoid called the *rating set of ρ* . That is, ρ is a map from 2^{A^*} to R satisfying the following properties:

- (1) $\rho(\emptyset) = 0_R$.
- (2) For all $K_1, K_2 \subseteq A^*$, $\rho(K_1 \cup K_2) = \rho(K_1) + \rho(K_2)$.

For the sake of improved readability, when applying a rating map ρ to a singleton set $K = \{w\}$, we shall write $\rho(w)$ for $\rho(\{w\})$. Additionally, we write $\rho_* : A^* \rightarrow R$ for the restriction to ρ to A^* : for every $w \in A^*$, we have $\rho_*(w) = \rho(w)$ (this notation allows us to write $\rho_*^{-1}(r) \subseteq A^*$ for the language of all words $w \in A^*$ such that $\rho(w) = r$).

Most of the statements involved in our framework make sense for arbitrary rating maps. However, we shall often have to work with special rating maps which satisfy additional properties. We present them now.

Nice rating maps. A rating map $\rho : 2^{A^*} \rightarrow R$ is nice when, for every language $K \subseteq A^*$, there exist finitely many words $w_1, \dots, w_n \in K$ such that $\rho(K) = \rho(w_1) + \dots + \rho(w_n)$.

Observe that in this case, ρ is characterized by the canonical map $\rho_* : A^* \rightarrow R$. Indeed, for every language K , we may consider the sum of all elements $\rho(w)$ for $w \in K$: while it may be infinite, it boils down to a finite one since R is commutative and idempotent. The hypothesis that ρ is nice implies that $\rho(K)$ is equal to this sum.

Multiplicative rating maps. A rating map $\rho : 2^{A^*} \rightarrow R$ is multiplicative when its rating set R has more structure: it needs to be an *idempotent semiring*. Moreover, ρ has to satisfy an additional property connecting this structure to language concatenation. Namely, it has to be a morphism of semirings.

A *semiring* is a tuple $(R, +, \cdot)$ where R is a set and “+” and “ \cdot ” are two binary operations called addition and multiplication, such that the following axioms are satisfied:

- $(R, +)$ is a commutative monoid (the neutral element is denoted by 0_R).
- (R, \cdot) is a monoid (the neutral element is denoted by 1_R).
- Multiplication distributes over addition: $r \cdot (s + t) = (r \cdot s) + (r \cdot t)$ and $(r + s) \cdot t = (r \cdot t) + (s \cdot t)$ for every $r, s, t \in R$.
- The neutral element of $(R, +)$ is a zero for (R, \cdot) : $0_R \cdot r = r \cdot 0_R = 0_R$ for every $r \in R$.

We say that a semiring R is *idempotent* when $r + r = r$ for every $r \in R$, *i.e.*, when the additive monoid $(R, +)$ is idempotent (on the other hand, there is no additional constraint on the multiplicative monoid (R, \cdot)).

Example 4.3. *A key example of infinite idempotent semiring is the set 2^{A^*} of all languages over A . Union is the addition (with \emptyset as neutral element) and concatenation is the multiplication (with $\{\varepsilon\}$ as neutral element).*

Clearly, any finite idempotent semiring $(R, +, \cdot)$ is in particular a rating set: $(R, +)$ is an idempotent and commutative monoid. In particular, one may verify that the canonical ordering “ \leq ” on R , is compatible with multiplication as well.

We may now define multiplicative rating maps: as expected they are semiring morphisms. Let $\rho : 2^{A^*} \rightarrow R$ be a rating map. By definition, this means that the rating set $(R, +)$ is a finite idempotent commutative monoid and that ρ is a monoid morphism from $(2^{A^*}, \cup)$ to $(R, +)$. We say that ρ is multiplicative when the rating set R is equipped with a second binary operation “ \cdot ” such that $(R, +, \cdot)$ is an idempotent semiring and ρ is also a monoid morphism from $(2^{A^*}, \cdot)$ to (R, \cdot) . In other words, the two following additional axioms hold:

- (3) $\rho(\varepsilon) = 1_R$.
- (4) For all $K_1, K_2 \subseteq A^*$, we have $\rho(K_1 K_2) = \rho(K_1) \cdot \rho(K_2)$.

Remark 4.4. *A key point is that the rating maps which are both nice and multiplicative are finitely representable. As we explained above, a nice rating map $\rho : 2^{A^*} \rightarrow R$ is characterized by the canonical map $\rho_* : A^* \rightarrow R$. Moreover, when ρ is multiplicative as well, ρ_* is finitely representable: it is a morphism into a finite monoid. Thus, we may consider algorithms taking nice multiplicative rating maps as input. Let us point out that the rating maps which are not nice and multiplicative remain important. We often deal with them in our proofs.*

4.2. Imprints and optimal covers. We now explain how we use rating maps to measure the quality of covers. This involves an additional notion: “imprints”. Consider a rating map $\rho : 2^{A^*} \rightarrow R$. For any finite set of languages \mathbf{K} , the ρ -imprint of \mathbf{K} (denoted by $\mathcal{I}[\rho](\mathbf{K}) \subseteq R$) is the following set:

$$\begin{aligned} \mathcal{I}[\rho](\mathbf{K}) &= \downarrow_R \{ \rho(K) \mid K \in \mathbf{K} \} \subseteq R \\ &= \{ r \in R \mid \text{there exists } K \in \mathbf{K} \text{ such that } r \leq \rho(K) \}. \end{aligned}$$

When using this notion, we shall have some language $L \subseteq A^*$ in hand: our goal is to find the “best possible” cover \mathbf{K} of L . Intuitively, ρ -imprints measure the “quality” of candidate covers \mathbf{K} (the smaller the ρ -imprint, the better the quality).

This leads to the notion of optimality. Let \mathcal{D} be an arbitrary lattice. Given a language L , an *optimal \mathcal{D} -cover of L for ρ* is a \mathcal{D} -cover of L which has the smallest possible ρ -imprint (with respect to inclusion). That is, \mathbf{K} is an optimal \mathcal{D} -cover of L for ρ if and only if,

$$\mathcal{I}[\rho](\mathbf{K}) \subseteq \mathcal{I}[\rho](\mathbf{K}') \quad \text{for every } \mathcal{D}\text{-cover } \mathbf{K}' \text{ of } L$$

Furthermore, in the special case when $L = A^*$, we speak of optimal *universal \mathcal{D} -cover* for ρ .

In general, given an arbitrary language L , there are infinitely many optimal \mathcal{D} -covers of L for ρ . However, there always exists at least one (this requires the hypothesis that \mathcal{D} is a lattice, see Lemma 4.15 in [PZ18a] for the proof).

Lemma 4.5. *Let \mathcal{D} be a lattice. Then, for any rating map $\rho : 2^{A^*} \rightarrow R$ and any language $L \subseteq A^*$, there exists an optimal \mathcal{D} -cover of L for ρ .*

A key point is that the proof of Lemma 4.5 is non-constructive: given L and $\rho : 2^{A^*} \rightarrow R$, computing an actual optimal \mathcal{D} -cover of L for ρ is a difficult problem in general. As seen below, getting such algorithm (in the special case when ρ is nice and multiplicative) yields a procedure for \mathcal{D} -covering. Before we can establish this connection precisely, we require a key observation about optimal \mathcal{D} -covers.

Optimal imprints. By definition, given a language L , all optimal \mathcal{D} -covers of L for ρ have the same ρ -imprint. Hence, this unique ρ -imprint is a *canonical* object for \mathcal{D} , L and ρ . We call it the *\mathcal{D} -optimal ρ -imprint on L* and we denote it by $\mathcal{I}_{\mathcal{D}}[L, \rho]$:

$$\mathcal{I}_{\mathcal{D}}[L, \rho] = \mathcal{I}[\rho](\mathbf{K}) \subseteq R \quad \text{for every optimal } \mathcal{D}\text{-cover } \mathbf{K} \text{ of } L \text{ for } \rho.$$

Additionally, in the particular case when $L = A^*$, we shall speak of *\mathcal{D} -optimal universal ρ -imprint* and write $\mathcal{I}_{\mathcal{D}}[\rho]$ for $\mathcal{I}_{\mathcal{D}}[A^*, \rho]$.

We complete these definitions with a few properties of optimal imprints. We start with a straightforward fact which compares the optimal imprints on languages which are comparable with inclusion. The proof is available in [PZ18a, Fact 4.17].

Fact 4.6. *Consider a rating map $\rho : 2^{A^*} \rightarrow R$ and \mathcal{D} a lattice. Let H, L be two languages such that $H \subseteq L$. Then, $\mathcal{I}_{\mathcal{D}}[H, \rho] \subseteq \mathcal{I}_{\mathcal{D}}[L, \rho]$.*

More precisely, the following fact connects optimal imprints with union of languages.

Fact 4.7. *Let $\rho : 2^{A^*} \rightarrow R$ be a rating map and consider two languages H, L . Then, for every lattice \mathcal{D} , we have $\mathcal{I}_{\mathcal{D}}[H \cup L, \rho] = \mathcal{I}_{\mathcal{D}}[H, \rho] \cup \mathcal{I}_{\mathcal{D}}[L, \rho]$.*

Proof. We already know that $\mathcal{I}_{\mathcal{D}}[H, \rho] \subseteq \mathcal{I}_{\mathcal{D}}[H \cup L, \rho]$ and $\mathcal{I}_{\mathcal{D}}[L, \rho] \subseteq \mathcal{I}_{\mathcal{D}}[H \cup L, \rho]$ by Fact 4.6. Therefore, the inclusion $\mathcal{I}_{\mathcal{D}}[H, \rho] \cup \mathcal{I}_{\mathcal{D}}[L, \rho] \subseteq \mathcal{I}_{\mathcal{D}}[H \cup L, \rho]$ is immediate. We prove the converse one. Let $r \in \mathcal{I}_{\mathcal{D}}[H \cup L, \rho]$. We let \mathbf{K}_H and \mathbf{K}_L as optimal \mathcal{D} -covers of H and L respectively (for ρ). Clearly, $\mathbf{K}_H \cup \mathbf{K}_L$ is a cover $H \cup L$. Therefore, $r \in \mathcal{I}_{\mathcal{D}}[H \cup L, \rho]$ implies that $r \in \mathcal{I}[\rho](\mathbf{K}_H \cup \mathbf{K}_L)$. Hence, there exists $K \in \mathbf{K}_H \cup \mathbf{K}_L$ such that $r \leq \rho(K)$. Then, either $K \in \mathbf{K}_H$ which implies $r \in \mathcal{I}[\rho](\mathbf{K}_H) = \mathcal{I}_{\mathcal{D}}[H, \rho]$ or $K \in \mathbf{K}_L$ which implies $r \in \mathcal{I}[\rho](\mathbf{K}_L) = \mathcal{I}_{\mathcal{D}}[L, \rho]$. Altogether, we get $r \in \mathcal{I}_{\mathcal{D}}[H, \rho] \cup \mathcal{I}_{\mathcal{D}}[L, \rho]$, finishing the proof. \square

4.3. Connection with the covering problem. Finally, we explain how rating maps are used for handling the covering problem.

Given a lattice \mathcal{D} , it turns out that the \mathcal{D} -covering problem reduces to another problem whose input is a nice multiplicative rating map. Let us point out that *two* reductions of this kind are presented in [PZ18a]. The first one is simpler but restricted to classes \mathcal{D} which are Boolean algebras. On the other hand, the second one applies to all lattices but requires working with more involved objects.

In the paper, we investigate classes of the form $BPol(\mathcal{C})$ which are Boolean algebras. Hence, we shall mostly work with the first variant whose statement is as follows (we refer the reader to [PZ18a] for the proof).

Proposition 4.8 ([PZ18a]). *Let \mathcal{D} be a Boolean algebra. Assume that there exists an algorithm for the following computational problem:*

Input: *A nice multiplicative rating map $\rho : 2^{A^*} \rightarrow R$.*
Output: *Compute the \mathcal{D} -optimal universal ρ -imprint, $\mathcal{I}_{\mathcal{D}}[\rho]$.*

Then, \mathcal{D} -covering is decidable.

Additionally, we shall need in Section 6 to apply a theorem of [Pla18] for classes of the form $Pol(\mathcal{C})$ (which are lattices but not Boolean algebras) as a sub-result. Thus, we also recall the terminology associated to the generalized reduction which holds for arbitrary lattices, since we need it to state this theorem.

When working with an arbitrary lattice, one needs to consider slightly more involved objects. Given a lattice \mathcal{D} , a map $\alpha : A^* \rightarrow M$ into a finite set M (in practice, α will be a monoid morphism but this is not required for the definition) and a rating map $\rho : 2^{A^*} \rightarrow R$, we write $\mathcal{P}_{\mathcal{D}}^{\alpha}[\rho]$ for the following set,

$$\mathcal{P}_{\mathcal{D}}^{\alpha}[\rho] = \{(s, r) \in M \times R \mid r \in \mathcal{I}_{\mathcal{D}}[\alpha^{-1}(s), \rho]\} \subseteq M \times R$$

We call $\mathcal{P}_{\mathcal{D}}^{\alpha}[\rho]$ the α -pointed \mathcal{D} -optimal ρ -imprint. Clearly, it encodes all sets $\mathcal{I}_{\mathcal{D}}[\alpha^{-1}(s), \rho]$ for $s \in M$. The following statement is [Pla18, Proposition 5.18].

Proposition 4.9. *Consider a lattice \mathcal{D} and some finite quotient-closed Boolean algebra \mathcal{C} . Assume that there exists an algorithm for the following computational problem:*

Input: *A \mathcal{C} compatible morphism $\alpha : A^* \rightarrow M$ and
a nice multiplicative rating map $\rho : 2^{A^*} \rightarrow R$.*
Output: *Compute the α -pointed \mathcal{D} -optimal ρ -imprint, $\mathcal{P}_{\mathcal{D}}^{\alpha}[\rho]$.*

Then, \mathcal{D} -covering is decidable.

5. CHARACTERIZATION OF $BPol(\mathcal{C})$ -OPTIMAL IMPRINTS

We present a generic characterization of $BPol(\mathcal{C})$ -optimal imprints which holds when \mathcal{C} is a finite quotient-closed Boolean algebra. For the sake of avoiding clutter, we assume that the finite quotient-closed Boolean algebra \mathcal{C} is fixed for the whole section.

Given a nice multiplicative rating map $\rho : 2^{A^*} \rightarrow R$, we want to characterize the set $\mathcal{I}_{BPol(\mathcal{C})}[\rho] \subseteq R$. An important point is that we do not work directly with this set. Instead, we characterize the family of all sets $\mathcal{I}_{BPol(\mathcal{C})}[D, \rho] \subseteq R$ where $D \subseteq A^*$ is a $\sim_{\mathcal{C}}$ -class. A key point is that this family of sets record more information than just the set $\mathcal{I}_{BPol(\mathcal{C})}[\rho]$. Indeed, by Fact 4.7, we have,

$$\mathcal{I}_{BPol(\mathcal{C})}[\rho] = \mathcal{I}_{BPol(\mathcal{C})}[A^*, \rho] = \bigcup_{D \in A^*/\sim_{\mathcal{C}}} \mathcal{I}_{BPol(\mathcal{C})}[D, \rho].$$

For the sake of convenience, we shall encode this family as a set of pairs in $(A^*/\sim_{\mathcal{C}}) \times R$. Given a multiplicative rating map $\rho : 2^{A^*} \rightarrow R$, we define:

$$\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho] = \{(D, r) \in (A^*/\sim_{\mathcal{C}}) \times R \mid r \in \mathcal{I}_{BPol(\mathcal{C})}[D, \rho]\}.$$

When ρ is nice, we characterize $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ as the *greatest* subset of R satisfying specific properties. From the statement, it is straightforward to obtain a greatest fixpoint procedure for computing $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ from ρ . In turns, this allows to compute $\mathcal{I}_{BPol(\mathcal{C})}[\rho]$ using the above equality. By Proposition 4.8, this yields an algorithm for $BPol(\mathcal{C})$ -covering, thus proving our main result: Theorem 3.7.

Remark 5.1. *This characterization is rather unique among the results that have been obtained for other classes in [PZ18a] and [Pla18]. Typically, optimal imprints are characterized as **least** subsets, not greatest ones.*

Notation. *In our statements, we shall frequently manipulate subsets of $(A^*/\sim_{\mathcal{C}}) \times R$. When doing so, the following notation will be convenient. Given $S \subseteq (A^*/\sim_{\mathcal{C}}) \times R$ and $D \in A^*/\sim_{\mathcal{C}}$, we write,*

$$S(D) = \{r \in R \mid (D, r) \in S\}$$

In particular, observe that $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho](D) = \mathcal{I}_{BPol(\mathcal{C})}[D, \rho]$ by definition.

Given a multiplicative rating map $\rho : 2^{A^*} \rightarrow R$ we define a notion of $BPol(\mathcal{C})$ -saturated subset of $(A^*/\sim_{\mathcal{C}}) \times R$ (for ρ). Our theorem then states that when ρ is nice, the greatest such subset is exactly $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$.

Remark 5.2. *The definition of $BPol(\mathcal{C})$ -saturated sets makes sense regardless of whether ρ is nice. However, we need this hypothesis for the greatest one to be $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$.*

The definition is based on an intermediary notion. For every set $S \subseteq (A^*/\sim_{\mathcal{C}}) \times R$, we associate another set $\mathcal{R}_S^{\rho} \subseteq (A^*/\sim_{\mathcal{C}}) \times R \times 2^R$. For the definition, we need to recall a few properties. Given a word $w \in A^*$, we denote its $\sim_{\mathcal{C}}$ class by $[w]_{\mathcal{C}}$. Moreover, since \mathcal{C} is closed under quotients, Lemma 2.7 yields that the equivalence $\sim_{\mathcal{C}}$ is a congruence. We denote by “ \bullet ” the multiplication of $\sim_{\mathcal{C}}$ -classes in the monoid $A^*/\sim_{\mathcal{C}}$. Additionally, since R is a semiring, 2^R is one as well for union as addition and the natural multiplication lifted from the one of R (for $U, V \in 2^R$, $UV = \{qr \mid q \in U \text{ and } r \in V\}$).

We may now present our definition. Consider a set $S \subseteq (A^*/\sim_{\mathcal{C}}) \times R$. We define \mathcal{R}_S^{ρ} as the least subset of $(A^*/\sim_{\mathcal{C}}) \times R \times 2^R$ (with respect to inclusion) which satisfies the following properties:

- *Trivial elements.* For every $w \in A^*$, we have $([w]_{\mathcal{C}}, \rho(w), \{\rho(w)\}) \in \mathcal{R}_S^{\rho}$.
- *Multiplication.* For every $(C, r, U), (D, q, V) \in \mathcal{R}_S^{\rho}$, we have $(C \bullet D, qr, UV) \in \mathcal{R}_S^{\rho}$.
- *S-restricted closure.* For every triple of idempotents $(E, f, F) \in \mathcal{R}_S^{\rho}$, we have $(E, f, F \cdot S(E) \cdot F) \in \mathcal{R}_S^{\rho}$.

