Information Processing Letters 108 (2008) 119–123

Contents lists available at ScienceDirect

Information Processing Letters

www.elsevier.com/locate/ipl

On handshakes in random graphs

Akka Zemmari

LaBRI, Université de Bordeaux - CNRS, 351 cours de la Libération, 33405 Talence, France

ARTICLE INFO

Article history:
Received 22 February 2008
Accepted 9 April 2008
Available online 24 April 2008
Communicated by L. Boasson

Keywords:
Analysis of algorithms
Randomized algorithms
Random graphs
Handshakes

ABSTRACT

In [Y. Métivier, N. Saheb, A. Zemmari, Analysis of a randomized rendezvous algorithm, Inform. and Comput. 184 (2003) 109–128], the authors introduce and analyze a randomized procedure to implement handshakes in graphs. In this paper, we investigate the same problem in random graphs. We prove results on the probability of success and we study the distribution of the random variable which counts the number of handshakes in the graph.

1. Introduction

Given a network of processors, a handshake algorithm enables the establishment of safe communications between two nodes, which guarantees that both the two nodes are being communicating with one other in exclusive mode. Distributed solutions of this problem are known in networks having a fixed topology [7,8,6,5,2]. What happens if we consider a network of processors with a random topology: links between processors exist with a given probability? In this paper, we consider networks with $n \geqslant 2$ nodes and for any pair of vertices, the probability that there is a communication link between this two vertices is equal to p $(0 \leqslant p \leqslant 1)$.

In [8], the authors proposed and studied the following randomized procedure (Algorithm 1) to implement handshakes in graphs. Let $G = (V, E)$ be an undirected graph which represents a distributed network with $|V| = n \geqslant 2$ processors.

The authors thoroughly study this procedure including the probability of having at least a handshake in the graph (this is called the probability of success), the expected number of handshakes and the probability distribution in some particular classes of graphs like chains, cycles and complete graphs.

In [5] and [6], the authors prove a conjecture from [8]: the probability of success in any graph with $n \geqslant 2$ vertices is lower bounded by the probability of success in the complete graph of size n. They also show that evaluating the probability of success in a given graph is a #P-complete problem.

1.1. Basic notations

We use the standard terminology of graph theory [1]. A *simple graph* $G = (V, E)$ is defined as a finite set of *vertices* together with a set E of *edges* which is a set of pairs of different vertices, $E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$. The cardinality of V is called the *size* of the graph. In the sequel, n will denote the size of the graph. For any $v \in V$, $N(v)$ denotes the set of neighbors of v, and $d(v) = |N(v)|$ the degree of v.

A *random graph* is a graph obtained by starting with a set of n vertices and adding edges between them at random. Different random graph models produce different probability distributions on graphs.

The most commonly studied model, called $G_{n,p}$, includes each possible edge independently with probability p (and so, the edge is not included with probability $q = 1 - p$). A closely related model, $G_{n,M}$ assigns equal

E-mail address: zemmari@labri.fr.

0020-0190/$ – see front matter
doi:10.1016/j.ipl.2008.04.018

1: Each vertex v repeats forever:
2: the vertex v selects one of its neighbors $c(v)$ chosen at random;
3: the vertex v sends 1 to $c(v)$;
4: the vertex v sends 0 to its neighbors different from $c(v)$;
5: the vertex v receives messages from all its neighbors.;

Algorithm 1. A randomized algorithm to obtain handshakes.

probability to all graphs with exactly M edges. In this paper, we consider the first model, with $p > 0$. It is straightforward that we have some edges.

1.2. Basics

In this section, we recall some fundamental definitions and results from [8,5] and [6] to make the present paper self-contained enough. The reader is referred to [8] and [6] for more details.

Let $G = (V, E)$ be a graph of size $|V| = n \geqslant 2$, and let $e = \{u, v\} \in E$. If we denote by $p(e)$ the probability of handshake over e then $p(e) = 1/d(u)d(v)$ where $d(u)$ denotes the degree of the vertex u. An execution of the randomized procedure in Algorithm 1 is called a *call* and is a function $c : V \to V$ which maps each vertex $v \in V$ to one of its neighbors.

Definition 1. (See [8].) A call c is a *success* if there is at least one handshake in the graph, otherwise, it is a *failure*. For any graph G, $s(G)$ denotes the probability of success and $f(G)$ the probability of failure.

