Computing in Distributed Systems
Chapter 3: The Colouring Problem

Akka Zemmari

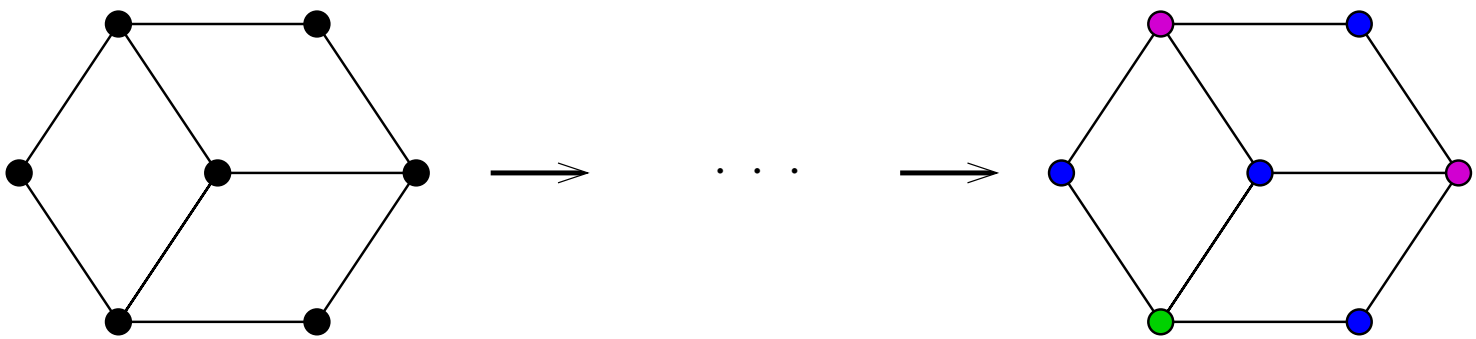
LaBRI - Université de Bordeaux

Content

The Colouring Problem

Definition.

Let $G = (V, E)$ be a simple connected undirected graph. A proper vertex colouring for G is an assignment of a colour $c(v)$ to each vertex v, such that any two adjacent vertices have a different colour, i.e., $c(v) \neq c(u)$ for every $\{u, v\} \in E$.

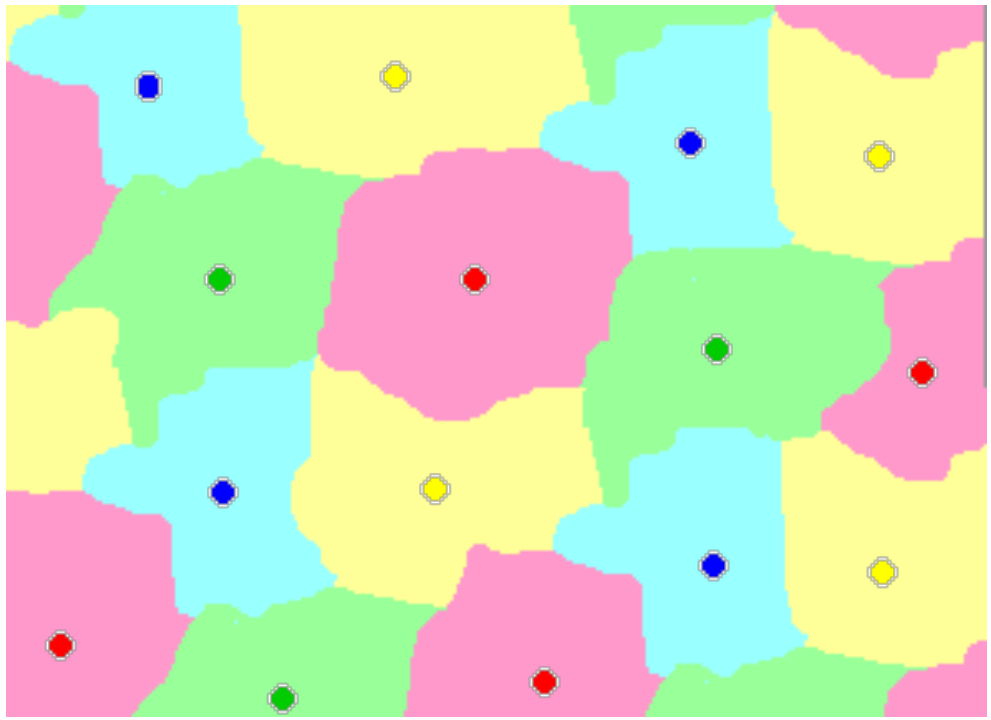


Remark

Distributed system $\rightarrow$ Nodes of the system have to colour themselves.

Why Color a Network

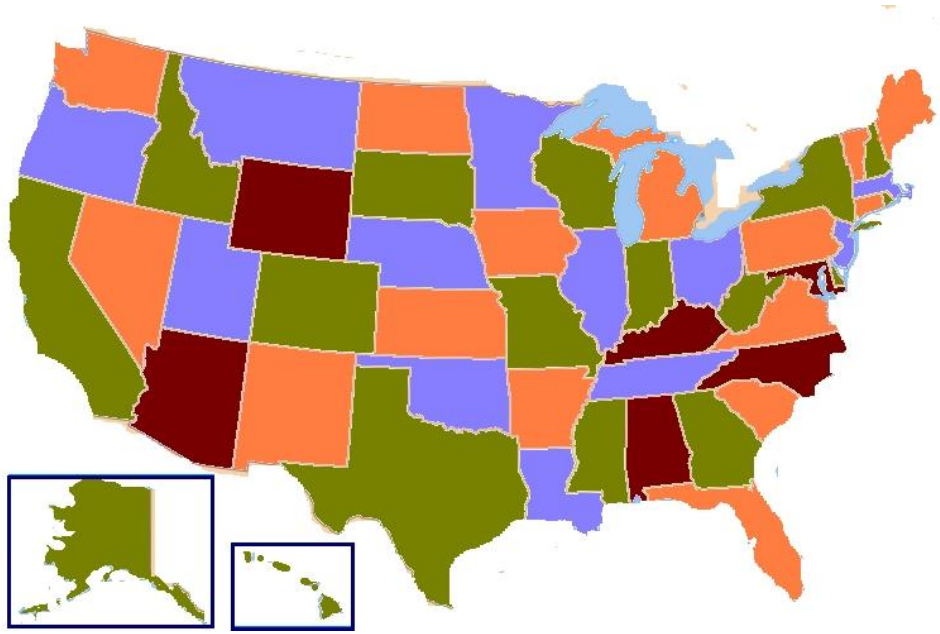
Medium access: reuse frequencies in wireless networks at certain spatial distance such that there is no interference.



Why Color a Network

4-Color Theorem

Can color each map using 4 colours only: no two adjacent countries have same color. ?

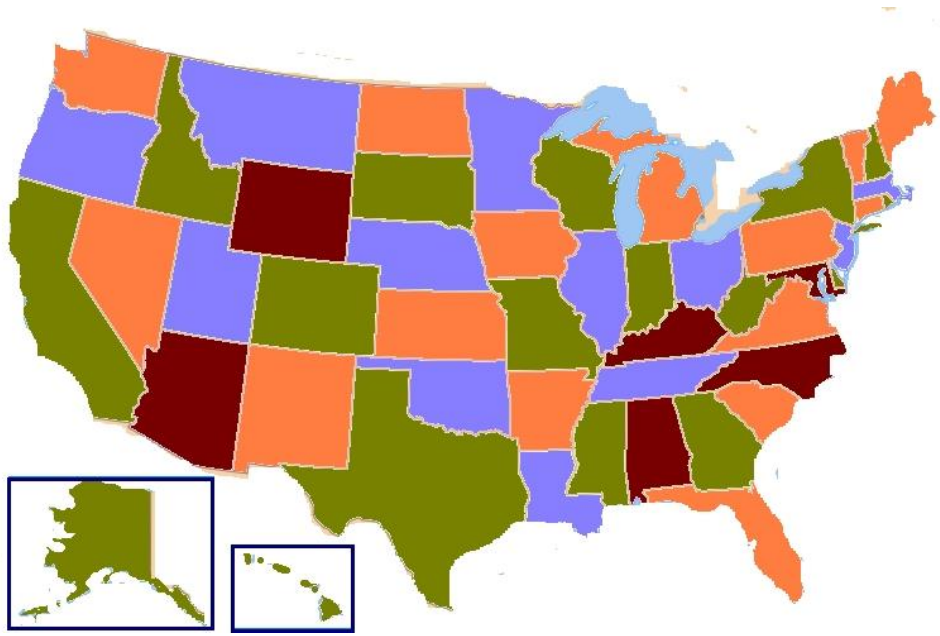


Why Color a Network

4-Color Theorem

Can color each map using 4 colours only: no two adjacent countries have same color.