We are ready to define the $BPol(\mathcal{C})$ -saturated subsets of $(A^*/\sim_{\mathcal{C}}) \times R$. Consider a set $S \subseteq (A^*/\sim_{\mathcal{C}}) \times R$. We say that S is $BPol(\mathcal{C})$ -saturated for ρ if and only if the following property holds for every $(D, r) \in S$:

$$\begin{aligned} &\text{we have } r_1, \dots, r_k \in R \text{ such that } r \leq r_1 + \dots + r_k \text{ and for every } i \leq k, \\ &\text{there exists } U_i \in 2^R \text{ satisfying } (D, r_i, U_i) \in \mathcal{R}_S^{\rho} \text{ and } r_1 + \dots + r_k \in \downarrow_R U_i \end{aligned} \quad (5.1)$$

We now state our characterization of $BPol(\mathcal{C})$ -optimal imprints. We do so in the following theorem.

Theorem 5.3. *Let $\rho : 2^{A^*} \rightarrow R$ be a nice multiplicative rating map. Then, $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ is the greatest $BPol(\mathcal{C})$ -saturated subset of $(A^*/\sim_{\mathcal{C}}) \times R$ for ρ .*

It is immediate from Theorem 5.3 that given as input a nice multiplicative rating map $\rho : 2^{A^*} \rightarrow R$, one may compute $\mathcal{I}_{BPol(\mathcal{C})}[\rho]$. Indeed, computing the greatest $BPol(\mathcal{C})$ -saturated subset of $(A^*/\sim_{\mathcal{C}}) \times R$ is achieved with a greatest fixpoint algorithm. One starts from the set $S_0 = (A^*/\sim_{\mathcal{C}}) \times R$ and computes a sequence $S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots$ of subsets. For every $n \in \mathbb{N}$, S_{n+1} is the set of all pairs $(D, r) \in S_n$ satisfying (5.1) for $S = S_n$. That is,

$$\begin{aligned} &\text{we have } r_1, \dots, r_k \in R \text{ such that } r \leq r_1 + \dots + r_k \text{ and for every } i \leq k, \\ &\text{there exists } U_i \in 2^R \text{ satisfying } (D, r_i, U_i) \in \mathcal{R}_{S_n}^{\rho} \text{ and } r_1 + \dots + r_k \in \downarrow_R U_i \end{aligned}$$

Clearly, $S_{n+1} \subseteq S_n$ for every $n \in \mathbb{N}$. Therefore, the computation eventually reaches a fixpoint which is the greatest $BPol(\mathcal{C})$ -saturated subset of $(A^*/\sim_{\mathcal{C}}) \times R$ by definition. Let us point out that the computation of S_{n+1} from S_n involves computing $\mathcal{R}_{S_n}^\rho$ which is achieved with a least fixpoint procedure by definition.

Altogether, it follows that Theorem 5.3 yields an algorithm for computing $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ (and therefore, $\mathcal{I}_{BPol(\mathcal{C})}[\rho]$ as well by Fact 4.7) which alternates between a greatest fixpoint and a least fixpoint.

Remark 5.4. *The hypothesis that ρ is nice in Theorem 5.3 is mandatory: the result fails otherwise. This is actually apparent on the definition of $BPol(\mathcal{C})$ -saturated sets. One may verify from the definition of \mathcal{R}_S^ρ that for every triple $(D, q, U) \in \mathcal{R}_S^\rho$ there exists a word $w \in A^*$ such that $D = [w]_{\mathcal{C}}$ and $q = \rho(w)$. Therefore, it follows from (5.1) that every $r \in R$ which belongs to a $BPol(\mathcal{C})$ -saturated subset must satisfy $r \leq \rho(w_1) + \dots + \rho(w_k)$ for some words $w_1, \dots, w_k \in A^*$. Intuitively, this means that $BPol(\mathcal{C})$ -saturated subsets only depend on the image of singletons. Therefore, using the notion only makes sense when ρ is characterized by these images: this is exactly the definition of nice rating maps.*

This might seem to be a minor observation. Indeed, by Proposition 4.8, being able to compute $\mathcal{I}_{BPol(\mathcal{C})}[\rho]$ from a nice multiplicative rating map suffices to meet our goal: getting an algorithm for $BPol(\mathcal{C})$ -covering. Actually, it does not even make sense to speak of an algorithm which takes arbitrary multiplicative rating maps as input since we are not able to finitely represent them. However, from a theoretical point of view, the fact that we only manage to get a description of $\mathcal{I}_{BPol(\mathcal{C})}[\rho]$ when ρ is nice is significant. In the proof of Theorem 5.3, we use a theorem of [Pla18] as a sub-result. Specifically, this theorem is a characterization of $Pol(\mathcal{C})$ -optimal pointed imprints: given a \mathcal{C} -compatible morphism $\alpha : A^ \rightarrow M$ and a multiplicative rating map $\tau : 2^{A^*} \rightarrow Q$, it describes the set $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\tau]$. A key point is that this characterization does **not** require τ to be nice. This is crucial: in the proof of Theorem 5.3, we consider auxiliary rating maps (built from ρ) which need not be nice. Altogether, this means that we are able to handle $Pol(\mathcal{C})$ for all multiplicative rating maps and this is crucial in order to handle $BPol(\mathcal{C})$ but we are only able to do so for nice multiplicative rating maps (the situation is similar for $Pol(BPol(\mathcal{C}))$ as shown in in [Pla18]). This explains why the results presented in this paper and in [Pla18] cannot be lifted to higher levels in concatenation hierarchies (at least not in a straightforward manner).*

We turn to the proof of Theorem 5.3. It spans the remaining four sections of the paper. Given a nice multiplicative rating map $\rho : 2^{A^*} \rightarrow R$, we have to show that $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ is the greatest $BPol(\mathcal{C})$ -saturated subset of $(A^*/\sim_{\mathcal{C}}) \times R$ for ρ . The main argument involves two directions which are proved independently. They correspond to soundness and completeness of the greatest fixpoint algorithm computing $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$.

- The soundness argument shows that $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ contains every $BPol(\mathcal{C})$ -saturated subset (this implies that the greatest fixpoint procedure only computes elements of $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$). We present it in Section 8.
- The completeness argument shows that $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ itself is $BPol(\mathcal{C})$ -saturated (this implies that the greatest fixpoint procedure computes all elements of $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$). We present it in Section 9.

When put together, these two results yield as desired that $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ is the greatest $BPol(\mathcal{C})$ -saturated subset of R , proving Theorem 5.3.

However, before we may start presenting the main argument, we require some additional material about rating maps. For both directions, we shall introduce auxiliary rating maps (built from ρ) and apply a characterization of $Pol(\mathcal{C})$ -optimal imprints to them (taken from [Pla18]). These auxiliary rating maps are built using generic constructions which are not specific to Theorem 5.3. We present them in Section 6. Then, we recall the $Pol(\mathcal{C})$ -theorem of [Pla18] in Section 7 (actually, we slightly generalize this theorem since we shall apply it for rating maps that are more general than the ones considered in [Pla18]).

6. NESTING OF RATING MAPS

In this section, we present two generic constructions. Both of them build a new rating map out of an already existing one and a lattice \mathcal{D} . The constructions are new: they do not appear in [PZ18a] (however, one of them generalizes and streamlines a technical construction used in [Pla18]).

Remark 6.1. *As announced, we shall later rely on these constructions in the proof of Theorem 5.3 (we use them in the special cases when \mathcal{D} is either $Pol(\mathcal{C})$ or $BPol(\mathcal{C})$). However, this section is independent from Theorem 5.3: all definitions are presented in a general context.*

We first present the constructions and then investigate the properties of the output rating maps they produce.

6.1. Definition. We present two constructions. The first one involves two objects: a lattice \mathcal{D} and a rating map $\rho : 2^{A^*} \rightarrow R$. We build a new rating map $\xi_{\mathcal{D}}[\rho]$ whose rating set is $(2^R, \cup)$.

$$\begin{aligned} \xi_{\mathcal{D}}[\rho] : (2^{A^*}, \cup) &\rightarrow (2^R, \cup) \\ K &\mapsto \mathcal{I}_{\mathcal{D}}[K, \rho] \end{aligned}$$

The fact that $\xi_{\mathcal{D}}[\rho]$ is indeed a rating map is shown below in Proposition 6.2. The second construction involves an additional object: a map $\alpha : A^* \rightarrow M$ where M is some arbitrary finite set (in practice, α will be a monoid morphism, but this is not required for the definition). We build a new rating map $\zeta_{\mathcal{D}}^{\alpha}[\rho]$ with rating set $(2^{M \times R}, \cup)$:

$$\begin{aligned} \zeta_{\mathcal{D}}^{\alpha}[\rho] : (2^{A^*}, \cup) &\rightarrow (2^{M \times R}, \cup) \\ K &\mapsto \{(s, r) \mid r \in \mathcal{I}_{\mathcal{D}}[\alpha^{-1}(s) \cap K, \rho]\}. \end{aligned}$$

Let us prove that these two maps are indeed rating maps. We state this result in the following proposition.

Proposition 6.2. *Consider a lattice \mathcal{D} , a map $\alpha : A^* \rightarrow M$ into a finite set M and a rating map $\rho : 2^{A^*} \rightarrow R$. Then, $\xi_{\mathcal{D}}[\rho]$ and $\zeta_{\mathcal{D}}^{\alpha}[\rho]$ are rating maps.*

Proof. We start with $\xi_{\mathcal{D}}[\rho]$. It is immediate that $\xi_{\mathcal{D}}[\rho](\emptyset) = \mathcal{I}_{\mathcal{D}}[\emptyset, \rho] = \emptyset$. Moreover, we obtain from Fact 4.7 that for every $H, L \subseteq A^*$,

$$\xi_{\mathcal{D}}[\rho](H \cup L) = \mathcal{I}_{\mathcal{D}}[H \cup L, \rho] = \mathcal{I}_{\mathcal{D}}[H, \rho] \cup \mathcal{I}_{\mathcal{D}}[L, \rho] = \xi_{\mathcal{D}}[\rho](H) \cup \xi_{\mathcal{D}}[\rho](L).$$

We conclude that $\xi_{\mathcal{D}}[\rho]$ is indeed a rating map. We turn to $\zeta_{\mathcal{D}}^{\alpha}[\rho]$. Clearly,

$$\zeta_{\mathcal{D}}^{\alpha}[\rho](\emptyset) = \{(s, r) \mid r \in \mathcal{I}_{\mathcal{D}}[\emptyset, \rho]\} = \{(s, r) \mid r \in \emptyset\} = \emptyset.$$

Moreover, given $H, L \subseteq A^*$,

$$\zeta_{\mathcal{D}}^{\alpha}[\rho](H \cup L) = \{(s, r) \mid r \in \mathcal{I}_{\mathcal{D}}[\alpha^{-1}(s) \cap (H \cup L), \rho]\}.$$

By Fact 4.7, this yields,

$$\zeta_{\mathcal{D}}^{\alpha}[\rho](H \cup L) = \{(s, r) \mid r \in \mathcal{I}_{\mathcal{D}}[\alpha^{-1}(s) \cap H, \rho] \cup \mathcal{I}_{\mathcal{D}}[\alpha^{-1}(s) \cap L, \rho]\}$$

This exactly says that $\zeta_{\mathcal{D}}^{\alpha}[\rho](H \cup L) = \zeta_{\mathcal{D}}^{\alpha}[\rho](H) \cup \zeta_{\mathcal{D}}^{\alpha}[\rho](L)$, finishing the proof that $\zeta_{\mathcal{D}}^{\alpha}[\rho]$ is a rating map. \square

A crucial observation is that the rating maps $\xi_{\mathcal{D}}[\rho]$ and $\zeta_{\mathcal{D}}^{\alpha}[\rho]$ are not nice in general, even when the original rating map ρ is. Let us present a counter-example.

Example 6.3. *Let \mathcal{D} be the Boolean algebra containing all languages which are either finite or co-finite (i.e., their complement is finite). Moreover, let $T = \{0, 1\}$ and $R = 2^T$. We define a nice rating map $\rho : 2^{A^*} \rightarrow R$ as follows (actually, it is simple to verify from the definition that ρ is also multiplicative). Since we are defining a nice rating map, it suffices to specify the evaluation of words: for any $w \in A^*$, we let $\rho(w) = \{0\}$ if w has even length and $\rho(w) = \{1\}$ if w has odd length. We show that the rating map $\xi_{\mathcal{D}}[\rho] : 2^{A^*} \rightarrow 2^R$ is not nice.*

By definition, $\xi_{\mathcal{D}}[\rho](A^) = \mathcal{I}_{\mathcal{D}}[A^*, \rho]$. Recall that \mathcal{D} contains only finite and co-finite languages. Moreover, covers may only contain finitely many languages. Hence, it is immediate that if \mathbf{K} is an optimal \mathcal{D} -cover \mathbf{K} of A^* for ρ , then there exists $K \in \mathbf{K}$ containing a word of even length and a word of odd length. Therefore $\rho(K) = \{0, 1\}$ by definition of ρ and it is immediate that $\xi_{\mathcal{D}}[\rho](A^*) = \mathcal{I}[\rho](\mathbf{K}) = \{\{0, 1\}, \{0\}, \{1\}, \emptyset\} = R$.*

Now observe that for any $w \in A^$, $\{w\} \in \mathcal{D}$ by definition (it is a finite language). Hence, $\{\{w\}\}$ is a \mathcal{D} -cover of $\{w\}$ and since $\xi_{\mathcal{D}}[\rho](w) = \mathcal{I}_{\mathcal{D}}[\{w\}, \rho]$, we know that $\xi_{\mathcal{D}}[\rho](w) = \{\{0\}, \emptyset\}$ if w has even length and $\xi_{\mathcal{D}}[\rho](w) = \{\{1\}, \emptyset\}$ if w has odd length. Altogether, we obtain that,*

$$\bigcup_{w \in A^*} \xi_{\mathcal{D}}[\rho](w) = \{\{0\}, \{1\}, \emptyset\} \neq \xi_{\mathcal{D}}[\rho](A^*)$$

We conclude that $\xi_{\mathcal{D}}[\rho]$ is not nice.

Another important question is whether $\xi_{\mathcal{D}}[\rho]$ and $\zeta_{\mathcal{D}}^{\alpha}[\rho]$ are multiplicative. The remainder of the section is devoted to discussing this point.

6.2. Multiplication. When $\alpha : A^* \rightarrow M$ is a morphism into a finite monoid and $\rho : 2^{A^*} \rightarrow R$ is a multiplicative rating map, (M, \cdot) is a monoid and $(R, +, \cdot)$ is an idempotent semiring. We may lift the multiplication of R to 2^R in the natural way: for $U, V \in 2^R$, $UV = \{qr \mid q \in U \text{ and } r \in V\}$. One may verify that $(2^R, \cup, \cdot)$ is an idempotent semiring. Similarly, we may lift the componentwise multiplication on $M \times R$ to $2^{M \times R}$ and $(2^{M \times R}, \cup, \cdot)$ is an idempotent semiring. Whenever we consider semiring structures for 2^R and $2^{M \times R}$, this is the additions and multiplications that we shall use.

Unfortunately, even though 2^R and $2^{M \times R}$ are semirings, **neither** $\xi_{\mathcal{D}}[\rho] : 2^{A^*} \rightarrow 2^R$ nor $\zeta_{\mathcal{D}}^{\alpha}[\rho] : 2^{A^*} \rightarrow 2^{M \times R}$ are multiplicative: they are not monoid morphisms for multiplication. However, it turns out that they behave almost as multiplicative rating maps when the class \mathcal{D} satisfies appropriate properties related to closure under concatenation. We formalize this with a new notion: quasi-multiplicative rating maps.

Quasi-multiplicative rating maps. Consider an arbitrary lattice \mathcal{G} and a rating map $\rho : (2^{A^*}, \cup) \rightarrow (R, +)$ whose rating set is an idempotent semiring $(R, +, \cdot)$ (but ρ need not be multiplicative). We say that ρ is \mathcal{G} -multiplicative when there exists an endomorphism μ_{ρ} of the monoid $(R, +)$ satisfying the following properties,

- (1) For every $q, r, s \in R$, $\mu_{\rho}(q \cdot \mu_{\rho}(r) \cdot s) = \mu_{\rho}(qrs)$.

- (2) We have $1_R \leq \rho(\varepsilon)$.
(3) For every $H, K \in \mathcal{G}$ and $a \in A$, we have,

$$\begin{aligned}\rho(H) &= \mu_\rho(\rho(H)) \\ \rho(HK) &= \mu_\rho(\rho(H) \cdot \rho(K)) \\ \rho(HaK) &= \mu_\rho(\rho(H) \cdot \rho(a) \cdot \rho(K))\end{aligned}$$

Observe that only the third axiom depends on \mathcal{G} . When considering a \mathcal{G} -multiplicative rating map, we always assume implicitly that the endomorphism μ_ρ is fixed. Additionally, when \mathcal{G} is the class of all languages (i.e., $\mathcal{G} = 2^{A^*}$), we say that ρ is *quasi-multiplicative*.

Remark 6.4. *When \mathcal{G} is the class of all languages (i.e., for quasi-multiplicative rating maps), Axiom 3 can be simplified. Indeed, in that case, it is implied by Axiom 1 and the following weaker variant:*

- (3) *For every $H, K \subseteq A^*$, we have $\rho(H) = \mu_\rho(\rho(H))$ and $\rho(HK) = \mu_\rho(\rho(H) \cdot \rho(K))$.*

Let us explain why this simplified axiom implies the original one. Given $H, K \subseteq A^$ and $a \in A$, we know from Axiom 1 that,*

$$\mu_\rho(\rho(H) \cdot \rho(a) \cdot \rho(K)) = \mu_\rho(\mu_\rho(\rho(H) \cdot \rho(a)) \cdot \rho(K))$$

Using the replacement of Axiom 3, we then obtain,

$$\mu_\rho(\mu_\rho(\rho(H) \cdot \rho(a)) \cdot \rho(K)) = \mu_\rho(\rho(Ha) \cdot \rho(K)) = \rho(HaK)$$

Altogether, we get $\mu_\rho(\rho(H) \cdot \rho(a) \cdot \rho(K)) = \rho(HaK)$.

Remark 6.5. *Clearly, a true multiplicative rating map is always quasi-multiplicative. Indeed, in this case, it suffices to choose μ_ρ as the identity; $\mu_\rho(r) = r$ for all $r \in R$.*

Additionally, we shall need the following lemma which is a straightforward adaptation of a result proved in [PZ18a] (see Lemma 5.8) to quasi-multiplicative rating maps.

Lemma 6.6. *Let \mathcal{G} be a quotient-closed lattice and $\rho : 2^{A^*} \rightarrow R$ be a \mathcal{G} -multiplicative rating map. For every $H, L \subseteq A^*$, $q \in \mathcal{I}_\mathcal{G}[H, \rho]$ and $r \in \mathcal{I}_\mathcal{G}[L, \rho]$, we have $\mu_\rho(qr) \in \mathcal{I}_\mathcal{G}[HL, \rho]$.*

Proof. Let $q \in \mathcal{I}_\mathcal{G}[H, \rho]$ and $r \in \mathcal{I}_\mathcal{G}[L, \rho]$. By definition, it suffices to prove that for every \mathcal{G} -cover \mathbf{K} of HL , we have $\mu_\rho(qr) \in \mathcal{I}[\rho](\mathbf{K})$. Let \mathbf{K} be a \mathcal{G} -cover of HL , we have to find $K \in \mathbf{K}$ such that $\mu_\rho(qr) \leq \rho(K)$. We use the following claim which is based on the Myhill-Nerode theorem.