In [8], the authors proved that for any $n \geqslant 2$, if K_n is the complete graph on n vertices then $s(K_n) \to 1 - e^{-1/2}$ as $n \to \infty$ and asked whether it is true that $s(G) \geqslant s(K_n)$ for all n-vertices graph G with at least an edge. In [5,6], Dietzfelbinger and Tamaki proved this conjecture:

Theorem 2. *(See [5,6].) If G is a graph with $n \geqslant 2$ nonisolated vertices, than $s(G) \geqslant s(K_n)$ where K_n denotes the complete graph with n vertices.*

1.3. Our contribution

In this paper, we study the randomized procedure in Algorithm 1 in random graphs. Clearly, if a vertex v has no neighbor then v cannot apply the algorithm, hence, we consider here that v performs Algorithm 1 if and only if its degree is at least 1.

The main results concern the probability of success in $G_{n,p}$ and the probability distribution. We prove that with high probability (w.h.p.)[1] the random variable that counts the number of handshakes in $G_{n,p}$ is asymptotically distributed as a Poisson random variable with parameter $\lambda = 1/2p$.

1.4. Outline

In Section 2, we obtain some preliminary results including the probability of having a handshake over an existing edge and the probability for a vertex to be involved in a handshake. In Section 3, we study the probability of success, that is, the probability of at least one handshake in the graph. Section 4 is devoted to the study of the probability distribution, we end the paper by a conclusion and some open problems.

2. Analysis of the algorithm

2.1. The probability of a handshake

Let $G_{n,p} = (V, E)$ a random graph and let u and v be two different vertices. If there is no edge between u and v, i.e., $\{u, v\} \notin E$ then there is no handshake between u and v. Hence, we assume that $e = \{u, v\} \in E$ and we study the probability of having a handshake on e conditioned on the event $e \in E$. Let $\mathcal{H}(e)$ denote the event *"there is a handshake on e"*. Then, we have the following formula

$$\begin{aligned}&\mathbb{P}r\big(\mathcal{H}(e) \mid e \in E\big)\\ &\quad= \sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-1} \mathbb{P}r\big(\mathcal{H}(e) \mid e \in E \wedge d(u) = k_1 \wedge d(v) = k_2\big)\\ &\qquad\times \mathbb{P}r\big(d(u) = k_1 \wedge d(v) = k_2 \mid e \in E\big).\end{aligned}$$

On the other hand, we have

$$\mathbb{P}r\big(\mathcal{H}(e) \mid e \in E \wedge d(u) = k_1 \wedge d(v) = k_2\big) = \frac{1}{k_1 k_2}$$

and

$$\begin{aligned}&\mathbb{P}r\big(d(u) = k_1 \wedge d(v) = k_2 \mid e \in E\big)\\ &\quad= \mathbb{P}r\big(d(u) = k_1 \mid e \in E\big)\,\mathbb{P}r\big(d(v) = k_2 \mid e \in E\big).\end{aligned} \tag{1}$$

A probabilistic reasoning yields:

$$\mathbb{P}r\big(d(u) = k \mid e \in E\big) = \binom{n-2}{k-1} p^{k-1} q^{n-k-1}. \tag{2}$$

Then, from (1), we obtain:

$$\begin{aligned}&\mathbb{P}r\big(d(u) = k_1 \wedge d(v) = k_2 \mid e \in E\big)\\ &\quad= \binom{n-2}{k_1-1}\binom{n-2}{k_2-1} p^{k_1+k_2-2} q^{2n-k_1-k_2-2}.\end{aligned} \tag{3}$$

Therefore

$$\mathbb{P}r\big(\mathcal{H}(e) \mid e \in E\big) = \left(\sum_{k=1}^{n-1} \frac{1}{k}\binom{n-2}{k-1} p^{k-1} q^{n-k-1}\right)^2. \tag{4}$$

A straightforward computation involving formal power series yields the following lemma.

[1] With high probability means with probability $1 - O(1/n)$.

Lemma 3. *Let $G_{n,p} = (V, E)$ be a random graph, and let u and v be two vertices. Then,*

$$\mathbb{P}r\big(\mathcal{H}(e)\big) = \frac{(1-q^{n-1})^2}{(n-1)^2 p}. \tag{5}$$

Then, we obtain:

Corollary 4. *Let $G_{n,p}$ be a random graph, and let v be a vertex. If we denote by $\mathcal{HS}(v)$ the event "v is involved in a handshake", then*

$$\mathbb{P}r\big(\mathcal{HS}(v)\big) = \frac{(1-q^{n-1})^2}{(n-1)p}.$$

Proof. Let $v \in V$ be a nonisolated vertex and let $u \in N(v)$ such that $c(v) = u$. There is a handshake between v and u iff $c(u) = v$. Then,

$$\begin{aligned}
&\mathbb{P}r\big(\mathcal{HS}(v) \mid v \text{ is not isolated}\big)\\
&= \sum_{k=1}^{n-1} \mathbb{P}r\big(c(u) = v \mid (u,v) \in E \wedge d(u) = k\big)\\
&\quad \times \mathbb{P}r\big(d(u) = k \mid (u,v) \in E\big)\\
&= \sum_{k=1}^{n-1} \frac{1}{k}\binom{n-2}{k-1} p^{k-1} q^{n-k-1} = \frac{1-q^{n-1}}{(n-1)p}.
\end{aligned}$$

Then, since $\mathbb{P}r(v \text{ in not isolated}) = 1 - q^{n-1}$, we obtain

$$\mathbb{P}r\big(\mathcal{HS}(v)\big) = \frac{(1-q^{n-1})^2}{(n-1)p}.$$

The corollary follows. □

2.2. The expected number of handshakes

Let $\{u, v\} \in V \times V$ such that $u \neq v$, and let $\chi_{u,v}$ denote the random variable defined as follows. $\chi_{u,v} = 1$ if there is a handshake between u and v and $\chi_{u,v} = 0$ otherwise. Then by the linearity of the expectation, if we denote by $X_{n,p}$ the number of handshakes in $G_{n,p}$ then

$$\begin{aligned}
\mathbb{E}(X_{n,p}) &= \frac{1}{2} \sum_{(u,v)\in V\times V \text{ s.t. } u\neq v} \mathbb{E}(\chi_{u,v})\\
&= \frac{1}{2} \sum_{(u,v)\in V\times V \text{ s.t. } u\neq v} \mathbb{P}r(\chi_{u,v} = 1),
\end{aligned}$$

where $\mathbb{E}$ denotes the mathematical expectation.

Hence, from the result of Lemma 3, we obtain:

Lemma 5.

$$\mathbb{E}(X_{n,p}) = \frac{n(n-1)}{2} \frac{(1-q^{n-1})^2}{(n-1)^2 p} = \frac{n}{2(n-1)p}(1-q^{n-1})^2.$$

Remark 6.

(1) If p is a constant, then $(1-q^{n-1})^2 \to 1$ as $n \to \infty$. Hence, $\mathbb{E}(X_{n,p}) \sim 1/2p$ as $n \to \infty$.

(2) If $np = \lambda + o(1) > 0$, that is the average degree of any vertex v is a constant, then $q^{n-1} \sim e^{-\lambda}$. Hence,

$$\exists \alpha > 0 \quad \text{such that } \mathbb{E}(X_{n,p}) \sim \alpha n.$$

(3) If $n^2 p = \lambda + o(1)$, then

$$\exists \beta > 0 \quad \text{such that } \mathbb{E}(X_{n,p}) \sim \beta.$$

(4) If $n^\gamma p = \lambda + o(1)$, with $\gamma > 2$ then

$$\mathbb{E}(X_{n,p}) \to 0, \quad \text{as } n \to \infty.$$

3. Probability of success

Let $G_{n,p} = (V, E)$ be a random graph, and let $v \in V$ be any vertex. If v is not an isolated vertex, then all its neighbors have the same chance $1/d(v)$ to be chosen, where $d(v)$ is the degree of the vertex v. In this section, we study the probability of success in $G_{n,p}$, i.e., the probability of at least a handshake in $G_{n,p}$.

We start by calculating the probability for $G_{n,p}$ to contain at least an isolated vertex (a vertex with degree 0).

Lemma 7. *Let $G_{n,p}$ a random graph. If we denote by $\mathcal{P}(G_{n,p})$ the probability for $G_{n,p}$ to contain at least an isolated vertex then*

$$\mathcal{P}(G_{n,p}) \leqslant nq^{n-1}.$$

Proof. Let v be any vertex in V. The probability for v to be an isolated vertex is given by $\mathbb{P}r(d(v) = 0) = q^{n-1}$. Then, if $V = \{v_1, v_2, \ldots, v_n\}$, then

$$\mathcal{P}(G_{n,p}) = \mathbb{P}r\left(\bigvee_{i=1}^{n} d(v_i) = 0\right).$$

On the other hand, for any $i \geqslant 1$, let $\{v_{j_1}, v_{j_2}, \ldots, v_{j_i}\}$ be i different vertices. We have:

$$\begin{aligned}
&\mathbb{P}r\big(d(v_{j_1}) = 0 \wedge d(v_{j_2}) = 0 \wedge \cdots \wedge d(v_{j_i}) = 0\big)\\
&= \mathbb{P}r\big(d(v_{j_1}) = 0\big)\, \mathbb{P}r\big(d(v_{j_2}) = 0 \mid d(v_{j_1}) = 0\big) \cdots\\
&\quad \times \mathbb{P}r\left(d(v_{j_i}) = 0 \,\middle|\, \bigwedge_{l=1}^{i-1} d(v_{j_l}) = 0\right)\\
&= q^{n-1} q^{n-2} \cdots q^{n-i}\\
&= q^{i(i-1)/2 + i(n-i)}.
\end{aligned}$$

Then using the Sieve principle [9], we obtain

$$\mathcal{P}(G_{n,p}) = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} q^{i(i-1)/2 + i(n-i)}.$$

Hence, if we truncate after the first order term, we obtain the result claimed in the lemma. □

Then, we can deduce the following corollary:

Corollary 8. *Let $\alpha \geqslant 2$. If $p = (\alpha \log n)/(n-1)$ then w.h.p. $G_{n,p}$ contains no isolated vertex.*

Proof. By the result of the previous lemma, we have $\mathcal{P}(G_{n,p}) \leqslant nq^{n-1} = n(1-p)^{n-1} \leqslant ne^{-(n-1)p}$. Then, if $p = (\alpha \log n)/(n-1)$ then

$$\mathcal{P}(G_{n,p}) \leqslant \frac{1}{n^{\alpha-1}}.$$

Meaning that the probability for $G_{n,p}$ to contain no isolated vertex is at least $1-o(1/n)$. The corollary follows. □

In the rest of this section, we suppose that $p = (\alpha \log n)/(n-1)$, that is, $G_{n,p}$, w.h.p., contains no isolated vertex. Let $(V_i)_{i=1,\dots,k}$ and $(E_i)_{i=1,\dots,k}$ such that

- $\forall i \neq j$, $V_i \cap V_j = \emptyset$ and $E_i \cap E_j = \emptyset$,
- $V = V_1 \cup V_2 \cup \cdots \cup V_k$, $E = E_1 \cup E_2 \cup \cdots \cup E_k$, and
- $\forall i$, $G_i = (V_i, E_i)$ is a connected graph.

It is easy to see that $s(G_{n,p}) = \max\{s(G_i) \mid i \in \{1,2,\dots,k\}\}$ and by the result in Theorem 2, $s(G_i) \geqslant s(K_{|V_i|})$, for any $i \in \{1,2,\dots,k\}$.

On the other hand, in [5], the author proved also that

$$s(K_n) \geqslant s(K_{n+1}), \quad \text{for any } n \geqslant 2.$$

Hence

$$s(K_n) \geqslant 1 - e^{-1/2}, \quad \text{for any } n \geqslant 2.$$

Then we have:

Lemma 9. *If $p = (\alpha \log n)/n$ for any $\alpha \geqslant 2$ then, w.h.p., the probability of success in $G_{n,p}$ is lower bounded by a constant*:

$$s(G_{n,p}) \geqslant 1 - e^{1/2}.$$

4. The handshake number distribution

In this section, we study the distribution of the number of handshakes in $G_{n,p}$. Let $X_{n,p}$ denote the r.v. which counts the number of this handshakes. We show, in particular that, asymptotically, $X_{n,p}$ has a Poisson distribution with parameter $\lambda = 1/2p$.

We give the fundamental theorem:

Theorem 10. *The probability of having exactly m handshakes in $G_{n,p}$ is given by*:

$$\mathbb{P}r(X_{n,p} = m) \sim \sum_{k=m}^{n/2} (-1)^{k+m} \binom{k}{m} \frac{n!}{k!2^k(n-2k)!} p^k \times \left(\frac{1-q^{n-1}}{(n-1)p}\right)^{2k}.$$

Proof. Let M denote a matching of size $l \geqslant 1$. Then if $M = \{e_1, e_2, \dots, e_l\}$, we have

$$\begin{aligned}\mathbb{P}r(M) &= \mathbb{P}r(e_1 \wedge e_2 \wedge \cdots \wedge e_l) \\ &= \mathbb{P}r(e_1)\,\mathbb{P}r(e_2 \mid e_3) \cdots \mathbb{P}r(e_l \mid e_1 \wedge e_2 \cdots \wedge e_{l-1}),\end{aligned}$$

where, for any $i \in \{1,2,\dots,l\}$, e_i denotes the event: there is a handshake over e_i.