$\Rightarrow$ First conjecture 1852, first proof with 5 colours 1890. First computer proof 1976 (Appel+Haken). Another proof using Coq assistant 2005.



A Simple (Sequential) Colouring Algorithm

Algorithme 1.

1. While $\exists$ a non-coloured vertex v do
2. color v using the smallest available colours (the smallest colour which is not used by any of the neighbours)
3. End While

Lemma.

The algorithm is correct, terminates in at most n rounds and uses at most $\Delta + 1$ colours (Δ is the maximum degree of the corresponding graph).

Solving the colouring problem in a distributed system ?

- Which assumptions we make ? $\rightarrow$ can we do it deterministically ? shall we use some randomisation ? ...
- Design a (randomised) distributed algorithm (an algorithm that nodes of the system will execute)
- Analyse the designed algorithm:
  - Time complexity: number of rounds ?
  - Message complexity ?
  - Size of the messages ?
  - Number of colours ?
  - ...

A First Distributed Algorithm (Wattenhofer)

We consider the following procedure:

FirstFree

Require: Nodes have different IDs

Colour v using the smallest available colour.

A First Distributed Algorithm (Wattenhofer)

Now we consider the following algorithm:

Reduce

Require: Nodes have different IDs

1. v sends its ID to its neighbours;
2. v receives the IDs of its neighbours
3. While v has an uncoloured neighbour with higher ID

3.1. v sends "undecided" to its neighbours;

4. v executes FirstFree;
5. v sends its decision to its neighbours.

A First Distributed Algorithm (Wattenhofer)

Analysis of the algorithm:

Theorem

The algorithm is correct and terminates in n steps. It uses $\Delta + 1$ colours.

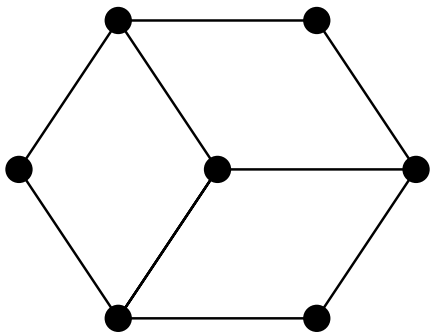
Observations

- The network is not anonymous (we use the node's IDs)
- Uses messages of size $O(\log n)$ (assuming nodes have IDs ranging from 1 to a polynomial on n)
- Its time complexity is $O(n)$.
+ Distributed algorithm.
+ Uses $\Delta + 1$ colours.

Colouring Anonymous Networks

- Assumption: The network is anonymous and nodes can send messages of size $O(\log n)$ bits
- **Theorem.** [Pel00] *There is no deterministic distributed algorithm for anonymous graphs for solving the colouring problem assuming all vertices wake up simultaneously.*
  $\Rightarrow$ use randomisation.

Johansson's Algorithm



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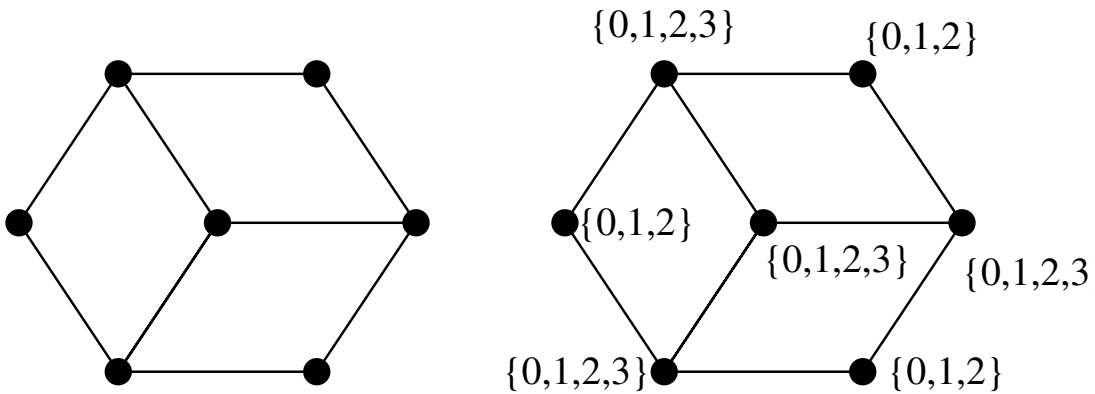

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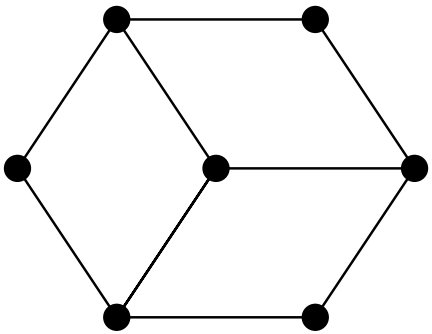

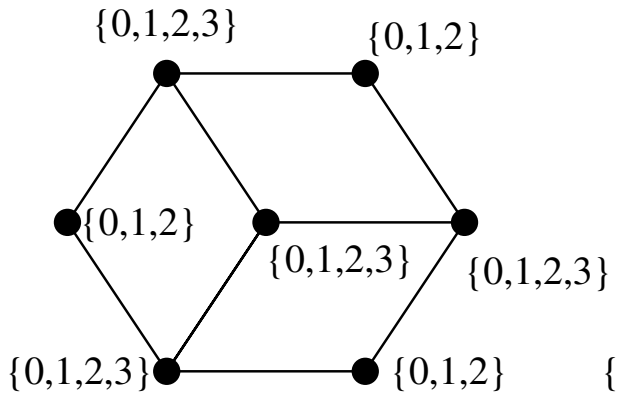

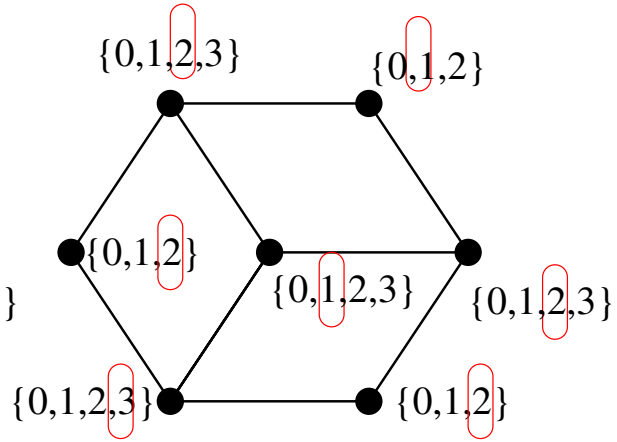

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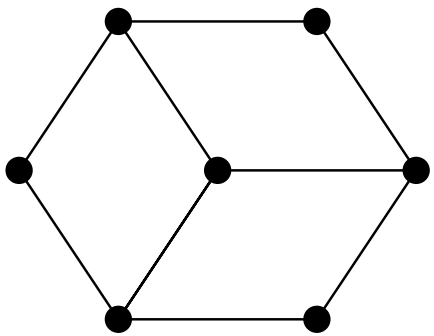