Claim. *There exists a language $G \in \mathcal{G}$ which satisfies the following two properties:*

- (1) *For all $u \in H$, there exists $K \in \mathbf{K}$ such that $G \subseteq u^{-1}K$.*
(2) *$r \leq \rho(G)$*

Proof. For every $u \in H$, we let $\mathbf{Q}_u = \{u^{-1}K \mid K \in \mathbf{K}\}$. Clearly, \mathbf{Q}_u is a \mathcal{G} -cover of L since \mathbf{K} is a cover of HL and \mathcal{G} is closed under quotients. Moreover, we know by hypothesis on \mathcal{G} that all languages in \mathbf{K} are regular. Therefore, it follows from the Myhill-Nerode theorem that they have finitely many quotients. Thus, while there might be infinitely many words $u \in H$, there are only finitely many distinct sets \mathbf{Q}_u . It follows that we may use finitely many intersections to build a \mathcal{G} -cover \mathbf{Q} of L such that for every $Q \in \mathbf{Q}$ and every $u \in H$, there exists $K \in \mathbf{K}$ satisfying $Q \subseteq u^{-1}K$. This means that all $Q \in \mathbf{Q}$ satisfy the first item in the claim, we now pick one which satisfies the second one as well.

Since $r \in \mathcal{I}_\mathcal{G}[L, \rho]$, and \mathbf{Q} is a \mathcal{G} -cover of L , we have $r \in \mathcal{I}[\rho](\mathbf{Q})$. Thus, we get $G \in \mathbf{Q}$ such that $r \leq \rho(G)$ by definition. This concludes the proof of the claim. \square

We may now finish the proof of Lemma 6.6. Let $G \in \mathcal{G}$ be as defined in the claim and consider the following set:

$$\mathbf{G} = \left\{ \bigcap_{v \in G} Kv^{-1} \mid K \in \mathbf{K} \right\}.$$

Observe that all languages in \mathbf{G} belong to \mathcal{G} . Indeed, by hypothesis on \mathcal{G} , every $K \in \mathbf{K}$ is regular. Thus, it has finitely many right quotients by the Myhill-Nerode theorem and the language $\bigcap_{v \in H} Kv^{-1}$ is the intersection of finitely many quotients of languages in \mathcal{G} . By closure under intersection and quotients, it follows that $\bigcap_{v \in G} Kv^{-1} \in \mathcal{G}$. Moreover, \mathbf{G} is a \mathcal{G} -cover of H . Indeed, given $u \in H$, we have $K \in \mathbf{K}$ such that $G \subseteq u^{-1}K$ by the first assertion in the claim. Hence, for every $v \in G$, we have $u \in Kv^{-1}$ and we obtain that $u \in \bigcap_{v \in G} Kv^{-1}$, which is an element of \mathbf{G} .

Therefore, since $q \in \mathcal{I}_{\mathcal{G}}[H, \rho]$ by hypothesis, we have $q \in \mathcal{I}[\rho](\mathbf{G})$ and we obtain $G' \in \mathbf{G}$ such that $q \leq \rho(G')$. Hence, since $r \leq \rho(G)$ by the second item in the claim, we have $qr \leq \rho(G') \cdot \rho(G)$. Since ρ is quasi-multiplicative over \mathcal{G} and $G, G' \in \mathcal{G}$, it follows from Axiom 3 in the definition of quasi-multiplicative rating maps that,

$$\rho(G'G) = \mu_{\rho}(\rho(G') \cdot \rho(G))$$

Moreover, since μ_{ρ} is an endomorphism of $(R, +)$ and $qr \leq \rho(G') \cdot \rho(G)$, we have,

$$\mu_{\rho}(qr) \leq \mu_{\rho}(\rho(G') \cdot \rho(G))$$

Altogether, we get $\mu_{\rho}(qr) \leq \rho(G'G)$. Finally, observe that $G'G \subseteq K$ for some $K \in \mathbf{K}$. Indeed, if $w \in G'G$, we have $w = uv$ with $u \in G'$ and $v \in G$. Moreover, $G' = \bigcap_{v \in G} Kv^{-1}$ for some $K \in \mathbf{K}$ by definition of \mathbf{G} . Hence, $u \in Kv^{-1}$ which yields $w = uv \in K$. Altogether, we get that $\mu_{\rho}(qr) \leq \rho(G'G) \leq \rho(K)$, which concludes the proof. \square

We now prove that when \mathcal{D} is a quotient-closed lattice closed under concatenation, the rating map $\zeta_{\mathcal{D}}^{\alpha}[\rho]$ is quasi-multiplicative provided that α is a morphism and ρ is already quasi-multiplicative (actually, this is also true for $\xi_{\mathcal{D}}[\rho]$ as well but we do not need this result). This result is tailored to the situation in which we shall later use $\zeta_{\mathcal{D}}^{\alpha}[\rho]$: $\mathcal{D} = \text{Pol}(\mathcal{C})$.

Lemma 6.7. *Let \mathcal{D} be a quotient-closed lattice closed under concatenation, $\alpha : A^* \rightarrow M$ be a morphism and $\rho : 2^{A^*} \rightarrow R$ be a quasi-multiplicative rating map. Then, $\zeta_{\mathcal{D}}^{\alpha}[\rho]$ is quasi-multiplicative for the following associated endomorphism $\mu_{\zeta_{\mathcal{D}}^{\alpha}[\rho]}$ of $(2^{M \times R}, \cup)$:*

$$\mu_{\zeta_{\mathcal{D}}^{\alpha}[\rho]}(T) = \downarrow_R \{(s, \mu_{\rho}(r)) \mid (s, r) \in T\} \quad \text{for every } T \in 2^{M \times R}.$$

Proof. We already know from Proposition 6.2 that $\zeta_{\mathcal{D}}^{\alpha}[\rho]$ is a rating map. Hence, we have to prove that the axioms of quasi-multiplicative rating maps hold for the endomorphism $\mu_{\zeta_{\mathcal{D}}^{\alpha}[\rho]}$ of $(2^{M \times R}, \cup)$ described in the lemma (it is clear from the definition that this is indeed an endomorphism). For the sake of avoiding clutter, we write μ for $\mu_{\zeta_{\mathcal{D}}^{\alpha}[\rho]}$.

We start with the first axiom. Consider $T, U, V \in 2^{M \times R}$. We have to show that $\mu(T\mu(U)V) = \mu(TUV)$. Assume first that $(s, r) \in \mu(T\mu(U)V)$. By definition of μ , this yields $(s_1, r_1) \in T$, $(s_2, r_2) \in \mu(U)$ and $(s_3, r_3) \in V$ such that $s = s_1s_2s_3$ and $r \leq \mu_{\rho}(r_1r_2r_3)$. Since $(s_2, r_2) \in \mu(U)$, we have $(s_2, r'_2) \in U$ such that $r_2 \leq \mu_{\rho}(r'_2)$. It follows that $r_1r_2r_3 \leq r_1\mu_{\rho}(r'_2)r_3$ and since μ_{ρ} is an endomorphism of $(R, +)$, we obtain,

$$r \leq \mu_{\rho}(r_1r_2r_3) \leq \mu_{\rho}(r_1\mu_{\rho}(r'_2)r_3).$$

Since ρ is quasi-multiplicative, the first axiom in the definition yields that $\mu_\rho(r_1\mu_\rho(r'_2)r_2) = \mu_\rho(r_1r'_2r_3)$ and we get $r \leq \mu_\rho(r_1r'_2r_3)$. Since $(s_1, r_1) \in T$, $(s_2, r'_2) \in U$ and $(s_3, r_3) \in V$. This yields $(s, r) = (s_1s_2s_3, r) \in \mu(TUV)$.

Conversely, assume that $(s, r) \in \mu(TUV)$. By definition of μ , we get $(s_1, r_1) \in T$, $(s_2, r_2) \in U$ and $(s_3, r_3) \in V$ such that $s = s_1s_2s_3$ and $r \leq \mu_\rho(r_1r_2r_3)$. The first axiom in the definition of quasi-multiplicative rating maps yields that $\mu_\rho(r_1\mu_\rho(r_2)r_3) = \mu_\rho(r_1r_2r_3)$. Therefore, $r \leq \mu_\rho(r_1\mu_\rho(r_2)r_3)$. Moreover, since $(s_2, r_2) \in U$, it is immediate that $(s_2, \mu_\rho(r_2)) \in \mu(U)$ by definition of μ . Altogether, this implies that $(s, r) \in \mu(T\mu(U)V)$.

We turn to the second axiom. The neutral element of $2^{M \times R}$ for multiplication is $\{(1_M, 1_R)\}$. Thus, we have to prove that $\{(1_M, 1_R)\} \subseteq \zeta_{\mathcal{D}}^\alpha[\rho](\varepsilon)$. By definition,

$$\zeta_{\mathcal{D}}^\alpha[\rho](\varepsilon) = \{(s, r) \mid r \in \mathcal{I}_{\mathcal{D}}[\alpha^{-1}(s) \cap \{\varepsilon\}, \rho]\}.$$

Clearly, $\rho(\varepsilon) \in \mathcal{I}_{\mathcal{D}}[\alpha^{-1}(1_M) \cap \{\varepsilon\}, \rho]$. Moreover, since ρ is quasi-multiplicative, we have $1_R \leq \rho(\varepsilon)$ and we get $1_R \in \mathcal{I}_{\mathcal{D}}[\alpha^{-1}(1_M) \cap \{\varepsilon\}, \rho]$ which yields $(1_M, 1_R) \in \zeta_{\mathcal{D}}^\alpha[\rho](\varepsilon)$, finishing the proof.

It remains to handle the third axiom. Since we are proving that ρ is quasi-multiplicative, Remark 6.4 implies that it suffices to show that for every $H, K \subseteq A^*$, we have $\zeta_{\mathcal{D}}^\alpha[\rho](HK) = \mu(\zeta_{\mathcal{D}}^\alpha[\rho](H) \cdot \zeta_{\mathcal{D}}^\alpha[\rho](K))$. We start with the former.

Let $H \subseteq A^*$. We prove that $\zeta_{\mathcal{D}}^\alpha[\rho](H) = \mu(\zeta_{\mathcal{D}}^\alpha[\rho](H))$. For $(s, r) \in M \times R$, we show that $(s, r) \in \zeta_{\mathcal{D}}^\alpha[\rho](H)$ if and only if $(s, r) \in \mu(\zeta_{\mathcal{D}}^\alpha[\rho](H))$. Let \mathbf{K} be an optimal \mathcal{D} -cover of $H \cap \alpha^{-1}(s)$ for ρ . By definition, $(s, r) \in \zeta_{\mathcal{D}}^\alpha[\rho](H)$ implies that $r \in \mathcal{I}[\rho](\mathbf{K})$. Hence, we have $K \in \mathbf{K}$ such that $r \leq \rho(K)$. By definition of \mathbf{K} , we have $(s, \rho(K)) \in \zeta_{\mathcal{D}}^\alpha[\rho](H)$. Moreover, since ρ is quasi-multiplicative, we get $\rho(K) = \mu_\rho(\rho(K))$ which yields, $r \leq \mu_\rho(\rho(K))$. Altogether, we get $(s, r) \in \mu(\zeta_{\mathcal{D}}^\alpha[\rho](H))$ by definition of μ .

Conversely, assume that $(s, r) \in \mu(\zeta_{\mathcal{D}}^\alpha[\rho](H))$. By definition of μ , we have $(s, r') \in \zeta_{\mathcal{D}}^\alpha[\rho](H)$ such that $r \leq \mu_\rho(r')$. By definition of \mathbf{K} , we have $r' \in \mathcal{I}[\rho](\mathbf{K})$ and we get $K' \in \mathbf{K}$ such that $r' \leq \rho(K')$. Hence, since μ_ρ is an endomorphism and ρ is quasi-multiplicative, we get $r \leq \mu_\rho(r') \leq \mu_\rho(\rho(K')) = \rho(K')$. This implies that $r \in \mathcal{I}[\rho](\mathbf{K})$ which yields $(s, r) \in \zeta_{\mathcal{D}}^\alpha[\rho](H)$ by definition of \mathbf{K} .

We turn to the second equality. Consider $K_1, K_2 \subseteq A^*$, we show that $\zeta_{\mathcal{D}}^\alpha[\rho](K_1K_2) = \mu(\zeta_{\mathcal{D}}^\alpha[\rho](K_1) \cdot \zeta_{\mathcal{D}}^\alpha[\rho](K_2))$.

We start with the inclusion $\mu(\zeta_{\mathcal{D}}^\alpha[\rho](K_1) \cdot \zeta_{\mathcal{D}}^\alpha[\rho](K_2)) \subseteq \zeta_{\mathcal{D}}^\alpha[\rho](K_1K_2)$. Let $(s, r) \in \mu(\zeta_{\mathcal{D}}^\alpha[\rho](K_1) \cdot \zeta_{\mathcal{D}}^\alpha[\rho](K_2))$. By definition of μ we have $(s_1, r_1) \in \zeta_{\mathcal{D}}^\alpha[\rho](K_1)$ and $(s_2, r_2) \in \zeta_{\mathcal{D}}^\alpha[\rho](K_2)$ such that $s = s_1s_2$ and $r \leq \mu_\rho(r_1r_2)$. By definition of $\zeta_{\mathcal{D}}^\alpha[\rho]$, this means that $r_1 \in \mathcal{I}_{\mathcal{D}}[K_1 \cap \alpha^{-1}(s_1), \rho]$ and $r_2 \in \mathcal{I}_{\mathcal{D}}[K_2 \cap \alpha^{-1}(s_2), \rho]$. Since \mathcal{D} is a quotient-closed lattice, the following is immediate from Lemma 6.6:

$$\mu_\rho(r_1r_2) \in \mathcal{I}_{\mathcal{D}}[(K_1 \cap \alpha^{-1}(s_1)) \cdot (K_2 \cap \alpha^{-1}(s_2)), \rho]$$

Observe that $(K_1 \cap \alpha^{-1}(s_1)) \cdot (K_2 \cap \alpha^{-1}(s_2)) \subseteq K_1K_2 \cap \alpha^{-1}(s_1)\alpha^{-1}(s_2)$. Moreover, since α is a morphism, it is clear that $\alpha^{-1}(s_1)\alpha^{-1}(s_2) \subseteq \alpha^{-1}(s_1s_2) = \alpha^{-1}(s)$. Altogether, this means that we have $(K_1 \cap \alpha^{-1}(s_1)) \cdot (K_2 \cap \alpha^{-1}(s_2)) \subseteq K_1K_2 \cap \alpha^{-1}(s)$. Therefore, we obtain from Fact 4.6 that $\mu_\rho(r_1r_2) \in \mathcal{I}_{\mathcal{D}}[K_1K_2 \cap \alpha^{-1}(s), \rho]$. Since $r \leq \mu_\rho(r_1r_2)$, this yields that $r \in \mathcal{I}_{\mathcal{D}}[K_1K_2 \cap \alpha^{-1}(s), \rho]$ as well, which exactly says that $(s, r) \in \zeta_{\mathcal{D}}^\alpha[\rho](K_1K_2)$, concluding the proof for the inclusion from left to right.

We finish with the converse inclusion. Let us point that this is where we use the hypothesis that \mathcal{D} is closed under concatenation. Let $(s, r) \in \zeta_{\mathcal{D}}^\alpha[\rho](K_1K_2)$. We have to show

that $(s, r) \in \mu(\zeta_{\mathcal{D}}^{\alpha}[\rho](K_1) \cdot \zeta_{\mathcal{D}}^{\alpha}[\rho](K_2))$. By definition, we have $r \in \mathcal{I}_{\mathcal{D}}[K_1 K_2 \cap \alpha^{-1}(s), \rho]$. For every $t \in M$ and $i \in \{1, 2\}$, we define $\mathbf{H}_{i,t}$ as an optimal \mathcal{D} -cover of $K_i \cap \alpha^{-1}(t)$. Consider the following finite set of languages \mathbf{H} ,

$$\mathbf{H} = \{H_1 H_2 \mid \text{there exists } s_1, s_2 \in M \text{ s.t. } s_1 s_2 = s, H_1 \in \mathbf{H}_{1,s_1} \text{ and } H_2 \in \mathbf{H}_{2,s_2}\}$$

We prove that \mathbf{H} is a \mathcal{D} -cover of $K_1 K_2 \cap \alpha^{-1}(s)$. Clearly all languages in \mathbf{H} belong to \mathcal{D} since \mathcal{D} is closed under concatenation by Theorem 3.2. Let us show that \mathbf{H} is a cover of $K_1 K_2 \cap \alpha^{-1}(s)$. Consider $w \in K_1 K_2 \cap \alpha^{-1}(s)$, we exhibit $H \in \mathbf{H}$ such that $w \in H$. Since $w \in K_1 K_2$, we have $w = w_1 w_2$ with $w_1 \in K_1$ and $w_2 \in K_2$. Let $s_1 = \alpha(w_1)$ and $s_2 = \alpha(w_2)$. Altogether, this means that $w_1 \in K_1 \cap \alpha^{-1}(s_1)$ and $w_2 \in K_2 \cap \alpha^{-1}(s_2)$. Therefore, we have $H_1 \in \mathbf{H}_{1,s_1}$ and $H_2 \in \mathbf{H}_{2,s_2}$ such that $w_1 \in H_1$ and $w_2 \in H_2$. This yields $w \in H_1 H_2$. Finally, $s_1 s_2 = \alpha(w) = s$ which yields that $H_1 H_2 \in \mathbf{H}$ by definition.

We may now finish the argument and show that $(s, r) \in \mu(\zeta_{\mathcal{D}}^{\alpha}[\rho](K_1) \cdot \zeta_{\mathcal{D}}^{\alpha}[\rho](K_2))$. Recall that $r \in \mathcal{I}_{\mathcal{D}}[K_1 K_2 \cap \alpha^{-1}(s), \rho]$. Thus, since \mathbf{H} is a \mathcal{D} -cover of $K_1 K_2 \cap \alpha^{-1}(s)$, we have $r \in \mathcal{I}[\rho](\mathbf{H})$. It follows that there exists $H \in \mathbf{H}$ such that $r \leq \rho(H)$. By definition of \mathbf{H} , we have $H = H_1 H_2$ with $H_1 \in \mathbf{H}_{1,s_1}$ and $H_2 \in \mathbf{H}_{2,s_2}$ where $s_1, s_2 \in M$ satisfy $s_1 s_2 = s$. Let $r_1 = \rho(H_1)$ and $r_2 = \rho(H_2)$. Since \mathbf{H}_{1,s_1} and \mathbf{H}_{2,s_2} are optimal \mathcal{D} -covers of $K_1 \cap \alpha^{-1}(s_1)$ and $K_2 \cap \alpha^{-1}(s_2)$ respectively, we have $r_1 \in \mathcal{I}_{\mathcal{D}}[K_1 \cap \alpha^{-1}(s_1), \rho]$ and $r_2 \in \mathcal{I}_{\mathcal{D}}[K_2 \cap \alpha^{-1}(s_2), \rho]$. It follows that $(s_1, r_1) \in \zeta_{\mathcal{D}}^{\alpha}[\rho](K_1)$ and $(s_2, r_2) \in \zeta_{\mathcal{D}}^{\alpha}[\rho](K_2)$. Consequently, $(s, r_1 r_2) = (s_1 s_2, r_1 r_2) \in \zeta_{\mathcal{D}}^{\alpha}[\rho](K_1) \cdot \zeta_{\mathcal{D}}^{\alpha}[\rho](K_2)$. Finally, by hypothesis and since ρ is quasi-multiplicative, we have,

$$r \leq \rho(H) = \rho(H_1 H_2) = \mu_{\rho}(\rho(H_1) \cdot \rho(H_2)) = \mu_{\rho}(r_1 r_2).$$

By definition of μ , this yields that $(s, r) \in \mu(\zeta_{\mathcal{D}}^{\alpha}[\rho](K_1) \cdot \zeta_{\mathcal{D}}^{\alpha}[\rho](K_2))$ which concludes the proof. \square

We now present a final result which applies to $\xi_{\mathcal{D}}[\rho]$. This lemma involves hypotheses that are slightly different from those of the previous one. Moreover, it states a weaker result. Under these hypotheses, $\xi_{\mathcal{D}}[\rho]$ is not “fully” quasi-multiplicative: it is only \mathcal{G} -multiplicative for a particular lattice \mathcal{G} (from which \mathcal{D} is built). In this case as well, the result is tailored to how we intend to use $\xi_{\mathcal{D}}[\rho]$. We consider the case when $\mathcal{D} = BPol(\mathcal{C})$ and $\mathcal{G} = Pol(\mathcal{C})$. Note that this result is where we use the weak concatenation principle that we presented for Boolean closure in Lemma 3.4.