On the other hand, since M is a matching, all edges in M are not adjacent. Hence, having a handshake over any edge in M do not affect the probability of having a handshake on the other edges.

Then, we have:

$$\mathbb{P}r(M) = \prod_{i=1}^{l} \mathbb{P}r(e_i).$$

From results of Section 2, we have, for any i,

$$\mathbb{P}r(e_i) = \left(\frac{1-q^{n-1}}{(n-1)p}\right)^2.$$

Then, we obtain

$$\mathbb{P}r(M) = \left(\frac{1-q^{n-1}}{(n-1)p}\right)^{2l}.$$

Now, let $\mathcal{M}_k$ denotes the set of all k-matchings, and define the sequence q_k as

$$q_k = \sum_{M \in \mathcal{M}_k} \mathbb{P}r(M).$$

Then

$$q_k = |\mathcal{M}_k| \left(\frac{1-q^{n-1}}{(n-1)p}\right)^{2k}.$$

On the other hand, for any $k \geqslant 1$, we can find a k-matching in $G_{n,p} = (V, E)$ iff $|E| \geqslant k$ and $\forall e_1, e_2 \in E$, e_1 and e_2 are not adjacent. A combinatorial reasoning yields the probability and the number of such k-matchings which is $\frac{n!}{k!2^k(n-2k)!}p^k$. Hence,

$$q_k = \frac{n!}{k!2^k(n-2k)!} p^k \left(\frac{1-q^{n-1}}{(n-1)p}\right)^{2k}.$$

A direct application of the Sieve principle [9] yields to

$$\mathbb{P}r(X_{n,p} = m) = \sum_{k=m}^{n/2} (-1)^{k+m} \binom{k}{m} \frac{n!}{k!2^k(n-2k)!} p^k \times \left(\frac{1-q^{n-1}}{(n-1)p}\right)^{2k}. \qquad \square$$

Corollary 11. *If p is a constant, then for every positive integer m, the probability for $X_{n,p}$ to be equal to m tends to $\frac{1}{m!}(1/2p)^m e^{-1/2p}$, as $n \to \infty$.*

5. Conclusion and perspectives

In this paper, we studied the handshake problem in random graphs. We proved that if $p = (\alpha \log n)/n$ then, with high probability, the probability of success is lower bounded by $1 - e^{-1/2}$. We also studied the asymptotic distribution of the number of handshakes and showed that, if p is a constant then this r.v. is distributed as a Poisson variable.

In [3,4], the authors studied the broadcast process in handshake communication based networks. It will be interesting to investigate the same problem in random graphs.

Acknowledgements

The author is grateful to Nasser Saheb for stimulating discussions and to the anonymous referees for constructive comments.

References

[1] C. Berge, Graphs and Hypergraphs, North-Holland, 1985.

[2] B. Derbel, Efficient distributed handshake using mobile agents, in: S. Chaudhuri, S.R. Das, H.S. Paul, S. Tirthapura (Eds.), ICDCN, in: Lecture Notes in Computer Science, vol. 4308, Springer, 2006, pp. 294–305.

[3] P. Duchon, N. Hanusse, N. Saheb, A. Zemmari, Broadcast in the rendezvous model, in: V. Diekert, M. Habib (Eds.), STACS, in: Lecture Notes in Computer Science, vol. 2996, Springer, 2004, pp. 559–570.

[4] P. Duchon, N. Hanusse, N. Saheb, A. Zemmari, Broadcast in the rendezvous model, Inform. and Comput. 204 (5) (2006) 697–712.

[5] M. Dietzfelbinger, The probability of a rendezvous is minimal in complete graphs, in: P. Bose, P. Morin (Eds.), ISAAC, in: Lecture Notes in Computer Science, vol. 2518, Springer, 2002, pp. 55–66.

[6] M. Dietzfelbinger, H. Tamaki, On the probability of rendezvous in graphs, Random Struct. Algorithms 26 (3) (2005) 266–288.

[7] Y. Métivier, N. Saheb, A. Zemmari, Randomized rendezvous, in: Mathematics and Computer Science: Algorithms, Trees, Combinatorics and Probabilities, in: Trends in Mathematics, Springer, 2000, pp. 183–194.

[8] Y. Métivier, N. Saheb, A. Zemmari, Analysis of a randomized rendezvous algorithm, Inform. and Comput. 184 (2003) 109–128.

[9] K.H. Rosen, Handbook of Discrete and Combinatorial Mathematics, CRC Press, 2000.