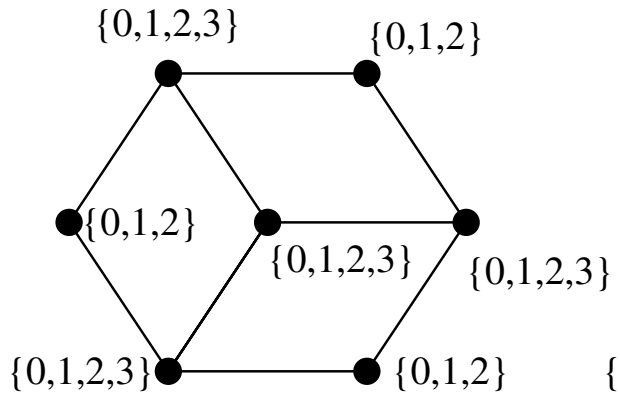

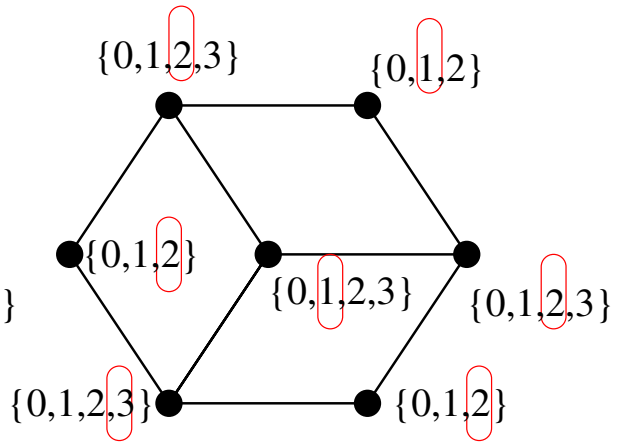

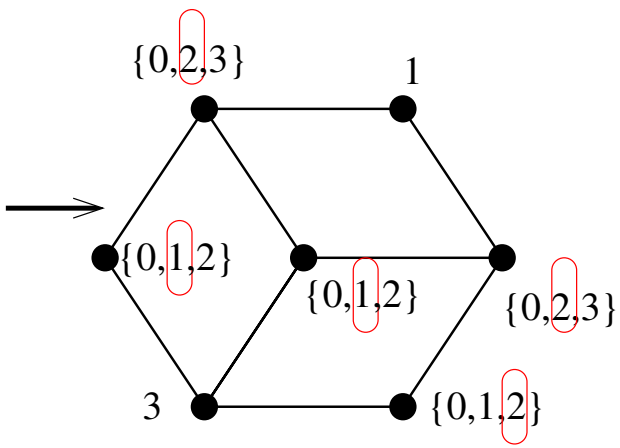

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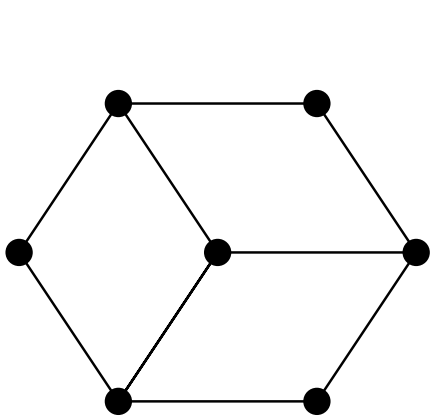

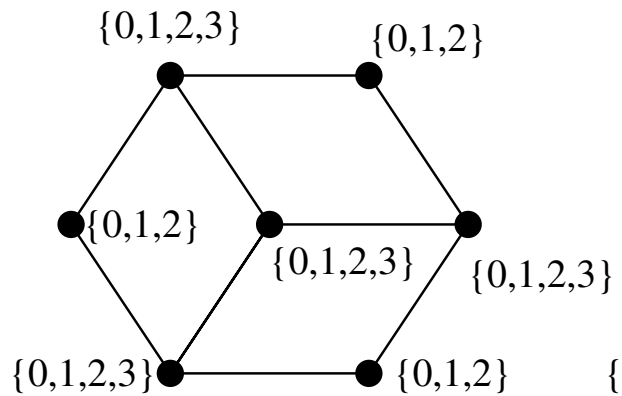

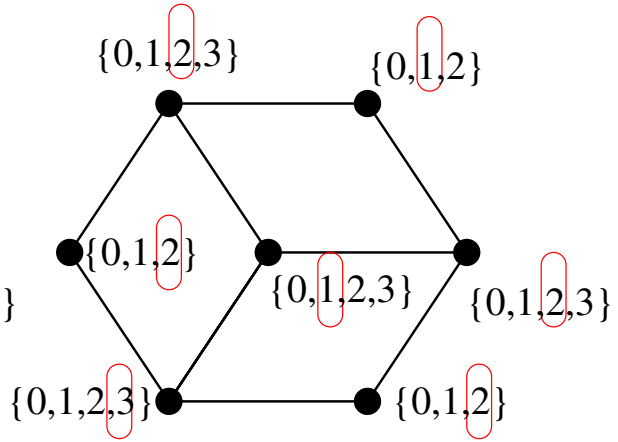

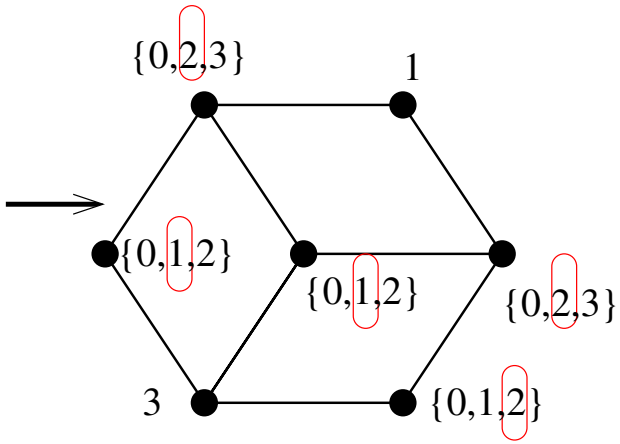

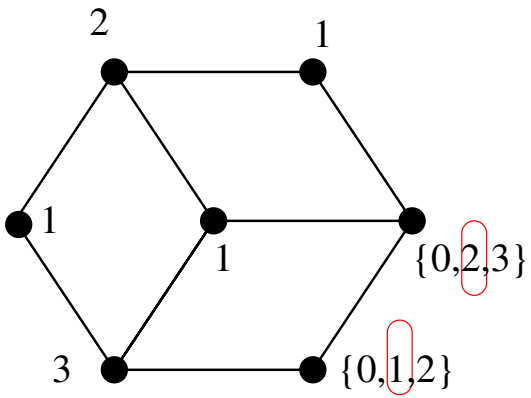

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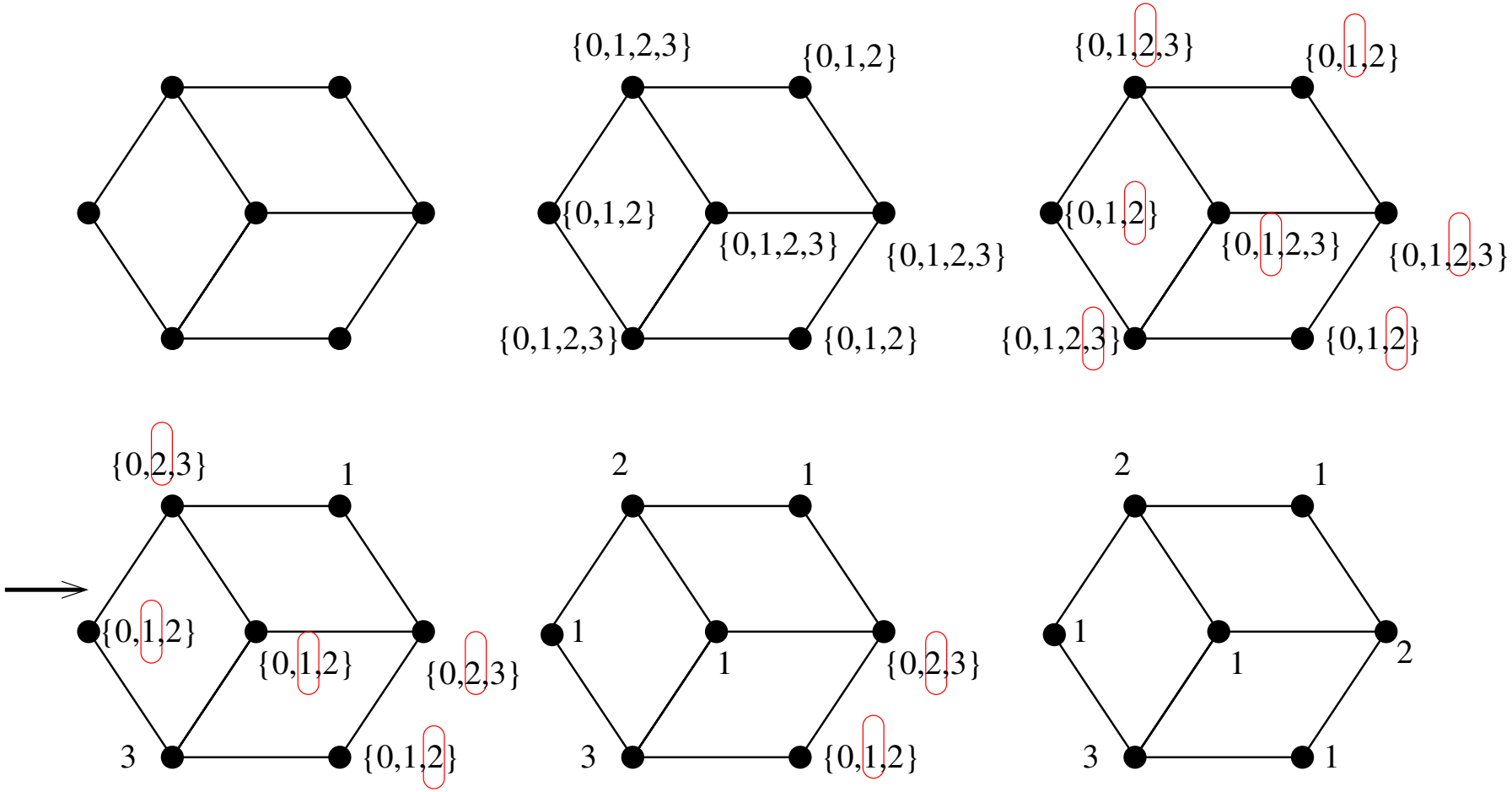