Lemma 6.8. *Let \mathcal{G} be a quotient-closed lattice closed under concatenation and marked concatenation and let $\mathcal{D} = Bool(\mathcal{G})$. Moreover, let $\rho : 2^{A^*} \rightarrow R$ be a multiplicative rating map. Then, $\xi_{\mathcal{D}}[\rho]$ is \mathcal{G} -multiplicative and the associated endomorphism $\mu_{\xi_{\mathcal{D}}[\rho]}$ of $(2^R, \cup)$ is as follows:*

$$\mu_{\xi_{\mathcal{D}}[\rho]}(T) = \downarrow_R T \quad \text{for every } T \in 2^R.$$

Proof. We already know from Proposition 6.2 that $\xi_{\mathcal{D}}[\rho]$ is a rating map. Hence, we have to prove that the axioms of \mathcal{G} -multiplicative rating maps hold for the endomorphism $\mu_{\xi_{\mathcal{D}}[\rho]}$ of $(2^R, \cup)$ described in the lemma (it is clear from the definition that this is indeed an endomorphism). The first axiom is immediate by definition: clearly, for every $T, U, V \in 2^R$, we have $\downarrow_R(T(\downarrow_R U)V) = \downarrow_R(TUV)$, by definition of μ again.

We turn to the second axiom. The neutral element of 2^R for multiplication is $\{1_R\}$. Thus, we have to prove that $\{1_R\} \subseteq \xi_{\mathcal{D}}[\rho](\varepsilon)$. By definition $\xi_{\mathcal{D}}[\rho](\varepsilon) = \mathcal{I}_{\mathcal{D}}[\{\varepsilon\}, \rho]$. Therefore, we have $\rho(\varepsilon) \in \xi_{\mathcal{D}}[\rho](\varepsilon)$. Since ρ is multiplicative, we have $\rho(\varepsilon) = 1_R$, which yields $1_R \in \xi_{\mathcal{D}}[\rho](\varepsilon)$, finishing the proof.

It remains to prove the third axiom. There are three equalities to handle. First, by definition, for every $H \subseteq A^*$, we have $\xi_{\mathcal{D}}[\rho](H) = \mathcal{I}_{\mathcal{D}}[H, \rho]$ and $\downarrow_R \mathcal{I}_{\mathcal{D}}[H, \rho] = \mathcal{I}_{\mathcal{D}}[H, \rho]$. Therefore, it is immediate that $\xi_{\mathcal{D}}[\rho](H) = \downarrow_R \xi_{\mathcal{D}}[\rho](H)$.

We turn to the second equality: we show that for every $K_1, K_2 \in \mathcal{G}$, we have $\xi_{\mathcal{D}}[\rho](K_1 K_2) = \downarrow_R(\xi_{\mathcal{D}}[\rho](K_1) \cdot \xi_{\mathcal{D}}[\rho](K_2))$. The right to left inclusion is immediate from the following fact (it actually holds for every $K_1, K_2 \subseteq A^*$).

Fact 6.9. *For every $K_1, K_2 \subseteq A^*$, we have $\downarrow_R(\xi_{\mathcal{D}}[\rho](K_1) \cdot \xi_{\mathcal{D}}[\rho](K_2)) \subseteq \xi_{\mathcal{D}}[\rho](K_1 K_2)$.*

Proof. Consider $r \in \downarrow_R(\xi_{\mathcal{D}}[\rho](K_1) \cdot \xi_{\mathcal{D}}[\rho](K_2))$. By definition, we have $r_1 \in \mathcal{I}_{\mathcal{D}}[K_1, \rho]$ and $r_2 \in \mathcal{I}_{\mathcal{D}}[K_2, \rho]$ such that $r \leq r_1 r_2$. Since \mathcal{D} is a quotient-closed Boolean algebra (it is the Boolean closure of a quotient-closed lattice by definition) and ρ is a (true) multiplicative rating map, Lemma 6.6 yields,

$$r_1 r_2 \in \mathcal{I}_{\mathcal{D}}[K_1 K_2, \rho].$$

Since imprints are closed under downset and $r \leq r_1 r_2$, this yields $r \in \mathcal{I}_{\mathcal{D}}[K_1 K_2, \rho]$. This exactly says that $r \in \xi_{\mathcal{D}}[\rho](K_1 K_2)$ finishing the proof. \square

We now concentrate on the converse inclusion. Consider $r \in \xi_{\mathcal{D}}[\rho](K_1 K_2)$. We show that $r \in \downarrow_R(\xi_{\mathcal{D}}[\rho](K_1) \cdot \xi_{\mathcal{D}}[\rho](K_2))$. Let us underline that for this inclusion, we do need the hypothesis that $K_1, K_2 \in \mathcal{G}$. We let $\mathbf{H}_1, \mathbf{H}_2$ be optimal \mathcal{D} -covers (for ρ) of K_1 and K_2 respectively. By definition, we have $\mathcal{I}[\rho](\mathbf{H}_1) = \xi_{\mathcal{D}}[\rho](K_1)$ and $\mathcal{I}[\rho](\mathbf{H}_2) = \xi_{\mathcal{D}}[\rho](K_2)$. Since \mathcal{G} is closed under concatenation and $\mathcal{D} = \text{Bool}(\mathcal{G})$, we may apply Lemma 3.4 to get a \mathcal{D} -cover \mathbf{H} of $K_1 K_2$ such that for every $H \in \mathbf{H}$, there exist $H_1 \in \mathbf{H}_1$ and $H_2 \in \mathbf{H}_2$ such that $H \subseteq H_1 H_2$ (note that this where we need the hypothesis that $K_1, K_2 \in \mathcal{G}$, this is required to apply Lemma 3.4). By definition $r \in \xi_{\mathcal{D}}[\rho](K_1 K_2)$ means that $r \in \mathcal{I}_{\mathcal{D}}[K_1 K_2, \rho]$. Therefore, since \mathbf{H} is a \mathcal{D} -cover of $K_1 K_2$, we have $r \leq \rho(H)$ for some $H \in \mathbf{H}$. The definition of \mathbf{H} then yields $H_1 \in \mathbf{H}_1$ and $H_2 \in \mathbf{H}_2$ such that $H \subseteq H_1 H_2$. This implies that $r \leq \rho(H) \leq \rho(H_1) \cdot \rho(H_2)$. Finally, $\rho(H_1) \in \xi_{\mathcal{D}}[\rho](K_1)$ and $\rho(H_2) \in \xi_{\mathcal{D}}[\rho](K_2)$ by definition of $\mathbf{H}_1, \mathbf{H}_2$. Consequently, we get $r \in \downarrow_R(\xi_{\mathcal{D}}[\rho](K_1) \cdot \xi_{\mathcal{D}}[\rho](K_2))$, which concludes the proof of this inclusion.

Finally, it remains to prove that for every $K_1, K_2 \in \mathcal{G}$, we have $\xi_{\mathcal{D}}[\rho](K_1 a K_2) = \downarrow_R(\xi_{\mathcal{D}}[\rho](K_1) \cdot \xi_{\mathcal{D}}[\rho](a) \cdot \xi_{\mathcal{D}}[\rho](K_2))$. The right to left inclusion is immediate from Fact 6.9. For the converse one, the proof is identical to what we did for proving that $\xi_{\mathcal{D}}[\rho](K_1 K_2) = \downarrow_R(\xi_{\mathcal{D}}[\rho](K_1) \cdot \xi_{\mathcal{D}}[\rho](K_2))$. The only difference is that instead of using Lemma 3.4, one needs to use Lemma 3.6, the variant which considers *marked* concatenation (this is where we need the hypothesis that \mathcal{G} is closed under marked concatenation). The detailed proof is left to the reader. \square

7. CHARACTERIZATION OF $\text{Pol}(\mathcal{C})$ -OPTIMAL IMPRINTS

In this section, we recall the theorem of [Pla18] for classes of the form $\text{Pol}(\mathcal{C})$ (when \mathcal{C} is a finite quotient-closed Boolean algebra). It states a characterization of $\text{Pol}(\mathcal{C})$ -optimal imprints: for a \mathcal{C} -compatible morphism $\alpha : A^* \rightarrow M$ and a multiplicative rating map $\rho : 2^{A^*} \rightarrow R$, $\mathcal{P}_{\text{Pol}(\mathcal{C})}^{\alpha}[\rho]$ is characterized as the least subset of $M \times R$ satisfying specific properties. When used in the special case when ρ is nice, one obtains a least fixpoint procedure for computing $\mathcal{P}_{\text{Pol}(\mathcal{C})}^{\alpha}[\rho]$ from α and ρ , thus implying the decidability of $\text{Pol}(\mathcal{C})$ -covering by Proposition 4.9. However, deciding $\text{Pol}(\mathcal{C})$ -covering is not our motivation for recalling this theorem here: we need the characterization in order to use it as a sub-result when proving Theorem 5.3.

Unfortunately, there are technical complications. When we apply the characterization as a sub-result, we shall do so for rating maps which are not multiplicative, only quasi-multiplicative (as expected, we build them using the constructions presented in Section 6). This case is not covered by the statement of [Pla18], which only considers true multiplicative rating maps. Consequently, we have to prove a generalization of this statement. We manage to avoid redoing the whole proof: while this involves some technical work, we prove our generalized statement as a corollary of the original one presented in [Pla18]. Additionally, we take this opportunity to slightly tweak the original statement in order to better accommodate the usage we shall make of the theorem later.

We first present the theorem and then focus on its proof. We fix an arbitrary finite quotient-closed Boolean algebra \mathcal{C} for the presentation.

7.1. Statement. Consider a \mathcal{C} -compatible morphism $\alpha : A^* \rightarrow M$ and a rating map $\rho : 2^{A^*} \rightarrow R$ which is $Pol(\mathcal{C})$ -multiplicative (there is no other constraint on ρ , in particular, it needs not be nice). In particular, recall that since α is \mathcal{C} -compatible, we know that for every $s \in M$, $[s]_{\mathcal{C}}$ is well-defined as a $\sim_{\mathcal{C}}$ -class containing $\alpha^{-1}(s)$. We say that a subset $S \subseteq M \times R$ is $Pol(\mathcal{C})$ -saturated (for α and ρ) when it satisfies the following properties:

- (1) *Trivial elements:* For every $w \in A^*$, $(\alpha(w), \rho(w)) \in S$.
- (2) *Multiplication:* For every $(s_1, r_1), (s_2, r_2) \in S$, we have $(s_1 s_2, r_1 r_2) \in S$.
- (3) *$Pol(\mathcal{C})$ -closure:* For every pair of multiplicative idempotents $(e, f) \in S$, we have,

$$(e, f \cdot \rho([e]_{\mathcal{C}}) \cdot f) \in S.$$

Remark 7.1. *This definition of $Pol(\mathcal{C})$ -saturated subsets is slightly different from the one used in [Pla18]. Specifically, we removed one property: closure under downset (i.e., $S = \downarrow_R S$). More precisely, we moved it to Theorem 7.2 below. This change is harmless and will be convenient when we use this notion later.*

We prove the following statement as a corollary of a theorem presented in [Pla18] (specifically, Theorem 6.5).

Theorem 7.2 ([Pla18]). *Consider a \mathcal{C} -compatible morphism $\alpha : A^* \rightarrow M$ and a $Pol(\mathcal{C})$ -multiplicative rating map $\rho : 2^{A^*} \rightarrow R$. Moreover, let S be the least $Pol(\mathcal{C})$ -saturated subset of $M \times R$. Then,*

$$\mathcal{P}_{Pol(\mathcal{C})}^{\alpha}[\rho] = \downarrow_R \{(s, \mu_{\rho}(r)) \mid (s, r) \in S\}.$$

Recall that a (true) multiplicative rating map $\rho : 2^{A^*} \rightarrow R$ is also quasi-multiplicative for the endomorphism μ_{ρ} defined as the identity on R (see Remark 6.5). Thus, in this case, Theorem 7.2 yields that $\mathcal{P}_{Pol(\mathcal{C})}^{\alpha}[\rho] = \downarrow_R S$ where S is the least $Pol(\mathcal{C})$ -saturated subset of $M \times R$. This is essentially the original statement of [Pla18]. When ρ is a nice multiplicative rating map, it is clear that one may compute the least $Pol(\mathcal{C})$ -saturated subset of $M \times R$ with a least fixpoint algorithm. Therefore, we get an algorithm for $Pol(\mathcal{C})$ -covering by Proposition 4.9.

Remark 7.3. *There is an important difference between Theorem 7.2 for $Pol(\mathcal{C})$ and Theorem 5.3 for $BPol(\mathcal{C})$: the latter is restricted to **nice** rating maps while this is not the case for the former. As we explained in Remark 5.4, while it is easy to miss, this difference is crucial.*

7.2. Proof of Theorem 7.2. We fix a \mathcal{C} -compatible morphism $\alpha : A^* \rightarrow M$ and a rating map $\rho : 2^{A^*} \rightarrow R$ which is $Pol(\mathcal{C})$ -multiplicative. Since ρ is $Pol(\mathcal{C})$ -multiplicative, we have a semiring structure $(R, +, \cdot)$ on R and an endomorphism μ_ρ of $(R, +)$ satisfying the appropriate axioms.

Our objective is to prove that if S is the least $Pol(\mathcal{C})$ -saturated subset of $M \times R$, then $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho] = \downarrow_R\{(s, \mu_\rho(r)) \mid (s, r) \in S\}$. As announced, we prove this result as a corollary of the original theorem presented in [Pla18]. We proceed in two steps:

- (1) First, we define a submonoid $(Q, +)$ of $(R, +)$ and a multiplication “ \odot ” on Q (distinct from “ \cdot ” on R) such that $(Q, +, \odot)$ is a semiring. Then, we define a true multiplicative rating map $\gamma : (2^{A^*}, \cup, \cdot) \rightarrow (Q, +, \odot)$ which coincides with ρ over languages in $Pol(\mathcal{C})$.
- (2) We apply the theorem of [Pla18] to get a description of $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$ and then use it to prove the desired result for $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$.

Step 1. Definition of the multiplicative rating map γ . Let us first present our new multiplication “ \odot ” which we define on the whole set R . For every $r, r' \in R$, we let $r \odot r' = \mu_\rho(rr')$. Clearly, “ \odot ” is associative by Axiom 1 in the definition of quasi-multiplicative rating maps.

Remark 7.4. *In general, (R, \odot) is only a semigroup, not a monoid: there exists no neutral element for “ \odot ”. We shall later define a submonoid $(Q, +)$ of $(R, +)$ such that Q is closed under “ \odot ” and $(Q, +, \odot)$ is a semiring.*

The definition of our submonoid $(Q, +)$ of $(R, +)$ is based on the following fact.

Fact 7.5. *For every $K \subseteq A^*$, there exists a unique element $q_K \in R$ such that:*

- *for every $H \in Pol(\mathcal{C})$ satisfying $K \subseteq H$, we have $q_K \leq \rho(H)$.*
- *there exists $K' \in Pol(\mathcal{C})$ such that $q_K = \rho(K')$.*

In particular, if $K \in Pol(\mathcal{C})$, then $q_K = \rho(K)$.

Proof. Let $R_K \subseteq R$ be the set of all elements $r \in R$ such that $r = \rho(H)$ for some $H \in Pol(\mathcal{C})$ such that $K \subseteq H$. Note that since $A^* \subseteq R_K$, we have $R_K \neq \emptyset$. For each $r \in R_K$, we fix $H_r \in Pol(\mathcal{C})$ such that $\rho(H_r) = r$. Finally, let $K' = \bigcap_{r \in R_K} H_r$. Clearly, $K' \in Pol(\mathcal{C})$ since $Pol(\mathcal{C})$ is a lattice by Theorem 3.2. Hence, the properties described in the fact hold for $q_K = \rho(K')$ by definition. \square

In view of Fact 7.5, we define $Q = \{q_K \mid K \subseteq A^*\}$. Moreover, we define $\gamma : 2^{A^*} \rightarrow Q$ as the following map:

$$\begin{aligned} \gamma : 2^{A^*} &\rightarrow Q \\ K &\mapsto q_K \end{aligned}$$

We now have to verify that this definition satisfies the desired property. We do so in the following lemma.

Lemma 7.6. *The pair $(Q, +)$ is a submonoid of $(R, +)$. Moreover, $(Q, +, \odot)$ is a semiring and the map $\gamma : (2^{A^*}, \cup, \cdot) \rightarrow (Q, +, \odot)$ is a multiplicative rating map.*

Proof. Since $Q \subseteq R$, we may view γ as a map from 2^{A^*} to R . We show that this map is a monoid morphism from $(2^{A^*}, \cup)$ to $(R, +)$ and a semigroup morphism from $(2^{A^*}, \cdot)$ to (R, \odot) . Since γ is surjective and $(2^{A^*}, \cup, \cdot)$ is a semiring, this implies the lemma.

We first show that γ is a monoid morphism from $(2^{A^*}, \cup)$ to $(R, +)$. Clearly $\gamma(\emptyset) = q_\emptyset = \rho(\emptyset) = 0_R$ by Fact 7.5 since $\emptyset \in Pol(\mathcal{C})$. Consider $H, K \subseteq A^*$. We show that $\gamma(H \cup K) = \gamma(H) + \gamma(K)$.

First, we prove that $\gamma(H) + \gamma(K) \leq \gamma(H \cup K)$. By definition in Fact 7.5, we have $L \in \text{Pol}(\mathcal{C})$ such that $H \cup K \subseteq L$ and $\gamma(H \cup K) = \rho(L)$. This implies that $H \subseteq L$ and $K \subseteq L$. Consequently, $\gamma(H) \leq \rho(L)$ and $\gamma(K) \leq \rho(L)$ by definition of γ . It follows that $\gamma(H) + \gamma(K) \leq \rho(L) = \gamma(H \cup K)$. It remains to show that $\gamma(H \cup K) \leq \gamma(H) + \gamma(K)$. By definition in Fact 7.5, we have $H', K' \in \text{Pol}(\mathcal{C})$ such that $H \subseteq H', K \subseteq K', \gamma(H) = \rho(H'), \gamma(K) = \rho(K')$. Hence, since ρ is a rating map, we have $\gamma(H) + \gamma(K) = \rho(H') + \rho(K') = \rho(H' \cup K')$. Moreover, $H' \cup K' \in \text{Pol}(\mathcal{C})$ by definition and it is clear that $H \cup K \subseteq H' \cup K'$. Hence, the definition of γ yields that $\gamma(H \cup K) \leq \rho(H' \cup K') = \gamma(H) + \gamma(K)$, finishing the proof for addition.

We turn to multiplication. We show that γ is a semigroup morphism from $(2^{A^*}, \cdot)$ to (R, \odot) . Consider $H, K \subseteq A^*$. We show that $\gamma(H) \odot \gamma(K) = \gamma(HK)$.

We first show that $\gamma(HK) \leq \gamma(H) \odot \gamma(K)$. By definition in Fact 7.5, we have $H', K' \in \text{Pol}(\mathcal{C})$ such that $H \subseteq H', K \subseteq K', \gamma(H) = \rho(H'), \gamma(K) = \rho(K')$. Hence, by definition of “ \odot ”, $\gamma(H) \odot \gamma(K) = \mu_\rho(\rho(H) \cdot \rho(K))$. Therefore, since ρ is $\text{Pol}(\mathcal{C})$ -multiplicative and $H', K' \in \text{Pol}(\mathcal{C})$, Axiom 3 in the definition of quasi-rating maps yields that $\gamma(H) \odot \gamma(K) = \rho(H'K')$. Finally, we have $HK \subseteq H'K'$ and $H'K' \in \text{Pol}(\mathcal{C})$ (since $\text{Pol}(\mathcal{C})$ is closed under concatenation, see Theorem 3.2). Therefore, we get $\gamma(HK) \leq \rho(H'K')$ by definition of γ . Altogether, this yields $\gamma(HK) \leq \gamma(H) \odot \gamma(K)$.

It remains to prove that $\gamma(H) \odot \gamma(K) \leq \gamma(HK)$. Using Fact 7.5, we get $G \in \text{Pol}(\mathcal{C})$ such that $KH \subseteq G$ and $\gamma(KH) = \rho(G)$. Consider the two following languages:

$$U_1 = \bigcap_{v \in H} Gv^{-1} \quad \text{and} \quad U_2 = \bigcap_{u \in U_1} u^{-1}G.$$

Since $\text{Pol}(\mathcal{C})$ is a quotient-closed lattice (see Theorem 3.2) and $G \in \text{Pol}(\mathcal{C})$, it follows from the Myhill-Nerode theorem (which states that regular languages have finitely many quotients) that $U_1, U_2 \in \text{Pol}(\mathcal{C})$. Moreover, since $KH \subseteq G$, one may verify from the definitions that $K \subseteq U_1$ and $H \subseteq U_2$. Therefore, we have $\gamma(K) \leq \rho(U_1)$ and $\gamma(H) \leq \rho(U_2)$ by definition of γ . Since $(R, +, \cdot)$ is a semiring and μ_ρ an endomorphism of $(R, +)$, it follows that,

$$\gamma(K) \odot \gamma(H) = \mu_\rho(\gamma(K) \cdot \gamma(H)) \leq \mu_\rho(\rho(U_1) \cdot \rho(U_2)).$$

Moreover, since $U_1, U_2 \in \text{Pol}(\mathcal{C})$, and ρ is $\text{Pol}(\mathcal{C})$ -multiplicative, it follows from Axiom 3 in the definition of quasi-multiplicative rating maps that $\mu_\rho(\rho(U_1) \cdot \rho(U_2)) = \rho(U_1U_2)$. Finally, it is immediate by definition of U_2 that $U_1U_2 \subseteq G$. Therefore, we obtain $\rho(U_1U_2) \leq \rho(G) = \gamma(KH)$. Altogether, this yields $\gamma(K) \odot \gamma(H) \leq \gamma(KH)$, finishing the proof. \square

This concludes the presentation of γ . We complete the definition with the following lemma which connected $\mathcal{P}_{\text{Pol}(\mathcal{C})}^\alpha[\rho] \subseteq M \times R$ to $\mathcal{P}_{\text{Pol}(\mathcal{C})}^\alpha[\gamma] \subseteq M \times Q \subseteq M \times R$.

Lemma 7.7. *We have $\mathcal{P}_{\text{Pol}(\mathcal{C})}^\alpha[\rho] = \downarrow_R \mathcal{P}_{\text{Pol}(\mathcal{C})}^\alpha[\gamma]$.*

Proof. We start with the left to right inclusion. Consider $(s, r) \in \mathcal{P}_{\text{Pol}(\mathcal{C})}^\alpha[\rho]$. By definition, we have $r \in \mathcal{I}_{\text{Pol}(\mathcal{C})}[\alpha^{-1}(s), \rho]$. Let \mathbf{K} be an optimal $\text{Pol}(\mathcal{C})$ -cover of $\alpha^{-1}(s)$ for γ . We have $\mathcal{I}_{\text{Pol}(\mathcal{C})}[\alpha^{-1}(s), \rho] \subseteq \mathcal{I}[\rho](\mathbf{K})$ which yields $K \in \mathbf{K}$ such that $r \leq \rho(K)$. Since $K \in \text{Pol}(\mathcal{C})$, it is immediate by definition in Fact 7.5 that $\gamma(K) = \rho(K)$. Moreover, since \mathbf{K} is optimal for γ , we have $\gamma(K) \in \mathcal{I}_{\text{Pol}(\mathcal{C})}[\alpha^{-1}(s), \gamma]$ which yields $(s, \gamma(K)) \in \mathcal{P}_{\text{Pol}(\mathcal{C})}^\alpha[\gamma]$. Thus, since $r \leq \rho(K) = \gamma(K)$, we have $(s, r) \in \downarrow_R \mathcal{P}_{\text{Pol}(\mathcal{C})}^\alpha[\gamma]$.

Conversely, assume that $(s, r) \in \downarrow_R \mathcal{P}_{\text{Pol}(\mathcal{C})}^\alpha[\gamma]$. We have $q \in Q$ such that $r \leq q$ and $(s, q) \in \mathcal{P}_{\text{Pol}(\mathcal{C})}^\alpha[\gamma]$. It follows that $q \in \mathcal{I}_{\text{Pol}(\mathcal{C})}[\alpha^{-1}(s), \gamma]$. Let \mathbf{H} be an optimal $\text{Pol}(\mathcal{C})$ -cover of

$\alpha^{-1}(s)$ for ρ . We have $\mathcal{I}_{Pol(\mathcal{C})}[\alpha^{-1}(s), \gamma] \subseteq \mathcal{I}[\gamma](\mathbf{H})$ which yields $H \in \mathbf{H}$ such that $q \leq \gamma(H)$. Since $H \in Pol(\mathcal{C})$, it is immediate by definition in Fact 7.5 that $\gamma(H) = \rho(H)$. Moreover, since \mathbf{H} is optimal for ρ , we have $\rho(H) \in \mathcal{I}_{Pol(\mathcal{C})}[\alpha^{-1}(s), \rho]$ and since $r \leq q \leq \gamma(H) = \rho(H)$, this implies that $r \in \downarrow_R \mathcal{I}_{Pol(\mathcal{C})}[\alpha^{-1}(s), \rho] = \mathcal{I}_{Pol(\mathcal{C})}[\alpha^{-1}(s), \rho]$. This exactly says that $(s, r) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$. \square

Step 2: Main argument. We may now apply the $Pol(\mathcal{C})$ -theorem of [Pla18, Theorem 6.5] to α and γ to get a description of $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma] \subseteq M \times Q$. This yields the following proposition:

Proposition 7.8 (Application of Theorem 6.5 in [Pla18]). *Let P be the least subset of $M \times Q$ satisfying the following properties:*

- (1) Trivial elements: For every $w \in A^*$, $(\alpha(w), \gamma(w)) \in P$.
- (2) Downset: $P = \downarrow_Q P$.
- (3) Multiplication (for \odot): For every $(s_1, q_1), (s_2, q_2) \in P$, we have $(s_1 s_2, q_1 \odot q_2) \in P$.
- (4) $Pol(\mathcal{C})$ -closure (for \odot): For every pair $(e, f) \in P$ such that e is an idempotent of M and f an idempotent of (Q, \odot) , we have, $(e, f \odot \gamma([e]_{\mathcal{C}}) \odot f) \in P$.

Then, $P = \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$.

We are now ready to prove Theorem 7.2. We fix S as the least $Pol(\mathcal{C})$ -saturated subset of $M \times R$. Moreover, we let,

$$T = \downarrow_R \{(s, \mu_\rho(r)) \in M \times R \mid (s, r) \in S\}.$$

We have to prove that $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho] = T$.

We start with the inclusion $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho] \subseteq T$. We actually show that $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma] \subseteq T$. This will imply that,

$$\downarrow_R \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma] \subseteq \downarrow_R T.$$

Since $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho] = \downarrow_R \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$ by Lemma 7.7 and $T = \downarrow_R T$ by definition, this yields as desired that $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho] \subseteq T$. We handle the proof in the following proposition.

Proposition 7.9. *We have $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma] \subseteq T$.*

Proof. By Proposition 7.8, $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$ is the least subset of $M \times Q$ satisfying four properties. Hence, it suffices to show that T satisfies these four properties. We leave the proof that T contains the trivial elements (as stated in the proposition) for last since we need the other properties to prove this.

It is immediate by definition that T is closed under downset: we have $\downarrow_R T = T$ which implies $\downarrow_Q T = T$ since $(Q, +)$ is a submonoid of $(R, +)$. Let us consider multiplication for " \odot ". Let $(s_1, r_1), (s_2, r_2) \in T$, we prove that $(s_1 s_2, r_1 \odot r_2) \in T$. By definition, we have $(s_1, r'_1), (s_2, r'_2) \in S$ such that $r_1 \leq \mu_\rho(r'_1)$ and $r_2 \leq \mu_\rho(r'_2)$. Since S is $Pol(\mathcal{C})$ -saturated, we have $(s_1 s_2, r'_1 r'_2) \in S$. Moreover, since μ_ρ is an endomorphism of $(R, +)$, it follows from Axiom 1 in the definition of quasi-multiplicative rating maps that,

$$r_1 \odot r_2 = \mu_\rho(r_1 r_2) \leq \mu_\rho(\mu_\rho(r'_1) \mu_\rho(r'_2)) = \mu_\rho(r'_1 r'_2).$$

By definition of T , this implies $(s_1 s_2, r_1 \odot r_2) \in T$. We now turn to $Pol(\mathcal{C})$ -closure for " \odot ". Consider a pair $(e, f) \in T$ such that e is an idempotent of M and f an idempotent of (R, \odot) . We show that $(e, f \odot \gamma([e]_{\mathcal{C}}) \odot f) \in T$. By definition of T , we have $(e, r) \in S$ such that $f \leq \mu_\rho(r)$. Since (R, \odot) is a finite monoid, it is standard that there exists a number $p \geq 1$ such that r^p is an idempotent of (R, \cdot) . Thus, since S is closed under multiplication (it

is $Pol(\mathcal{C})$ -saturated) and $e \in M$ is an idempotent, we get $(e, r^p) \in S$. Moreover, since S satisfies $Pol(\mathcal{C})$ -closure, we get,

$$(e, r^p \cdot \rho([e]_{\mathcal{C}}) \cdot r^p) \in S.$$

Since $[e]_{\mathcal{C}} \in \mathcal{C} \subseteq Pol(\mathcal{C})$, we have $\gamma([e]_{\mathcal{C}}) = \rho([e]_{\mathcal{C}})$ by definition in Fact 7.5. Since $f \leq \mu_{\rho}(r)$ and f is an idempotent of (R, \odot) , one may then verify from Axiom 1 in the definition of quasi-multiplicative rating maps that $f \odot \gamma([e]_{\mathcal{C}}) \odot f \leq \mu_{\rho}(r^p \cdot \rho([e]_{\mathcal{C}}) \cdot r^p)$ which concludes the proof: we obtain that $(e, f \odot \gamma([e]_{\mathcal{C}}) \odot f) \in T$.

It remains to show that $(\alpha(w), \gamma(w)) \in T$ for every $w \in A^*$. We start with a preliminary fact that we require for this proof.

Fact 7.10. *We have $(1_M, \rho([\varepsilon]_{\mathcal{C}})) \in T$.*

Proof. We have $(1_M, \rho(\varepsilon)) = (\alpha(\varepsilon), \rho(\varepsilon)) \in S$ since S is $Pol(\mathcal{C})$ -saturated (this is a trivial element). Since (R, \cdot) is a finite semigroup, there exists $p \geq 1$ such that $(\rho(\varepsilon))^p$ is an idempotent of (R, \cdot) . By closure under multiplication for S , we get $(1_M, (\rho(\varepsilon))^p) \in S$. Clearly, this is an idempotent and $[1_M]_{\mathcal{C}} = [\varepsilon]_{\mathcal{C}}$. Thus, $Pol(\mathcal{C})$ -closure for S yields,

$$(1_M, (\rho(\varepsilon))^p \cdot \rho([\varepsilon]_{\mathcal{C}}) \cdot (\rho(\varepsilon))^p) \in S.$$

Consequently, we get,

$$(1_M, \mu_{\rho}((\rho(\varepsilon))^p \cdot \rho([\varepsilon]_{\mathcal{C}}) \cdot (\rho(\varepsilon))^p)) \in T.$$

By Axiom 2 in the definition of $Pol(\mathcal{C})$ -multiplicative rating maps, we know that $1_R \leq \rho(\varepsilon)$. Thus, $\rho([\varepsilon]_{\mathcal{C}}) \leq (\rho(\varepsilon))^p \cdot \rho([\varepsilon]_{\mathcal{C}}) \cdot (\rho(\varepsilon))^p$ and since μ_{ρ} is an endomorphism of $(R, +)$, we get that $\mu_{\rho}(\rho([\varepsilon]_{\mathcal{C}})) \leq \mu_{\rho}((\rho(\varepsilon))^p \cdot \rho([\varepsilon]_{\mathcal{C}}) \cdot (\rho(\varepsilon))^p)$. Since $\downarrow_R T = T$ by definition, this yields $(1_M, \mu_{\rho}(\rho([\varepsilon]_{\mathcal{C}}))) \in T$. Finally, $\mu_{\rho}(\rho([\varepsilon]_{\mathcal{C}})) = \rho([\varepsilon]_{\mathcal{C}})$ by Axiom 3 in the definition of $Pol(\mathcal{C})$ -multiplicative rating maps (we have $[\varepsilon]_{\mathcal{C}} \in \mathcal{C} \subseteq Pol(\mathcal{C})$). Altogether, this yields $(1_M, \rho([\varepsilon]_{\mathcal{C}})) \in T$ as desired. \square

We may now prove that $(\alpha(w), \gamma(w)) \in T$ for every $w \in A^*$. We first consider the case when $w = \varepsilon$. By definition of γ , we know that ρ and γ coincide over languages in $Pol(\mathcal{C})$. Since $[\varepsilon]_{\mathcal{C}}$ is such a language, Fact 7.10 yields that $(\alpha(\varepsilon), \gamma([\varepsilon]_{\mathcal{C}})) \in T$. Moreover, $\varepsilon \in [\varepsilon]_{\mathcal{C}}$ which implies that $\gamma(\varepsilon) \leq \gamma([\varepsilon]_{\mathcal{C}})$. Hence, since we already know that T is closed under downset, we get $(\alpha(\varepsilon), \gamma(\varepsilon)) \in T$ as desired.

It remains to handle the case when $w = a_1 \cdots a_n \in A^+$ for $a_1, \dots, a_n \in A$. This is a consequence of the following fact.

Fact 7.11. *For every $a \in A$, $(\alpha(a), \gamma(a)) \in T$.*

Let us first use Fact 7.11 to prove that $(\alpha(w), \gamma(w)) \in T$. It is immediate from the fact that $(\alpha(a_i), \gamma(a_i)) \in T$ for every $i \leq n$. Since we already established that T is closed under multiplication (for \odot) and γ is a multiplicative rating map, we get,

$$(\alpha(w), \gamma(w)) = (\alpha(a_1) \cdots \alpha(a_n), \gamma(a_1) \odot \cdots \odot \gamma(a_n)) \in T.$$

We finish with the proof of Fact 7.11.

Proof of Fact 7.11. Since S is $Pol(\mathcal{C})$ -saturated, we have $(\alpha(a_i), \rho(a_i)) \in S$. This yields $(\alpha(a_i), \mu_{\rho}(\rho(a_i))) \in T$. Moreover, we established that $(1_M, \rho([\varepsilon]_{\mathcal{C}})) \in T$ in Fact 7.10. Since ρ is $Pol(\mathcal{C})$ -multiplicative and $[\varepsilon]_{\mathcal{C}} \in Pol(\mathcal{C})$, this yields that $(1_M, \mu_{\rho}(\rho([\varepsilon]_{\mathcal{C}}))) \in T$. Thus, since we already established that T is closed under multiplication (for \odot), we get,

$$(\alpha(a_i), \mu_{\rho}(\rho([\varepsilon]_{\mathcal{C}}))) \odot \mu_{\rho}(\rho(a_i)) \odot \mu_{\rho}(\rho([\varepsilon]_{\mathcal{C}})) \in T.$$

By definition of “ \odot ” and Axiom 1 in the definition of quasi-multiplicative rating maps, this yields,

$$(\alpha(a_i), \mu_\rho(\rho([\varepsilon]_{\mathcal{C}}) \cdot \rho(a_i) \cdot \rho([\varepsilon]_{\mathcal{C}}))) \in T.$$

Additionally, since ρ is $Pol(\mathcal{C})$ -multiplicative and $[\varepsilon]_{\mathcal{C}} \in Pol(\mathcal{C})$, Axiom 3 yields,

$$(\alpha(a_i), \rho([\varepsilon]_{\mathcal{C}} a_i [\varepsilon]_{\mathcal{C}})) \in T.$$

Since $[\varepsilon]_{\mathcal{C}} a_i [\varepsilon]_{\mathcal{C}} \in Pol(\mathcal{C})$ and ρ and γ coincide for such languages, we get,

$$(\alpha(a_i), \gamma([\varepsilon]_{\mathcal{C}} a_i [\varepsilon]_{\mathcal{C}})) \in T.$$

Finally, it is clear that $a_i \in [\varepsilon]_{\mathcal{C}} a_i [\varepsilon]_{\mathcal{C}}$. Hence, $\gamma(a_i) \leq \gamma([\varepsilon]_{\mathcal{C}} a_i [\varepsilon]_{\mathcal{C}})$ and since we already established that T is closed under downset, we get $(\alpha(a_i), \gamma(a_i)) \in T$ as desired. \square

This concludes the proof of Proposition 7.9. \square

It remains to prove the inclusion $T \subseteq \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$. This is a consequence of the following proposition.

Proposition 7.12. *For every $(s, r) \in S$, we have $(s, \mu_\rho(r)) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$.*

Let us first use the proposition to show that $T \subseteq \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$. Consider $(s, r') \in T$. By definition, we have $(s, r) \in S$ such that $r' \leq \mu_\rho(r)$. By Proposition 7.12, it is immediate that $(s, \mu_\rho(r)) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$. Moreover, it is clear that $\downarrow_R \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho] = \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$. Therefore, we have $(s, r') \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$, finishing the proof.

Proof of Proposition 7.12. Consider $(s, r) \in S$. By definition, S is the least $Pol(\mathcal{C})$ -saturated subset of $M \times R$. Therefore, (s, r) is built from trivial elements using multiplication and $Pol(\mathcal{C})$ -closure, we proceed by induction on this construction.