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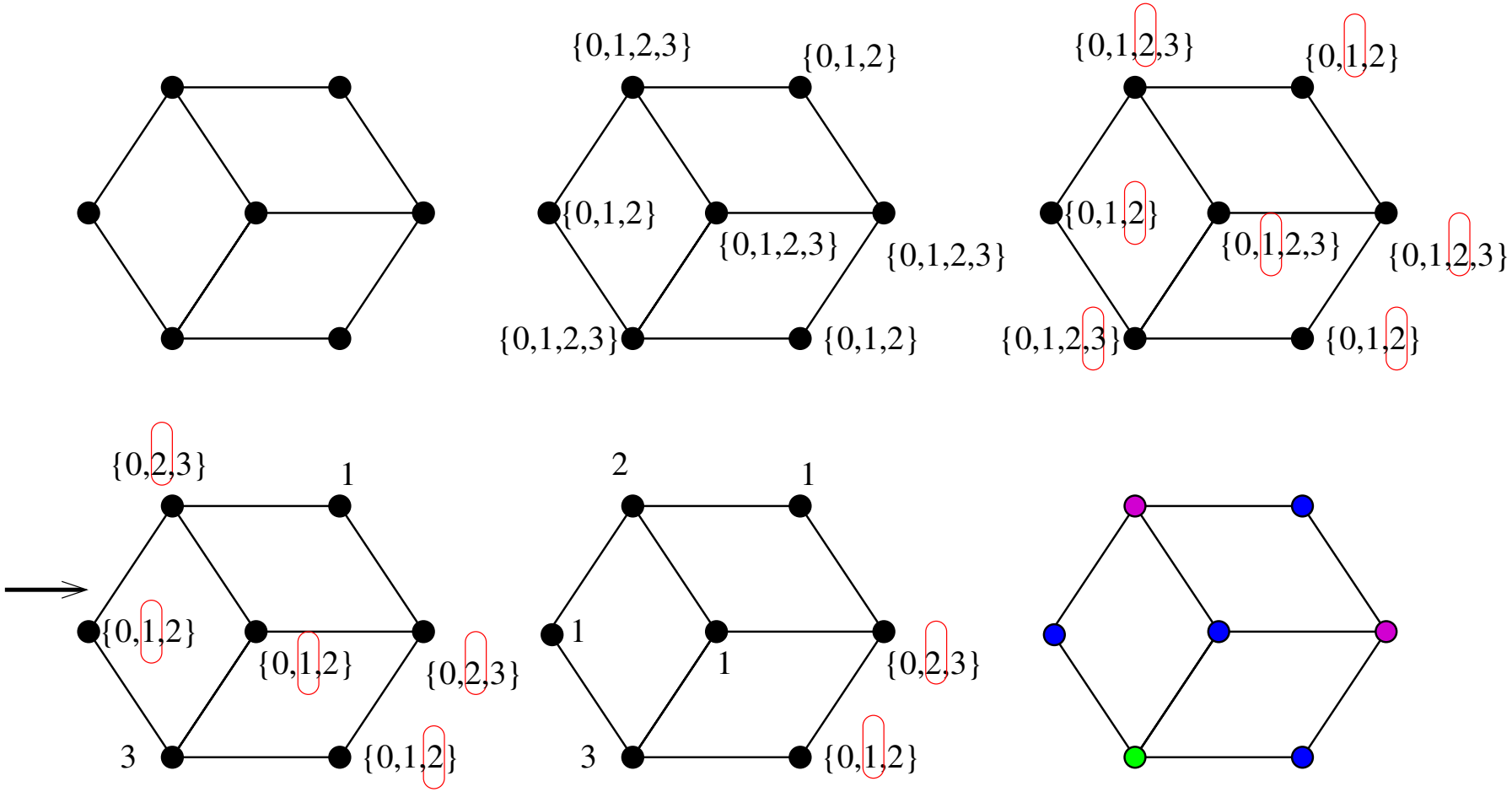

Johansson's Algorithm

Theorem.
Johansson's Algorithm runs in $O(\log n)$ rounds on average and w.h.p.

Remark.
Messages are of size $O(\log n)$.

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Messages are of size $O(\log n)$.

Question: Can we do the same using messages of size 1-bit ?

YES !

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YES !

Algorithm *Fast_Colour*

var:

$colour_v$: *word* **Init** *empty-word*;

$actives_v$: $\subseteq N(v)$ **Init** $N(v)$;

b_v: $\in \{0,1\}\}$;

While $actives_v \neq \emptyset$ **do**

$b_v \leftarrow flip(0,1)$;

$colour_v \leftarrow b_v \oplus colour_v$;

For all $u \in actifs_v$ **do**

send b_v to u;

receive b_u from u;

If $b_v \neq b_u$**then**

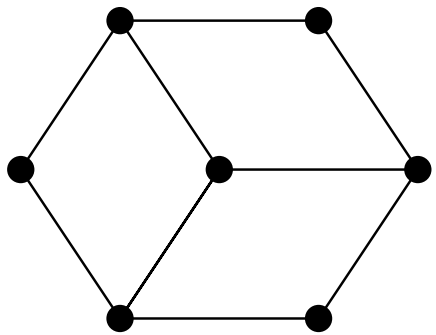
$active_v \leftarrow active_v \setminus \{u\}$;

End If

End For

End While

Algorithm *Fast_Colour*



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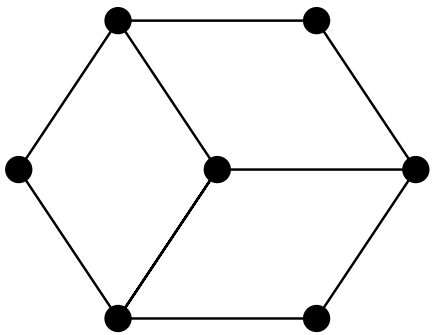

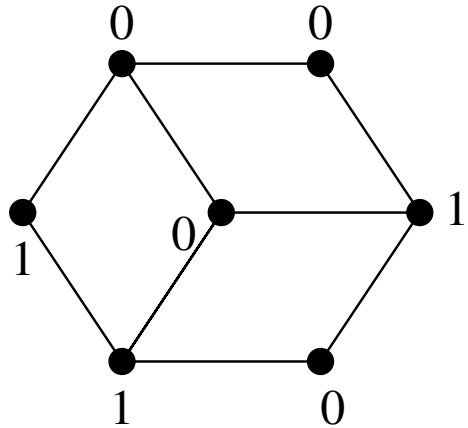

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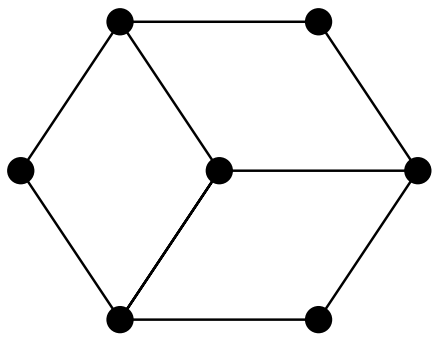