Assume first that (s, r) is a trivial element: we have $w \in A^*$ such that $s = \alpha(w)$ and $r = \rho(w)$. By Proposition 7.8, we know that $(s, \gamma(w)) = (\alpha(w), \gamma(w)) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$. Moreover, $\gamma(w) = \rho(K)$ for some $K \in Pol(\mathcal{C})$ such that $w \in K$ by Fact 7.5. Thus, since ρ is $Pol(\mathcal{C})$ -multiplicative, we have,

$$\mu_\rho(r) = \mu_\rho(\rho(w)) \leq \mu_\rho(\rho(K)) = \rho(K) = \gamma(w)$$

Since $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho] = \downarrow_R \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$ by Lemma 7.7, this implies that $(s, \mu_\rho(r)) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$.

Assume now that the last operation used to build $(s, r) \in S$ is multiplication: We have $(s_1, r_1), (s_2, r_2) \in S$ such that $(s, r) = (s_1 s_2, r_1 r_2)$. By induction, $(s_1, \mu_\rho(r_1)) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$ and $(s_2, \mu_\rho(r_2)) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$. By Lemma 7.7, this yields $q_1, q_2 \in Q$ such that $\mu_\rho(r_1) \leq q_1$, $\mu_\rho(r_2) \leq q_2$, and $(s_1, q_1), (s_2, q_2) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$. Moreover, $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$ is closed under multiplication (for \odot) by Proposition 7.8 which implies that $(s_1 s_2, q_1 \odot q_2) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$. By Axiom 1 in the definition of quasi-multiplicative rating maps, we have,

$$\mu_\rho(r_1 r_2) = \mu_\rho(\mu_\rho(r_1) \cdot \mu_\rho(r_2)) \leq \mu_\rho(q_1 q_2) = q_1 \odot q_2$$

Hence, using Lemma 7.7 again, we get $(s_1 s_2, \mu_\rho(r_1 r_2)) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$ which concludes the proof for this case.

It remains to handle $Pol(\mathcal{C})$ -closure. We have a pair of multiplicative idempotents $(e, f) \in S$ such that $(s, r) = (e, f \cdot \rho([e]_{\mathcal{C}}) \cdot f)$. By induction hypothesis, we know that $(e, \mu_\rho(f)) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$. By Lemma 7.7, this yields $q \in Q$ such that $\mu_\rho(f) \leq q$ and $(e, q) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$. It is standard that there exists a number $\ell \geq 1$ such that the multiplication of ℓ copies of $q \in Q$ with “ \odot ” is an idempotent g of (Q, \odot) . Since $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$ is closed under

multiplication (for \odot) by Proposition 7.8, we get that $(e, g) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$. Additionally, $Pol(\mathcal{C})$ -closure (for \odot) yields,

$$(e, g \odot \rho([e]_{\mathcal{C}}) \odot g) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\gamma]$$

Finally, since f is an idempotent of (R, \cdot) and $\mu_\rho(f) \leq g$, one may verify from Axiom 1 in the definition of quasi-multiplicative rating maps that,

$$\mu_\rho(r) = \mu_\rho(f \cdot \rho([e]_{\mathcal{C}}) \cdot f) \leq g \odot \rho([e]_{\mathcal{C}}) \odot g$$

Using Lemma 7.7, this yields $(s, \mu_\rho(r)) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\rho]$, finishing the proof. \square

8. SOUNDNESS IN THEOREM 5.3

We may now properly start the proof of Theorem 5.3. In this section, we handle the soundness direction of the statement. It is divided in two parts. First, we present a preliminary result which applies to the Boolean closure operation in general, i.e. to classes of the form $Bool(\mathcal{G})$ when \mathcal{G} is an arbitrary lattice. Then we apply this preliminary result in the special case when $\mathcal{G} = Pol(\mathcal{C})$ (for \mathcal{C} a finite quotient-closed Boolean algebra) to handle soundness in Theorem 5.3.

8.1. Preliminary result. We first introduce terminology that we need to state our result. We fix a lattice \mathcal{G} and let $\mathcal{D} = Bool(\mathcal{G})$. Moreover, we let $\rho : 2^{A^*} \rightarrow R$ as a rating map.

Using induction, we define a rating map $\tau_n : 2^{A^*} \rightarrow Q_n$ for every $n \in \mathbb{N}$. When $n = 0$, the rating set Q_0 is $(2^R, \cup)$ and τ_0 is defined as follows,

$$\begin{aligned} \tau_0 : (2^{A^*}, \cup) &\rightarrow (2^R, \cup) \\ K &\mapsto \{\rho(w) \mid w \in K\} \end{aligned}$$

It is immediate by definition that τ_0 is indeed a rating map (i.e. a monoid morphism).

Assume now that $n \geq 1$ and that $\tau_{n-1} : 2^{A^*} \rightarrow Q_{n-1}$ is defined. Recall that $\rho_* : A^* \rightarrow R$ denotes the canonical map associated to the rating map ρ . We define τ_n as the rating map $\zeta_{\mathcal{G}}^{\rho_*}[\tau_{n-1}]$ presented in Proposition 6.2. By definition, this means that for all $n \geq 1$, the rating set Q_n of τ_n is $(2^{R \times Q_{n-1}}, \cup)$.

We complete this definition with maps $f_n : Q_n \rightarrow 2^R$ for $n \in \mathbb{N}$. In this case as well, we use induction on n .

- For $n = 0$, let $T \in Q_0 = 2^R$. We let,

$$f_0(T) = \downarrow_R \{r_1 + \dots + r_k \mid r_1, \dots, r_k \in T\}$$

- For $n \geq 1$, let $T \in Q_n = 2^{R \times Q_{n-1}}$. We let,

$$f_n(T) = \downarrow_R \left\{ r_1 + \dots + r_k \mid \begin{array}{l} \text{we have } (r_1, T_1), \dots, (r_k, T_k) \in T \text{ such that} \\ r_1 + \dots + r_k \in f_{n-1}(T_i) \text{ for every } i \leq k \end{array} \right\}$$

We have the following simple fact which is immediate from the definition.

Fact 8.1. *For every $n \in \mathbb{N}$ and $U, U' \in Q_n$ such that $U \subseteq U'$, we have $f_n(U) \subseteq f_n(U')$.*

We may now state the preliminary result that we shall use in our soundness proof for Theorem 5.3.

Proposition 8.2. *Consider a language $L \in \mathcal{G}$. Then, the following inclusion holds,*

$$\bigcap_{n \in \mathbb{N}} f_n(\tau_n(L)) \subseteq \mathcal{I}_{Bool(\mathcal{G})}[L, \rho]$$

Proof. The proof is based on the following more involved statement which is handled by induction on $n \in \mathbb{N}$.

Lemma 8.3. *Let $n \in \mathbb{N}$ and $L, K_0, \dots, K_n, H_0, \dots, H_n \in \mathcal{G}$ such that $\{K_i \setminus H_i \mid i \leq n\}$ is a cover of L . Then, for every $s \in f_{2n}(\tau_{2n}(L))$, there exists $j \leq n$ such that $s \leq \rho(K_j \setminus H_j)$.*

Before we prove the lemma, let us use it to prove the first property described in Proposition 8.2. Let $L \in \mathcal{G}$ be a language. We write S for the set,

$$S = \bigcap_{n \in \mathbb{N}} f_n(\tau_n(L))$$

We show that $S \subseteq \mathcal{I}_{Bool(\mathcal{G})}[L, \rho]$. First, we prove the following fact which describes a special optimal $Bool(\mathcal{G})$ -cover of L for ρ .

Fact 8.4. *There exists $n \in \mathbb{N}$ and $K_0, \dots, K_n, H_0, \dots, H_n \in \mathcal{G}$ such that $\{K_i \setminus H_i \mid i \leq n\}$ is an optimal $Bool(\mathcal{G})$ -cover of L for ρ .*

Proof. Let \mathbf{H} be an arbitrary optimal $Bool(\mathcal{G})$ -cover of L for ρ . Since $Bool(\mathcal{G}) = Bool(\mathcal{G})$, each $V \in \mathbf{H}$ is the Boolean combination of languages in \mathcal{G} . We put it in disjunctive normal form. Each disjunct is an intersection languages belonging to \mathcal{G} , or whose complement belongs to \mathcal{G} . Since \mathcal{G} is lattice, both \mathcal{G} and the complement class $co\text{-}\mathcal{G}$ are closed under intersection. Therefore, each disjunct in the disjunctive normal form of V is actually of the form $K \setminus H$, where K, H both belong to \mathcal{G} . We let \mathbf{K} as the set of all languages $K \setminus H$ which are a disjunct in the disjunctive normal form of some $V \in \mathbf{H}$. Clearly, \mathbf{K} remains a $Bool(\mathcal{G})$ -cover of L since \mathbf{H} was one. Moreover, it is immediate that $\mathcal{I}[\rho](\mathbf{K}) \subseteq \mathcal{I}[\rho](\mathbf{H})$ since every language in \mathbf{K} is included in a language of \mathbf{H} . Hence, \mathbf{K} remains an optimal $Bool(\mathcal{G})$ -cover of L for ρ since \mathbf{H} was one. \square

We let $n \in \mathbb{N}$ and $K_0, \dots, K_n, H_0, \dots, H_n \in \mathcal{G}$ be as defined in Fact 8.4. We may now prove that $S \subseteq \mathcal{I}_{Bool(\mathcal{G})}[L, \rho]$. Let $s \in S$. By hypothesis on S , we have $s \in f_{2n}(\tau_{2n}(L))$. Therefore, since $\{K_i \setminus H_i \mid i \leq n\}$ is by definition a cover of L , it is immediate from Lemma 8.3 that there exists $j \leq n$ such that $s \leq \rho(K_j \setminus H_j)$. Since $\{K_i \setminus H_i \mid i \leq n\}$ is an optimal $Bool(\mathcal{G})$ -cover of L for ρ , this implies that $s \in \mathcal{I}_{Bool(\mathcal{G})}[L, \rho]$ which concludes the main proof.

We turn to the proof of Lemma 8.3. The argument is an induction on $n \in \mathbb{N}$. We start with the base case $n = 0$.

Base case. Consider $L, K_0, H_0 \in \mathcal{G}$ such that $\{K_0 \setminus H_0\}$ is a cover of L and let $s \in f_0(\tau_0(L))$. We have to show that $s \leq \rho(K_0 \setminus H_0)$. By definition of f_0 , we get $r_1, \dots, r_k \in \tau_0(L)$ such that $s \leq r_1 + \dots + r_k$. Moreover, the definition of τ_0 yields that for every $i \leq k$, $r_i = \rho(w_i)$ for some $w_i \in L$. Therefore, $r_i \leq \rho(L)$ for every $i \leq k$ and since R is idempotent for addition, $s \leq r_1 + \dots + r_k \leq \rho(L)$. Finally, since $\{K_0 \setminus H_0\}$ is a cover of L , we have $L \subseteq K_0 \setminus H_0$ and we get $s \leq \rho(K_0 \setminus H_0)$, finishing the argument for the base case.

Inductive step. We now assume that $n \geq 1$. Let $L, K_0, \dots, K_n, H_0, \dots, H_n \in \mathcal{G}$ such that $\{K_i \setminus H_i \mid i \leq n\}$ is a cover of L and let $s \in f_{2n}(\tau_{2n}(L))$. We have to exhibit $j \leq n$ such that $s \leq \rho(K_j \setminus H_j)$. Using the hypothesis that $s \in f_{2n}(\tau_{2n}(L))$, we prove the following fact.

Fact 8.5. *There exists $(r, U) \in \tau_{2n}(L)$ such that $s \in f_{2n-1}(U)$.*

Proof. By definition of f_{2n} , the hypothesis that $s \in f_{2n}(\tau_{2n}(L))$ yields $(r'_1, U'_1), \dots, (r'_k, U'_k) \in \tau_{2n}(L)$ such that $r'_1 + \dots + r'_k \in f_{2n-1}(U'_i)$ for every $i \leq k$ and $s \leq r'_1 + \dots + r'_k$. We let $(r, U) = (r'_1, U'_1) \in \tau_{2n}(L)$. We have $r'_1 + \dots + r'_k \in f_{2n-1}(U)$ and $s \leq r'_1 + \dots + r'_k$. Hence, since $f_{2n-1}(U)$ is closed under downset (by definition), this implies $s \in f_{2n-1}(U)$. \square

Recall that by definition, τ_{2n} is the rating map $\zeta_{\rho_*, \tau_{2n-1}}^{\mathcal{G}}[\cdot]$. Hence, $(r, U) \in \tau_{2n}(L)$ means that,

$$U \in \mathcal{I}_{\mathcal{G}} [\rho_*^{-1}(r) \cap L, \tau_{2n-1}]$$

By Fact 4.6, this yields,

$$U \in \mathcal{I}_{\mathcal{G}} [L, \tau_{2n-1}]$$

Since $\{K_i \setminus H_i \mid i \leq n\}$ is a cover of L , $L, K_1, \dots, K_n \in \mathcal{G}$ and \mathcal{G} is a lattice, it follows that $\{L \cap K_i \mid i \leq n\}$ is a \mathcal{G} -cover of L . Therefore, since $U \in \mathcal{I}_{\mathcal{G}} [L, \tau_{2n-1}]$, we obtain some $\ell \leq n$ such that $U \subseteq \tau_{2n-1}(L \cap K_\ell)$.

Furthermore, since $s \in f_{2n-1}(U)$, we may unravel the definition of f_{2n-1} which yields $(r_1, U_1), \dots, (r_k, U_k) \in U$ such that $r_1 + \dots + r_k \in f_{2(n-1)}(U_m)$ for every $m \leq k$ and $s \leq r_1 + \dots + r_k$. Note that by definition of $f_{2(n-1)}$, this also implies that $s \in f_{2(n-1)}(U_m)$ for every $m \leq k$. We now distinguish two sub-cases.

Sub-case 1: Assume that for every $m \leq k$, we have,

$$\rho_*^{-1}(r_m) \cap (K_\ell \setminus H_\ell) \neq \emptyset$$

This means that for every $m \leq k$, we have $w_m \in K_\ell \setminus H_\ell$ such that $\rho(w_m) = r_m$. In particular, $r_m = \rho(w_m) \leq \rho(K_\ell \setminus H_\ell)$ for every $m \leq k$. Finally, since R is idempotent for addition, we get $r_1 + \dots + r_k \leq \rho(K_\ell \setminus H_\ell)$. Since $s \leq r_1 + \dots + r_k$, we get $s \leq \rho(K_\ell \setminus H_\ell)$ and the Lemma 8.3 holds for $j = \ell$ in this case.

Sub-case 2: Conversely, assume that there exists $m \leq k$ such that,

$$\rho_*^{-1}(r_m) \cap (K_\ell \setminus H_\ell) = \emptyset$$

This implies that $K_\ell \cap \rho_*^{-1}(r_m) \subseteq K_\ell \cap H_\ell$. Recall that $(r_m, U_m) \in U$ and that $U \subseteq \tau_{2n-1}(L \cap K_\ell)$. Since τ_{2n-1} is the rating map $\zeta_{\rho_*, \tau_{2(n-1)}}^{\mathcal{G}}[b]$ by definition, it follows that,

$$U_m \in \mathcal{I}_{\mathcal{G}} [\rho_*^{-1}(r_m) \cap L \cap K_\ell, \tau_{2(n-1)}]$$

Combined with the inclusion $K_\ell \cap \rho_*^{-1}(r_m) \subseteq K_\ell \cap H_\ell$ and Fact 4.6, this yields that,

$$U_m \in \mathcal{I}_{\mathcal{G}} [L \cap K_\ell \cap H_\ell, \tau_{2(n-1)}]$$

Since \mathcal{G} is a lattice, it is clear that $\{L \cap K_\ell \cap H_\ell\}$ is a \mathcal{G} -cover of $L \cap K_\ell \cap H_\ell$. Thus, we obtain that $U_m \subseteq \tau_{2(n-1)}(L \cap K_\ell \cap H_\ell)$. Therefore, since $s \in f_{2(n-1)}(U_m)$ by definition of U_m , we obtain from Fact 8.1 that,

$$s \in f_{2(n-1)}(\tau_{2(n-1)}(L \cap K_\ell \cap H_\ell))$$

Finally, since $\{K_i \setminus H_i \mid i \leq n\}$ was a cover of L , it is clear that $\{K_i \setminus H_i \mid i \leq n \text{ and } i \neq \ell\}$ (of size $n-1$) is a cover of $L \cap K_\ell \cap H_\ell$. Therefore, it follows by induction on n in Lemma 8.3 that there exists $j \leq n$ (with $j \neq \ell$) such that $s \leq \rho(K_j \setminus H_j)$, finishing the proof. \square

8.2. Soundness proof for Theorem 5.3. We may now come back to our main objective: soundness in Theorem 5.3. We fix a finite quotient-closed Boolean algebra \mathcal{C} and a nice multiplicative rating map $\rho : 2^{A^*} \rightarrow R$. We show that for every $BPol(\mathcal{C})$ -saturated subset $S \subseteq (A^*/\sim_{\mathcal{C}}) \times R$ (for ρ), we have $S \subseteq \mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$.

Remark 8.6. *We do not use the hypothesis that ρ is nice for this direction. This is only needed for completeness.*

By Theorem 3.2, $Pol(\mathcal{C})$ is a lattice. Therefore, we may instantiate the definitions and results presented at the beginning of the section for our nice multiplicative rating map $\rho : 2^{A^*} \rightarrow R$ in the special case when $\mathcal{G} = Pol(\mathcal{C})$ and $\mathcal{D} = BPol(\mathcal{C})$. We keep using the same notations, we have the rating maps $\tau_n : 2^{A^*} \rightarrow Q_n$ (as we prove below, they are now quasi-multiplicative since $\mathcal{G} = Pol(\mathcal{C})$ and ρ is multiplicative) and the maps $f_n : Q_n \rightarrow 2^R$.

We complete Proposition 8.2 with another one specific to this special case. In fact, the proof of this second proposition is based on Theorem 7.2, the characterization of $Pol(\mathcal{C})$ -optimal imprints (we apply it to the rating maps τ_n). Together, these two results imply soundness in Theorem 5.3.

Proposition 8.7. *Consider $S \subseteq (A^*/\sim_{\mathcal{C}}) \times R$ which is $BPol(\mathcal{C})$ -saturated for ρ . Then, the following inclusion holds for every $D \in A^*/\sim_{\mathcal{C}}$,*

$$S(D) \subseteq \bigcap_{n \in \mathbb{N}} f_n(\tau_n(D))$$

When put together, Proposition 8.2 and Proposition 8.7 imply soundness in Theorem 5.3. Indeed, consider a $BPol(\mathcal{C})$ -saturated set $S \subseteq (A^*/\sim_{\mathcal{C}}) \times R$. We show that $S \subseteq \mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$. This amounts to proving that for every $D \in A^*/\sim_{\mathcal{C}}$, we have,

$$S(D) \subseteq \mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho](D) = \mathcal{I}_{BPol(\mathcal{C})}[D, \rho]$$

It is immediate from Proposition 8.7 that,

$$S(D) \subseteq \bigcap_{n \in \mathbb{N}} f_n(\tau_n(D))$$

Moreover, since $D \in A^*/\sim_{\mathcal{C}}$, we have $D \in \mathcal{C}$ which implies that $D \in Pol(\mathcal{C})$. Therefore, Proposition 8.2 yields that,

$$\bigcap_{n \in \mathbb{N}} f_n(\tau_n(D)) \subseteq \mathcal{I}_{BPol(\mathcal{C})}[D, \rho]$$

Altogether, we get the desired inclusion: $S(D) \subseteq \mathcal{I}_{BPol(\mathcal{C})}[D, \rho]$. This concludes the soundness proof.

It remains to prove Proposition 8.7. We start with a few additional results about the rating maps τ_n that we are able to prove using our new hypotheses (i.e. ρ is multiplicative and $\mathcal{G} = Pol(\mathcal{C})$).

Preliminaries. Recall that ρ is multiplicative. Hence, R is a semiring $(R, +, \cdot)$. Since $Q_0 = 2^R$ and $Q_n = 2^{R \times Q_{n-1}}$ for all $n \geq 1$, we may lift the multiplication of R to all the rating sets Q_n in the natural way. It is simple to verify that (Q_n, \cup, \cdot) is a semiring for every $n \in \mathbb{N}$. We first show that the rating maps τ_n are quasi-multiplicative for these multiplications.

Lemma 8.8. *The rating map $\tau_0 : 2^{A^*} \rightarrow Q_0$ is multiplicative. Moreover, for every $n \geq 1$, the rating map $\tau_n : 2^{A^*} \rightarrow Q_n$ is quasi-multiplicative and the associated endomorphism μ_{τ_n} of (Q_n, \cup) is as follows,*

$$\mu_{\tau_n}(T) = \downarrow_{Q_{n-1}} \{(r, \mu_{\tau_{n-1}}(V)) \mid (r, V) \in T\} \quad \text{for every } T \in Q_n = 2^{T \times Q_{n-1}}$$

Proof. Recall that $Q_0 = 2^R$ and $\tau_0(K) = \{\rho(w) \mid w \in K\}$ for every language K . It is immediate that τ_0 is multiplicative since ρ is multiplicative by hypothesis. The result for the rating maps τ_n for $n \geq 1$ is then immediate from Lemma 6.7 using induction on n since $\text{Pol}(\mathcal{C})$ is a quotient-closed lattice closed under concatenation by Theorem 3.2. \square

We complete this result with two lemmas which connect the hypothesis that the rating maps τ_n are quasi-multiplicative with the maps $f_n : Q_n \rightarrow 2^R$.

Lemma 8.9. *For every $n \in \mathbb{N}$ and $T \in Q_n$, we have $f_n(T) \subseteq f_n(\mu_{\tau_n}(T))$.*

Proof. We proceed by induction on $n \in \mathbb{N}$. When $n = 0$, τ_0 is multiplicative and the endomorphism μ_{τ_0} of (Q_0, \cup) is the identity on Q_0 . Hence, the lemma is immediate. Assume now that $n \geq 1$. Consider $T \in Q_n$. Let $r \in f_n(T) \subseteq R$. By definition, we have $(r_1, T_1), \dots, (r_k, T_k) \in T$ such that $r_1 + \dots + r_k \in f_{n-1}(T_i)$ for every $i \leq k$ and $r \leq r_1 + \dots + r_k$. By definition of μ_{τ_n} in Lemma 8.8, we have $(r_1, \mu_{\tau_{n-1}}(T_1)), \dots, (r_k, \mu_{\tau_{n-1}}(T_k)) \in \mu_{\tau_n}(T)$. Moreover, since $r_1 + \dots + r_k \in f_{n-1}(T_i)$ for every $i \leq k$ it is immediate by induction hypothesis that $r_1 + \dots + r_k \in f_{n-1}(\mu_{\tau_{n-1}}(T_i))$ for every $i \leq k$. Altogether, we obtain that $r \in f_n(\mu_{\tau_n}(T))$, finishing the proof for the first inclusion. \square

Lemma 8.10. *For every $n \in \mathbb{N}$ and $T, T' \in Q_n$, we have $f_n(T) \cdot f_n(T') \subseteq f_n(T \cdot T')$.*

Proof. We proceed by induction on $n \in \mathbb{N}$. We first handle the case $n = 0$. Let $r \in f_0(T) \cdot f_0(T')$. We have $s \in f_0(T)$ and $s' \in f_0(T')$ such that $r = ss'$. By definition, this yields $r_1, \dots, r_k \in T$ and $r'_1, \dots, r'_{k'} \in T'$ such that $s \leq r_1 + \dots + r_k$ and $s' \leq r'_1 + \dots + r'_{k'}$. It follows that $ss' \leq \sum_{i \leq k} \sum_{j \leq k'} r_i r'_j$. Since $r_i r'_j \in T \cdot T'$ for every $i \leq k$ and $j \leq k'$, it follows that $ss' \in f_0(T \cdot T')$.

Assume now that $n \geq 1$. Let $r \in f_n(T) \cdot f_n(T')$. We have $s \in f_n(T)$ and $s' \in f_n(T')$ such that $r = ss'$. By definition, this yields $(r_1, T_1), \dots, (r_k, T_k) \in T$ and $(r'_1, T'_1), \dots, (r'_{k'}, T'_{k'}) \in T'$ such that,

- $s \leq r_1 + \dots + r_k$ and $r_1 + \dots + r_k \in f_{n-1}(T_i)$ for every $i \leq k$.
- $s' \leq r'_1 + \dots + r'_{k'}$ and $r'_1 + \dots + r'_{k'} \in f_{n-1}(T'_j)$ for every $j \leq k'$.

Clearly, we have $ss' \leq \sum_{i \leq k} \sum_{j \leq k'} r_i r'_j$. Moreover, for every $i \leq k$ and $j \leq k'$, we have,

$$\sum_{i \leq k} \sum_{j \leq k'} r_i r'_j \in f_{n-1}(T_i) \cdot f_{n-1}(T'_j)$$

By induction hypothesis, this yields,

$$\sum_{i \leq k} \sum_{j \leq k'} r_i r'_j \in f_{n-1}(T_i \cdot T'_j)$$

Finally, it is immediate that for every $i \leq k$ and $j \leq k'$, we have $(r_i r'_j, T_i \cdot T'_j) \in T \cdot T'$. Altogether, this yields that $ss' \in f_n(T \cdot T')$ by definition. \square

Proof of Proposition 8.7. We now turn to the main argument. We fix $S \subseteq (A^*/\sim_{\mathcal{C}}) \times R$ which is $BPol(\mathcal{C})$ -saturated (for ρ). We have to show that $S(D) \subseteq f_n(\tau_n(D))$ for every $n \in \mathbb{N}$ and $D \in A^*/\sim_{\mathcal{C}}$. The argument is an induction on $n \in \mathbb{N}$.

Base case. We start with the case $n = 0$. Consider $D \in A^*/\sim_{\mathcal{C}}$. We show that $S(D) \subseteq f_0(\tau_0(D))$. Consider $r \in S(D)$, i.e. $(D, r) \in S$. Since S is $BPol(\mathcal{C})$ -saturated, we know that (D, r) satisfies (5.1). We have $r_1, \dots, r_k \in R$ such that $r \leq r_1 + \dots + r_k$ and for every $i \leq k$, there exists $U_i \in 2^R$ satisfying $(D, r_i, U_i) \in \mathcal{R}_S^\rho$ and $r_1 + \dots + r_k \in \downarrow_R U_i$. Given $i \leq k$, one may verify from the definition of \mathcal{R}_S^ρ that since $(D, r_i, U_i) \in \mathcal{R}_S^\rho$, we have $w_i \in A^*$ such that $[w_i]_{\mathcal{C}} = D$ and $\rho(w_i) = r_i$. In particular, $w_1, \dots, w_k \in D$ and it is therefore immediate from the definition of τ_0 that we have $r_1, \dots, r_k \in \tau_0(D)$. Hence, since $s \leq r_1 + \dots + r_k$, we obtain that $s \in f_0(\tau_0(D))$ by definition of f_0 which concludes the proof.

Inductive step. We now assume that $n \geq 1$. The argument is based on the following lemma which is where we use the characterization of $Pol(\mathcal{C})$ -optimal imprints (i.e. Theorem 7.2): we apply it to the quasi-multiplicative rating map τ_{n-1} . Moreover, this is also where we apply induction on n in Proposition 8.7.

Lemma 8.11. *For every $(D, q, U) \in \mathcal{R}_S^\rho$, there exists $T \in Q_{n-1}$ satisfying the two following properties:*

$$T \in \mathcal{I}_{Pol(\mathcal{C})} [\rho_*^{-1}(q) \cap D, \tau_{n-1}] \text{ and } \downarrow_R U \subseteq f_{n-1}(T)$$

We start by applying Lemma 8.11 to complete the main proof. Consider $D \in A^*/\sim_{\mathcal{C}}$. We show that $S(D) \subseteq f_n(\tau_n(D))$. Let $r \in S(D)$ (i.e. $(D, r) \in S$), we prove that $r \in f_n(\tau_n(D))$.

Since S is $BPol(\mathcal{C})$ -saturated, we know that (D, r) satisfies (5.1). We have $r_1, \dots, r_k \in R$ such that $r \leq r_1 + \dots + r_k$ and for every $i \leq k$, there exists $U_i \in 2^R$ satisfying $(D, r_i, U_i) \in \mathcal{R}_S^\rho$ and $r_1 + \dots + r_k \in \downarrow_R U_i$. Therefore, we obtain from Lemma 8.11 that for all $i \leq k$, there exists $T_i \in Q_{n-1}$ such that $T_i \in \mathcal{I}_{Pol(\mathcal{C})} [\rho_*^{-1}(r_i) \cap D, \tau_{n-1}]$ and $r_1 + \dots + r_k \in f_{n-1}(T_i)$. Recall that by definition, τ_n is the rating map $\zeta_{Pol(\mathcal{C})}^{\rho_*}[\tau_{n-1}] : 2^{A^*} \rightarrow (2^{Q_{n-1}})^R$. Thus, if we unravel the definition, for every $i \leq k$, the hypothesis that $T_i \in \mathcal{I}_{Pol(\mathcal{C})} [\rho_*^{-1}(r_i) \cap D, \tau_{n-1}]$ exactly says that,

$$(r_i, U_i) \in \tau_n(D)$$

Since we also have $r \leq r_1 + \dots + r_k$ and $r_1 + \dots + r_k \in f_{n-1}(T_i)$ for every $i \leq k$, it is immediate by definition of f_n that $r \in f_n(\tau_n(D))$, finishing the proof of Proposition 8.7. It remains to prove Lemma 8.11.

Proof of Lemma 8.11. We first apply Theorem 7.2 and then use the result to prove the lemma. Clearly, the Cartesian product $(A^*/\sim_{\mathcal{C}}) \times R$ is a monoid when equipped with the componentwise multiplication. Let $\alpha : A^* \rightarrow (A^*/\sim_{\mathcal{C}}) \times R$ be the morphism defined by $\alpha(w) = ([w]_{\mathcal{C}}, \rho(w))$. Clearly, α is a \mathcal{C} -compatible morphism: for every pair $(D, r) \in (A^*/\sim_{\mathcal{C}}) \times R$, it suffices to define $[(D, r)]_{\mathcal{C}} = D$. Consequently, since we also know that τ_{n-1} is quasi-multiplicative by Lemma 8.8, we may apply Theorem 7.2 to obtain a description of the set $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\tau_{n-1}] \subseteq (A^*/\sim_{\mathcal{C}}) \times R \times Q_{n-1}$. Consider the least $Pol(\mathcal{C})$ -saturated subset X of $(A^*/\sim_{\mathcal{C}}) \times R \times Q_{n-1}$ for α and τ_{n-1} . Theorem 7.2 yields that,

$$\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\tau_{n-1}] = \downarrow_{Q_{n-1}} \{(D, q, \mu_{\tau_{n-1}}(P)) \mid (D, q, P) \in X\} \quad (8.1)$$

The proof of Lemma 8.11 is now based on the following lemma which we shall prove by induction on the definition of \mathcal{R}_S^ρ .

Lemma 8.12. *For every $(D, q, U) \in \mathcal{R}_S^\rho$. There exists $P \in Q_{n-1}$ such that $(D, q, P) \in X$ and $U \subseteq f_{n-1}(P)$.*

Let us first use Lemma 8.12 to handle the main proof. Let $(D, q, U) \in \mathcal{R}_S^\rho$. We have to exhibit $T \in Q_{n-1}$ such that $(D, q, T) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\tau_{n-1}]$ and $\downarrow_R U \subseteq f_{n-1}(T)$.

Using Lemma 8.12, we get $P \in Q_{n-1}$ such that $(D, q, P) \in X$ and $U \subseteq f_{n-1}(P)$. In view of (8.1), this yields,

$$(D, q, \mu_{\tau_{n-1}}(P)) \in \mathcal{P}_{Pol(\mathcal{C})}^\alpha[\tau_{n-1}]$$

By definition of the set $\mathcal{P}_{Pol(\mathcal{C})}^\alpha[\tau_{n-1}]$ and of the morphism α , this exactly says that,

$$\mu_{\tau_{n-1}}(P) \in \mathcal{I}_{Pol(\mathcal{C})}[\rho_*^{-1}(q) \cap D, \tau_{n-1}]$$

Moreover, Lemma 8.9 yields that $U \subseteq f_{n-1}(P) \subseteq f_{n-1}(\mu_{\tau_{n-1}}(P))$. Finally, $f_{n-1}(\mu_{\tau_{n-1}}(P)) = \downarrow_R f_{n-1}(\mu_{\tau_{n-1}}(P))$ by definition of f_{n-1} and we obtain, $\downarrow_R U \subseteq f_{n-1}(\mu_{\tau_{n-1}}(P))$. Hence, the desired property holds for $T = \mu_{\tau_{n-1}}(P)$ and we are finished.

It remains to prove Lemma 8.12. Consider $(D, q, U) \in \mathcal{R}_S^\rho$. By definition of \mathcal{R}_S^ρ , we know that $(D, q, U) \in \mathcal{R}_S^\rho$ is built from trivial elements using two operations: multiplication and S -restricted $Pol(\mathcal{C})$ -closure. We proceed by induction on this construction. There are three cases depending on the last operation used to build $(D, q, U) \in \mathcal{R}_S^\rho$.

Base case: trivial elements. In that case, there exists $w \in A^*$ such that $D = [w]_{\mathcal{C}}$, $q = \rho(w)$ and $U = \{\rho(w)\}$. Since X is $Pol(\mathcal{C})$ -saturated for ρ_* and τ_{n-1} , we know that $([w]_{\mathcal{C}}, \rho(w), \tau_{n-1}(w)) \in X$. Hence, it remains to prove that $U = \{\rho(w)\} \subseteq f_{n-1}(\tau_{n-1}(w))$. It will then be immediate that Lemma 8.12 holds for $P = \tau_{n-1}(w)$. This is immediate from the following fact.

Fact 8.13. *For every $m \in \mathbb{N}$, we have $\rho(w) \in f_m(\tau_m(w))$.*

Proof. This is a simple induction on m . When $m = 0$, we have $\rho(w) \in \tau_0(w)$ by definition of τ_0 . It follows that $\rho(w) \in f_0(\tau_0(w))$ by definition of f_0 . Assume now that $m \geq 1$. Recall that τ_m is $\zeta_{Pol(\mathcal{C})}^{\rho_*}[\tau_{m-1}] : 2^{A^*} \rightarrow 2^{R \times Q_{m-1}}$ by definition. Therefore,

$$\tau_m(w)(\rho(w)) = \mathcal{I}_{Pol(\mathcal{C})}[\rho_*^{-1}(\rho(w)) \cap \{w\}, \tau_{m-1}] = \mathcal{I}_{Pol(\mathcal{C})}[\{w\}, \tau_{m-1}]$$

Thus, $\tau_{m-1}(w) \in \tau_m(w)(\rho(w))$. Moreover induction yields $\rho(w) \in f_{m-1}(\tau_{m-1}(w))$. Therefore, by definition of f_m , we have $\rho(w) \in f_m(\tau_m(w))$. \square

Inductive case 1: multiplication. We have $(D_1, r_1, U_1) \in \mathcal{R}_S^\rho$ and $(D_2, r_2, U_2) \in \mathcal{R}_S^\rho$ such that $D = D_1 \bullet D_2$, $q = r_1 r_2$ and $U = U_1 U_2$. By induction, we obtain $P_1, P_2 \in Q_{n-1}$ such that $(D_1, r_1, P_1), (D_2, r_2, P_2) \in X$, $U_1 \subseteq f_{n-1}(P_1)$ and $U_2 \subseteq f_{n-1}(P_2)$. Since X is $Pol(\mathcal{C})$ -saturated, it is closed under multiplication and we get,

$$(D, q, P_1 P_2) = (D_1 \bullet D_2, r_1 r_2, P_1 P_2) \in X$$

Moreover, Lemma 8.10 yields that,

$$U = U_1 U_2 \subseteq f_{n-1}(P_1) \cdot f_{n-1}(P_2) \subseteq f_{n-1}(P_1 P_2)$$

Altogether, it follows that Lemma 8.12 holds for $P = P_1 P_2$.

Inductive case 2: S -restricted closure. In that case, we have idempotents $(E, f, F) \in \mathcal{R}_S^\rho$ such that $D = E$, $q = f$ and $U = F \cdot S(E) \cdot F$.

By induction, we get $V \in Q_{n-1}$ such that $(E, f, V) \in X$ and $F \subseteq f_{n-1}(V)$. Since Q_{n-1} is a finite monoid, there exists a number $n \geq 1$ such that V^n is a multiplicative idempotent of Q_{n-1} . Hence, since $(E, f, V) \in X$ which is closed under multiplication (it is

$Pol(\mathcal{C})$ -saturated), we get that $(E, f, V^n) \in X$. We may again use the hypothesis that X is $Pol(\mathcal{C})$ -saturated to apply $Pol(\mathcal{C})$ -closure and obtain,

$$(E, f, V^n \cdot \tau_{n-1}(E) \cdot V^n) \in X$$

Since $D = E$ and $q = f$, it now remains to show that $U = F \cdot S(E) \cdot F \subseteq f_{n-1}(V^n \cdot \tau_{n-1}(E) \cdot V^n)$. It will then be immediate that Lemma 8.12 holds for $P = V^n \cdot \tau_{n-1}(E) \cdot V^n$. It is immediate by induction in Proposition 8.7 that,

$$S(E) \subseteq f_{n-1}(\tau_{n-1}(E))$$

Moreover, since we already know that $F \subseteq f_{n-1}(V)$ is an idempotent by definition, it follows from Lemma 8.10 that,

$$F \cdot S(E) \cdot F = F^n \cdot S(E) \cdot F^n \subseteq f_{n-1}(V^n \cdot \tau_{n-1}(E) \cdot V^n)$$

This concludes the proof of Lemma 8.12. \square

9. COMPLETENESS IN THEOREM 5.3

In this section, we prove the completeness direction in Theorem 5.3. As for the soundness proof, the section is divided in two parts. We start with a preliminary result which applies to Boolean closure in general, i.e. to classes of the form $Bool(\mathcal{G})$ when \mathcal{G} is an arbitrary lattice. We then use it in the special case $\mathcal{G} = Pol(\mathcal{C})$ to handle the completeness proof.

9.1. Preliminary result. We fix a lattice \mathcal{G} and let $\mathcal{D} = Bool(\mathcal{G})$. Moreover, we let $\rho : 2^{A^*} \rightarrow R$ as a **nice** rating map. Finally, we write τ for the rating map $\xi_{\mathcal{D}}[\rho] : 2^{A^*} \rightarrow 2^R$ introduced in Proposition 6.2.