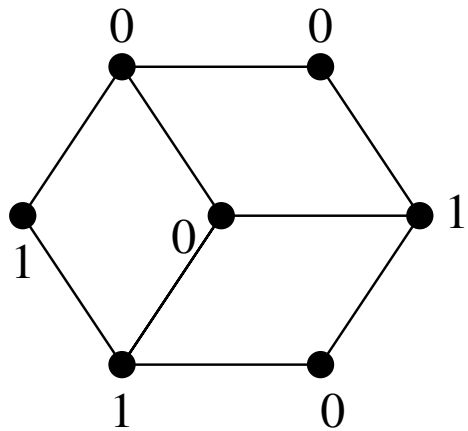

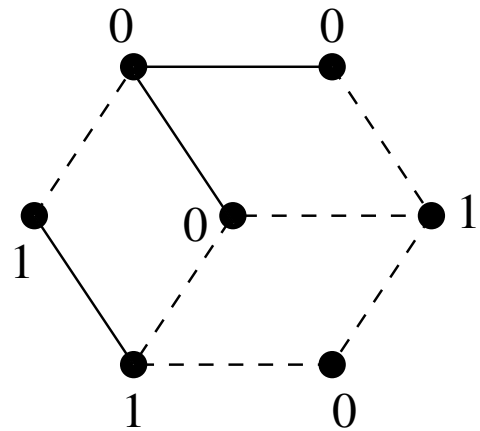

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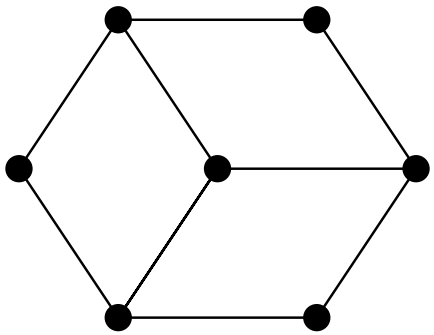

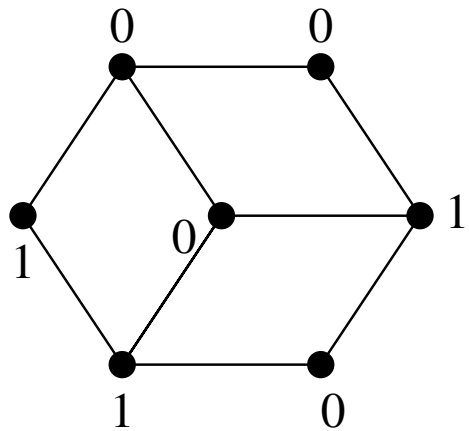

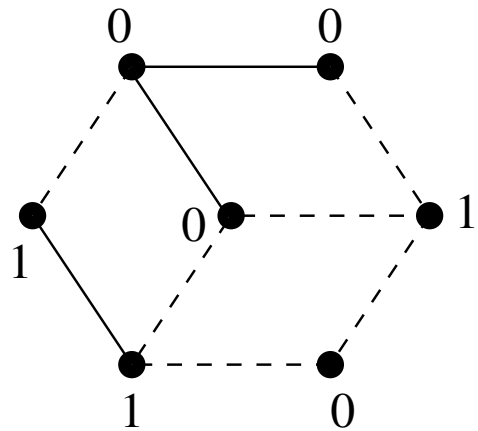

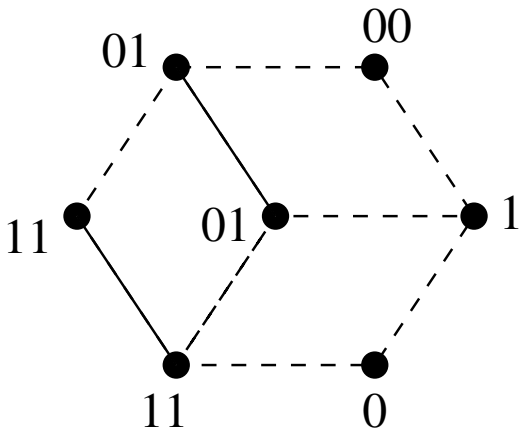

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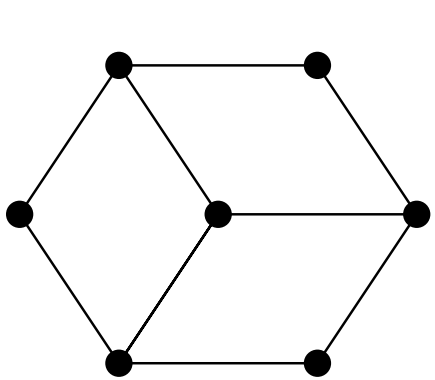

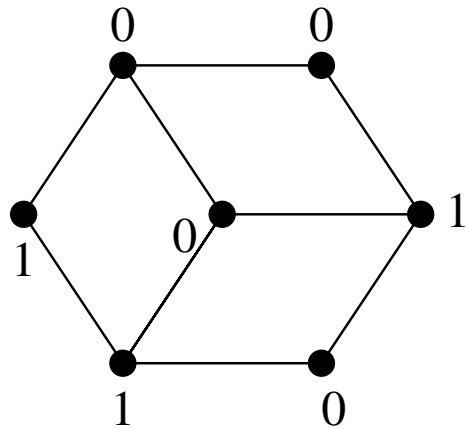

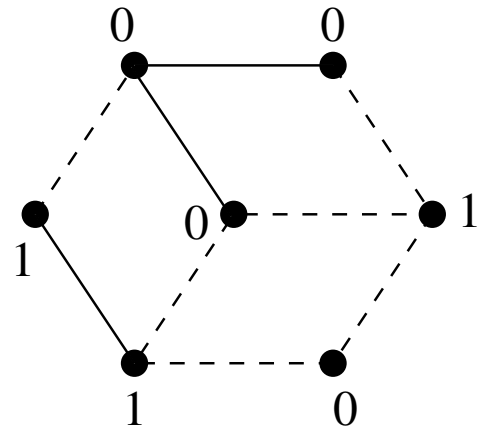

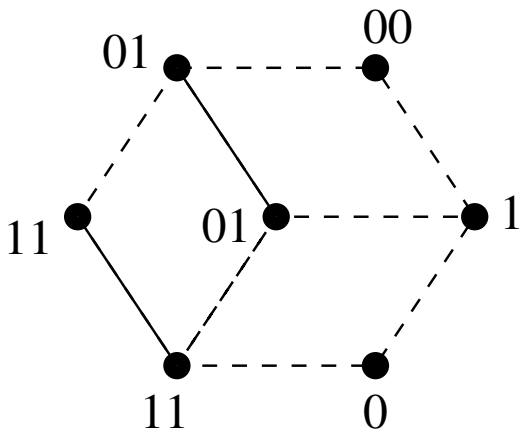

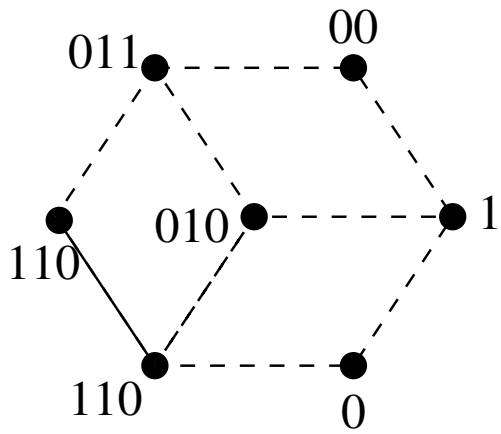

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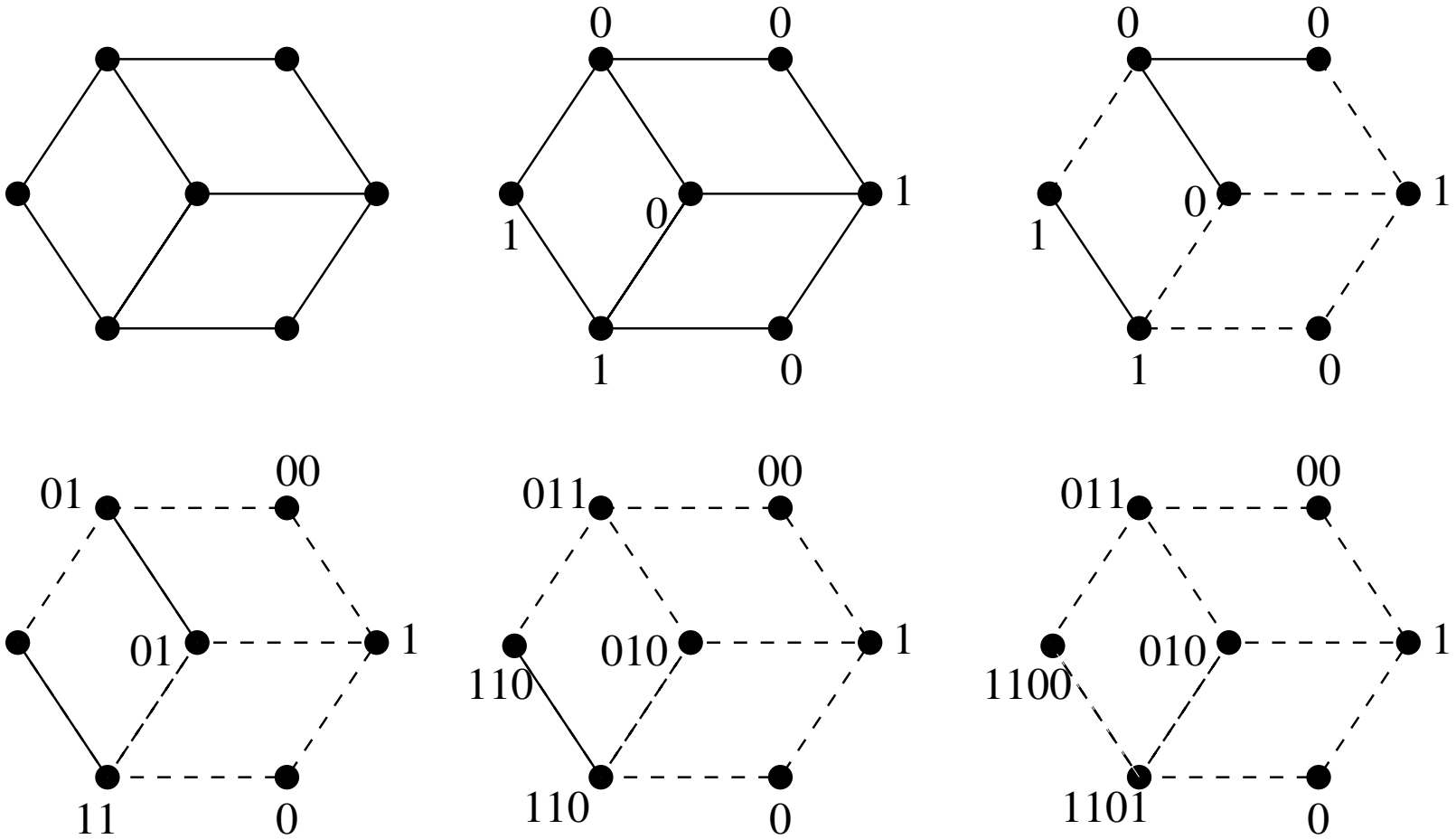

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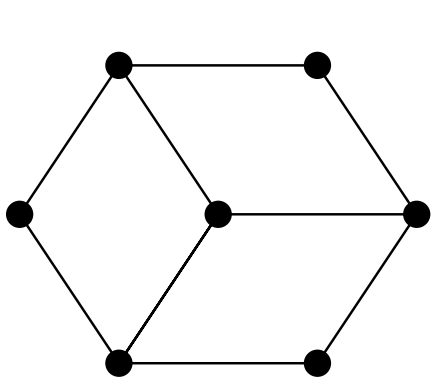

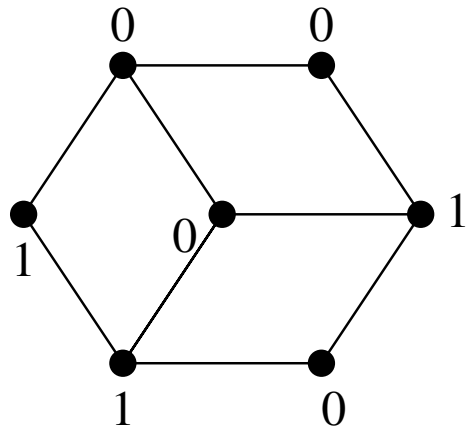

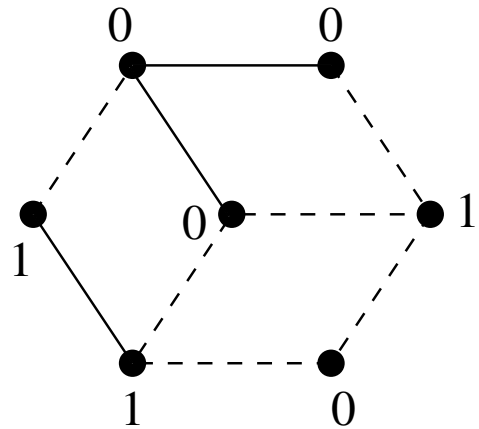

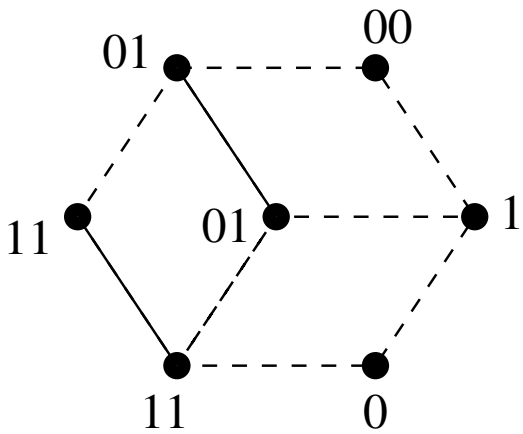

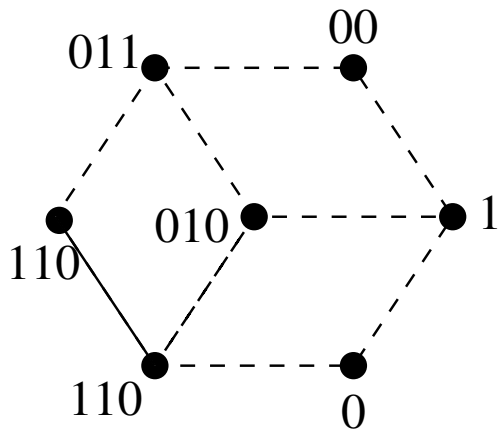

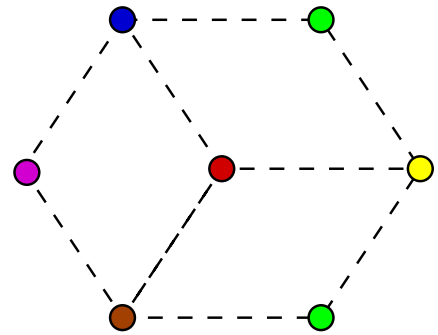

Analysis of the Algorithm

Lemma.
In any phase of the algorithm, the expected number of edges removed from the residual graph G is half the number of its edges.

Corollary.
There are constants k_1 and K_1 such that for any graph G of n vertices, the number of phases to remove all edges from G is:

- less than $k_1 \log n$ on average,
- less than $K_1 \log n$ w.h.p.

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Theorem.
Algorithm *Fast_Colour* computes a colouring for any arbitrary graph of size n in time $O(\log n)$ w.h.p., each message containing 1 bit.

- What about the number of colours ?
- Can we do better ?
  - YES !

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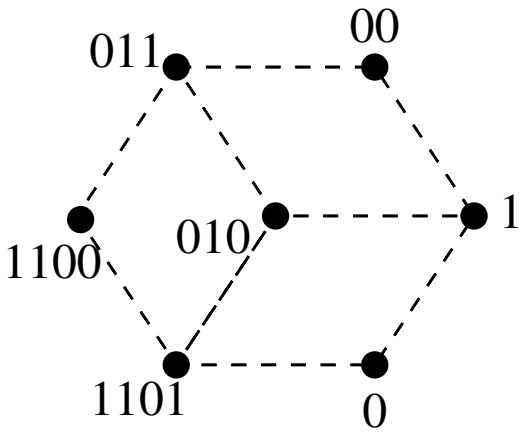
Reducing the Number of Colours - Algorithm $\mathcal{R}$

var:
  FC_u : integer **Init** $couleur_v$; (* final colour of v *)
  IN_v : $\subset N(v)$ **Init** $\emptyset$;
  OUT_v : $\subset N(u)$ **Init** $\emptyset$;
  $Colours_v$: set of possible colours **Init** $\{0, 1, \cdots, d(v)\}$;
For each $v \in N(v)$
  If $(x(u) < x(v))$ **then** $OUT_v = OUT_v \cup \{v\}$;
  Else $IN_v = IN_v \cup \{v\}$;
  End If;
End For
While $IN_v \neq \emptyset$ **do**
  receive $< c, w >$ from a neighbour w;
  $IN_v = IN_v \setminus \{w\}$;
  $Colours_v = Colours_v \setminus \{c\}$;
End While
$FC_v = \min\{Colours_v\}$;
For Each $u \in OUT_v$
  send $< FC_v, u >$ to v;
End For;