We may now state our preliminary result. Note that the statement involves the rating map $\zeta_{\mathcal{G}}^{\rho_*}[\tau] : 2^{A^*} \rightarrow 2^{R \times 2^R}$ introduced in Proposition 6.2. It is built from the canonical map $\rho_* : A^* \rightarrow R$ associated to ρ and the rating map $\tau = \xi_{\mathcal{D}}[\rho]$.

Proposition 9.1. *For every $L \in \mathcal{D}$ and $s \in \tau(L)$, we have $r_1, \dots, r_k \in R$ such that $(r_i, \{r_1 + \dots + r_k\}) \in \zeta_{\mathcal{G}}^{\rho_*}[\tau](L)$ for every $i \leq k$ and $s \leq r_1 + \dots + r_k$.*

Proof. We fix a language $L \in \mathcal{D}$ and $s \in \tau(L)$ for the proof. We exhibit $r_1, \dots, r_k \in R$ such that $(r_i, \{r_1 + \dots + r_k\}) \in \zeta_{\mathcal{G}}^{\rho_*}[\tau](L)$ for every $i \leq k$ and $s \leq r_1 + \dots + r_k$.

For every $q \in R$, we let \mathbf{H}_q as an optimal \mathcal{G} -cover of $L \cap \rho_*^{-1}(q)$ for τ . Since $\mathcal{D} = Bool(\mathcal{G})$, we have the following fact.

Fact 9.2. *There exists a \mathcal{D} -cover \mathbf{K} of L such that for every $K \in \mathbf{K}$ and every $q \in R$, if $K \cap \rho_*^{-1}(q) \neq \emptyset$, then there exists $H \in \mathbf{H}_q$ such that $K \subseteq H$.*

Proof. Let $\mathbf{H} = \bigcup_{q \in R} \mathbf{H}_q$ and consider the following equivalence \sim defined on L . For every $u, v \in L$, we let $u \sim v$ if and only if $u \in H \Leftrightarrow v \in H$ for every $H \in \mathbf{H}$. We let \mathbf{K} as the partition of L into \sim -classes. By definition, \mathbf{K} is a cover of L . Moreover, it is a \mathcal{D} -cover. Indeed, by definition, \mathbf{K} only contains Boolean combinations of $L \in \mathcal{D}$ with languages in \mathbf{H} (which are in \mathcal{G} and $\mathcal{D} = Bool(\mathcal{G})$). It remains to show that \mathbf{K} satisfies the property described in the fact.

Let $q \in R$ and assume that there exists $w \in K \cap \rho_*^{-1}(q)$. By definition of \mathbf{K} , we have $K \subseteq L$ which means that $w \in L \cap \rho_*^{-1}(q)$. Therefore, since \mathbf{H}_q is a cover of $L \cap \rho_*^{-1}(q)$ by definition, we have $H \in \mathbf{H}_q$ such that $w \in H$. Consequently, $K \cap H \neq \emptyset$. Finally, since K is a \sim -class by definition of \mathbf{K} , it follows from the definition of \sim that $K \subseteq H$. \square

We let \mathbf{K} as the \mathcal{D} -cover of L described in Fact 9.2. By definition, we have $L \subseteq \bigcup_{K \in \mathbf{K}} K$. Hence, since $\tau = \xi_{\mathcal{D}}[\rho]$ is a rating map, it follows that,

$$\tau(L) \subseteq \bigcup_{K \in \mathbf{K}} \tau(K)$$

Consequently, since $s \in \tau(L)$, we get some $K \in \mathbf{K}$ such that $s \in \tau(K)$. We let \mathbf{G} as an optimal \mathcal{D} -cover of K for ρ . Since $K \in \mathcal{D}$, we may choose \mathbf{G} such that $G \subseteq K$ for all $G \in \mathbf{G}$. By definition of \mathbf{G} , we have $\mathcal{I}[\rho](\mathbf{G}) = \mathcal{I}_{\mathcal{D}}[K, \rho]$ and since $\tau(K) = \mathcal{I}_{\mathcal{D}}[K, \rho]$ by definition, we get that $s \in \mathcal{I}[\rho](\mathbf{G})$. Therefore, there exists $G \in \mathbf{G}$ such that $s \leq \rho(G)$.

Since ρ is nice by hypothesis, we have $w_1, \dots, w_k \in G$ such that $\rho(G) = \rho(w_1) + \dots + \rho(w_k)$. We let $r_i = \rho(w_i)$ for every $i \leq k$. by definition, we have $s \leq r_1 + \dots + r_k$. Therefore, it remains to show that $(r_i, \{r_1 + \dots + r_k\}) \in \zeta_{\mathcal{G}}^{\rho^*}[\tau](L)$ for every $i \leq k$.

We fix $i \leq k$ for the proof. By definition, $\rho(w_i) = r_i$ and $w_i \in G \subseteq K$. Hence, $w_i \in K \cap \rho_*^{-1}(r_i)$ and it follows from the definition of \mathbf{K} in Fact 9.2 that there exists a language $H \in \mathbf{H}_{r_i}$ such that $K \subseteq H$. Recall that $r_1 + \dots + r_k = \rho(G)$. Moreover, $\rho(G) \in \mathcal{I}[\rho](\mathbf{G})$ and we have $\mathcal{I}[\rho](\mathbf{G}) = \tau(K)$ by definition. Consequently,

$$r_1 + \dots + r_k \in \tau(K) \subseteq \tau(H)$$

Therefore, since $H \in \mathbf{H}_{r_i}$, we have $\{r_1 + \dots + r_k\} \in \mathcal{I}[\tau](\mathbf{H}_{r_i})$. Recall that we defined \mathbf{H}_{r_i} as an optimal \mathcal{G} -cover of $L \cap \rho_*^{-1}(r_i)$ for τ . Hence, we obtain,

$$\{r_1 + \dots + r_k\} \in \mathcal{I}_{\mathcal{G}}[L \cap \rho_*^{-1}(r_i), \tau]$$

By definition of $\zeta_{\mathcal{G}}^{\rho^*}[\tau]$, this yields that $(r_i, \{r_1 + \dots + r_k\}) \in \zeta_{\mathcal{G}}^{\rho^*}[\tau](L)$ which concludes the proof. \square

9.2. Completeness proof for Theorem 5.3. We may now come back to our main objective: completeness in Theorem 5.3. We fix a finite quotient-closed Boolean algebra \mathcal{C} and a \mathcal{C} -compatible nice multiplicative rating map $\rho : 2^{A^*} \rightarrow R$. We need to prove that $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ is $BPol(\mathcal{C})$ -saturated for ρ . Since, we already showed in the previous section that every $BPol(\mathcal{C})$ -saturated subset is included in $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$, Theorem 5.3 will follow: $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ is the greatest $BPol(\mathcal{C})$ -saturated subset of R . For the sake of avoiding clutter, we shall write S for $\mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho] \subseteq (A^*/\sim_{\mathcal{C}}) \times R$.

Remark 9.3. *Contrary to the soundness direction, we do need the hypothesis that ρ is nice here. This is required for applying the above preliminary result: Proposition 9.1.*

By Theorem 3.2, $Pol(\mathcal{C})$ is a lattice. Therefore, we may instantiate the definitions and the result presented at the beginning of the section for our nice multiplicative rating map $\rho : 2^{A^*} \rightarrow R$ in the special case when $\mathcal{G} = Pol(\mathcal{C})$ and $\mathcal{D} = BPol(\mathcal{C})$. In particular, recall that in this case, we write τ for the rating map $\xi_{BPol(\mathcal{C})}[\rho]$.

We complete Proposition 9.1 with another one specific to this special case. As for the soundness direction, its proof is based on Theorem 7.2: the characterization of $Pol(\mathcal{C})$ -optimal imprints. Together, these two results imply that $S = \mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ is $BPol(\mathcal{C})$ -saturated for ρ .

Proposition 9.4. *Let $D \in A^*/\sim_{\mathcal{C}}$ and $r \in R$. For every $U \in \mathcal{I}_{Pol(\mathcal{C})}[D \cap \rho_*^{-1}(r), \tau]$, there exists $U' \in 2^R$ such that $(D, r, U') \in \mathcal{R}_{\mathcal{G}}^{\rho}$ and $U \subseteq \downarrow_R U'$.*

We first combine Proposition 9.1 and Proposition 9.4 to prove that $S = \mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ is $BPol(\mathcal{C})$ -saturated for ρ and finish the completeness proof. Consider $(D, r) \in \mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$, we have to show that (D, r) satisfy (5.1). That is, we have to exhibit $r_1, \dots, r_k \in R$ such that $r \leq r_1 + \dots + r_k$ and for every $i \leq k$, there exists $U_i \in 2^R$ satisfying $(D, r_i, U_i) \in \mathcal{R}_S^\rho$ and $r_1 + \dots + r_k \in \downarrow_R U_i$

Since $D \in A^*/\sim_{\mathcal{C}}$, we have $D \in \mathcal{C} \subseteq Pol(\mathcal{C})$. Moreover, $(D, r) \in \mathcal{P}_{PBPol(\mathcal{C})}^{\mathcal{C}, \alpha}[\rho]$ means that $r \in \mathcal{I}_{BPol(\mathcal{C})}[D, \rho]$ by definition (i.e. $r \in \tau(D)$ since τ is $\xi_{BPol(\mathcal{C})}[\rho]$). Hence, we may apply Proposition 9.1. This yields $r_1, \dots, r_k \in R$ such that $r \leq r_1 + \dots + r_k$ and for every $i \leq k$, we have $\{r_1 + \dots + r_k\} \in \mathcal{I}_{Pol(\mathcal{C})}[L \cap \rho_*^{-1}(r_i), \tau]$. Then, we obtain from Proposition 9.4 that for every $i \leq k$, there exists $U_i \in 2^R$ such that $(D, r, U_i) \in \mathcal{R}_S^\rho$ and $r_1 + \dots + r_k \in \downarrow_R U_i$. Altogether, this is exactly the property stated in (5.1). This concludes the completeness proof for Theorem 5.3. It remains to prove Proposition 9.4.

Proof of Proposition 9.4. We first apply Theorem 7.2 and then use the result to prove the proposition. Clearly, the Cartesian product $(A^*/\sim_{\mathcal{C}}) \times R$ is a monoid when equipped with the componentwise multiplication. Let $\alpha : A^* \rightarrow (A^*/\sim_{\mathcal{C}}) \times R$ be the morphism defined by $\alpha(w) = ([w]_{\mathcal{C}}, \rho(w))$. Clearly, α is a \mathcal{C} -compatible morphism: for every pair $(D, r) \in (A^*/\sim_{\mathcal{C}}) \times R$, it suffices to define $[(D, r)]_{\mathcal{C}} = D$.

We also know that $\tau = \xi_{BPol(\mathcal{C})}[\rho]$ is $Pol(\mathcal{C})$ -multiplicative By Lemma 6.8 since $Pol(\mathcal{C})$ is a quotient-closed lattice closed under marked concatenation. Moreover, the associated endomorphism of $(2^R, \cup)$ is the map $V \mapsto \downarrow_R V$. Therefore, we may apply Theorem 7.2 to get a description of $\mathcal{P}_{Pol(\mathcal{C})}^{\alpha}[\tau]$. Let X be the least $Pol(\mathcal{C})$ -saturated subset of $(A^*/\sim_{\mathcal{C}}) \times R \times 2^R$ for α and τ . Theorem 7.2 yields that,

$$\mathcal{P}_{Pol(\mathcal{C})}^{\alpha}[\tau] = \downarrow_{2^R} \{(D, r, \downarrow_R U) \mid (D, r, U) \in X\} \quad (9.1)$$

Proposition 9.4 is now a consequence of the following lemma.

Lemma 9.5. *For every $(D, r, U) \in X$, there exists $U' \in 2^R$ such that $(D, r, U') \in \mathcal{R}_S^\rho$ and $U \subseteq \downarrow_R U'$.*

Let $D \in A^*/\sim_{\mathcal{C}}$ and $r \in R$. Consider $U \in \mathcal{I}_{Pol(\mathcal{C})}[D \cap \rho_*^{-1}(r), \tau]$. We have to exhibit $U' \in 2^R$ such that $(D, r, U') \in \mathcal{R}_S^\rho$ and $U \subseteq \downarrow_R U'$.

By definition of α and $\mathcal{P}_{Pol(\mathcal{C})}^{\alpha}[\tau]$, $U \in \mathcal{I}_{Pol(\mathcal{C})}[D \cap \rho_*^{-1}(r), \tau]$ exactly says that $(D, r, U) \in \mathcal{P}_{Pol(\mathcal{C})}^{\alpha}[\tau]$. Hence, using (9.1), we get $V \in 2^R$ such that $(D, r, V) \in X$ and $U \subseteq \downarrow_R V$. Then, Lemma 9.5 yields $U' \in 2^R$ such that $(D, r, U') \in \mathcal{R}_S^\rho$ and $V \subseteq \downarrow_R U'$. It follows that $U \subseteq \downarrow_R \downarrow_R U' = \downarrow_R U'$ and Proposition 9.4 is proved.

It remains to prove Lemma 9.5. Consider $(D, r, U) \in X$. By definition, X is the least $Pol(\mathcal{C})$ -saturated subset of $(A^*/\sim_{\mathcal{C}}) \times R \times 2^R$ for α and τ . Hence, we know that (D, r, U) is built from trivial elements using multiplication and $Pol(\mathcal{C})$ -closure. We proceed by induction on this construction.

Base case: trivial elements. In that case, we have $w \in A^*$ such that $D = [w]_{\mathcal{C}}$, $r = \rho(w)$ and $U = \tau(w)$. Observe that $\{w\} \in BPol(\mathcal{C})$. Indeed, assume that $w = a_1 \cdots a_n$ with $a_1 \cdots a_n \in A$. Then, since $A^* \in \mathcal{C}$ (it is a Boolean algebra), we have,

$$\{w\} = A^* a_1 A^* \cdots a_n A^* \cap \left(A^* \setminus \left(\bigcup_{b_1, \dots, b_{n+1} \in A} A^* b_1 A^* \cdots b_{n+1} A^* \right) \right) \in BPol(\mathcal{C})$$

Hence, $\{\{w\}\}$ is a $BPol(\mathcal{C})$ -cover of $\{w\}$ and it is immediate that,

$$\tau(w) = \xi_{BPol(\mathcal{C})}[\rho](w) = \mathcal{I}_{BPol(\mathcal{C})}[\{w\}, \rho] = \downarrow_R\{\rho(w)\}$$

Moreover, we have $([w]_{\mathcal{C}}, \rho(w), \{\rho(w)\}) \in \mathcal{R}_S^\rho$ by definition (it is a trivial element). Clearly, Lemma 9.5 now holds for $U' = \{\rho(w)\}$: $U = \downarrow_R U'$.

Inductive case 1: multiplication. In that case, we have $(D_1, r_1, U_1), (D_2, r_2, U_2) \in X$ such that $D = D_1 \bullet D_2$, $r = r_1 r_2$ and $U = U_1 U_2$. Using induction, we get $U'_1, U'_2 \in 2^R$ such that $U_1 \subseteq \downarrow_R U'_1$, $U_2 \subseteq \downarrow_R U'_2$ and $(D_1, r_1, U'_1), (D_2, r_2, U'_2) \in \mathcal{R}_S^\rho$. Since \mathcal{R}_S^ρ is closed under multiplication, we have,

$$(D, r, U'_1 U'_2) = (D_1 \bullet D_2, r_1 r_2, U'_1 U'_2) \in \mathcal{R}_S^\rho$$

Moreover, since $U_1 \subseteq \downarrow_R U'_1$ and $U_2 \subseteq \downarrow_R U'_2$, it is immediate that $U_1 U_2 \subseteq \downarrow_R (U'_1 U'_2)$. Thus, Lemma 9.5 now holds for $U' = U'_1 U'_2$ which concludes this case.

Inductive case 2: $Pol(\mathcal{C})$ -closure. In that case, we have a triple of idempotents $(E, f, F) \in X$ such that $D = E$, $r = f$, and $U = F \cdot \tau(E) \cdot F$.

Using induction, we get $V \in 2^R$ such that $F \subseteq \downarrow_R V$ and $(E, f, V) \in \mathcal{R}_S^\rho$. Since 2^R is a finite monoid, there exists a number $n \geq 1$ such that V^n is a multiplicative idempotent of 2^R . Moreover, since \mathcal{R}_S^ρ is closed under multiplication, we have $(E, f, V^n) \in \mathcal{R}_S^\rho$ which is a triple of idempotents. Hence, we may apply S -restricted $Pol(\mathcal{C})$ -closure in the definition of \mathcal{R}_S^ρ . This yields,

$$(E, f, V^n \cdot S(E) \cdot V^n) \in \mathcal{R}_S^\rho$$

We prove that $U \subseteq \downarrow_R (V^n \cdot S(E) \cdot V^n)$. Since we have $D = E$ and $r = f$, it will follow that Lemma 9.5 holds for $U' = V^n \cdot S(E) \cdot V^n$, finishing the proof.

By definition, $S = \mathcal{P}_{BPol(\mathcal{C})}^{\mathcal{C}}[\rho]$ which means that $S(E) = \mathcal{I}_{BPol(\mathcal{C})}[E, \rho]$. Moreover, since τ is $\xi_{BPol(\mathcal{C})}[\rho]$ by definition, we also have $\tau(E) = \mathcal{I}_{BPol(\mathcal{C})}[E, \rho]$. Thus, $\tau(E) = S(E)$. Moreover, we also know that $F \subseteq \downarrow_R V$. Hence, since $F \in 2^R$ is an idempotent, this yields,

$$U' = F \cdot \tau(E) \cdot F \subseteq (\downarrow_R V)^n \cdot S(E) \cdot (\downarrow_R V)^n \subseteq \downarrow_R (V^n \cdot S(E) \cdot V^n)$$

This concludes the proof of Proposition 9.4. \square

10. CONCLUSION

We proved that separation and covering are decidable for all classes of the form $BPol(\mathcal{C})$ when \mathcal{C} is a finite quotient-closed Boolean algebra. This yields separation and covering algorithms for a whole family of classes. Arguably, the most important one is the level two in the Straubing-Thérien hierarchy (which corresponds to the logic $\mathcal{B}\Sigma_2(<)$). Moreover, using a known transfer theorem [PZ17a], this result can be lifted to dot-depth two.

An interesting consequence of our results is that since we proved the decidability of separation for the level two in the Straubing-Thérien hierarchy, the main theorem of [PZ14] is an immediate corollary: membership for this level is decidable. However, the algorithm of [PZ14] was actually based on a characterization theorem: languages of level two in the Straubing-Thérien hierarchy are characterized by a syntactic property of a canonical recognizer (*i.e.*, their syntactic monoid). It turns out that one can also deduce this characterization theorem from our results (this does require some combinatorial work however). In fact, one may generalize it to all classes $BPol(\mathcal{C})$ when \mathcal{C} is a finite quotient-closed Boolean algebra.

Finally, the main and most natural follow-up question is much harder: can our results be pushed to higher levels within concatenation hierarchies? For now, we know that given any finite quotient-closed Boolean algebra \mathcal{C} , $Pol(\mathcal{C})$, $BPol(\mathcal{C})$ and $Pol(BPol(\mathcal{C}))$ have decidable covering (the former and the latter are results of [Pla18]). Consequently, the next relevant levels are $BPol(BPol(\mathcal{C}))$ and $Pol(BPol(BPol(\mathcal{C})))$.

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