Reducing the Number of Colours - Algorithm $\mathcal{R}$



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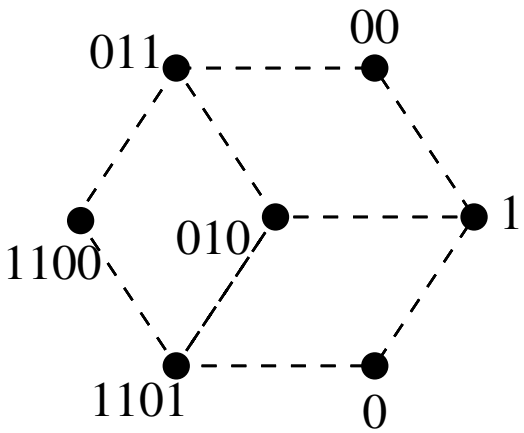
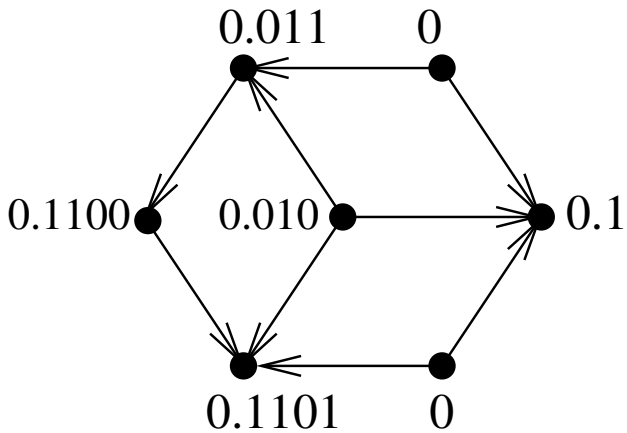
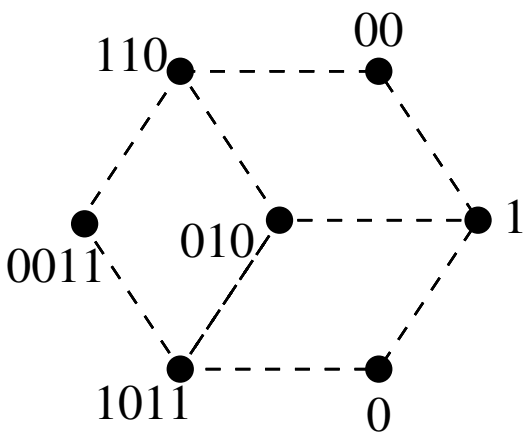
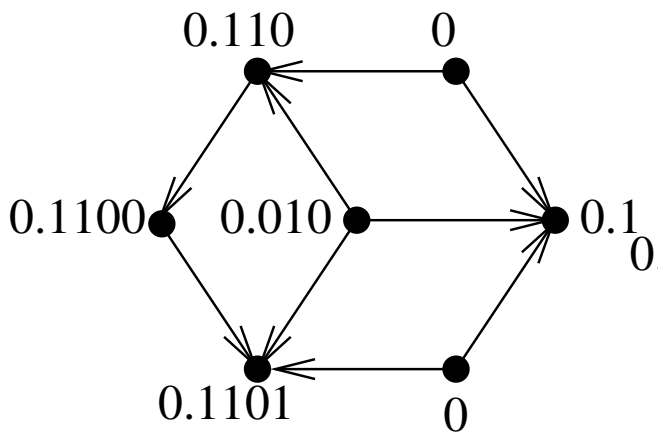
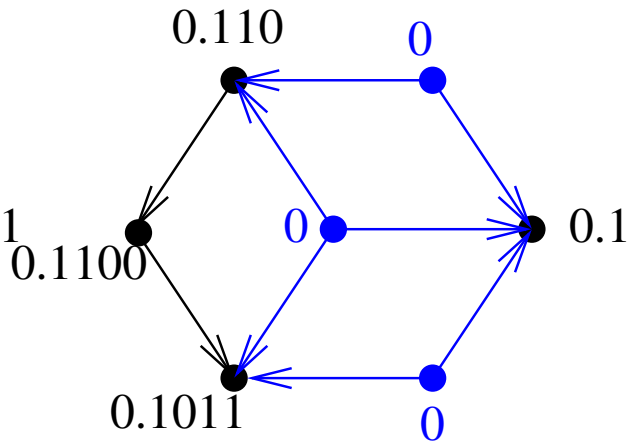




Reducing the Number of Colours - Algorithm $\mathcal{R}$







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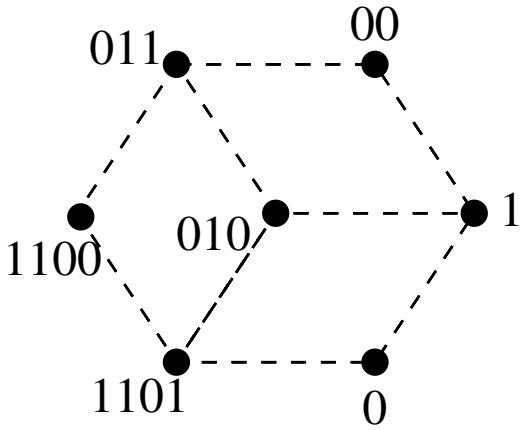
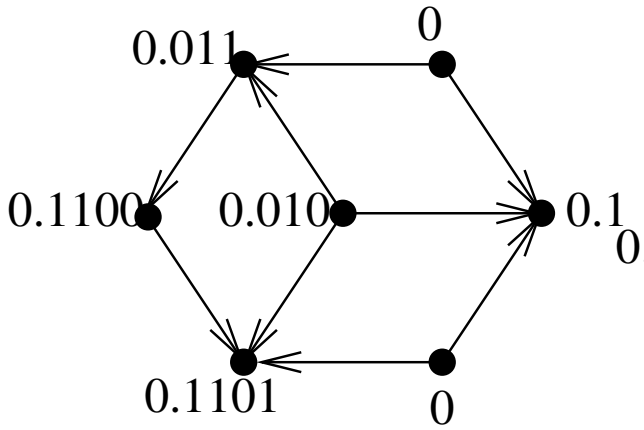
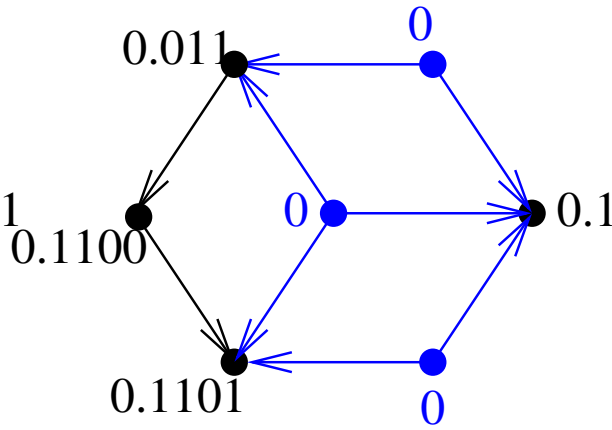
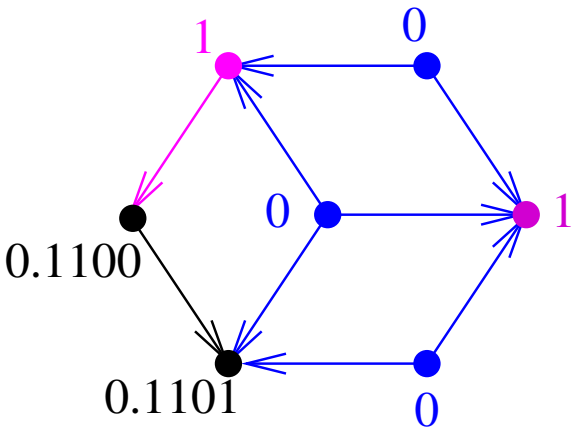
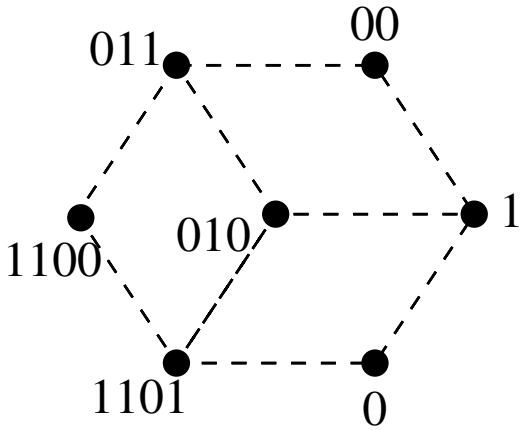


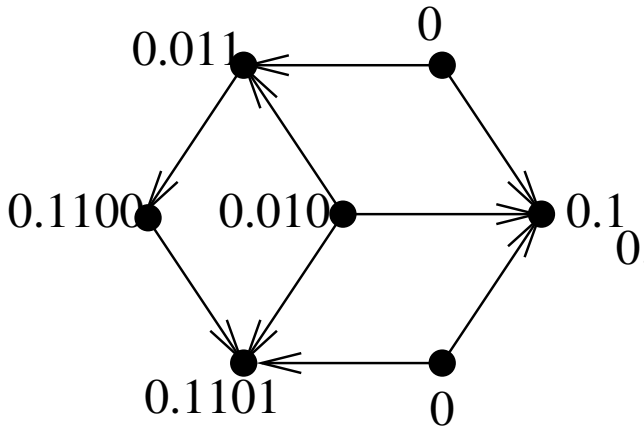


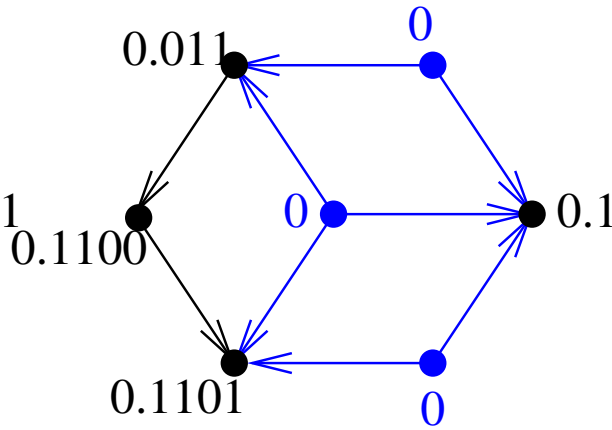


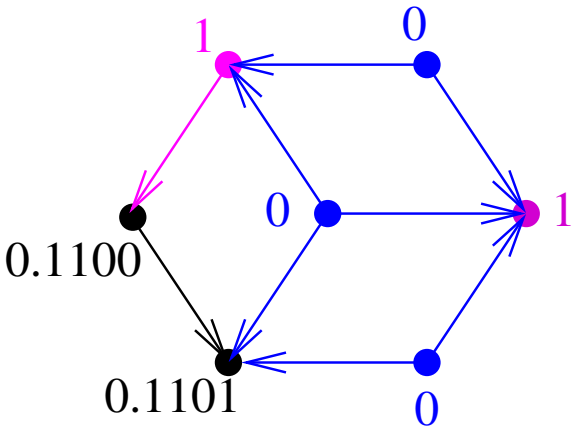


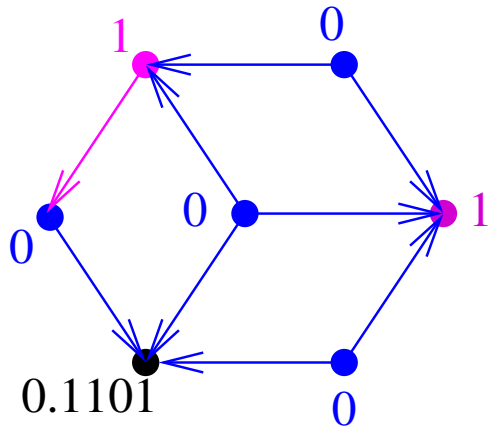
Reducing the Number of Colours - Algorithm $\mathcal{R}$











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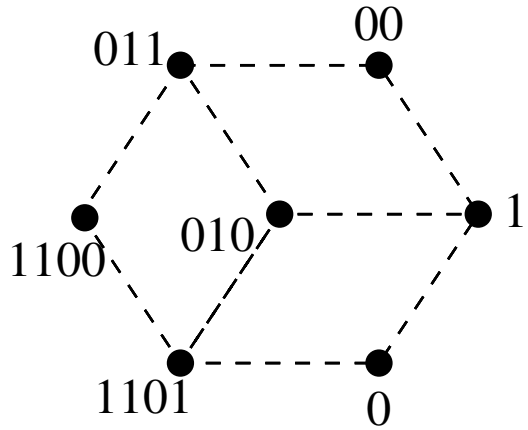

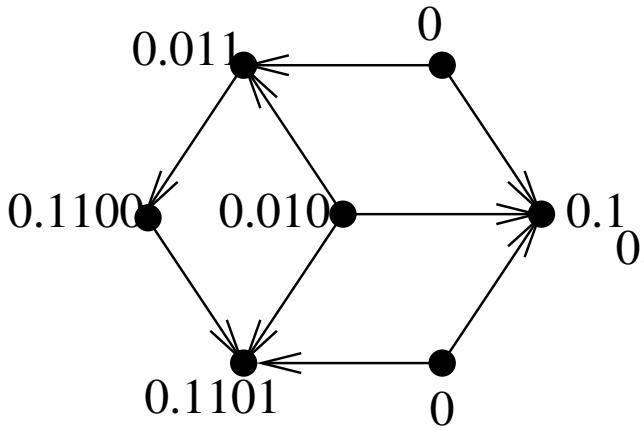

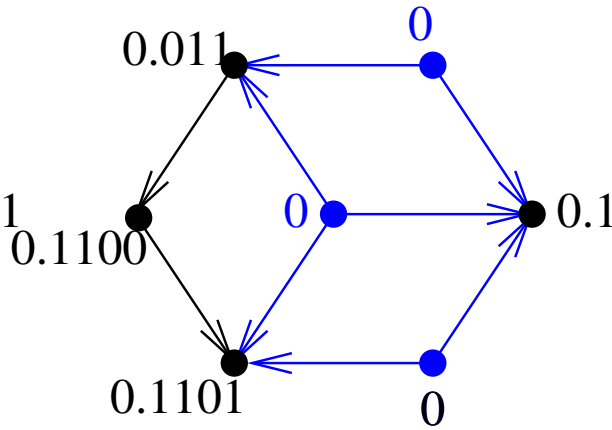

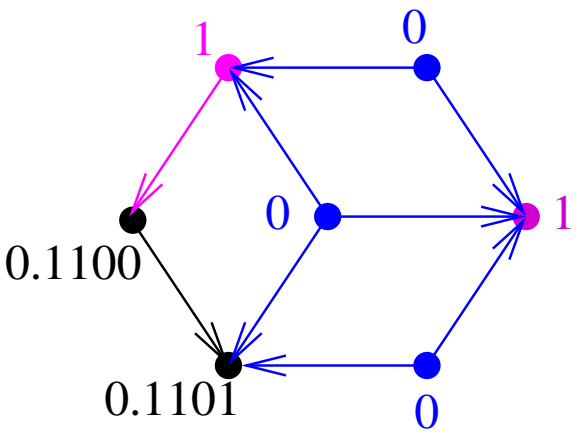

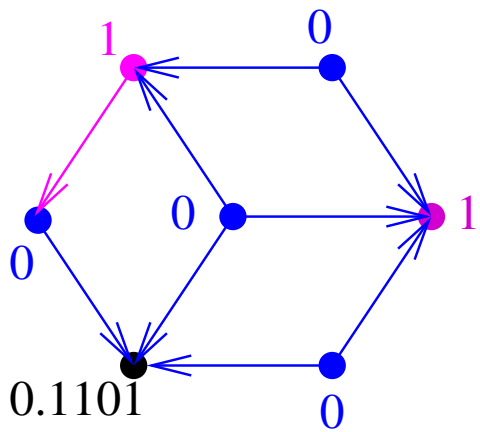

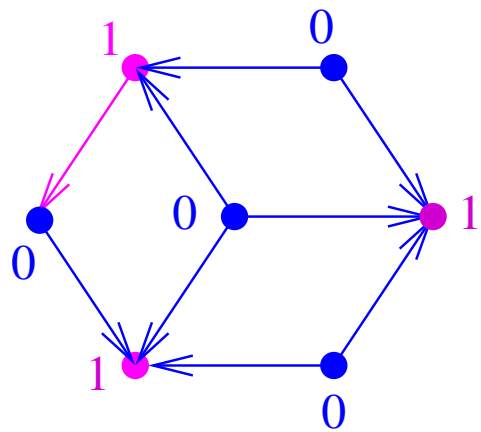

Analysis of the Algorithm

Theorem

For any graph $G = (V, E)$ having a maximum degree Δ, Algorithm $\mathcal{R}$ achieves a $(\Delta + 1)-$colouring of G in at most $e\Delta + 3 \log n$ rounds w.h.p.

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